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THEORETICAL MECHANICS.

◆

VOLUME I.

A MANUAL

OF THE

MECHANICS OF ENGINEERING

AND OF

THE CONSTRUCTION OF MACHINES,

WITH AN

INTRODUCTION TO THE CALCULUS.

DESIGNED AS A TEXT-BOOK FOR TECHNICAL SCHOOLS AND COLLEGES,
AND FOR THE USE OF ENGINEERS, ARCHITECTS, ETC.

BY

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OF THE IMPERIAL ACADEMY OF SCIENCES AT ST. PETERSBURG, ETC.

IN THREE VOLUMES.

VOLUME FIRST—THEORETICAL MECHANICS.

WITH 902 WOODCUTS IN THE TEXT.

TRANSLATED FROM THE FOURTH AUGMENTED AND IMPROVED GERMAN EDITION BY
ECKLEY B. COXE, A.M.,
MINING ENGINEER.

NEW YORK:
D. VAN NOSTRAND, PUBLISHER,
23 MURRAY ST. AND 27 WARREN ST.
1870.

Entered, according to Act of Congress, in the year 1869, by
D. VAN NOSTRAND,
In the Clerk's Office of the District Court of the United States for the Southern
District of New York.

Electrotyped by Smith & McDougal, 82 & 84 Beekman Street, N.Y.
Printed by C. S. Westcott & Co., 79 John Street.

5648 6730386

PREFACE TO THE FIRST EDITION.

IT is not without apprehension that I give to the public my elementary treatise upon the Mechanics of Engineering and of the Construction of Machines. Although I can say to myself that, in preparing this manual, I have gone to work with all possible care and attention, yet I fear that I have not been able to satisfy the wishes of every one. The ideas, wishes and requirements of the public are so various, that it is not possible to do so. Some may find the treatment of a particular subject too detailed, others perhaps too short; some will desire a more scientific discussion of certain subjects, while others would prefer one more popular. Many years of study, much experience in teaching and very varied observations and experiments have led me to adopt, as most suitable to the object in view, the method, according to which this work has been arranged. My principal effort has been to obtain the greatest simplicity in enunciation and demonstration, and to treat all the important laws, in their practical applications, without the aid of the higher mathematics. If we consider how many subjects a technical man must master in order to accomplish any thing very important in his profession, we must make it our business as teachers and authors for technical men to facilitate the thorough study of science by simplicity of diction, by removing whatever may be unnecessary, and by employing the best known and most practicable methods. For this reason I have entirely avoided the use of the Calculus in this work. Although at the present time the opportunities for acquiring a knowledge of it are no longer rare, yet it is an undeniable fact that, unless we are constantly making use of it, we soon lose that facility of calculation, which is indispensable; for this reason so many able engineers can no longer employ the Cal-

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culus which they learned in their youth. As I do not agree with those authors, who in popular treatises enunciate without proof the more difficult laws, I have preferred to deduce or demonstrate them in an elementary, although sometimes in a somewhat roundabout, manner.

Formulas without proof will therefore seldom be found in this work. We will assume that the reader has a general knowledge of certain principles of natural philosophy and a thorough knowledge of the elements of pure mathematics. My attention has been especially directed to preserving the proper mean between generalization and specialization. Although I appreciate the advantages of generalization, yet it is my opinion that in this work, as in all elementary treatises, too much generalization is to be avoided. The simple is oftener met with in practice than the complex. It is also undeniable that in considering the general case we often fail to attain a more profound knowledge of the special one, and that it is often easier to deduce the complex from the simple than the simple from the complex. The reader must not expect to find in this work a treatise upon the construction of machines, but only an introduction to or preparation for it. Mechanics should bear the same relation to the construction of machines that Descriptive Geometry does to Mechanical Drawing. When the pupil has acquired sufficient knowledge of Mechanics and of Descriptive Geometry, it appears better to combine the course of Construction of Machines with that of Mechanical Drawing.

It may be doubted whether it was advisable to divide my subject into two parts, theoretical and applied. If we remember that this work is intended to give instruction upon all the mechanical relations of the construction and of the theory of machines, the advantage, or rather, the necessity, of such a division becomes evident. In order to judge of a structure or of a machine, we must have a knowledge of mechanical principles of a very varied character, E.G., those of friction, strength, inertia, impact, efflux, &c.; the material for the mechanical study of a structure or of a machine must, therefore, be gathered from almost all the divisions of mechanics. Now, since it is better to study all the mechanical principles of a machine at once than to collect them from all the different parts of mechanics, the advantage of such a division is apparent.

Having practical application always in view, I have endeavored, in preparing my work, to illustrate the principles laid down in it by examples taken from every-day life. I am justified in asserting that this work contrasts favorably with any other of the same character in the number of appropriate examples, which are solved in it. I also hope that the great number of carefully-prepared figures will contribute to the object in view. My thanks are due to the publishers for having given the book in all respects the best appearance. Particular care has been taken to have the calculations correct; generally every example has been calculated three times, and not by the same person. It is, therefore, improbable that any gross errors will be found in them. In the examples, as in the formulas, I have employed the Prussian weights and measures, as they are probably familiar to the majority of my readers. The printing (in this case so difficult) is open to little complaint. The mistakes in copying, or of impression, which have been observed, are noted at the end of the book.

I do not think that many additions to this list need be made. An attentive examination of the illustrations will show that they have been prepared with care. The larger illustrations, particularly those representing bodies in three dimensions, are drawn according to the method of *Axonometric Projection*, first treated by me (see Polytechn. Mittheilungen Band I. Tubingen, 1845). This method of drawing possesses all the advantages of Isometric Projection, while in addition the pictures, which it furnishes, are not only more beautiful in themselves, but more easily awaken in us distinct conceptions of the objects represented. The drawings in this work are made in such a way that the dimensions of the width or depth appear but one-half as large as those of the height and length of the same size. I cannot omit thanking Mr. Ernest Röting, student at the academy in Freiberg, whose revision has essentially contributed to the accuracy of the work

It is necessary to inform the reader that he will find much new matter, which is peculiar to the author. Without stopping to mention many small articles, which occur in almost every chapter, I would call attention to the following comprehensive discussions: A general and easy determination of the centre of gravity of plane surfaces and of polyhedra, limited by plane surfaces, will be found in paragraphs 107, 112, and 113; an approximate formula for the catenary in paragraph 148; additional remarks upon the friction of axles in paragraphs 167, 168, 169, 172, and 173. Important additions to the theory of impact have been made, particularly in paragraphs 277 and 278; for heretofore the impact of imperfectly elastic bodies has been too little considered, and that of a perfectly elastic with an imperfectly elastic body has not been treated at all. Very important additions, and in some cases entirely new laws, will be found in the chapter upon hydraulics, a subject to which I have for a number of years devoted special study. The laws of incomplete contraction, first observed by the author, will be found for the first time in a manual of mechanics. The author has also incorporated in it the principal results, so important in practice, of his experiments upon the efflux of water through oblique short pipes, elbows, curved and long pipes, etc., although the third number of his "Untersuchungen im Gebiete der Mechanik und Hydraulik" has not yet appeared. The chapter upon running water, upon hydrometry and upon the impact of water contains some original matter. The theories of the reaction of water discharging from a vessel and of the impact of water, which are treated according to the principle of mechanical effect, are original.

I cannot, however, conceal from the reader that, since the volume has been finished, I have wished that some few subjects had been treated differently; but I must add that as yet I have observed no great imperfections. If at times the reader should miss something, he is referred to the second volume, which will supply both the accidental and the intentional omissions, as has been noted in many places in the first volume.

The printing of the second volume will now go on without interruption, so that we may expect the complete work to be in the hands of the reader before the end of the year. The pocket-book, the "Ingenieur," cited in the Mechanics, which contains a collection of formulas, rules and tables of arithmetic, geometry and mechanics, will soon appear.

It will be a source of great pleasure and satisfaction to me, if I have accomplished the purposes for which this work has been undertaken, namely, to give to the practical man a useful counsellor in questions of application, to the teacher of practical mechanics a serviceable text-book for instruction, and to the student of engineering a welcome aid in the study of mechanics.

JULIUS WEISBACH.

FREIBERG, *March 19th*, 1846.

PREFACE TO THE SECOND EDITION.

THE present (second) edition of the Mechanics of Engineering and of the Construction of Machines has undergone no essential alterations either in method or arrangement. The internal construction of the work has been changed in many places, and its size has been considerably increased. The author has also endeavored, as much as possible, to correct the errors and omissions of the first edition. The great increase in size is mainly due to three additions. The first consists of a condensed Introduction to the Calculus, which has been made as popular as possible, and has been prefixed to the main work. The object of introducing it was to avoid too complicated and too artificial demonstrations by means of the lower mathematics, and also to render the reader more independent in his study of mechanics, and to place him upon a higher stand-point in this important branch of science. By making use of the principles explained in the Introduction, it was possible to discuss many subjects of great practical importance, which previously we could not treat at all, or, at least, only imperfectly with the aid of elementary algebra and geometry. In order to avoid interruptions to those who have not made themselves familiar with the Elements of the Calculus, prefixed to the work, all the paragraphs, in which it is applied, are designated by a parenthesis ( ).

The second addition consists of a new chapter on Hydrostatics, in which the molecular action of water is treated. Since a knowledge of the molecular forces (capillarity) is of importance in experiments and observations in hydraulics and pneumatics, the author has thought it advisable to treat the fundamental principles of these forces in a separate chapter. Finally, a chapter has been added to the work in the form of an appendix, which treats of oscillation and wave motion. The author found himself compelled to do this in consequence of the importance to the engineer of a more accurate knowledge of the theory of oscillation. The great influence of vibration upon the working and durability of machines is a subject to which too much attention cannot be given. It is also to observations of oscillations that we owe the latest determination of the modulus of elasticity, which is of such importance in practice. I have mentioned in the Appendix the magnetic force, principally because it is of great use to the engineer in determining directions in mines, where the access to daylight is not easy. The theory of water-waves, which closes the volume, is a part of hydraulics; its presence in this work requires, therefore, no explanation. Unfortunately, it is far from complete. The changes in the other parts of the work are the following: the chapter upon elasticity and strength has been much extended and altered, the subject of hydraulics has been treated more at length, and some modifications in it have been made, in consequence of the continued experiments of the author.

I trust that the present edition will be received with the same favor as the last, by which the author was encouraged to continue his preparation of the work.

JULIUS WEISBACH.

FREIBERG, *May* 15*th*, 1850.

PREFACE TO THE THIRD EDITION.

THE third edition of the first volume of my Mechanics of Engineering and of the Construction of Machines, which I now give to the public, has, compared with its predecessors, not only been improved, but also augmented and completed. The changes are due principally to the advance of science, and in some cases to the results of more recent investigations. When not withheld by some good reason, I have endeavored, so far as possible, to satisfy the wishes which have been communicated to me from different quarters in regard to the work. From the extraordinary favor, with which it has been received both in and out of Germany, on this as well as on the other side of the Atlantic, I flatter myself that it has suited both in method and size the greater portion of the public for whom it was intended, and my efforts in preparing the new edition have been naturally directed to removing any errors or omissions, that have been observed, and to incorporating in it the latest experiments, treated in the same manner and as concisely as possible. I am sorry to be obliged to remark that the work has been subjected to unjust criticism. Thus, E.G., Professor Wiebe, of Berlin, in a remark upon pages 245 and 246 of his work upon "die Lehre von der Befestigung der Machinentheile," (Berlin, 1854), states that I have given coefficients of torsion for square shafts in my Mechanics (first edition), as well as in the "Ingenieur," 16 times greater than those given by Morin. The Professor has here committed an oversight; for in my formulas, as is expressly stated in both works, the fourth power of the half length of the side occurs, while the formulas of Morin and Wiebe, as well as those of my second edition, contain the fourth power of the whole length of the side of the cross-section. Now since 2^4 is equal to 16, the error observed by Professor Wiebe proceeds from a mistake on his part.

I shall make no reply to the partial criticism contained in *Grunert's Archiv der Mathematik*, as I do not wish to enter upon a useless controversy here. Besides, Professor Grunert has already printed in his *Archiv* enough nonsense about Physics and Practical Mechanics (as I can easily prove) to demonstrate his unfitness for criticising works on those subjects.

It would have been easier for me to have given my book a more scientific form; but it would then have met with less favor, as it is intended for practical men.

From another stand-point also the book can easily and with equal injustice be found fault with. Any one, who has had some practical experience, will have observed how little theory is made use of, and how often it is put in the back-ground and looked upon with disfavor by practical men. The fault of this is no doubt due in great measure to that method of instruction, which condemns the study of *science* for the sake of its applications.

This edition, which has been augmented principally by the revision of the theory of elasticity and strength, and by the introduction of the latest hydraulic experiments, excels its predecessors not only in substance, but also in appearance, all the illustrations being new. The printing of the second volume will continue uninterruptedly.

JULIUS WEISBACH.

FREIBERG, *July*, 1856.

PREFACE TO THE FOURTH EDITION.

THE fourth edition of my Mechanics of Engineering and of the Construction of Machines has undergone no change either in method or arrangement. As three large editions have been exhausted in a comparatively short time, as two have been published in the English language, one in England and one in North America, and as the work has been translated into Swedish, Polish, and Russian, I may well hope that this manual has met the wishes and needs of that great practical public for whom it is intended. I have, therefore, in preparing this edition, endeavored simply to remove any errors or omissions, which may have been observed, and to introduce the results of the latest practically important experiments, together with the newest developments of theory. Thus, E.G., in the chapter upon friction I have included the results of the latest experiments by Bochet, and the section upon elasticity and strength has been rewritten in accordance with the present stand-point of science, in doing which I have made use of the recent works of Lamé, Rankine, Bresse, etc. The section upon hydraulics has been augmented, improved and completed. The later researches of the author are here discussed. I will mention more particularly the experiments upon the efflux of water under great and very great pressures, as well as upon the heights of jets, those upon the efflux of air, and the comparative experiments upon the impact of streams of air and water. The chapter upon the efflux of air has been entirely rewritten, as the author is of the opinion that the ordinary formulas for the efflux of air under high pressures do not represent the law of efflux. The formulas obtained are very simple, since, without materially affecting its accuracy, I have substituted in the well-known formula for heat

$$\frac{1 + \delta \tau_1}{1 + \delta \tau} = \left(\frac{\gamma_1}{\gamma}\right)^{0,42}$$

0,50 instead of the exponent 0,42, by which I obtain

$$\frac{1 + \delta \tau_1}{1 + \delta \tau} = \sqrt{\frac{\gamma_1}{\gamma}} \text{ (see § 461).}$$

The practical value of a formula does not depend upon its correctness even at extreme limits, but rather upon the fact that, within given limits, it furnishes values which agree sufficiently well with the results of experiment.

Several new paragraphs, in which Phoronomics and Aerostatics are treated with the aid of the Calculus, have been added. In hydraulics the pressure of water flowing through pipes, on account of its practical importance, has been treated separately in two new paragraphs (§ 439 and § 440). In the chapter upon the force and resistance of water I have treated the theory of the simple reaction wheel, as well as its application as an instrument for proving the theory of the impact and resistance of water. The more recent gas and water meters are also discussed, since these instruments are set in motion by the reaction of the issuing fluid, the intensity of which can easily be determined by the foregoing theory.

Finally, the Appendix has been slightly augmented by the introduction of the report of the recent experiments of Geh. Oberbaurath Hagen upon waves of water.

* * * * * * * * * * * *

In answer to the criticism, which has been made in some quarters, that a more scientific treatment of the subject, based upon the Calculus, would have been more in accordance with the object of the book, I would state that my book is intended for the use of practical men, who often do not possess either the requisite knowledge of the Calculus or sufficient facility in the use of it. Having labored during upwards of thirty years as instructor in a technical institution, during which time I have been engaged in practical works of various kinds and have made many journeys for the purpose of technical studies, I can confidently give an opinion upon this subject.

As I consider my reputation as an author of much more importance than any mere pecuniary advantage, it is always a pleasure to me to find my "Mechanics" made use of in works of a similar character; but when writers avail themselves of it without the slightest acknowledgment, I can only appeal to the judgment of the public.

JULIUS WEISBACH.

FREIBERG, *May*, 1868.

TRANSLATOR'S PREFACE.

THE favor, with which both the English and American editions of the Mechanics of Engineering and of the Construction of Machines were received, would sufficiently justify the appearance of a new one, even if the original work had undergone no change. But as the first two volumes of the last (fourth) German edition contain more than twice as much matter as those of the first, and since a third volume of about fifteen hundred pages has been added, the translator feels not only that the work may be considered a new one, but also that, in offering it to the public, he is supplying a real want. The text of this edition has been, to a great extent, rewritten and rearranged, and the translation is entirely original.

Weisbach's Mechanics is now so well known, wherever that science is taught, that any eulogy on our part would be superfluous.

A large number of typographical errors, observed in the German edition, have been corrected with the approbation of the author, who has also communicated to the translator some slight modifications in the text. The work of translation was begun with the author's approval, while the translator was a student of the Mining Academy at Freiberg, but the work was delayed by his professional engagements. He hopes that it will now appear without interruption.

At the suggestion of the author, an Appendix has been added containing an account of the articles upon the subjects treated in this volume, which have been published by him since the appearance of the last German edition.

All the tables, formulas, examples, etc., in which the Prussian weights and measures occur, have been transformed so as to be applicable to the English system. Where the metrical system was employed in the original work, it has been retained in the translation, as the meter is now much used both in England and America.

The "Ingenieur," which is so often quoted in this work, has, unfortunately, not been translated into English, but all the references to it have been preserved, as the work is a valuable one, even to those who have little or no knowledge of German, and perhaps an English edition of it may be published.

A list of errors and omissions observed in this volume will be given in the succeeding one, and the translator will be glad to be informed of any typographical errors.

He would call attention to the illustrations, which are printed from electrotype copies of the wood-cuts prepared for the German edition, and his thanks are due to the publisher and stereotypers for the excellent appearance of the work.

ECKLEY B. COXE.

PREFACE TO THE REVISED EDITION.

The only changes in the text of this edition have been the corrections of typographical and other errors. A list of the important errors in the first edition, which have been corrected in this, is in preparation, and will be furnished, upon application to the publisher, to those who have that edition.

The translator wishes to return his thanks to Prof. Sheldon, of Brooklyn, and to Prof. Frank and Mr. R. H. Sanders, of Philadelphia, for the corrections they have sent him.

A list of the errors in this edition will be given in the second volume.

ECKLEY B. COXE.

JULY 5, 1870.

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THEORETICAL MECHANICS.

INTRODUCTION

TO

THE CALCULUS.

Art. 1. The dependence of a quantity y upon another quantity x is expressed by a mathematical formula: E.G., $y = 3x^2$, or $y = a x^m$, etc. We write $y = f(x)$ or $z = \phi(y)$ etc., and we call y a function of x, and z a function of y. The symbols f and ϕ, etc., indicate in general that y is dependent upon x, or z upon y, but leave the dependence of these quantities upon one another entirely undetermined, and do not give the algebraical operation by which y can be deduced from x, or z from y. A function $y = f(x)$ is an indeterminate equation; it gives an unlimited number of values for x and y, which correspond to it. If one of them (x) is given, the other (y) is determined by the function, and if one of them is changed, the other also undergoes a change. Therefore the indeterminate quantities x and y are called VARIABLES, or variable quantities; and the quantities which are given, or are to be regarded as given, and indicate the operation by which y is to be deduced from x, are called CONSTANTS, or constant quantities. That one of the variables which can be chosen at pleasure is called *the independent variable*, and the other, which is determined by means of a given operation from the first, is called *the dependent variable.* In $y = a x^m$, a and m are constants, x is the independent and y the dependent variable.

The dependence of z upon two other quantities, x and y, is expressed by the equation $z = f(x, y)$. In this case z is at the same time a function of x and y, and we have here two *independent variables.*

ART. **2.** Every dependence of a quantity y upon another quantity x, expressed by a function or formula $y = f(x)$ can be represented by means of a curve, $A\ P\ Q$, Fig. 1 and Fig. 2.

FIG. 1.

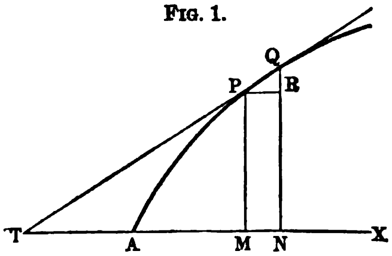
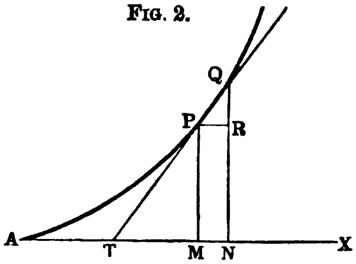


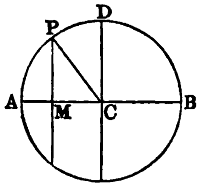
FIG. 2.



The different values of the independent variable x answer to the abscissas $A\ M$, $A\ N$, etc., and the different values of the dependent variable to the ordinates $M\ P$, $N\ Q$, etc., of the curve. The co-ordinates (abscissas and ordinates) represent then the two variables of the function.

The graphic representation of a function, or the referring of the same to a curve, presents several advantages. It furnishes us in the first place with a general view of the connexion between the two variable quantities; secondly, it replaces a table or summary of every two values of the function belonging together; and thirdly, it affords us a knowledge of the different properties and relations of the function. If with the radius $C\,A = C\,B = r$ we describe a circle $A\,D\,B$ (Fig. 3), corresponding to the function $y = \sqrt{2\ r\ x - x^2}$ where x and y indicate the co-ordinates $A\ M$, $M\ P$, this curve affords us not only a general view of the different values that the function can assume, but also makes us acquainted with other peculiarities of this function, for the properties of the circle have also their meaning in the function. We know, E.G., without farther research, that y becomes equal to zero, not only when $x = 0$ but also when $x = 2\ r$, and that y is a maximum and $= r$ when $x = r$.

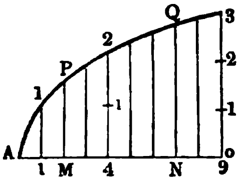
FIG. 3.



ART. 3. The Laws of Nature can generally be expressed by functions between two or more quantities, and are therefore in most cases capable of a graphic representation.

(1) When a body falls freely *in vacuo*, we have for the velocity y, which corresponds to the height of fall x, $y = \sqrt{2\,g\,x}$, but this formula corresponds to the equation $y = \sqrt{p\,x}$ of the parabola, when the parameter (p) of the latter is made equal to the double acceleration ($2\,g$) of gravity. We can therefore represent graphically the laws of the free fall of a body by the parabola $A\,P\,Q$ (Fig. 4), whose parameter $p = 2\,g$. The abscissas $A\,M$, $A\,N$, of this curve are the space traversed by the body in its fall, and the ordinates $M\,P$, and $N\,Q$, the corresponding velocities.

FIG. 4.



(2) If a is a certain volume of air under the pressure of one atmosphere, we have according to Marriotte's Law, the volume of the same mass of air under a pressure of x atmospheres, $y = \frac{a}{x}$, and we have, for $x=1, y = a$; for $x = 2$, $y = \frac{a}{2}$, for $x = 4$, $y = \frac{a}{4}$, for $x=10$, $y=\frac{a}{10}$; for $x=100, y=\frac{a}{100}$, for $x=\infty, y=0$.

We see in this manner that the volume becomes smaller as the tension becomes greater, and that if the law of Marriotte were correct for all tensions an infinitely great tension would correspond to an infinitely small volume.

Further, for $x = \frac{1}{2}$, we have $y= 2\,a$; for $x=\frac{1}{4}$, we have $y=4\,a$;
" $x = \frac{1}{10}$, " $y=10\,a$; " $x=0$, " $y=\infty\,a$;

so that the smaller the tension, the greater the volume becomes; and if the tension is infinitely small the volume is infinitely great.

The curve which corresponds to this law is drawn in Fig. 5. $A\,M$, $A\,N$, are the tensions or abscissas x, $M\,P$, $N\,Q$, the corresponding volumes or ordinates y. We see that this curve approaches gradually the axes $A\,X$ and $A\,Y$ without ever reaching them.

(3) The dependence of the expansive force of saturated steam

upon its temperature x can be expressed, at least within certain limits, by the formula

$$y=\left(\frac{a+x}{b}\right)^m$$

and by experiment we have within certain limits $a=75$, $b=175$, and $m=6$. If we put

$$y=\left(\frac{75+x}{175}\right)^6$$

FIG. 5.

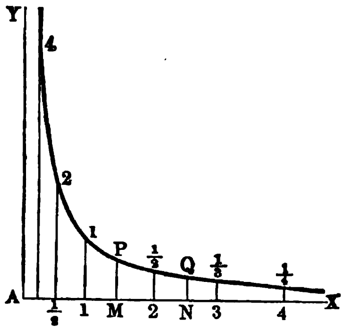
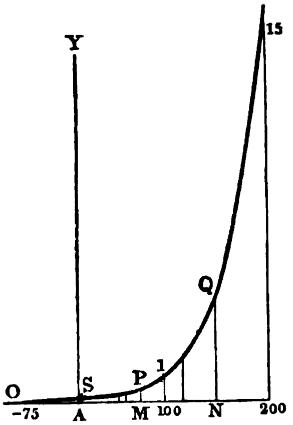


FIG. 6.



and assume the formula to be correct without limit, we obtain

$$\begin{array}{lrl}
\text{for } x = & 100^\circ, & y=\left(\frac{175}{175}\right)^6 = 1{,}000 \text{ atmosphere,} \\
\text{“ } x = & 50^\circ, & y=\left(\frac{125}{175}\right)^6 = 0{,}133 \quad \text{“} \\
\text{“ } x = & 0^\circ, & y=\left(\frac{75}{175}\right)^6 = 0{,}006 \quad \text{“} \\
\text{“ } x = & -75^\circ, & y=\left(\frac{0}{175}\right)^6 = 0{,}000 \quad \text{“} \\
\text{“ } x = & 120^\circ, & y=\left(\frac{195}{175}\right)^6 = 1{,}914 \quad \text{“} \\
\text{“ } x = & 150^\circ, & y=\left(\frac{225}{175}\right)^6 = 4{,}517 \quad \text{“} \\
\text{“ } x = & 200^\circ, & y=\left(\frac{275}{175}\right)^6 = 15{,}058 \quad \text{“}
\end{array}$$

PQ, Fig. 6, presents to the eye the corresponding curve. It passes at a distance $AO = -75$ from the origin of co-ordinates

A through the axis of abscissas and at a distance $A S = 0{,}006$ cuts the axis of ordinates; an abscissa $A M < 100$ corresponds to an ordinate $M P < 1$, and an abscissa $A N > 100$ belongs to an ordinate $N Q > 1$; and we can also see that not only y augments as x increases to infinity, but also that the curve becomes steeper and steeper as x becomes greater.

ART. **4.** A function $z = f\,(x.y)$ with two independent variables can be represented by means of a curved surface $B\,C\,D$, Fig. 7, in which the independent variables x and y are given by the abscissas $A\,M$ and $A\,N$ on the axes $A\,X$ and $A\,Y$, and the dependent variable z by the ordinate $O\,P$ of a point P in the surface $A\,B\,C$. If for a definite value of x we give different values to y, the values of z deduced furnish us with the ordinates of the points of a curve $E\,P\,F$ parallel to the co-ordinate plane $Y\,Z$; if on the contrary for a given value of y we take different values of x, we determine the ordinates z of the points of a curve $G\,P\,H$ parallel to the co-ordinate plane $X\,Z$. We can consequently consider the whole curved surface $B\,C\,D$ as the union of a series of curves parallel to the co-ordinate planes. The law of Marriotte and Gay-Lussac $z = \frac{a\,(1 + \delta\,y)}{x}$, by means of which we can calculate the volume z of a mass of air from the pressure x and the temperature y, is graphically represented by the curved surface $C\,K\,P\,H$, Fig. 8. $A\,M$ is the pressure x, $A\,N$ or $M\,O$ the temperature y, and $O\,P$ the corresponding volume z: the co-ordinates of the curve $P\,G\,H$ give the volumes for a temperature $A\,N = y$, and those of the right line $K\,P$ the volumes for the same pressure $A\,M = x$.

FIG. 7.

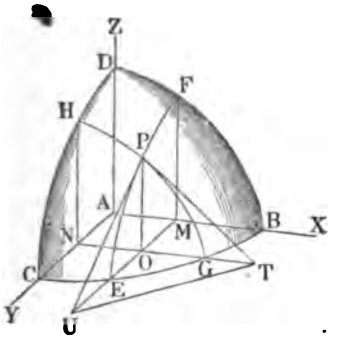
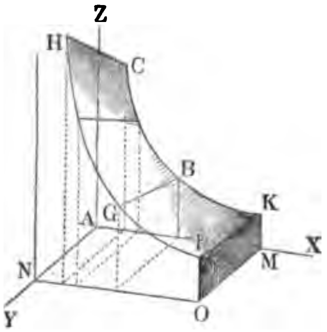


FIG. 8.



ART. **5.** When we increase the independent variable of a function or the abscissa $A\,M = x$ (Fig. 9 and Fig. 10) of the corresponding curve an infinitely small quantity $M\,N$, which we will in future designate by $d\,x$, the corresponding dependent variable or ordinate $MP = y$ becomes $N\,Q = y'$, being increased by an infinitely small quantity $R\,Q = N\,Q - M\,P$, to be designated by $d\,y$. Both these increments $d\,x$ and $d\,y$ of x and y are called the Differentials of the Variables or Co-ordinates x and y, and our principal problem now is to determine for the functions that most commonly occur the differentials, or rather the ratio of the differentials of the variables x and y belonging together. If in the function $y = f\,(x)$, where x represents the abscissa $A\,M$, and y the ordinate $M\,P$, we substitute, instead of x, $x + d\,x = A\,M + M\,N = A\,N$, we obtain, instead of y, $y + d\,y = M\,P + R\,Q = N\,Q$; therefore

$$y + d\,y = f\,(x + d\,x),$$

and subtracting the first value of y from it, the differential of the variable y remains, *i. e.*

$$d\,y = d\,f\,(x) = f\,(x + d\,x) - f\,(x)$$

FIG. 9.

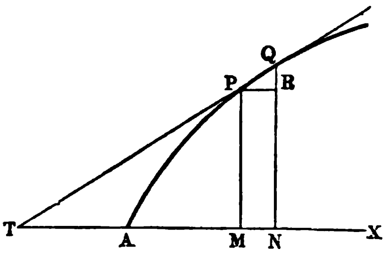
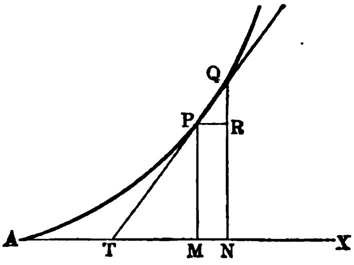


FIG. 10.



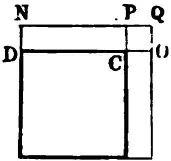
This is the general rule for the determination of the differential of a function, which when applied to different functions furnishes several rules more or less general: E.G., if $y = x^2$, we have

$$d\,y = (x + d\,x)^2 - x^2$$
$$(x + d\,x)^2 = x^2 + 2\,x\,d\,x + d\,x^2$$
$$d\,y = 2\,x\,d\,x + d\,x^2 = (2\,x + d\,x)\,d\,x;$$

and more simply since $d\,x$, being infinitely small compared to $2\,x$, disappears, or since $2\,x$ is not sensibly changed by the addition of $d\,x$, and the latter can therefore be disregarded,

$$d\,y = d\,(x)^2 = 2\,x\,d\,x.$$

FIG. 11.



The formula $y = x^2$ corresponds to the contents of a square, $ABCD$, Fig. 11, whose side is $AB = AD = x$, and we see from the figure that, by the addition to the side of $BM = DN = dx$, the square is increased by two rectangles BO and $DP = 2\,x\,dx$, and by a square (dx^2), so that by an infinitely small increase dx of x the square $y = x^2$ is increased by the differential quantity $2\,x\,dx$.

ART. 6. The right line, TPQ, Fig. 9 and Fig. 10, passing through two points P and Q of the curve, which are at an infinitely small distance from each other, is called the Tangent to this curve, and determines the direction of the curve between these two points. The direction of the tangent is given by the angle $PTM = a$ at which the axes of abscissas AX is cut by the line. When the curve is concave, as APQ, Fig. 9, the tangent lies beyond the curve and the axis of abscissas; but when it is convex, as APQ, Fig. 10, the line lies between the curve and the axis of abscissas.

In the infinitely small right-angled triangle PQR (Fig. 9 and Fig. 10), with the base $PR = dx$, and the altitude $RQ = dy$, the angle QPR is equal to the tangential angle $PTM = a$, and we have

$$tang.\, QPR = \frac{QR}{PR},$$

whence

$$tang.\, a = \frac{dy}{dx};$$

therefore the ratio or quotient of the two differentials dy and dx gives the trigonometrical tangent of the tangential angle; E.G., for the parabola whose equation is $y^2 = px$ we have, putting $y^2 = px = z$,

$$dz = (y+dy)^2 - y^2 = y^2 + 2\,y\,dy + dy^2 - y^2 = 2\,y\,dy + dy^2,$$

or as dy^2 vanishes before $2\,y\,dy$, or what is the same thing, dy before $2\,y$,

$$dz = 2\,y\,dy,$$

and also

$$dz = p\,(x + dx) - px,$$

therefore $2\,y\,dy = p\,dx$, whence for the tangential angle of the parabola we have

$$tang.\, a = \frac{dy}{dx} = \frac{p}{2y} = \frac{y^2}{2xy} = \frac{y}{2x}$$

The definite portion $P\,T$ of the tangent between the point of tangency P and the point T where it cuts the axes of abscissas

Fig. 12.

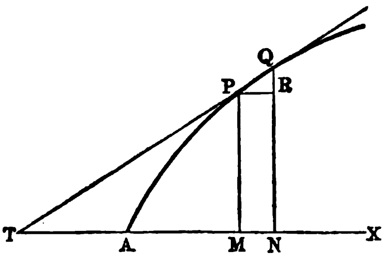
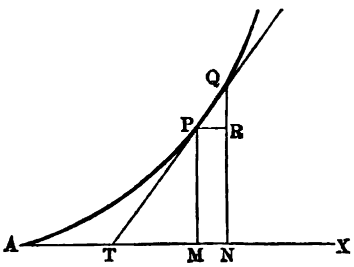


Fig. 13.



is generally called the *Tangent*, and the projection $T\,M$ of the same upon the axes of abscissas the *Sub-tangent*; hence we have,

$$subtang. = P\,M\,cot.\,P\,T\,M$$

$$= y\ cot.\ a = y\frac{d\,x}{d\,y},$$

E.G., for the parabola, $subiang. = y\frac{2\,x}{y} = 2\,x.$

The subtangent is therefore equal to the double abscissa, and from it the position of the tangent for any point P of the parabola is easily found.

For the curved surface $B\,C\,D$, Fig. 7, the angles of inclination a and β of the tangents $P\,T$ and $P\,U$ at a point P are determined by the formulas:

$$tang.\ a = \frac{d\,z}{d\,x} \qquad tang.\ \beta = \frac{d\,z}{d\,y}$$

The plane $P\,T\,U$ passing through $P\,T$ and $P\,U$ is the tangent-plane of the curved surface.

Art. 7. For a function $y = a + mf\,(x)$ we have

$$d\,y = [a + mf\,(x + d\,x)] - [a + mf\,(x)];$$
$$= a - a + mf\,(x + d\,x) - mf\,(x$$
$$= m\,[f\,(x + d\,x) - f\,(x)];$$

i. e.

I.) . . . . . $d\,[a + mf\,(x)] = m\,d\,f\,(x),$

E.G., $d\,(5 + 3\,x^2) = 3\,[(x + d\,x)^2 - x^2] = 3\,.\,2\,x\,d\,x = 6\,x\,d\,x.$

In like manner:

$$d\,(4 - \tfrac{1}{2}\,x^3), = -\tfrac{1}{2}\,d\,(x)^3 = -\tfrac{1}{2}\,[(x + d\,x)^3 - x^3]$$
$$= -\tfrac{1}{2}\,(x^3 + 3\,x^2\,d\,x + 3\,x\,d\,x^2 + d\,x^3 - x^3)$$
$$= -\tfrac{1}{2}\,.\,3\,x^2\,d\,x = -\tfrac{3}{2}\,x^2\,d\,x.$$

Hence we can establish the following important rule: The constant member (a, 5) of a function disappears by differentiation, and the constant factors remain unchanged.

The correctness of this rule can be graphically represented. For the curve $A\ P\ Q$, Fig. 14, whose co-ordinates in one case are

FIG. 14.

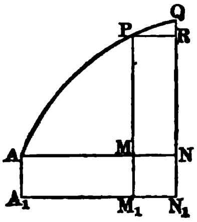
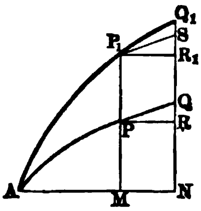


FIG. 15.



$A\,M = x$ and $M\,P = y = f\,(x)$, and in the other $A_1\,M_1 = x$ and $M_1\,P = a + y = a + f(x)$, we have $P\,R = d\,x$ and $R\,Q = d\,y = d\,f\,(x)$ and also $= d\ (a + y) = d\ [a + f\,(x)]$; and for the curves $A\ P_1\ Q_1$ and $A\ P\ Q$, Fig. 15, whose corresponding ordinates $M\,P_1$ and $M\,P$ as well as $N\,Q_1$ and $N\,Q$ have a certain relation to one another, the relation between the differentials $R_1\ Q_1 = N\,Q_1 - M\,P_1$ and $R\,Q = N\,Q - M\,P$ is the same; for if we put $MP_1 = m\,.\,MP$ and $N\,Q_1 = m\,.\,N\,Q$, it follows that $R_1\,Q_1 = N\,Q_1 - M\,P_1 = m.\,(N\,Q - M\,P) = m\,.\,Q\,R.$

i. e. $$d\,[m\,f\,(x)] = m\,d\,f\,(x).$$

If $y = u + v$, or the sum of two variables u and v, we have

$d\,y = u + d\,u + v + d\,v - (u + v)$, *i. e.*, according to Art. 5.

II.) . . . $d\,(u + v) = d\,u + d\,v$, and in like manner,

$$d\,[f\,(x) + \phi\,(x)] = d\,f\,(x) + d\,\phi\,(x).$$

The differential of the sum of several functions is then equal to the sum of the differentials of the separate function; E.G.

$$d\,(2\,x + 3\,x^2 - \tfrac{1}{2}\,x^3) = 2\,d\,x + 6\,x\,d\,x - \tfrac{3}{2}\,x^2\,d\,x = (2 + 6\,x - \tfrac{3}{2}\,x^2)\,d\,x.$$

The correctness of this formula can also be made evident by the consideration of the curve $A\ P\ Q$, Fig. 15. If $M\,P = f\,(x)$ and $P\,P_1 = \phi\,(x)$ we have

$$M\,P_1 = y = f\,(x) + \phi\,(x) \text{ and}$$

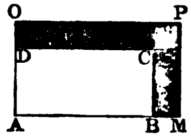
$$d\,y = R_1\ Q_1 = R_1\ S + S\,Q_1 = R\,Q + S\,Q_1 = d\,f\,(x) + d\,\phi\,(x);$$

for $P_1 S$ can be drawn parallel to $P Q$, and therefore we can put $R_1 S = R Q$ and $Q S = P P_1$.

ART. 8. If $y = u v$ or the product of two variables, E.G. the contents of the rectangle $A B C D$, Fig. 16, with the variable sides $A B = u$ and $B C = v$, we have

$$dy = (u + du)(v + dv) - uv = uv + udv + vdu + du\,dv - uv,$$
$$= udv + vdu + dudv = udv + (v + dv)\,du.$$

FIG. 16.



But in $v + dv$, dv is infinitely small compared to v, and we can put

$$v + dv = v, \text{ and } (v + dv)\,du = vdu,$$

and also

$$udv + (v + dv)\,du = udv + vdu,$$

so that

III.) . . . $d(uv) = udv + vdu$,

it follows therefore that

$$d[f(x) . \phi(x)] = f(x)\,d\phi(x) + \phi(x)\,df(x).$$

The differential of the product of two variables is then equal to the sum of the products of each variable by the differential of the other.

When the sides of the rectangle $A B C D$ are increased by $BM = du$ and $D O = dv$ its contents $y = A B \times A D = uv$ is augmented by the rectangles $C O = udv$ and $CM = vdu$ and $C P = dudv$, the latter, being infinitely small, compared with the others, disappears; the differential of this surface is only equal to the sum $udv + vdu$ of the contents of the two rectangles $C O$ and $C M$.

In conformity with this rule we have for $y = x(3x^2 + 1)$:

$$dy = xd(3x^2 + 1) + (3x^2 + 1)\,dx = 3xd(x^2) + (3x^2 + 1)\,dx$$
$$= 3x . 2xdx + 3x^2\,dx + dx = (9x^2 + 1)\,dx.$$

Further, if w be a third variable factor, we have

$$d(uvw) = u\,d(vw) + vw\,du,$$

or since $d(vw) = v\,dw + w\,dv$,

$d(uvw) = uv\,dw + uw\,dv + vw\,du$, and in like manner

$$d(uvwz) = uvw\,dz + uvz\,dw + uwz\,dv + vwz\,du;$$

if $w = v = w = z$, it follows that $d(u^4) = 4u^3\,du$, and in general

IV.) . . . $d(x^m) = m x^{m-1}\,dx$, if m is a positive integer, E.G.

$$d(x^7) = 7x^6\,dx, \quad d\tfrac{3}{4}x^8 = 6x^7\,dx.$$

If $y = x^{-m}$, m being again a positive integer, we have also

$yx^m = 1$ and $d(yx^m) = 0$, *i. e.*

$yd(x^m) + x^m\,dy = 0$, and therefore

$$dy = -\frac{yd(x^m)}{x^m} = \frac{x^{-m} m x^{m-1}\,dx}{x^m} = -m x^{-m-1}\,dx,$$

or, if we put $-m = n$,

$$d(x^n) = n\,x^{n-1}\,dx.$$

The Rule IV. applies also to powers, whose exponents are negative whole numbers, as E.G.,

$$d(x^{-3}) = -3x^{-4}\,dx = -\frac{3\,dx}{x^4}, \text{ and}$$

$$d\,(3x^2+1)^{-2} = -2\,(3x^2+1)^{-3}\,d\,(3x^2) = -\frac{12\,x\,dx}{(3x^2+1)^3}.$$

If in $y = x^{\frac{m}{n}}$, $\frac{m}{n}$ is a fraction whose denominator n and whose numerator m are integers, we have also $y^n = x^m$ and $d\,(y^n) = d\,(x^m)$, I.E.,

$$n\,y^{n-1}\,dy = m\,x^{m-1}\,dx, \text{ therefore}$$

$$dy = \frac{m}{n}\,\frac{x^{m-1}\,dx}{y^{n-1}} = \frac{m}{n}\,\frac{x^{m-1}\,dx}{x^{m-\frac{m}{n}}} = \frac{m}{n}\,x^{\frac{m}{n}-1}\,dx.$$

If we put $\frac{m}{n} = p$, it follows that

$dy = d\,(x^p) = p\,x^{p-1}\,dx$, which agrees with Rule IV., which can now be considered as general.

Also $d\;(u^p) = p\,u^{p-1}\,d\,u$, when u denotes any function dependent upon x.

Hence we have, E.G., $d\,(\sqrt{x^3}) = d\,(x^{\frac{3}{2}}) = \frac{3}{2}\,x^{\frac{1}{2}}\,dx = \frac{3}{2}\,\sqrt{x}\,dx$,

$$d\,\sqrt{2\,r\,x - x^2} = d\,\sqrt{u} = d\,(u^{\frac{1}{2}}) = \tfrac{1}{2}\,u^{-\frac{1}{2}}\,d\,u$$

$$= \tfrac{1}{2}\,\frac{d\,(2\,r\,x - x^2)}{u^{\frac{1}{2}}} = \frac{2\,r\,dx - 2\,x\,dx}{2\,\sqrt{u}} = \frac{(r-x)\,dx}{\sqrt{2\,rx - x^2}}.$$

In order to find the differential of a quotient $y = \frac{u}{v}$, we put $u = v\,y$, whence $d\,u = v\,d\,y + y\,d\,v$, and

$$dy = \frac{d\,u - y\,d\,v}{v} = \frac{d\,u - \frac{u}{v}\,d\,v}{v}, \text{ I.E.,}$$

V.) $$d\left(\frac{u}{v}\right) = \frac{v\,d\,u - u\,d\,v}{v^2}.$$

According to this Rule, E.G.,

$$d\left(\frac{x^2-1}{x+2}\right) = \frac{(x+2)\,d\,(x^2-1) - (x^2-1)\,d\,(x+2)}{(x+2)^2}$$

$$= \frac{(x+2)\,.\,2\,x\,dx - (x^2-1)\,.\,d\,x}{(x+2)^2} = \left(\frac{x^2+4x+1}{(x+2)^2}\right)d\,x.$$

We have also:

$$d\left(\frac{a}{v}\right) = -\frac{a\,dv}{v^2};\ \text{E.G.,}\ d\left(\frac{4}{x^2}\right) = -\frac{4\,d\,(x^2)}{x^4} = -\frac{8\,dx}{x^3}.$$

ART. **9.** The function $y = x^n$ is the most important in the whole analysis, for we meet it in all researches. When we give the exponent n all possible values, positive and negative, whole and fractional, etc., it furnishes the different kinds of curves, which are represented in Fig. 17. A is here the point of origin of the co-ordinates, $X\,\overline{X}$ the axis of abscissas, and $Y\,\overline{Y}$ that of the ordinates.

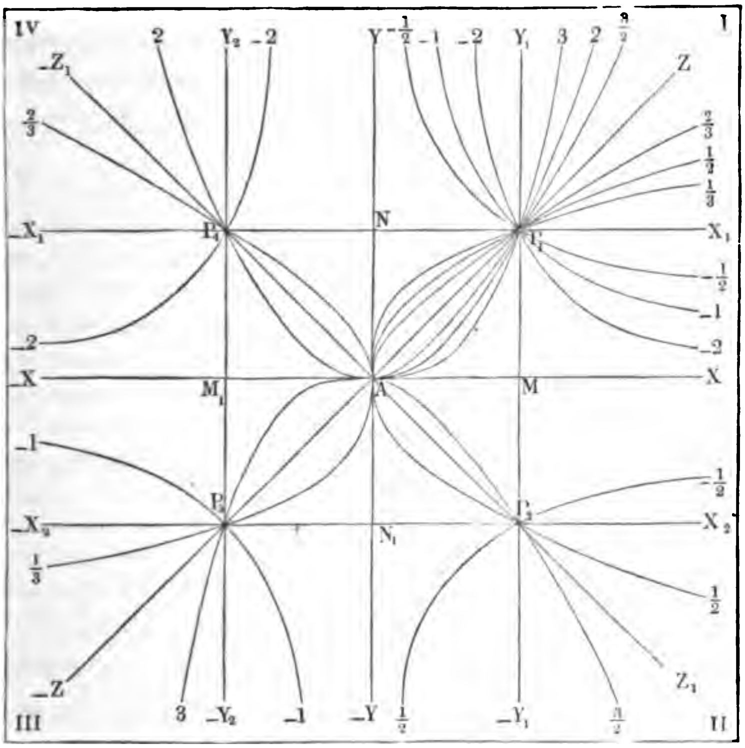
If on both sides of the co-ordinate axes at the distances $x = \pm 1$ and $y = \pm 1$ from the point A we draw the parallels $X_1\,\overline{X}_1, X_2\,\overline{X}_2$, $Y_1\,\overline{Y}_1$, $Y_2\,\overline{Y}_2$ to the axes, and join the points P_1, P_2, P_3, and P_4, where they cut each other, by means of the diagonals $Z\,\overline{Z}$, $Z_1\,\overline{Z}_1$, we obtain a diagram which contains all the curves, given by the equation $y = x^n$. For every point on the axis of abscissas $X\,\overline{X}$ we have $y = 0$, and for every point on the axis of ordinates $Y\,\overline{Y}$, $x = 0$; and for the points in the axes $X_1\,\overline{X}_1$ and $X_2\,\overline{X}_2$, $y = \pm 1$, and for the points in the axes $Y_1\,\overline{Y}_1$ and $Y_2\,\overline{Y}_2$, $x = \pm 1$.

If in the equation $y = x^n$ we put $x = 1$, we obtain for all possible values of n, $y = 1$, and for certain values of n, also $y = -1$; consequently all the curves belonging to the equation $y = x^n$ pass through the point P_1, whose co-ordinates are $A\,M = 1$ and $A\,N = 1$. If we take $n = 1$ we have $y = x$ and we obtain the right line $Z\,A\,\overline{Z}$, which is equally inclined to the two axes $X\,\overline{X}$ and $Y\,\overline{Y}$, and which rises on one side of A at an angle of $45°\left(\frac{\pi}{4}\right)$, and on the other side dips at the same angle. On the contrary, for $y = -x$ we obtain the right line $Z_1\,A\,\overline{Z}_1$ which dips on one side of A at an angle of 45°, and rises on the other side at the same angle.

If, however, $n > 1$, $y = x^n$ becomes smaller for $x < 1$, and for $x > 1$ greater, than x, and when $n < 1$, $y = x^n$ is greater for $x < 1$ and smaller for $x > 1$ than x. The first case ($n > 1$) corresponds to convex curves, which run in the beginning under, and from P_1 over the right line ($Z\,A\,\overline{Z}$), and the second case ($n < 1$) to concave curves, where the reverse takes place.

When, in the first case, we take n smaller and smaller until at last it disappears, or becomes equal to zero, the ordinates approach the constant value $y = x^0 = 1$ and the corresponding curve approaches more and more to the broken line $A\ N\ P_1\ X_1$; if, on the contrary, in the second case, n becomes greater and greater, the values of the ordinates approach the limit $y = x^\infty = x^{\frac{1}{0}} = \infty$, and

FIG. 17.



those of the abscissas, on the contrary, approach the value $x = y^0 = 1$, and the corresponding curve approximates more and more to the broken line $A\ M\ P_1\ Y_1$.

If we take $n = -1$, whence $y = x^{-1} = \frac{1}{x}$, for $x = 0$, we have $y = \infty$, and for $x = \infty$, $y = 0$ and we obtain curve, which has been discussed in Art. 3, and drawn in Fig. 5 ($\bar{1}\ P\ \bar{1}$); it approaches on one side the axes of ordinates, and on the other the axes of abscissas without ever reaching them.

If the exponent $(-n)$ of the function $y = x^{-n} = \frac{1}{x^n}$ is a proper fraction, for $x < 1$, we have $y < \frac{1}{x}$, and on the contrary for $x > 1$, $y > \frac{1}{x}$, and if this exponent is greater than unity, we have on the contrary for $x < 1$, $y > \frac{1}{x}$, and for $x > 1$, $y < \frac{1}{x}$. The curve corresponding to $y = x^{-n}$, according as n is greater or smaller than unity, runs in the beginning below or above, and from P_1 above or below, the curve $y = x^{-1} = \frac{1}{x}$. While those curves, which correspond to the positive values of n, are placed in the beginning below, and from P_1 on above, the right line $X_1\ \bar{X}_1$, the curves of the negative exponents $(-n)$ run first above, and from P_1 on below, $X_1\ \bar{X}_1$. For the former curves we have, for $y = 0$, $x = 0$, and for $x = \infty$, $y = \infty$, and for the latter, for $x = 0$, $y = \infty$, and for $x = \infty$, $y = 0$. While the former diverge more and more from the co-ordinate axes $X\ \bar{X}$ and $Y\ \bar{Y}$, the farther we follow them from the origin A_1, the latter approach more and more on one side the axis $X\ \bar{X}$, and on the other axis $Y\ \bar{Y}$, without ever reaching them.

The last system of curves approach nearer and nearer the broken line $Y\ N\ P_1\ X_1$ or the broken line $Y_1\ P_1\ M\ X$ as the exponent approaches nearer and nearer the limit $n = 0$ or $n = \infty$.

If in $y = x^{\pm m}$, m is an entire uneven number $(1, 3, 5, 7 \ldots)$, y and x have the same sign. Positive values of x correspond to positive values of y, and negative values of x to negative values of y. If on the contrary m is an entire even number (2, 4, 6, etc.), y becomes positive for all values of x, positive or negative. Therefore the curves in the first case, as E.G., $(3\ P_1\ A\ P_3\ 3)$ or $(\bar{1}\ P_1\ \bar{1}, \bar{1}\ P_3\ \bar{1})$, run on one side of the axis of ordinates above, and on the other side below, the axis of abscissas $X\ A\ \bar{X}$; on the contrary the curves in the second case, as E.G., $(2\ P_1\ A\ P_4\ 2)$ or $(\bar{2}\ P_1\ \bar{2}, \bar{2}\ P_4\ \bar{2})$, are placed above the axis of abscissas only, and are contained in the first and fourth quadrants; the former corresponds for $m = \pm \infty$ to the limiting lines $Y_1\ M\ A\ M_1\ \bar{Y}_2$ and $X\ M\ Y_1$, $\bar{X}\ M_1\ \bar{Y}_2$, the latter on the contrary to the limiting lines $Y_1\ M\ A\ M_1\ Y_2$ and $X\ M\ Y_1$ $\bar{X}\ M\ Y_2$.

If we have $y = x^{\pm\frac{1}{n}}$, n being an entire uneven number, y and x have the same signs, and if n is an entire even number, every positive value of x gives two equal values for y, one of which is positive and the other negative, and on the contrary for every negative value of x, y is imaginary or impossible. The curves, as E.G. ($\frac{1}{3}$ $P_1\,A\,P_3$ $\frac{1}{3}$), which correspond to the first case, are found only in the first and third quadrants, and the curves of the second case, as E.G. ($\frac{1}{2}$ $P_1\,A\,P_2$ $\frac{1}{2}$), only in the first and second quadrants: the former become for $m = \infty$ the limiting lines $X_1\,N\,A\,N_1\,\overline{X}_2$ and $X_1\,N\,Y, \overline{X}_2\,N_1\,\overline{Y}$, and the latter the limiting lines $X_1\,N\,A\,N_1\,X_2$ and $X_1\,N\,Y, X_2\,N_1\,\overline{Y}$.

Since $y = x^{\pm\frac{1}{n}}$ involves $x = y^{\pm n}$, it follows, that the latter system of curves $\left(y = x^{\pm\frac{1}{n}}\right)$ differs from the former ($y = x^{\pm m}$) in its position only, and that by causing them to revolve, the curves of one system may be made to coincide with those of the other.

Since $y = x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m = (x^m)^{\frac{1}{n}}$ we can always give from what has gone before the general course of a curve. E.G., the curve for

$$y = x^{\frac{2}{3}} = (x^{\frac{1}{3}})^2 = \left(\sqrt[3]{x}\right)^2$$

has, for both positive and negative values of x, positive ordinates; on the contrary, the curve for

$$y = x^{\frac{3}{2}} = (x^{\frac{1}{2}})^3 = \left(\sqrt{x}\right)^3$$

has, for positive values of x only, real ordinates, and they are equal in magnitude, but with opposite signs. Further, for the curve

$$y = x^{\frac{3}{5}} = \left(\sqrt[5]{x}\right)^3,$$

y and x have the same sign, since neither the fifth root nor the cube causes a change of sign.

Finally, the curves, which correspond to the equation $y = -x^{\frac{m}{n}}$, differ from those of the equation $y = x^{\frac{m}{n}}$ only by their reversed position in regard to the axis of abscissas $X\,\overline{X}$, and they form the symmetrical halves of a complete curve.

ART. **10.** From the important formula $d\,(x^n) = n\,x^{n-1}\,d\,x$ we obtain the formula for the tangential angle of the corresponding curves represented in Fig. 18. It is

$$tang.\ a = \frac{d\,y}{d\,x} = n\,x^{n-1},$$

and therefore we have the subtangent of these curves

$$= y \frac{dx}{dy} = \frac{x^n}{n x^{n-1}} = \frac{x}{n}.$$

Hence, for the so-called parabola of Neil, the equation of which is $a y^2 = x^3$ or $y = \sqrt{\frac{x^3}{a}}$, we have

$$tang.\, \alpha = \frac{1}{\sqrt{a}} \frac{d(x^{\frac{3}{2}})}{dx} = \frac{1}{\sqrt{a}} \cdot \tfrac{3}{2} x^{\frac{1}{2}} = \tfrac{3}{2} \sqrt{\frac{x}{a}},$$

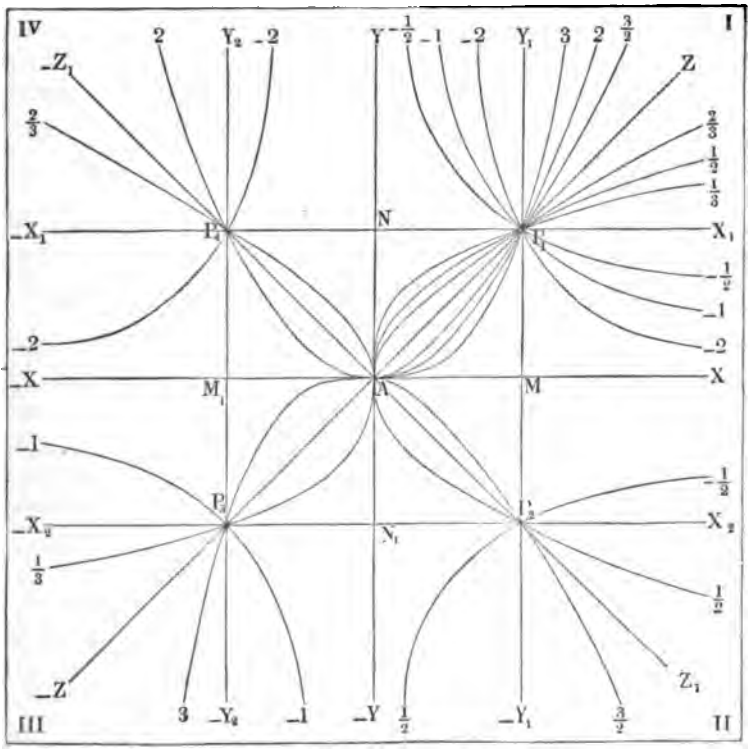
and the subtangent $= \frac{2}{3} x$.

Farther, for the curve already discussed $y = \frac{a^2}{x} = a^2 x^{-1}$,

$$tang.\, \alpha = a^2 \frac{d\,(x^{-1})}{dx} = -\frac{a^2}{x^2} = -\left(\frac{a}{x}\right)^2,$$

and the subtangent $= \frac{x}{-1} = -x$. (See Fig. 5.)

FIG. 18.



Consequently, we have for $x = 0$, $tang.\, a = -\infty$ and $a = 90°$,
for $x = a$, $tang.\, a = -1$ and $a = 135°$
and for $x = \infty$, $tang.\, a = 0$ and $a = 0°$, etc.

ART. 11. When a right line $A\,O$, Fig. 19, cuts the axis of abscissas at an angle $O\,A\,X = a$, and is at a distance $C\,K = n$ from the origin of co-ordinates C, the equation between the co-ordinates $C\,M = N\,P = x$ and $C\,N = M\,P = y$ of a point in the same is $y\, cos.\, a - x\, sin.\, a = n$, since $n = M\,R - M\,L$, $M\,R = y\, cos.\, a$ and $M\,L = x\, sin.\, a$.

For $x = 0$, y becomes $C\,B = b = \frac{n}{cos.\, a}$, therefore we have $n = b\, cos.\, a$, and $y\, cos.\, a - x\, sin.\, a = b.\, cos.\, a$ or

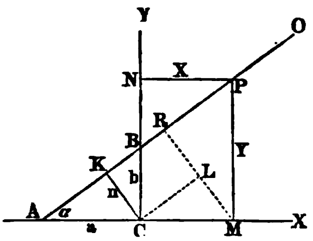
$$y = b + x\, tang.\, a.$$

Generally the lines $C\,A$ and $C\,B$, which measure the distances from the points where the line cuts the co-ordinate axes $C\,X$ and $C\,Y$ to the origin of co-ordinates, are called the *parameters* of the line, and are designated by the letters a and b. According to the figure $C\,A = -a$, therefore

$$tang.\, a = \frac{C\,B}{C\,A} = -\frac{b}{a},$$

and consequently the equation of the straight line becomes

FIG. 19.



$$y = b - \frac{b}{a}x, \text{or } \frac{x}{a} + \frac{y}{b} = 1. \quad \text{(See Ingenieur, page 164.)}$$

When a curve approaches more and more a line, which is situated at a finite distance from the origin of co-ordinates, without ever attaining it, the line is called the ASYMPTOTE OF THE CURVE.

The asymptote can be considered as the tangent to a point of the curve situated at an infinite distance. Its angle of inclination to the axis of abscissas can be determined by

$$tang.\, a = \frac{d\,y}{d\,x},$$

and its distance n from the origin of co-ordinates by the equation

$$n = y\, cos.\, a - x\, sin.\, a = (y - x\, tang.\, a)\, cos.\, a$$

$$= \frac{y - x\, tang.\, a}{\sqrt{1 + (tang.\, a)^2}} = \left(y - x\frac{d\,y}{d\,x}\right) : \sqrt{1 + \left(\frac{d\,y}{d\,x}\right)^2},$$

as well as by the formula $n = (y \, cotg. \, a - x) \, sin. \, a = \frac{y \, cotg. \, a - x}{\sqrt{1 + (cotg. \, a)^2}}$

$$= \left(y \frac{dx}{dy} - x\right) : \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

when we substitute x and $y = \infty$ in them.

In order that a tangent to a point infinitely distant can be an asymptote, it is necessary, that for x or $y = \infty$, $y - x \, tang. \, a$ or $y \, cos. \, a - x$ shall not become infinitely great.

For a curve whose equation is $y = x^{-m} = \frac{1}{x^m}$

$tang. \, a = - \frac{m}{x^{m+1}}$ and $y - x \, tang. \, a = x^{-m} + \frac{m}{x^m} = \frac{m+1}{x^m}$

and also $y \, cotg. \, a - x = - \frac{x}{m} - x = - (m + 1) \frac{x}{m}$, therefore

1) for $x = \infty$, $y = 0$, $tang. \, a = 0$, $y - x$, $tang. \, a = 0$ and $n = 0$, and 2) for $y = \infty$, $x = 0$, $tang. \, a = \infty$, $y \, cotg. - x = 0$ and $n = 0$.

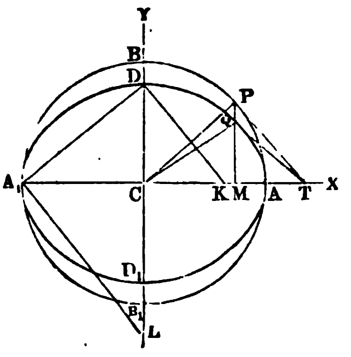
The axis of abscissas $X \overline{X}$ corresponds to the conditions $tang. \, a = \infty$ and $n = 0$, the axis of ordinates $Y \overline{Y}$ to the conditions $tang. \, a = 0$ and $n = 0$; therefore these axes are the asymptotes of the curve, corresponding to the equation $y = x^{-m}$. (Compare the curves $\overline{1} \, P_1 \, \overline{1}$, $\overline{2} \, P_1 \, \overline{2}$, and $\overline{\tfrac{1}{2}} \, P_1 \, \overline{\tfrac{1}{2}}$ in Fig. 18, page 48.)

ART. 12. The equation of an ellipse $A \, D \, A_1 \, D_1$, Fig. 20, can be deduced from the equation

$$x^2 + y_1^2 = a^2$$

of the circle $A \, B \, A_1 \, B_1$, whose radius is $C A = C B = C P = a$ and whose co-ordinates are $C M = x$ and $M P = y_1$, when we consider, that the ordinate $M Q = y$ of the ellipse is to the ordinate $M P = y_1$ of the circle, as the lesser semi-axis $C D = b$ of the ellipse is to the greater semi-axis, which is equal to the radius of the circle $C B = a$. We have then

FIG. 20.



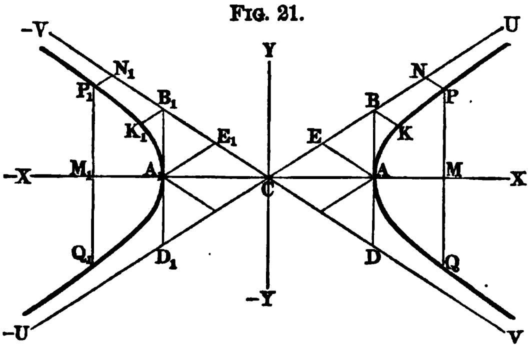
$\frac{y}{y_1} = \frac{b}{a}$, whence $y_1 = \frac{a}{b} y$ and $x^2 + \frac{a^2}{b^2} y^2 = a^2$, I.E.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ the equation of the ellipse.}$$

If we substitute in this equation for $+ b^2$, $- b^2$, we obtain the equation
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$
which is that of the hyperbola formed by the two branches $P\,A\,Q$ and $P_1\,A_1\,Q_1$, Fig. 21.

When in the formula
$$y = \frac{b}{a}\sqrt{x^2 - a^2}$$
deduced from the latter equation we take x infinitely great, a^2 disappears before x^2, and we have
$$y = \frac{b}{a}\sqrt{x^2} = \pm\frac{bx}{a} = \pm\, x \, tang.\, a,$$
the equation of two right lines $C\,U$ and $C\,V$ passing through the origin of co-ordinates C. Since the ordinates
$$\pm\frac{b}{a}x = \frac{b}{a}\sqrt{x^2} \text{ and } \frac{b}{a}\sqrt{x^2 - a^2}$$

FIG. 21.



tend to become equal as x becomes greater, it follows that the right lines $C\,U$ and $C\,V$ are *the asymptotes of the Hyperbola.*

If we take $C\,A = a$, the perpendicular $A\,B = + b$ and $A\,D = - b$, we can determine the two asymptotes; for the tangent of the angle $\pm\, a$, formed by the asymptotes with the axis of abscissas, is
$$tang.\, A\,C\,B = \frac{A\,B}{C\,A}, \text{ I.E. } tang.\, a = \frac{b}{a}, \text{ and}$$
in like manner
$$tang.\, A\,C\,D = \frac{A\,D}{C\,A}, \text{ I.E. } tang.\,(-a) = -\frac{b}{a}.$$

If we take the asymptotes $U\,\overline{U}$ and $V\,\overline{V}$ as axes of co-ordinates, and put the abscissa or co-ordinate $C N$ in the direction of the one axis $= u$, and the ordinate or co-ordinate $N P$ in the direction of the other $= v$, we have, since the direction of u varies from the axis of abscissas by the angle a, and that of v by the angle $- a$

$$C M = x = C N \cos. a + N P \cos. a = (u + v) \cos. a, \text{ and}$$
$$M P = y = C N \sin. a - N P \sin. a = (u - v) \sin. a.$$

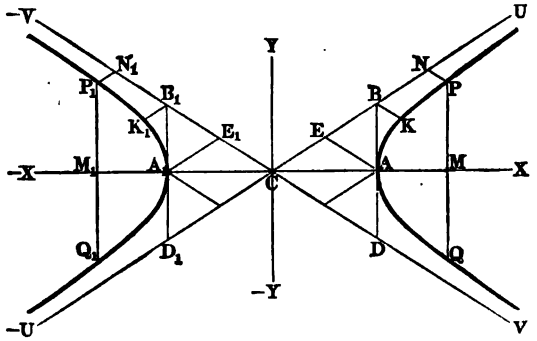
If we designate the hypothenuse $C B = \sqrt{a^2 + b^2}$ by e, we have
$$\cos. a = \frac{a}{e} \text{ and } \sin. a = \frac{b}{e},$$
and consequently $\frac{\cos. a}{a} = \frac{\sin. a}{b} = \frac{1}{e}$, and
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{(u^2 + 2 u v + v^2)}{a^2} \cos.^2 a - \frac{(u^2 - 2 u v + v^2)}{b^2} \sin.^2 a$$
$$= \frac{u^2 + 2 u v + v^2}{e^2} - \frac{u^2 - 2 u v + v^2}{e^2} = \frac{4 u v}{e^2} = 1.$$

From the latter we obtain what is known as the equation of the hyperbola referred to its asymptotes
$$u v = \frac{e^2}{4} \text{ or } v = \frac{e^2}{4 u}.$$

According to this it is easy to draw the hyperbola between the two given asymptotes.

The co-ordinates of the vertex A are $C E = E A = \frac{e}{2}$, and

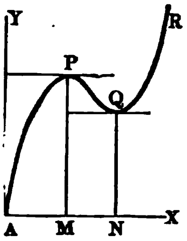
FIG. 22.



the co-ordinates for the point K are $C B = e$ and $B K = \frac{e}{4}$; further, for the abscissas $2 e$, $3 e$, $4 e$, etc., the ordinates are $\frac{1}{2} \frac{e}{4}$, $\frac{1}{3} \frac{e}{4}$, $\frac{1}{4} \frac{e}{4}$, etc.

Art. 13. If in the ratio of the differentials $\frac{dy}{dx}$, or in the formula for the tangent *tang.* α of the tangential angle, we substitute successively the different values of x, we obtain all the different positions of the tangent to the corresponding curve. If we take $x = 0$, we obtain the tangent of the tangential angle at the origin of co-ordinates, and if on the contrary we take $x = \infty$, we have the same for a point infinitely distant. The most important points are those where the tangent to the curve runs parallel to one or other of the co-ordinate axes, because here one or other of the co-ordinates x and y have their greatest or smallest value, or, as we say, is a maximum or minimum. When the curve is parallel to the axis of abscissas we have $\alpha = 0$, and *tang.* $\alpha = 0$; when parallel to the axis of ordinates $\alpha = 90°$, or *tang.* $\alpha = \infty$, whence we deduce the following Rule:

Fig. 23.



To find the values of the abscissa or independent variable x, which correspond to the maximum or minimum value of the ordinate or dependent variable y, we must put the ratio of the differentials $\frac{dy}{dx} = 0$, or $= \infty$ and resolve the resulting equation in regard to x; e.g., for the equation $y = 6x - \frac{9}{2}x^2 + x^3$, which corresponds to the curve $A\,P\,Q\,R$ in Fig. 23.

$$\frac{dy}{dx} = 6 - 9x + 3x^2 = 3(2 - 3x + x^2) = 3(1 - x)(2 - x);$$

consequently, in placing $\frac{dy}{dx} = 0$, we have

$$1 - x = 0 \text{ and } 2 - x = 0,$$

i.e. $x = 1$ and $x = 2$.

Substituting these values in the formula

$$y = 6x - \tfrac{9}{2}x^2 + x^3,$$

we have the maximum value of y, $MP = 6 - \frac{9}{2} + 1 = \frac{5}{2}$, and the minimum value, $NQ = 12 - 18 + 8 = 2$.

Farther, for the curve $K\,O\,P\,Q\,R$, Fig. 24, whose equation is

$$y = x + \sqrt[3]{(x-1)^2}, \text{ we have}$$

$$\frac{dy}{dx} = tang.\ \alpha = 1 + \tfrac{2}{3}(x - 1)^{-\frac{1}{3}} = 1 + \frac{2}{3\sqrt[3]{x-1}},$$

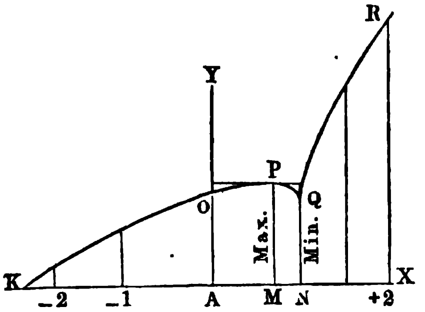
which becomes $= 0$, for $\frac{2}{3\sqrt[3]{x-1}} = -1$, i.e. for $AM = x = 1 - (\frac{2}{3})^3 = \frac{19}{27} = 0.7037$, and on the contrary $= \infty$, for $AN = x = 1$.

The first case corresponds to the maximum value,

$$M P = y_m = 1 - (\tfrac{2}{3})^3 + (\tfrac{2}{3})^2 = \tfrac{31}{27} = 1.148,$$

and the last to the minimum value, $N Q = y_n = 1$.

FIG. 24.



We have also $A\ O = y = 1$ for $x = 0$, and $y = 0$ for the abscissa $A\ K = x$, corresponding to the cubic equation $x^3 + x^2 - 2x + 1$, whose value is $x = -2.148$.

ART. 14. Since in the equation of a curve which starts from the origin of co-ordinates A, and rises above the axis of abscissas, y increases with x, $d\ y$ is always positive, and since when the curve on the contrary descends towards that axis, y decreases when x increases, $d\ y$ becomes negative. Finally at the point where the curve runs parallel to the co-ordinate axis $A\ X$, $d\ y$ becomes equal to zero, and the differentials of the ordinates, corresponding to the equal differentials $d\ x = M\ N = N\ O = P\ S = Q\ T$ of the abscissas, are

$$S\ Q = P\ S\ tang.\ Q\ P\ S, \text{ I.E., } d\ y_1 = d\ x\ tang.\ a_1,$$

$$T R = Q\ T\ tang.\ R\ Q\ T, \text{ I.E., } d\ y_2 = d\ x\ tang.\ a_2, \text{ etc.}$$

The tangential angles a_1, a_2, etc., also increase for a convex curve $A\ P\ R$, Fig. 25, and decrease for a concave curve $A\ P\ R$, Fig. 26; consequently in the first case

FIG. 25.

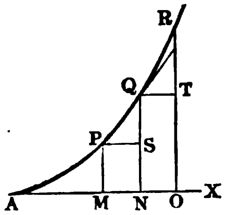


FIG. 26.

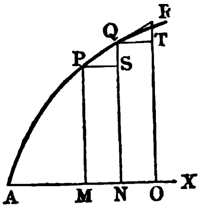
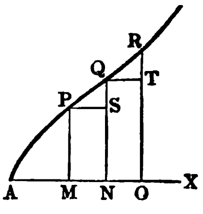


FIG. 27.



$$d\ (tang.\ a) = d \left(\frac{d\ y}{d\ x}\right) \text{ is positive,}$$

and in the second $d\ (tang.\ a) = d \left(\frac{d\ y}{d\ x}\right)$ is negative, and for the points of inflexion Q, Fig. 27, I.E. for the places Q where the convexity changes into concavity, or where the contrary takes place, we have $S\,Q = T\,R$, and therefore $d\ (tang.\ a) = d \left(\frac{d\,y}{d\,x}\right) = 0$.

Hence we have the following Rule:

If the differential of the tangential angle is positive, the curve is convex, if it is negative, the curve is concave, and if it is equal to zero we have a point of inflexion of the curve to deal with. From the foregoing we can easily make the following deductions:

The place, where the curve runs parallel with the axis of abscissas and for which $tang.\ a = 0$, corresponds either to a minimum or to a maximum, or to a point of inflexion of the curve, according as the curve is convex, concave, or neither, I.E., as $d\ (tang.\ a)$ is *positive*, *negative*, or *equal to zero*. On the contrary, the point, where the curve runs parallel with the axis of ordinates and for which we have $tang.\ a = \infty$, corresponds to a minimum, or maximum, or to a point of inflexion of the curve, according as the latter is concave, convex, or in part concave, or in part convex: I.E., as $d\ (tang.\ a)$ is negative or positive on each side of this point, or has a different sign on different sides of it.

A portion of a curve with a point of inflexion of the first kind is shown in Fig. 28, and a curve with one of the second kind in Fig. 29. We perceive that the corresponding ordinate $N\,Q$ is neither a maximum nor a minimum, for in this case both of the neighboring ordinates $M\,P$ and $O\,R$ are larger or smaller than $N\,Q$. In Geometry, Physics, Mechanics, etc., the determination of the maximum and minimum, or the so-called eminent, values of a function, is often of the greatest importance. Since in the course of this work various determinations of such values of functions will be met with, we will here treat only the following geometrical problem.

FIG. 28.

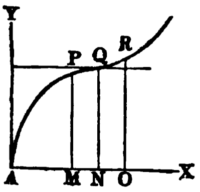


FIG. 29

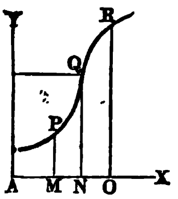
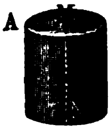


FIG. 30.



To determine the dimensions of a circular cylinder $A\,N$, Fig. 30, which for a given contents V has the smallest surface O, let us designate the diameter of the base of the cylinder by x and the height of the same by y; here we have

$$V = \frac{\pi}{4} x^2 y$$

and the surface or the area of the two bases plus that of the curved portion

$$O = \frac{2 \pi x^2}{4} + \pi x y,$$

but from the first equation we have

$$\pi y = \frac{4 V}{x^2} \text{ or } \pi x y = 4 V x^{-1}$$

substituting this value of $\pi x y$, we obtain

$$O = \frac{\pi x^2}{2} + 4 V x^{-1},$$

and since we can treat O and x as the co-ordinates of a curve, we have

$$tang.\ a = \frac{d O}{d x} = \pi x - 4 V x^{-2}.$$

Putting this quotient equal to zero, we obtain the equation of condition

$$\pi x = \frac{4 V}{x^2} \text{ or } \pi x^3 = 4 V.$$

Resolving the equation in reference to x, we have

$$x = \sqrt[3]{\frac{4 V}{\pi}}, \text{ and}$$

$$y = \frac{4 V}{\pi x^2} = \sqrt[3]{\frac{64 V^3}{\pi^3} \cdot \frac{\pi^2}{16 V^2}} = \sqrt[3]{\frac{4 V}{\pi}} = x.$$

Since $d\ (tang.\ a) = \left(\pi + \frac{8 V}{x^3}\right) d x$ is positive, the value found furnishes the required minimum. We can employ the same method when we wish to determine the dimensions of a cylindrical vessel which for a given contents will need the smallest amount of material. They are already determined directly when the vessel besides its circular bottom is to have a circular cover, but when the latter is not needed we have

$$O = \frac{\pi x^2}{4} + 4 V x^{-1}, \text{ consequently}$$

$$\frac{\pi x}{2} = \frac{4 V}{x^2}, \text{ whence it follows that}$$

$$x = 2 \sqrt[3]{\frac{V}{\pi}} \text{ and } y = \sqrt[3]{\frac{V^3}{\pi^3} \cdot \frac{\pi^2}{V^2}} = \sqrt[3]{\frac{V}{\pi}} = \tfrac{1}{2} x.$$

While in the first case we must make the height equal to the width of the cylinder, in the second we must make it but one-half the width of the latter.

ART. **15.** By successive differentiations of a function $y = f(x)$, we obtain a whole series of new functions of the independent variable x, which are

$$f_1(x) = \frac{dy}{dx} = \frac{df(x)}{dx}$$

$$f_2(x) = \frac{df_1(x)}{dx}, f_3(x) = \frac{df_2(x)}{dx}, \text{ etc.},$$

E.G., for $y = f(x) = x^{\frac{5}{3}}$, we have

$$f_1(x) = \tfrac{5}{3} x^{\frac{2}{3}}, f_2(x) = \tfrac{10}{9} x^{-\frac{1}{3}}, f_3(x) = -\tfrac{10}{27} x^{-\frac{4}{3}}, \text{ etc.}$$

For a function which is developed according to a series of the ascending powers of x

$$y = f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + \text{ etc., we have}$$

$$f_1(x) = A_1 + 2 A_2 x + 3 A_3 x^2 + 4 A_4 x^3 + \text{ etc.}$$

$$f_2(x) = 2 A_2 + 2 . 3 A_3 x + 3 . 4 A_4 x^2 + \text{ etc.}$$

$$f_3(x) = 2 . 3 A_3 + 2 . 3 . 4 A_4 x + \text{ etc.}$$

Substituting in these series $x = 0$ we obtain a series of expressions suitable for the determination of the constants $A_0, A_1, A_2 \ldots$ viz.

$$f(0) = A_0, f_1(0) = 1 A_1, f_2(0) = 2 A_2, f_3(0) = 2 . 3 . A_3,$$

etc., whence we deduce these co-efficients themselves.

$$A_0 = f(0), A_1 = f_1(0), A_2 = \tfrac{1}{2} f_2(0), A_3 = \frac{1}{2 . 3} f_3(0),$$

$$A_4 = \frac{1}{2 . 3 . 4} f_4(0) \text{ etc.}$$

Thus we can develop a function into the following series, known as McLaurin's.

$$f(x) = f(0) + f_1(0) \cdot \frac{x}{1} + f_2(0) \cdot \frac{x^2}{1 . 2} + f_3(0) \cdot \frac{x^3}{1 . 2 . 3}$$
$$+ f_4(0) \cdot \frac{x^4}{1 . 2 . 3 . 4} + \ldots\ldots$$

For the binomial function $y = f(x) = (1 + x)^n$ we have

$$f_1(x) = n(1 + x)^{n-1}, f_2(x) = n(n - 1)(1 + x)^{n-2}$$
$$f_3(x) = n(n - 1)(n - 2)(1 + x)^{n-3}, \text{ etc.}$$

When we put $x = 0$, we obtain

$$f(0) = 1, f_1(0) = n, f_2(0) = n(n - 1)$$
$$f_3(0) = n(n - 1)(n - 2), \text{ etc., whence the binomial series.}$$

$$\text{I.)}\ (1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \text{etc.}$$

We have also

$$(1-x)^n = 1 - \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 - \frac{n(n-1)(n-2)}{1.2.3}x^3 + \text{etc.,}$$

as well as

$$(1+x)^{-n} = 1 - \frac{n}{1}x + \frac{n(n+1)}{1.2}x^2 - \frac{n(n+1)(n+2)}{1.2.3}x^3 + \dots.$$

Farther, putting $1 + x = (1 - z)^{-1} = \frac{1}{1-z}$, we have $z = \frac{x}{1+x}$ and

$$(1+x)^n = (1-z)^{-n} = 1 + nz + \frac{n(n+1)}{1.2}z^2 + \frac{n(n+1)(n+2)}{1.2.3}z^3 + \dots\ \text{I.E.}$$

$$\text{II.)}\ (1+x)^n = 1 + \frac{n}{1}\left(\frac{x}{1+x}\right) + \frac{n(n+1)}{1.2}\left(\frac{x}{1+x}\right)^2 + \frac{n(n+1)(n+2)}{1.2.3}\left(\frac{x}{1+x}\right)^3 + \dots$$

The series I. is finite for entire positive values of n, and the series II. for entire negative values of the same.

E.G., $(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$, and

$$(1+x)^{-5} = 1 - 5\left(\frac{x}{1+x}\right) + 10\left(\frac{x}{1+x}\right)^2 - 10\left(\frac{x}{1+x}\right)^3 + 5\left(\frac{x}{1+x}\right)^4 - \left(\frac{x}{1+x}\right)^5.$$

Since $a + x = a\left(1 + \frac{x}{a}\right)$, it follows also that

$$(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n = a^n\left[1 + \frac{n}{1}\left(\frac{x}{a}\right) + \frac{n(n-1)}{1.2}\left(\frac{x}{a}\right)^2 + \dots\right]\ \text{I.E.}$$

$$\text{III.)}\ (a+x)^n = a^n + \frac{n}{1}a^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3 + \dots$$

E.G., $$\sqrt[3]{1009^2} = (1000 + 9)^{\frac{2}{3}} = 100\,(1 + 0{,}009)^{\frac{2}{3}}$$
$$= 100\left(1 + \tfrac{2}{3}\,.\,0{,}009 + \frac{\frac{2}{3}\,(\frac{2}{3} - 1)}{2}\,.\,(0{,}009)^2 + \dots\right)$$
$$= 100\,(1 + 0{,}006 - 0{,}000009) = 100{,}5991.$$

We have also
$$(x + 1)^n = x^n + n\,x^{n-1} + \frac{n\,(n - 1)}{1\,.\,2}\,x^{n-2} + \dots \text{etc.}$$
and approximately for very great values of x,
$$(x + 1)^n = x^n + n\,x^{n-1}.$$

From this it follows that
$$x^{n-1} = \frac{(x + 1)^n - x^n}{n}, \text{ further}$$
$$(x - 1)^{n-1} = \frac{x^n - (x - 1)^n}{n},$$
$$(x - 2)^{n-1} = \frac{(x - 1)^n - (x - 2)^n}{n},$$
$$(x - 3)^{n-1} = \frac{(x - 2)^n - (x - 3)^n}{n},$$
$$\vdots \quad = \quad \vdots$$
and finally $$1^{n-1} = \frac{2^n - 1^n}{n};$$
adding the two members of these equations together, we have
$$x^{n-1} + (x - 1)^{n-1} + (x - 2)^{n-1} + (x - 3)^{n-1} + \dots + 1$$
$$= \frac{(x + 1)^n - 1^n}{n},$$
or, putting $n - 1 = m$, and writing the series in the reversed order, we have
$$1^m + 2^m + 3^m + \dots + (x - 1)^m + x^m = \frac{(x + 1)^{m+1} - 1}{m + 1}.$$

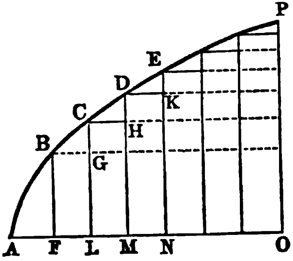
Now since x is very great, or properly infinitely great, we can put $(x + 1)^{m+1} = x^{m+1}$, and we then obtain the sum of the powers of the natural series of numbers.

IV.) $$1^m + 2^m + 3^m + \dots + x^m = \frac{x^{m+1}}{m + 1},$$

E.G., $\sqrt[3]{1^2} + \sqrt[3]{2^2} + \sqrt[3]{3^2} + \sqrt[3]{4^2} + \dots + \sqrt[3]{1000^2}$ approximately
$$= \frac{1000^{\frac{5}{3}}}{\frac{5}{3}} = \tfrac{3}{5}\sqrt[3]{1000^5} = 60000.$$

ART. **16.** The ordinate $OP = y$, Fig. 31, corresponding to the abscissa $AO = x$, can be considered as composed of an infinite number of unequal elements dy, as FB, GC, HD, KE...., which correspond to the equal differentials $dx = AF, = FL = LM = MN$.... of the abscissa. If therefore $dy = \phi(x) \cdot dx$ were given, we could determine y by summing all the values of dy, which we obtain, by substituting successively in $\phi(x)\,dx$ for x, dx, $2\,dx$, $3\,dx$.... to $n\,dx = x$. This summing is indicated by the so-called *sign of integration* $\int$, which is placed before the general expression of the differential to be summed. Thus we write, instead of

FIG. 81.



$$y = [\phi(dx) + \phi(2\,dx) + \phi(3\,dx) + \ldots + \phi(x)]\,dx,$$

$$y = \int \phi(x)\,dx.$$

In this case we call y the *integral* of $\phi(x)\,dx$, and $\phi(x)\,dx$ the differential of y. Sometimes we can obtain the integral $\int \phi(x)\,dx$, by really summing up the series $\phi(dx)$, $\phi(2\,dx)$, $\phi(3\,dx)$, etc.; but it is always simpler in the determination of an integral to employ one of the Rules of what is known as the Integral Calculus, which will be the next subject treated.

If n is the number of differentials dx of x, we have $x = n\,dx$ or $dx = \frac{x}{n}$, and we can put

$$\int \phi(x)\,dx = \left[\phi\left(\frac{x}{n}\right) + \phi\left(\frac{2\,x}{n}\right) + \phi\left(\frac{3\,x}{n}\right) + \ldots + \phi\left(\frac{n\,x}{n}\right)\right]\frac{x}{n}.$$

Thus for the differential $dy = a\,x\,dx$, we have

$$y = \int a\,x\,dx = a\,dx\,(dx + 2\,dx + 3\,dx + \ldots + n\,dx)$$
$$= (1 + 2 + 3 + \ldots + n)\,a\,dx^2,$$

or since according to Art. 15, IV., for $n = \infty$ we have the sum of the natural series of numbers

$$1 + 2 + 3 + 4 + 5 \ldots + n = \tfrac{1}{2}\,n^2 \text{ and } dx^2 = \frac{x^2}{n^2}$$

$$y = \int a\,x\,dx = \tfrac{1}{2}\,n^2\,a\,\frac{x^2}{n^2} = \tfrac{1}{2}\,a\,x^2.$$

In a similar way we find, if $x = n\,dx$ or if x is composed of n elements dx,

$$y = \int \phi(x)\,dx = \int \frac{x^2\,dx}{a} = \left[(dx)^2 + (2\,dx)^2 + (3\,dx)^2 + \ldots + (ndx)^2\right]\frac{dx}{a}$$
$$= (1 + 2^2 + 3^2 + \ldots + n^2)\frac{dx^3}{a}.$$

But from § 15, IV., for $n = \infty$, we have

$1 + 2^2 + 3^2 + \ldots . + n^2 = \frac{n^3}{3}$, whence it follows that

$$\int \frac{x^2\,d\,x}{a} = \frac{n^3}{3} \cdot \frac{d\,x^3}{a} = \frac{(n\,d\,x)^3}{3\,a} = \frac{x^3}{3\,a}.$$

ART. 17. From the formula $d\,(a + m\,f\,(x)) = m\,d\,f\,(x)$, we obtain by inversion

$\int m\,d\,f\,(x) = a + m\,f\,(x) = a + m \int d\,f\,(x)$, or putting

$$d\,f\,(x) = \phi\,(x)\,.\,d\,x$$

I.) $\int m\,\phi\,(x)\,d\,x = a + m \int \phi\,(x)\,d\,x,$

and hence it follows that the constant factor m remains, in the *Integration* as in the *Differentiation*, unchanged, and that a constant member such as a can not be determined by mere integration; the integration furnishes only an *indefinite* integral.

In order to find the constant member, a pair of corresponding values of x and $y = \int \phi\,(x)\,d\,x$ must be known. If for $x = c, y = k$, and we have found $y = \int \phi\,(x)\,d\,x = a + f\,(x)$ then we must also have $k = a + f\,(c)$, and by subtraction we obtain $y - k = f\,(x) - f\,(c)$; therefore in this case we have

$$y = \int \phi\,(x)\,d\,x = k + f\,(x) - f\,(c) = f\,(x) + k - f\,(c),$$

and the constant factor $a = k - f\,(c)$.

When, E.G., we know that the indefinite integral $y = \int x\,d\,x = \frac{x^2}{2}$ gives, for $x = 1, y = 3$ we have the necessary constant $a = 3 - \frac{1}{2} = \frac{5}{2}$, and therefore the integral

$$y = \int x\,d\,x = a + \frac{x^2}{2} = \frac{5 + x^2}{2}.$$

Even the determination of the constant leaves the integral still indefinite, for we can assume any value for the independent variable x; but if we wish to have the definite value k_1 of the integral corresponding to the definite value c_1 of x, we must substitute this value in the integral which we have found, or, $k_1 = k + f\,(c_1) - f\,(c)$.

$$\text{E.G., } y = \int x\,d\,x = \frac{5 + x^2}{2} \text{ gives, for } x = 5, y = 15.$$

Generally the value of x for which y becomes $= 0$ is known; in this case we have $k = 0$, and the indefinite integral of the form

$\int \phi x \, (d x) = f(x)$ leads to the definite one $k_1 = f(c_1) - f(c)$, which can also be found by substituting in the expression $f(x)$ of the indefinite integral the two given limits, c_1 and c, of x, and by subtracting the values found from one another. In order to indicate this we write instead of $\int \phi(x)\, d x$, $\int_c^{c_1} \phi(x)\, d x$,

$$\text{if, E.G., } \int \phi \, d x = \frac{x^2}{2}, \int_c^{c_1} \phi(x) d x = \frac{c_1^2 - c^2}{2}.$$

By the inversion of the differential formula

$d\,[f(x) + \phi(x)] = d f(x) + d \phi(x)$ we obtain the integral formula $\int [d f(x) + d \phi(x)] = f(x) + \phi(x)$, or putting

$d f(x) = \psi(x)\, d x$ and $d \phi(x) = \chi(x)\, d x$,

II.) $\int [\psi(x)\, d x + \chi(x)\, d x] = \int \psi(x)\, d x + \int \chi(x)\, d x.$

Therefore *the integral of the sum of several differentials is equal to the sum of the integrals of each of the differentials.*

E.G. $\int (3 + 5x)\, d x = \int 3\, d x + \int 5 x\, d x = 3x + \frac{5}{2} x^2.$

ART. **18.** The most important differential formula, IV., Art. 8, $d(x^n) = n x^{n-1}\, d x$, gives by inversion an integral formula which is equally important.

It is $\int n x^{n-1}\, d x = x^n$, or $n \int x^{n-1}\, d x = x^n$, whence

$$\int x^{n-1}\, d x = \frac{x^n}{n};$$

substituting $n - 1 = m$, and $n = m + 1$, we obtain the following important integral:

$$\int x^m\, d x = \frac{x^{m+1}}{m+1},$$

which is employed at least as often as all the other formulas together.

The form of this integral shows that it corresponds to the system of curves treated in Art. 9 and represented in Fig. 17.

From it we have, E.G.,

$$\int 5 x^3\, d x = 5 \int x^3\, d x = \frac{5}{4} x^4;$$
$$\int \sqrt[3]{x^4}\, d x = \int x^{\frac{4}{3}}\, d x = \frac{3}{7} x^{\frac{7}{3}} = \frac{3}{7} \sqrt[3]{x^7};$$
$$\int (4 - 6x^2 + 5x^4)\, d x = \int 4\, d x - \int 6 x^2\, d x + \int 5 x^4\, d x$$
$$= 4 \int d x - 6 \int x^2\, d x + 5 \int x^4\, d x = 4x - 2x^3 + x^5;$$ farther,

putting $3x - 2 = u$, $3\, d x = d u$, or $d x = \frac{d u}{3}$, we have

$$\int \sqrt{3\,x-2}\,.\,d\,x = \int u^{\frac{1}{2}}\,\frac{d\,u}{3} = \tfrac{1}{3}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}} = \tfrac{2}{9}\,\sqrt{u^3} = \tfrac{2}{9}\,\sqrt{(3\,x-2)^3};$$

and finally, substituting $2\,x^2 - 1 = u$ and $4\,x\,d\,x = d\,u$ or $x\,d\,x = \frac{d\,u}{4}$, we have

$$\int \frac{5\,x\,d\,x}{\sqrt[3]{2\,x^2-1}} = \int \frac{5\,d\,u}{4\sqrt[3]{u}} = \tfrac{5}{4}\int u^{-\frac{1}{3}}\,d\,u = \tfrac{5}{4}\,\frac{u^{\frac{2}{3}}}{\frac{2}{3}}$$

$$= \frac{15}{8}\,\sqrt[3]{u^2} = \frac{15}{8}\,\sqrt[3]{(2\,x^2-1)^2}.$$

By the substitution of the limits the indefinite integral can be changed into a definite one.

E.G. $\int_1^2 5\,x^3\,d\,x = \tfrac{5}{4}\,(2^4 - 1^4) = \tfrac{5}{4}\,.\,(16 - 1) = 18\tfrac{3}{4}.$

$$\int_4^9 \frac{d\,x}{2\,\sqrt{x}} = \sqrt{9} - \sqrt{4} = 1$$

$$\int_1^6 \sqrt{3\,x-2}\,.\,d\,x = \tfrac{2}{9}\,(\sqrt{16^3} - \sqrt{1^3}) = \tfrac{2}{9}\,(64-1) = 14.$$

If E.G. $\int(4 - 6\,x^2 + 5\,x^4)\,d\,x = 7$, for $x = 0$ we would have, in general,

$$\int (4 - 6\,x^2 + 5\,x^4)\,d\,x = 7 + 4\,x - 2\,x^3 + x^5.$$

ART. 19. The so-called *exponential function* $y = a^x$, which consists of a power with a variable exponent, can be developed as follows into a series by means of McLaurin's Theorem, and its differential can then be found.

Putting $a^x = A_0 + A_1\,x + A_2\,x^2 + A_3\,x^3 + \ldots.$ we have, for $x = 0$, $a^x = a^0 = 1$, whence $A_0 = 1$;

From $a^x = 1 + A_1\,x + A_2\,x^2 + A_3\,x^3 + \ldots.$ we have

$$a^{dx} = 1 + A_1\,d\,x + A_2\,d\,x^2 + A_3\,d\,x^3 + \ldots. \text{ and also}$$

$$\begin{aligned} d\,(a^x) &= a^{x+dx} - a^x = a^x\,a^{dx} - a^x = a^x\,(a^{dx} - 1) \\ &= a^x\,(A_1\,d\,x + A_2\,d\,x^2 + A_3\,d\,x^3 + \ldots.) \\ &= a^x\,(A_1 + A_2\,d\,x + \ldots.)\,d\,x = A_1\,a^x\,d\,x. \end{aligned}$$

Hence, by successive differentiation of the series, we have

$$f\,(x) = a^x = 1 + A_1\,x + A_2\,x^2 + A_3\,x^3 + \ldots$$

$$f_1\,(x) = \frac{d(a^x)}{d\,x} = A_1\,a^x = A_1 + 2\,A_2\,x + 3\,A_3\,x^2 + \ldots$$

$$f_2(x) = \frac{d\,(A_1\,a^x)}{d\,x} = A_1^2\,a^x = 2\,A_2 + 2\,.\,3\,.\,A_3 x + \ldots$$

$$f_3(x) = \frac{d\,(A_1^2\,a^x)}{d\,x} = A_1^3\,a^x = 2\,.\,3\,.\,A_3 + \ldots$$

Putting $x = 0$, it follows that

$$A_1 = A_1,\ 2\,A_2 = A_1^2,\ 2\,.\,3\,.\,A_3 = A_1^3$$

whence $A_2 = \frac{1}{1\,.\,2}\,A_1^2$, $A_3 = \frac{1}{1\,.\,2\,.\,3}\,A_1^3$, $A_4 = \frac{1}{1\,.\,2\,.\,3\,.\,4}\,A_1^4$, &c.

and the exponential series takes the form

$$\text{I.}\quad a^x = 1 + A_1\frac{x}{1} + A_1^2\frac{x^2}{1.2} + A_1^3\frac{x^3}{1.2.3} + A_1^4\frac{x^4}{1.2.3.4} + \ldots.$$

The constant coefficient A_1 is of course a definite function of the constant base, as the latter is a function of the former. If one of the two numbers be given, the other is then determined. The most simple, or the so-called natural series of powers, whose base (a) will be designated hereafter by e, is obtained by putting $A_1 = 1$. Then we have,

$$\text{II.)}\quad e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \ldots$$

and if we put $x = 1$ we obtain the *base* of the natural series of powers,

$$e^1 = e = 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \ldots. = 2{,}7182828.$$

If we put $e = a^m$, or $a = e^{\frac{1}{m}}$, we have $\frac{1}{m} = l\,a$, which is the Naperian or hyperbolic Logarithm of a, and

$$\text{III.)}\quad a^x = (e^{\frac{1}{m}})^x = e^{\frac{x}{m}} = 1 + \frac{1}{1}\left(\frac{x}{m}\right) + \frac{1}{1\,.\,2}\left(\frac{x}{m}\right)^2 + \frac{1}{1\,.\,2\,.\,3}\left(\frac{x}{m}\right)^3 + \ldots\,.$$

Since this series corresponds in its form to that of I, we have also $A_1 = \frac{1}{m}$, and,

$$\text{IV.)}\quad d\,(a^x) = A_1\,a^x\,d\,x = \frac{a^x\,d\,x}{m} = l\,a\,.\,a^x\,d\,x, \text{ as well as}$$

$$\text{V.)}\quad d\,(e^x) = e^x\,d\,x.$$

E.G. $d\,(e^{3x+1}) = e^{3x+1}\,d\,(3\,x + 1) = 3\,e^{3x+1}\,d\,x.$

If we put $y = a^x = e^{\frac{x}{m}}$ we have, on the contrary,

$$x = \log_e y \text{ and } \frac{x}{m} = l\,y.$$
$$\log_a y = m\,l\,y, \text{ and, on the contrary,}$$
$$l\,y, \text{ or } \log_e y = \frac{1}{m} \log_a y.$$

The number m is called the *modulus* of the system corresponding to the base a. By means of it we can transform the Naperian logarithm into any artificial one, or one of the latter into the former. For Brigg's system of Logarithms the base is $a = 10$, whence $\frac{1}{m} = l\,10 = 2{,}30258$, and, on the contrary, $m = \frac{1}{l\,10} = 0{,}43429$.

We have also $\log y = 0{,}43429\; l\,y$, and
$l\,y = 2{,}30258 \log y$.
(See Ingenieur, page 81, etc.)

ART. 20. The course of the curves which correspond to the exponential functions $y = e^x$, and $y = 10^x$, is represented by Fig. 32. For $x = 0$, we have in both cases $y = e^0 = a^0 = 1$. Hence both curves $O\,Q\,S$ and $O\,Q_1\,S_1$ pass through the same point (O) of the axis of ordinates $A\,Y$. For $x = 1$ we have,

$$y = e^1 = 2{,}718, \text{ and}$$
$$y = 10^1 = 10,$$

$x = 2$ gives

$$y = e^2 = 2{,}718^2 = 7{,}389, \text{ and}$$
$$y = 10^x = 10^2 = 100, \text{ \&c.}$$

Both curves rise on the positive side of the axis of abscissas very steeply, particularly the latter.

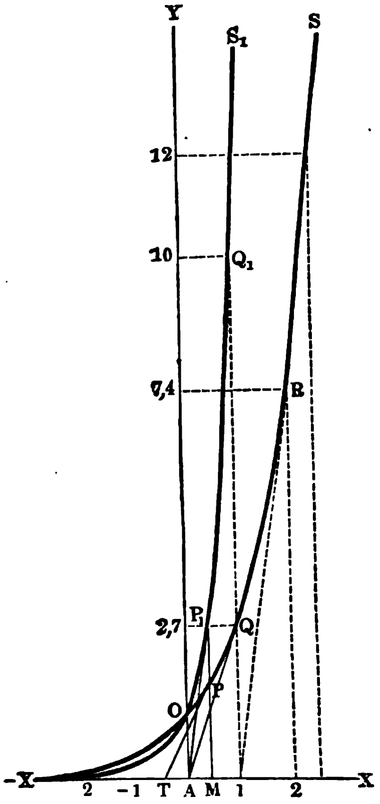
For $x = -1$ we have $e^x = e^{-1} = \frac{1}{2{,}718\ldots} = 0{,}368\ldots$, and
$$10^x = 10^{-1} = 0{,}1;$$
farther, for $x = -2$, we have
$$e^x = e^{-2} = \frac{1}{2{,}718^2} = 0{,}135,$$
and $\quad 10^x = 10^{-2} = 0{,}01;$

for $x = -\infty$ both equations give

$$e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{a^{\infty}} = 0,$$

The two curves approach nearer and nearer this axis of abscissas on the negative side of the axis of abscissas, the last more quickly than the first, but they never really meet this axis.

FIG. 82.



Since we deduce from the equation

$$y = e^x, x = l\, y$$

and also from

$$y = a^x, x = \log_a y$$

the abscissas of these curves furnish a scale for the Naperian and common logarithms; for the abscissas are the logarithms of the ordinates.

E.G. we have,

$$A\,M = l\,M\,P$$
$$= \log_a M\,P_1, \text{ etc.}$$

From the differential formula IV of the last article the tangential angle of the exponential curve is determined by the simple formula,

$$\text{tang. } a = \frac{d\,y}{d\,x} = \frac{a^x\,d\,x}{m\,d\,x}$$
$$= \frac{a^x}{m} = \frac{y}{m} = y\,l\,a.$$

Consequently for the curve $O\ P_1\ Q_1\ S_1$, Fig. 32, the subtangent $= y$ *cotg.* $a = m$, that is, is constant; and for the curve $O\ P\ Q\ S$ it is always $= 1$, E.G., for the point Q, $\overline{A\,1}$, $= 1$ for the point R, $\overline{12} = 1$, etc.

ART. 21. If $x = a^y$, we have also

$$d\,x = d\,(a^y) = \frac{a^y\,d\,y}{m},$$

and by inversion,

$$d\,y = \frac{m\,d\,x}{a^y} = \frac{m\,d\,x}{x}$$

But $y = log._a x$, that is, to the logarithm of the variable power x with the constant base a; therefore we have the following differential formula for the logarithmic functions,

$$y = log._a x \text{ and } y = l\,x:$$

$$\text{I.)}\quad d\,(log._a x) = \frac{m\,d\,x}{x} = \frac{1}{l\,a}.\frac{d\,x}{x},$$

$$\text{II.)}\quad d\,(l\,x) = \frac{d\,x}{x}.$$

If a is the tangential angle of the curve corresponding to the equation $y = log._a x$, we have $tang.\ a = \frac{m}{x}$, and the subtangent $= y\ cotg.\ a = \frac{x\,y}{m}$, or proportional to the area $x\,y$ of the rectangle constructed with the sides x and y.

By means of the differential formulas I. and II. we obtain

$$1)\quad d\,(l\sqrt[2]{x}) = \frac{d\sqrt[2]{x}}{\sqrt{x}} = \frac{d\,(x^{\frac{1}{2}})}{x^{\frac{1}{2}}} = \tfrac{1}{2}\frac{x^{-\frac{1}{2}}\,d\,x}{x^{\frac{1}{2}}} = \frac{d\,x}{2\,x}, \text{ or also}$$

$$= d\,(\tfrac{1}{2}\,l\,x) = \tfrac{1}{2}\,d\,(l\,x) = \tfrac{1}{2}\,.\,\frac{d\,x}{x}.$$

$$2)\quad d\,l\,\frac{2+x}{x^2} = d\,[l\,(2+x) - l\,x^2]$$

$$= d\,l\,(2+x) - d\,l\,(x^2)$$

$$= \frac{d\,x}{2+x} - 2\,\frac{d\,x}{x} = -\,\frac{(4+x)\,d\,x}{x\,(2+x)}.$$

$$3)\quad d\left(l\,\frac{e^x-1}{e^x+1}\right) = d\,[l(e^x-1)] - d\,[l(e^x+1)]$$

$$= \frac{d\,(e^x)}{e^x-1} - \frac{d\,(e^x)}{e^x+1} = \frac{e^x\,d\,x}{e^x-1} - \frac{e^x\,d\,x}{e^x+1} = \frac{2\,e^x\,d\,x}{e^{2x}-1}.$$

ART. 22. If we reverse the differential formulas of the foregoing article, we obtain the following important integral formulas.

From $d\,(a^x) = \frac{a^x\,d\,x}{m}$ it follows that $\int\frac{a^x\,d\,x}{m} = a^x$, I.E.,

$$\text{I.)}\quad \int a^x\,d\,x = m\,a^x = a^x : l\,a, \text{ and therefore}$$

$$\text{II.)}\quad \int e^x\,d\,x = e^x.$$

Farther, from $d\,(log._a x) = \frac{m\,d\,x}{x}$, it follows that $\int\frac{m\,d\,x}{x} = log._a x$, I.E.

III.) $\int \frac{d\,x}{x} = \frac{1}{m}\, log_e\, x = l\, x$, which is also given by the formula $d\,(l\,x) = \frac{d\,x}{x}$.

By their aid we can easily calculate the following examples:
$\int e^{5x-1}\, d\,x = \frac{1}{5} \int e^{5x-1}\, d\,(5\,x - 1) = \frac{1}{5}\, e^{5x-1}$.

$$\int \frac{3\,d\,x}{7x + 2} = \tfrac{3}{7} \int \frac{d\,(7\,x + 2)}{7\,x + 2} = \tfrac{3}{7}\, l\,(7\,x + 2).$$

$$\int \left(\frac{x^2 + 1}{x - 1}\right) d\,x = \int \left(x + 1 + \frac{2}{x - 1}\right) d\,x.$$

$$= \int x\,d\,x + \int d\,x + 2 \int \frac{d\,(x - 1)}{x - 1} = \frac{x^2}{2} + x + 2\,l\,(x - 1).$$

ART. **23.** The first integral formula $\int x^m\, d\,x = \frac{x^{m+1}}{m + 1}$ leaves the last integral undetermined; for putting $m = -1$, it follows that $\int \frac{d\,x}{x} = \int x^{-1}\, d\,x = \frac{x^0}{0}$ + a constant $= \infty$ + constant, but if we put $x = 1 + u$, and $d\,x = d\,u$, we have

$\frac{d\,x}{x} = \frac{d\,u}{1 + u} = (1 - u + u^2 - u^3 + u^4 - \ldots.)\,d\,u$; and therefore

$$\int \frac{d\,x}{x} = \int \frac{d\,u}{1 + u} = \int (1 - u + u^2 - u^3 + u^4 - \ldots.)\,d\,u$$
$$= \int d\,u - \int u\,d\,u + \int u^2\,d\,u - \int u^3\,d\,u + \ldots.$$
$$= u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \ldots.;$$

we can therefore also put $l\,(1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \ldots$, or

IV.) $l\,x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \ldots.$

With the aid of this series we can calculate the logarithm of all numbers which differ very little from 1; but if we require the logarithm of large numbers we must adopt the following method.

Taking u negative in the foregoing formula, we have

$$l\,(1 - u) = -u - \frac{u^2}{2} - \frac{u^3}{3} - \frac{u^4}{4} - \ldots;$$

and subtracting one series from the other, we have

$$l(1+u) - l\,(1-u) = 2\left(u + \frac{u^3}{3} + \frac{u^5}{5} + \ldots\right)$$

$$l\left(\frac{1+u}{1-u}\right) = 2\left(u + \frac{u^3}{3} + \frac{u^5}{5} + \ldots\right) \text{ or putting}$$

$$\frac{1+u}{1-u} = x, \text{ or } u = \frac{x-1}{x+1}, \text{ we have}$$

$$\text{V.)} \qquad l\,x = 2\left[\frac{x-1}{x+1} + \tfrac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \tfrac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \ldots\right]$$

This formula is to be employed for the determination of the logarithm of such numbers as differ sensibly from 1, since $\frac{x-1}{x+1}$ is always less than 1.

We have also $l\,(x+y) - l\,x = l\left(\frac{x+y}{x}\right) = l\left(1 + \frac{y}{x}\right)$

$$= \frac{y}{x} - \tfrac{1}{2}\left(\frac{y}{x}\right)^2 + \tfrac{1}{3}\left(\frac{y}{x}\right)^3 - \text{etc.}$$

$$= 2\left[\frac{y}{2x+y} + \tfrac{1}{3}\left(\frac{y}{2x+y}\right)^3 + \tfrac{1}{5}\left(\frac{y}{2x+y}\right)^5 + \ldots\right] \text{ whence}$$

$$\text{VI.)} \qquad l\,(x+y) = l\,x + 2\left[\frac{y}{2x+y} + \tfrac{1}{3}\left(\frac{y}{2x+y}\right)^3 + \ldots\right]$$

This formula is used to calculate from one logarithm, that of a somewhat greater number

E.G., $l2 = 2\left[\frac{2-1}{2+1} + \tfrac{1}{3}\cdot\left(\frac{2-1}{2+1}\right)^3 + \ldots\right]$

$$= 2\left(\tfrac{1}{3} + \tfrac{1}{3}\cdot\tfrac{1}{27} + \tfrac{1}{5}\cdot\tfrac{1}{243} + \ldots\right)$$

$$= 2\left\{\begin{matrix} 0{,}33333 \\ 0{,}01234 \\ 0{,}00082 \\ 0{,}00007 \end{matrix}\right\} = 2\,.\,0{,}34656 = 0{,}69312,$$

more exactly $= 0{,}69314718$.

Hence $l\,8 = l\,2^3 = 3\,l\,2 = 2{,}0794415$, and according to the last formula, $l\,10 = l\,(8+2)$

$$= l\,8 + 2\left[\frac{2}{16+2} + \tfrac{1}{3}\left(\frac{2}{16+2}\right)^3 + \ldots\right]$$

$$= 2{,}0794415 + 0{,}2231436 = 2{,}302585.$$

We can also put $l\,2 = l\,1 + 2\left[\frac{1}{2+1} + \frac{1}{3}\left(\frac{1}{2+1}\right)^3 + \ldots.\right]$

$$= 2\left(\tfrac{1}{3} + \tfrac{1}{3}\cdot\frac{1}{3^3} + \tfrac{1}{5}\cdot\frac{1}{3^5} + \ldots.\right) = 0{,}693147;$$

farther, $l\,5 = l\,(4+1) = 2\,l\,2 + 2\left(\tfrac{1}{9} + \tfrac{1}{3}\cdot\frac{1}{9^3} + \ldots.\right)$, and finally we can put $l\,10 = l\,2 + l\,5$.

(Compare Art. 19.)

ART. 24. The trigonometrical and circular functions, whose differentials will now be determined, are of practical importance.

The function of the sine, $y = sin.\, x$, gives for $x = 0$, $y = 0$;

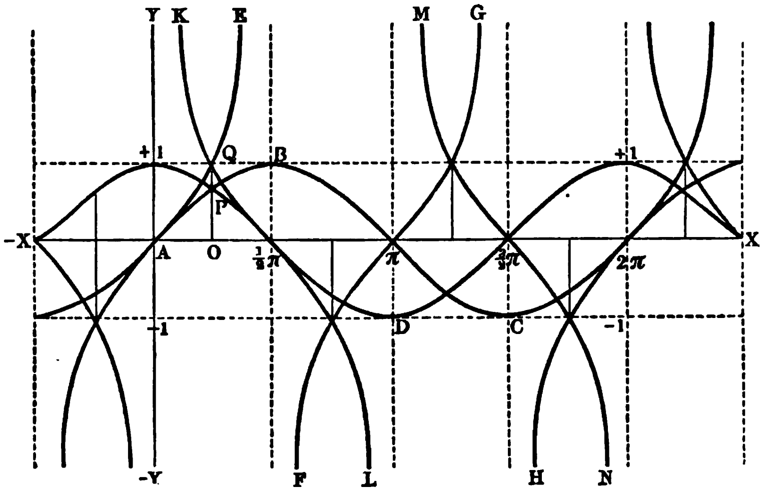
$$\text{for } x = \frac{\pi}{4} = \frac{3{,}1416}{4} = 0{,}7854\ldots, y = \sqrt{\tfrac{1}{2}} = 0{,}7071,$$

" $x = \frac{\pi}{2}$, $y = 1$, for $x = \pi$, $y = 0$;

" $x = \frac{3}{2}\pi$, $y = -1$, for $x = 2\pi$, $y = 0$, etc.

Taking x as the abscissa $A\ O$, and y as the corresponding ordinate $O\ P$, we obtain the serpentine curve $(A\ P\ B\ \pi\ C\ \overline{2\ \pi})$, Fig. 33, which continues to infinity on both sides of A.

FIG. 33.



The function of the cosine, $y = cos.\, x$, gives, for $x = 0$, $y = 1$; for $x = \frac{\pi}{4}$, $y = \sqrt{\tfrac{1}{2}}$; for $x = \frac{\pi}{2}$, $y = 0$; for $x = \pi$, $y = -1$; for $x = \frac{3}{2}\pi$, $y = 0$; for $x = 2\pi$, $y = 1$, etc.; it corresponds to exactly the same serpentine line $\left(+ 1\, P\, \frac{\pi}{2}\, D\, \frac{3\pi}{2} + 1\right)$ as the function of the sine, but it is always a distance $\frac{1}{2}\pi = 1,5708$ behind or in front of the curve of the sine.

The curves, corresponding to the function of the tangent or cotangent, $y = tang.\, a$ and $y = cotang.\, x$, are, however, of an entirely different form.

If we substitute in $y = tang.\, x$, $x = 0, \frac{1}{4}\pi, \frac{1}{2}\pi$, we obtain $y = 0, 1, \infty$, and therefore a curve $(A\ Q\ E)$ which approaches more and more, without ever attaining it, a line parallel to the axis of ordinates $A\ Y$, and cutting the axis of abscissas $A\ X$ at a distance $\frac{\pi}{2}$ from the origin of the co-ordinates. Now if we put $x = \frac{\pi}{2}, \pi, \frac{3}{2}\pi$, we obtain $y = -\infty, 0, +\infty$, and therefore a curve $(F\, \pi\ G)$, which continually approaches the parallel lines, passing through $\left(\frac{\pi}{2}\right)$ and $(\frac{3}{2}\pi)$, and for which these parallel lines are asymptotes. (See Art. 11.)

If we increase x still more, the same values of y are repeated, and therefore the function $y = tang.\, x$ corresponds to a series of curves which are separated from each other in the direction of the axis of abscissas by a distance $\pi = 3,1416$. On the contrary, the function $y = cot.\, x$ gives for $x = 0, \frac{\pi}{4}\ \frac{\pi}{2}, \pi$, $y = \infty, 1, 0, -\infty$, and therefore corresponds to a curve $\left(K\ Q\ \frac{\pi}{2}\ L\right)$ which differs from the tangential curve only by its position; it is also easy to perceive that an infinite number of branches of the curve, as, E.G., $\left(M\, \frac{3\pi}{2}\, N\right)$ correspond to this function.

While the curve of the Sine and Cosine forms a continuous, unbroken whole, the curve of the Tangent as well as that of the Cotangent is formed of separate branches; for the ordinates for certain values of x change from positive to negative infinity, in consequence of which the curve naturally loses its continuity.

ART. 25. The differentials of the trigonometrical lines or functions are given by the consideration of Fig. 34, in which

$C A = C P = C Q = 1$, arc $A P = x$, $P Q = d x$,
$P M = \sin. x$, $C M = \cos. x$, $A S = \operatorname{tang.} x$,
$O Q = N Q - M P = \sin. (x + d x) - \sin. x = d \sin. x$,
$O P = -(C N - C M) = -\cos. (x + d x) + \cos. x = -d \cos. x$, and
$S T = A T - A S = \operatorname{tang.} (x + d x) - \operatorname{tang.} x = d \operatorname{tang.} x$.

Since the elementary arc $P Q$ is perpendicular to the radius $C P$, and since the angle $P C A$ between the two lines $C P$ and $C A$ is equal to the angle $P Q O$ between the two perpendicular to them, $P Q$ and $O Q$, the triangles $C P M$ and $Q P O$ are similar, and we have

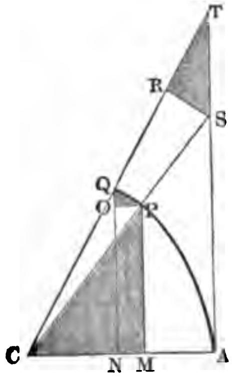
$$\frac{O Q}{P Q} = \frac{C M}{C P}, \text{ I.E. } \frac{d \sin. x}{d x} = \frac{\cos. x}{1}, \text{ whence}$$

I.) $d (\sin. x) = \cos. x \,.\, d x$, and in like manner,

$$\frac{O P}{P Q} = \frac{P M}{C P}, \text{ I.E. } \frac{-d \cos. x}{d x} = \frac{\sin. x}{1}, \text{ whence}$$

II.) $d (\cos. x) = -\sin. x \, d x$.

FIG. 34.



We see from this, that the influence of errors in the arc or angle upon the sine increases as $\cos. x$ becomes greater, or as the arc or angle becomes smaller, while on the contrary their influence upon the cosine increases as sine x becomes greater, that is, the more the arc approaches to $\frac{\pi}{2}$, and that finally the differential of the cosine has the opposite sign from that of the arc, for we know that an increase of x causes a decrease of $\cos. x$, and a decrease of x an increase of $\cos. x$.

Letting fall a perpendicular $S R$ upon $C T$ we form a triangle $S R T$ which is similar to the triangle $C P M$, since the angle $R T S$ is equal to $C Q N$ or $C P M$, and we have

$$\frac{S T}{S R} = \frac{C P}{C M}, \text{ I.E. } \frac{d \operatorname{tang.} x}{S R} = \frac{1}{\cos. x}; \text{ but we have also}$$

$$\frac{S R}{C S} = \frac{P Q}{C P}, \text{ I.E. } S R = \frac{C S \,.\, d x}{1} \text{ and}$$

$$C\,S = secant.\; x = \frac{1}{cos.\; x}, \text{ whence } S\,R = \frac{d\,x}{cos.\; x} \text{ and}$$

III.) $d\,(tang.\; x) = \dfrac{d\,x}{(cos.\; x)^2}$.

If instead of x we substitute $\frac{\pi}{2} - x$, and instead of $d\,x$, $d\left(\frac{\pi}{2} - x\right) = -\,d\,x$, we obtain

$$d\; tang.\left(\frac{\pi}{2} - x\right) = -\,\frac{d\,x}{\left[cos.\left(\frac{\pi}{2} - x\right)\right]^2}, \text{ I.E.,}$$

IV.) $d\,(cotang.\; x) = -\,\dfrac{d\,x}{(sin.\; x)^2}$.

By inversion this formula gives for the differential of the arc

$$d\,x = \frac{d\; sin.\; x}{cos.\; x} = -\,\frac{d\; cos.\; x}{sin.\; x} = (cos.\; x)^2\, d\; tang.\; x$$

$$= -\,(sin.\; x)^2\, d\; cotang.\; x, \text{ or}$$

$$d\,x = \frac{d\; sin.\; x}{\sqrt{1 - (sin.\; x)^2}} = \frac{d\; tang.\; x}{1 + (tang.\; x)^2}, \text{ as well as}$$

[*] $$d\,x = -\,\frac{d\; cos.\; x}{\sqrt{1 - (cos.\; x)^2}} = -\,\frac{d\; cotang.\; x}{1 + (cotang.\; x)^2}.$$

If we designate $sin.\; x$ by y, and x by $sin.^{-1}\, y$, we have

V.) $d\, sin.^{-1}\, y = \dfrac{d\,y}{\sqrt{1 - y^2}}$,

and in the same manner we find

VI.) $d\; cos.^{-1}\, y = -\,\dfrac{d\,y}{\sqrt{1 - y^2}}$,

VII.) $d\; tang.^{-1}\, y = \dfrac{d\,y}{1 + y^2}$

VIII.) $d\; cotang.^{-1}\, y = -\,\dfrac{d\,y}{1 + y^2}$.

ART. 26. By inversion the latter differential formulæ give

I.) $\int cos.\; x\; d\,x = sin.\; x$,

II.) $\int sin.\; x\; d\,x = -\; cos.\; x$,

III.) $\displaystyle\int \frac{d\,x}{cos.^2\, x} = tang.\; x$,

IV.) $\displaystyle\int \frac{d\,x}{sin.^2\, x} = -\; cotang.\; x$,

---

[*] $Sin.^{-1}\, y$, $tang.^{-1}\, y$, etc., designate the arc whose sine is y, whose tangent is y, etc.—TR.

V.) $\int \frac{d\,x}{\sqrt{1-x^2}} = \sin.^{-1} x = -\cos.^{-1} x$, and

VI.) $\int \frac{d\,x}{\sqrt{1+x^2}} = \text{tang.}^{-1} x = -\text{cotang.}^{-1} x.$

From the above, since we have $d\,(l\,\sin. x) = \frac{d\,\sin. x}{\sin. x} = \frac{\cos. x\,.\,d\,x}{\sin. x}$ $= \text{cotg.}\,x\,.\,d\,x$, we can easily deduce

VII.) $\int \text{cotg.}\,x\,d\,x = l\,\sin. x$, and also

VIII.) $\int \text{tang.}\,x\,d\,x = -\,l\,\cos. x$; further

$$d\,(l\,\text{tang.}\,x) = \frac{d\,\text{tang.}\,x}{\text{tang.}\,x} = \frac{d\,x}{\cos. x^2\,\text{tang.}\,x} = \frac{d\,x}{\sin. x\,\cos. x} = \frac{d\,(2\,x)}{\sin. 2\,x},$$

whence $d\,(l\,\text{tang.}\,\tfrac{1}{2}\,x) = \frac{d\,x}{\sin. x}$, and

IX.) $\int \frac{d\,x}{\sin. x} = l\,\text{tang.}\,\frac{x}{2},$

X.) $\int \frac{d\,x}{\cos. x} = l\,\text{tang.}\left(\frac{\pi}{4} + \frac{x}{2}\right) = l\,\text{cotg.}\left(\frac{\pi}{4} - \frac{x}{2}\right).$

Now putting $\frac{1}{1-x^2} = \frac{a}{1+x} + \frac{b}{1-x} = \frac{a\,(1-x)+b\,(1+x)}{(1+x)\,(1-x)}$, we have $1 = a\,(1-x) + b\,(1+x)$, and taking $1 + x = 0$, or $x = -1$, we obtain $1 = a\,(1+1)$ whence $a = \frac{1}{2}$, and putting $1 - x = 0$, or $x = 1$, we obtain $1 = 2\,b$ or $b = \frac{1}{2}$, whence

$$\frac{1}{1-x^2} = \frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x}; \text{ and finally}$$

$$\int \frac{d\,x}{1-x^2} = \tfrac{1}{2}\int \frac{d\,x}{1+x} + \tfrac{1}{2}\int \frac{d\,x}{1-x} = \tfrac{1}{2}\,l\,(1+x) - \tfrac{1}{2}\,l\,(1-x), \text{ I.E.,}$$

XI.) $\int \frac{d\,x}{1-x^2} = \frac{1}{2}\,l\left(\frac{1+x}{1-x}\right)$, and in like manner

XII.) $\int \frac{d\,x}{x^2-1} = \frac{1}{2}\,l\left(\frac{x-1}{x+1}\right).$

Putting $\sqrt{1+x^2} = xy$, we have $1 + x^2 = x^2y^2$ and $dx(1-y^2) = xy\,dy$, whence

$$\frac{dx}{\sqrt{1+x^2}} = \frac{dy}{1-y^2} = \tfrac{1}{2}\,d\,l\left(\frac{1+y}{1-y}\right), \text{ and}$$

XIII.) $$\int \frac{dx}{\sqrt{1+x^2}} = l\,(x + \sqrt{1+x^2}), \text{ and also}$$

XIV.) $$\int \frac{dx}{\sqrt{x^2-1}} = l\,(x + \sqrt{x^2-1}).$$

ART. 27. In order to find the integral of $tang.^{-1}x = \int \frac{dx}{1+x^2}$ we have only to change $\frac{1}{1+x^2}$ into a series, by division, and then integrate each member. We obtain thus

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \ldots, \text{ and}$$

$$\int \frac{dx}{1+x^2} = \int dx - \int x^2\,dx + \int x^4\,dx - \int x^6\,dx + \ldots, \text{ consequently}$$

I.) $$tang.^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \ldots \text{etc., E.G.,}$$

$\frac{\pi}{4} = tang.^{-1}1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$, and the half circumference

$\pi = 4\,(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots)$, or

$\frac{\pi}{6} = tang.^{-1}\sqrt{\frac{1}{3}} = \sqrt{\frac{1}{3}}\,[1 - \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{5}(\frac{1}{3})^2 - \frac{1}{7}(\frac{1}{3})^3 + \ldots]$,

whence $\pi = 6\sqrt{\frac{1}{3}}\,(1 - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \ldots) = 3{,}1415926\ldots.$

In the same manner we obtain from

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \tfrac{5}{16}x^6 + \ldots$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \int dx + \tfrac{1}{2}\int x^2\,dx + \tfrac{3}{8}\int x^4\,dx + \tfrac{5}{16}\int x^6\,dx + \ldots, \text{ I.E.,}$$

II.) $$sin.^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\,.\,3\,x^5}{2\,.\,4\,.\,5} + \frac{1\,.\,3\,.\,5\,x^7}{2\,.\,4\,.\,6\,.\,7} + \ldots,$$

E.G., $\frac{\pi}{6} = sin.^{-1}\frac{1}{2} = \frac{1}{2}\,(1 + \frac{1}{24} + \frac{3}{640} + \frac{5}{7168} + \ldots)$,

$$\pi = 3 \,.\left\{\begin{matrix} 1{,}04167 \\ 0{,}00469 \\ 0{,}00070 \\ 0{,}00012 \end{matrix}\right\} = 3{,}1416\,..$$

When we put $sin.\, x = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + \ldots$, etc., we obtain by successive differentiation

$$\frac{d\,(sin.\, x)}{d\,x} = cos.\, x = A_1 + 2\,A_2 x + 3\,A_3 x^2 + 4\,A_4 x^3 + \ldots$$

$$\frac{d\,(cos.\, x)}{d\,x} = -\,sin.\, x = 2\,A_2 + 2\,.\,3\,A_3 x + 3\,.\,4\,A_4 x^2 + \ldots$$

$$-\,\frac{d\,(sin.\, x)}{d\,x} = -\,cos.\, x = 2\,.\,3\,.\,A_3 + 2\,.\,3\,.\,4\,.\,A_4 x + \ldots$$

$$-\,\frac{d\,(cos.\, x)}{d\,x} = sin.\, x = 2\,.\,3\,.\,4\,.\,A_4 + \ldots$$

Now for $x = 0$ we have $sin.\, x = 0$, and $cos.\, x = 1$, therefore we obtain from the first series $A_0 = 0$, from the second $A_1 = cos.\, 0 = 1$, from the third $A_2 = 0$, from the fourth $A_3 = -\frac{1}{2\,.\,3}$, from the fifth $A_4 = 0$, etc. If we substitute these values in the supposed series, we have the series of the sine

III.) $$sin.\, x = \frac{x}{1} - \frac{x^3}{1\,.\,2\,.\,3} + \frac{x^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} - \frac{x^7}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7} \ldots\ldots$$

In the same way we obtain

IV.) $$cos.\, x = 1 - \frac{x^2}{1\,.\,2} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} - \frac{x^6}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6} + \ldots,$$

V.) $$tang.\, x = x + \frac{x^3}{3} + \frac{2\,x^5}{3\,.\,5} + \frac{17\,x^7}{3\,.\,5\,.\,7\,.\,3} + \ldots \text{ and}$$

VI.) $$cotang.\, x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{3\,.\,5\,.\,3} - \frac{2\,x^5}{3\,.\,5\,.\,7\,.\,9} -, \text{etc.}$$

(See Ingenieur, page 159.)

ART. **28.** When we integrate the differential formula $d\,(u\,v) = u\,d\,v + v\,d\,u$, of Art. 8, we obtain the expression $u\,v = \int u\,d\,v + \int v\,d\,u$, and the following formula for integration:

$$\int v\,d\,u = u\,v - \int u\,d\,v, \text{ or}$$

$$\int \phi\,(x)\,d\,f\,(x) = \phi\,(x)\,f\,(x) - \int f\,(x)\,d\,\phi\,(x).$$

This is known as the integration by parts.

This rule is always employed if the integral $\int v\,d\,u = \int \phi\,(x)\,d\,f\,(x)$ is not known, and if, on the contrary, $\int u\,d\,v =$

$\int f(x)\, d\, \phi\, x$ is. R.G. By means of this formula we can refer the integration of the formula,

$$d\,y = \sqrt{1+x^2}\,.\,d\,x$$

to another known integral. We must substitute

$$\phi\,(x) = \sqrt{1+x^2}, \text{ whence } d\,\phi\,(x) = \frac{x\,d\,x}{\sqrt{1+x^2}}$$

and $f\,(x) = x$, whence $d\,f\,(x) = d\,x$, then we have,

$$\int \sqrt{1+x^2}\,d\,x = x\sqrt{1+x^2} - \int \frac{x^2\,d\,x}{\sqrt{1+x^2}}, \text{ but}$$

$$\frac{x^2}{\sqrt{1+x^2}} = \frac{1+x^2}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1+x^2}} = \sqrt{1+x^2} - \frac{1}{\sqrt{1+x^2}},$$

whence it follows that

$$\int \sqrt{1+x^2}\,d\,x = x\sqrt{1+x^2} - \int \sqrt{1+x^2}\,d\,x + \int \frac{d\,x}{\sqrt{1+x^2}}, \text{ or}$$

$$2\int \sqrt{1+x^2}\,d\,x = x\sqrt{1+x^2} + \int \frac{d\,x}{\sqrt{1+x^2}},$$

and consequently,

$$\text{I.)} \quad \int \sqrt{1+x^2}\,d\,x = \tfrac{1}{2}\,x\,\sqrt{1+x^2} + \tfrac{1}{2}\int \frac{d\,x}{\sqrt{1+x^2}}$$

$$= \tfrac{1}{2}\,[x\,\sqrt{1+x^2} + l\,(x + \sqrt{1+x^2})].$$

In like manner,

$$\text{II.)} \quad \int \sqrt{1-x^2}\,d\,x = \tfrac{1}{2}\,x\sqrt{1-x^2} + \tfrac{1}{2}\int \frac{d\,x}{\sqrt{1-x^2}}$$

$$= \tfrac{1}{2}\,[x\sqrt{1-x^2} + \sin.^{-1} x], \text{ and}$$

$$\text{III.)} \quad \int \sqrt{x^2-1}\,d\,x = \tfrac{1}{2}\,x\sqrt{x^2-1} - \tfrac{1}{2}\int \frac{d\,x}{\sqrt{x^2-1}}$$

$$= \tfrac{1}{2}\,[x\sqrt{x^2-1} - l\,(x + \sqrt{x^2-1})].$$

We have also

$$\int (\sin.\,x)^2\,d\,x = \int \sin.\,x\,\sin.\,x\,d\,x = -\int \sin.\,x\,d\,(\cos.\,x) = -\sin.\,x\cos.\,x$$
$$+\int \cos.\,x\,d\,(\sin.\,x) = -\sin.\,x\cos.\,x + \int (\cos.\,x)^2\,d\,x$$
$$= -\sin.\,x\cos.\,x + \int [1 - (\sin.\,x)^2]\,d\,x,$$

whence it follows that

$$2 \int (\sin. x)^2\, d x = \int d x - \sin. x \cos. x, \text{ and}$$

IV.) $\int (\sin. x)^2\, d x = \frac{1}{2} (x - \sin. x \cos. x) = \frac{1}{2} (x - \frac{1}{2} \sin. 2 x).$

In like manner

V.) $\int (\cos. x)^2\, d x = \frac{1}{2} (x + \sin. x \cos. x) = \frac{1}{2} (x + \frac{1}{2} \sin. 2 x)$, and

VI.) $\int \sin. x \cos. x\, d x = \frac{1}{4} \int \sin. 2 x\, d (2 x) = - \frac{1}{4} \cos. 2 x,$

VII.) $\int (\tang. x)^2\, d x = \tang. x - x$, and

VIII.) $\int (\cotg. x)^2\, d x = - (\cotg. x + x).$

Finally we have

IX.) $\int x \sin. x\, d x = -x \cos. x + \int \cos. x\, d x = -x \cos. x + \sin. x,$

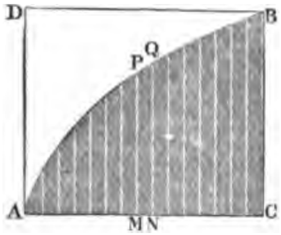
X.) $\int x e^x\, d x = \int x\, d (e^x) = x e^x - \int e^x\, d x = (x - 1) e^x,$

XI.) $\int l x . d x = x l x - \int x \frac{d x}{x} = x (l x - 1)$, and

XII.) $\int x l x . d x = \frac{x^2}{2} l x - \int \frac{x^2}{2} \frac{d x}{x} = (l x - \frac{1}{2}) \frac{x^2}{2}$

ART. 29. If we wish to find the quadrature of a curve, $A P B$, Fig. 35, I.E., to determine or express by a function of the abscissas of this curve the area of the surface $A B C$, which is enclosed by the curve $A P B$ and its co-ordinates $A C$ and $B C$, we imagine this surface divided by an infinite number of ordinates $M P$, $N Q$, etc., into elementary strips, like $M N P Q$, with the constant width $d x$, and the variable length $M P = y$. Since we can put the area of such an element of the surface

FIG. 35.



$$d F = \left(\frac{M P + N Q}{2}\right) . M N = (y + \tfrac{1}{2} d y)\, d x = y\, d x$$

we will find the area of the entire surface by integrating the differential $y\, d x$, and we have

$$F = \int y\, d x;$$

E.G., for the parabola whose parameter is p we have $y^2 = p x$, and, therefore, its surface

$$F = \int \sqrt{p x}\, d x = \sqrt{p} \int x^{\frac{1}{2}}\, d x = \frac{x \sqrt{p . x}}{\frac{3}{2}} = \tfrac{2}{3} x \sqrt{p x} = \tfrac{2}{3} x y.$$

The surface of the parabola $A\ B\ C$ is therefore two-thirds of the rectangle $A\ C\ B\ D$ which encloses it.

This formula holds good also for oblique co-ordinates inclined at an angle $X A\ Y = a$, E.G., for the surface $A\ B\ C$, Fig. 36, we have when we substitute instead of $B\ C = y$ the normal distance $B\ N = y\ sin.\ a$

$$F = sin.\ a \int y\ d\ x,$$

E.G., for the parabola when the axis of abscissas $A\ X$ is a diameter, and the axis of ordinates $A\ Y$ is tangent to the curve, we have

$$y^2 = p_1\ x = \frac{p\ x}{sin.^2\ a}.\quad \text{(See "Ingenieur," page 177.)}$$

and $F = \frac{2}{3}\ x\ y\ sin.\ a$,

I.E., the surface $A\ B\ C = \frac{2}{3}$ parallelogram $A\ B\ C\ D$.

FIG. 36.

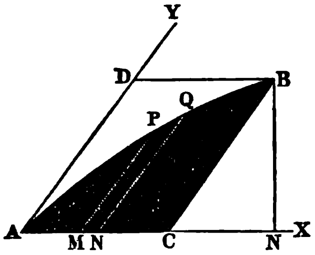
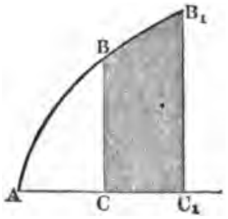


FIG. 37.



For a surface $B\ C\ C_1\ B_1 = F$, between the abscissa $A C_1 = c_1$, and $A\ C = c$, Fig. 37, we obtain, according to Art. 17,

$$F = \int_c^{c_1} y\ d\ x.$$

E.G., for $y = \frac{a^2}{x}$,

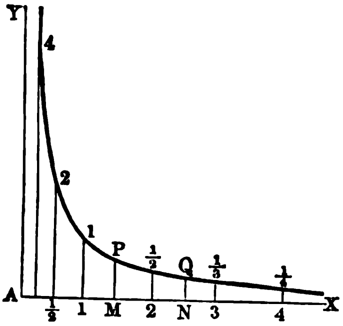
$$F = \int_c^{c_1} \frac{a^2\ d\ x}{x} = a^2\ (l\ c_1 - l\ c),$$

I.E., $F = a^2\ l\left(\frac{c_1}{c}\right).$

The equation $\frac{a^2}{x}$ corresponds to the curve $P\ Q$, Fig. 38, discussed in Art. 3, and if we have $A\ M = c$ and $A\ N = c_1$, the area of the surface $M\ N\ Q\ P$ is

$$F = a^2\, l\left(\frac{c_1}{c}\right)$$

FIG. 38.



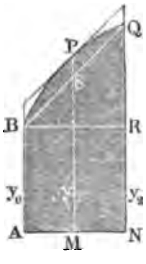
If we suppose, for simplicity, that $a = c = 1$, and $c_1 = x$, we obtain

$$F = l\,x;$$

hence the surfaces (1 $M\,P$ 1), (1 $N\,Q$ 1), etc., are the Naperian logarithms of the abscissas $A\,M$, $A\,N$, etc. The curve itself is the so-called *equilateral hyperbola* in which the two semi-axes a and b are equal; hence the angle formed by the asymptotes with the axes is $\alpha = 45°$; and the right lines $A\,X$ and $A\,Y$, which approach nearer and nearer the curve without ever attaining it, are its asymptotes. In consequence of the relation between the abscissas and the area of the surfaces, the Naperian logarithms are often styled *hyperbolic logarithms.*

ART. 30. We can put every integral $\int y\,d\,x = \int \phi\,(x)\,d\,x$ equal to the area of a surface F, and if the integration cannot be effected by means of one of the known rules, we can find it, at least approximately, by calculating the area of the corresponding surface by means of a well-known geometrical device.

FIG. 39.



If a surface $A\,B\,P\,Q\,N$, Fig. 39, is determined by the base $A\,N = x$, and by three equidistant ordinates $A\,B = y_0$, $M\,P = y_1$, $N\,Q = y_2$, we have the area of the trapezoid

$$A\,B\,Q\,N = F_1 = (y_0 + y_2)\,\frac{x}{2};$$

and that of the segment $B\,P\,Q\,S\,B$, if we consider $B\,P\,Q$ to be a parabola

$$F_2 = \tfrac{2}{3}\,P\,S.B\,R = \tfrac{2}{3}\,(M\,P - M\,S)\,.\,A\,N = \tfrac{2}{3}\left(y_1 - \frac{y_0 + y_2}{2}\right)x.$$

Hence the entire surface is

$$F = F_1 + F_2 = \left[\tfrac{1}{2}(y_0 + y_2) + \tfrac{2}{3}\left(y_1 - \frac{y_0 + y_2}{2}\right)\right]x$$

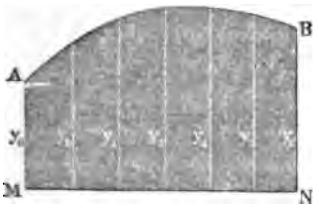
$$= [\tfrac{1}{6}(y_0 + y_2) + \tfrac{2}{3}y_1]x = (y_0 + 4y_1 + y_2)\cdot\frac{x}{6}.$$

If we introduce in the equation a mean ordinate y and put $F = x\,y$, we obtain

$$y = \frac{y_0 + 4y_1 + y_2}{6}.$$

In order to find the area of a surface, lying above a given base $M\,N = x$, and determined by an uneven number of ordinates $y_0, y_1, y_2, y_3 \dots y_n$, by which it is divided into an even number of equally wide strips, we have only to make repeated application of this rule. The width of a strip is $\frac{x}{n}$, and the area of the first pair of strips is

FIG. 40.



$$= \frac{y_0 + 4y_1 + y_2}{6}\cdot\frac{2x}{n},$$

of the second pair

$$= \frac{y_2 + 4y_3 + y_4}{6}\cdot\frac{2x}{n},$$

of the third pair,

$$= \frac{y_4 + 4y_5 + y_6}{6}\cdot\frac{2x}{n}, \text{ etc.};$$

and the area of the first six strips, or of the first three pair, for which $n = 6$, is

$$F = (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)\frac{x}{3.6}\cdot$$

$$[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]\frac{x}{18};$$

it is easy to perceive that the area of a surface divided in four pair of strips is

$$F = [y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]\frac{x}{3.8},$$

and in general, for a surface divided in n strips, we have

$$F = [y_0 + y_n + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})]\frac{x}{3\,n},$$

and the mean altitude of such a surface is

$$y = \frac{y_0 + y_n + 4\,(y_1 + y_3 + \dots + y_{n-1}) + 2\,(y_2 + y_4 + \dots + y_{n-2})}{3\,n}$$

in which n must be an even number.

This formula, well known under the name of Simpson's Rule (see "Ingenieur," page 190), can be employed for the determination of an integral $\int_c^{c_1} y\,d\,x = \int_c^{c_1} \phi\,(x)\,d\,x$, if we divide $x = c_1 - c$ into an even number n of equal parts, and calculate the ordinates

$$y_0 = \phi\,(c), y_1 = \phi\left(c + \frac{x}{n}\right), y_2 = \phi\left(c + \frac{2\,x}{n}\right),$$

$$y_3 = \phi\left(c + \frac{3\,x}{n}\right) \dots \text{up to } y_n = \phi\,(x),$$

and then substitute these values in the formula

$$\int_c^{c_1} y\,d\,x = \int_c^{c_1} \phi\,(x)\,d\,x$$

$$= [y_0 + y_n + 4\,(y_1 + y_3 + \dots + y_{n-1}) + 2\,(y_2 + y_4 + \dots + y_{n-2})\,]\,\frac{c_1 - c}{3\,n}$$

E.G., $\int_1^2 \frac{d\,x}{x}$ gives, since here $c_1 - c = 2 - 1 = 1$ and $y = \phi\,(x) = \frac{1}{x}$,

when we assume $n = 6$ or $\frac{x}{n} = \frac{c_1 - c}{6} = \frac{1}{6}$,

$$y_0 = \frac{1}{1} = 1{,}0000,\ y_1 = \frac{1}{\frac{7}{6}} = \tfrac{6}{7} = 0{,}8571,\ y_2 = \frac{1}{\frac{8}{6}} = \tfrac{3}{4} = 0{,}7500,$$

$$y_3 = \frac{1}{\frac{9}{6}} = \tfrac{6}{9} = 0{,}6666,\ y_4 = \frac{1}{\frac{10}{6}} = 0{,}6000,\ y_5 = \frac{6}{11} = 0{,}5454, \text{ and } y_6 = 0{,}5000,$$

therefore

$$y_0 + y_6 = 1{,}5000,\ y_1 + y_3 + y_5 = 2{,}0692, \text{ and } y_2 + y_4 = 1{,}3500,$$

and we have the required integral

$$\int_1^2 \frac{d\,x}{x} = (1{,}5000 + 4\,.\,2{,}0692 + 2\,.\,1{,}3500)\,.\,\tfrac{1}{18} = \frac{12{,}4768}{18} = 0{,}69315.$$

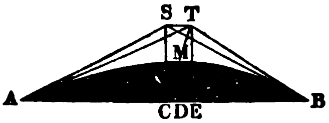
From Art. 22, III, we have

$$\int_1^2 \frac{d\,x}{x} = l\,2 - l\,1 = 0{,}693147.$$

We see that the results of the two methods agree very well.

ART. 31. Further on, another rule will be given which can be employed for an uneven number of strips. If we treat a very flat segment $A\,M\,B$, Fig 41, as a segment of a parabola, we have from Art. 29 the area of the same,

FIG. 41.



$$F = \tfrac{2}{3}\,A\,B\,.\,M\,D,$$

or, if $A\,T$ and $B\,T$ are the tangents at the ends A and B, and therefore $C\,T = 2\,C\,M$, we have $F = \frac{2}{3}\,.\,\frac{A\,B\,.\,T\,E}{2} = \frac{2}{3}$ of the isosceles triangle $A\,S\,B$ of the same height, and therefore $= \frac{2}{3}\,A\,C\,.\,C\,S = \frac{2}{3}\,\overline{A\,C}^2\ tang.\ S\,A\,C.$

The angle $S\,A\,C = S\,B\,C$ is $= T\,A\,C + T\,A\,S = T\,B\,C - T\,B\,S$; putting the small angles $T\,A\,S$ and $T\,B\,S$, equal to each other, we obtain for the same

$$T\,A\,S = T\,B\,S = \frac{T\,B\,C - T\,A\,C}{2}, \text{ and}$$

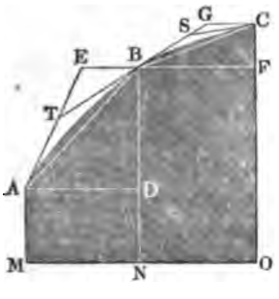
$$S\,A\,C = T\,A\,C + \frac{T\,B\,C - T\,A\,C}{2} = \frac{T\,A\,C + T\,B\,C}{2} = \frac{\delta + \varepsilon}{2},$$

when we denote the tangential angles $T\,A\,C$ and $T\,B\,C$ by δ and ε. Now since $A\,C = B\,C = \frac{1}{2}\,A\,B = \frac{1}{2}$ the chord s, we have

$$F = \tfrac{1}{6}\,s^2\ tang.\left(\frac{\delta + \varepsilon}{2}\right).$$

This formula can be employed for the portion of surface $M\,A\,B\,N$, Fig. 42, whose tangential angles $T\,A\,D = \alpha$ and $T\,B\,E = \beta$ are given; putting the angle formed by the chord $B\,A\,D = A\,B\,E = \sigma$, we have

FIG. 42.



$$T\,A\,B = \delta = T\,A\,D - B\,A\,D$$
$$= \alpha - \sigma \text{ and}$$
$$T\,B\,A = \varepsilon = A\,B\,E - T\,B\,E$$
$$= \sigma - \beta, \text{ whence}$$
$$\delta + \varepsilon = \alpha - \beta,$$

and the segment over $A\,B$

$$F = \tfrac{1}{6}\,s^2\ tang.\left(\frac{\alpha - \beta}{2}\right)$$

or, since $\alpha - \beta$ is small,

$$F = \frac{s^2}{12}\,tang.\,(\alpha - \beta) = \frac{s^2}{12}\left(\frac{tang.\,\alpha - tang.\,\beta}{1 + tang.\,\alpha\ tang.\,\beta}\right),$$

or since α and β differ but little from each other, and therefore we can substitute in *tang.* α *tang.* β instead of α and β the mean value σ, we have

$$F = \tfrac{1}{12}\, s^2 . \frac{tang.\,\alpha - tang.\,\beta}{1 + tang.\,\sigma^2} = \tfrac{1}{12}\, s^2\, cos.^2\, \sigma\, (tang.\,\alpha - tang.\,\beta),$$

and substituting for $s\ cos.\ \sigma$ the base $M\,N = x$,

$$F = \frac{x^2}{12}\,(tang.\,\alpha - tang.\,\beta),$$

therefore the area of the entire portion of surface $M\,A\,B\,N$, when y_0 and y_1 designate its ordinates $M\,A$ and $N\,B$, is

$$F_1 = (y_0 + y_1)\,\frac{x}{2} + (tang.\,\alpha - tang.\,\beta)\,\frac{x^2}{12}.$$

If another portion of the surface $N\,B\,C\,O$ adjoins the first and has a base $N\,O = x$, and the ordinates $B\,N$ and $C\,O = y_1$ and y_2, and the tangential angles $S\,B\,F = \beta$ and $S\,C\,G = \gamma$, we have for the area of the same

$$F_2 = (y_1 + y_2)\,\frac{x}{2} + (tang.\,\beta - tang.\,\gamma)\,\frac{x^2}{12},$$

and therefore for the whole surface, since $-tang.\,\beta$ cancels $+tang.\,\beta$,

$$F = F_1 + F_2 = (\tfrac{1}{2}\, y_0 + y_1 + \tfrac{1}{2}\, y_2)\, x + (tang.\,\alpha - tang.\,\gamma)\,\frac{x^2}{12}.$$

For a surface composed of strips of like width we have, when α is the tangential angle at the commencement and δ at the end,

$$F = (\tfrac{1}{2}\, y_0 + y_1 + y_2 + \tfrac{1}{2}\, y_3)\, x + (tang.\,\alpha - tang.\,\delta)\,\frac{x^2}{12},$$

and in general for a portion of surface, determined by the abscissas $\frac{x}{n}, \frac{2\,x}{n}, \frac{3\,x}{n} \ldots x$, and by the ordinates $y_0\ y_1, y_2 \ldots y_n$, and by the tangential angles α_0 and α_n of the ends,

$$F = (\tfrac{1}{2}\, y_0 + y_1 + y_2 + \ldots + y_{n-1} + \tfrac{1}{2}\, y_n)\,\frac{x}{n}$$
$$+ \tfrac{1}{12}\,(tang.\,\alpha - tang.\,\alpha_n)\left(\frac{x}{n}\right)^2$$

An Integral

$$\int_c^{x_1} y\, d\,x = \int_c^{x_1} \phi\,(x)\, d\,x$$
$$= (\tfrac{1}{2}\, y_0 + y_1 + y_2 + \ldots + y_{n-1} + \tfrac{1}{2}\, y_n)\,\frac{x}{n}$$
$$+ \tfrac{1}{12}\,(tang.\,\alpha - tang.\,\alpha_n)\left(\frac{x}{n}\right)^2$$

can be found by putting $x = c_1 - c$, calculating the values

$$y_0 = \phi(c), y_1 = \phi\left(c + \frac{x}{n}\right), y_2 = \phi\left(c + \frac{2x}{n}\right),$$

$$y_3 = \phi\left(c + \frac{3x}{n}\right) \dots, y_n = \phi\left(c + \frac{nx}{n}\right) = \phi(c_1),$$

as well as $tang.\, a = \frac{dy}{dx} = \psi(x) = \psi(c)$ and $tang.\, a_n = \psi(c_1)$, and substituting them in the equation.

E.G., for $\int_1^2 \frac{dx}{x}$ we have, if we take $n = 6$, since

$$x = c_1 - c = 2 - 1 \text{ and } y = \phi(x) = \frac{1}{x},$$

$$y_0 = \frac{1}{c} = 1, y_1 = \frac{1}{1 + \frac{1}{6}} = \tfrac{6}{7}, y_2 = \tfrac{6}{8}, y_3 = \tfrac{6}{9}, y_4 = \tfrac{6}{10}, y_5 = \tfrac{6}{11} \text{ and } y_6 = \tfrac{6}{12};$$

also, since $\frac{dy}{dx} = \frac{d(x^{-1})}{dx} = -\frac{1}{x^2}$,

$tang.\, a = -\frac{1}{1} = -1$ and $tang.\, \beta = -\left(\frac{1}{2}\right)^2 = -\frac{1}{4}$, and therefore

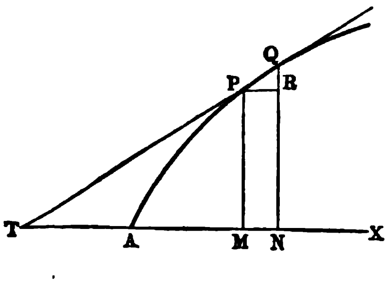
$$\int_1^2 \frac{dx}{x} = \left(\tfrac{1}{2} + \tfrac{6}{7} + \tfrac{6}{8} + \tfrac{6}{9} + \tfrac{6}{10} + \tfrac{6}{11} + \tfrac{1}{4}\right) \cdot \tfrac{1}{6} + \left(-1 + \tfrac{1}{4}\right) \cdot \tfrac{1}{12} \cdot \tfrac{1}{36}$$

$$= \frac{4{,}1692}{6} - \tfrac{3}{4} \cdot \tfrac{1}{12} \cdot \tfrac{1}{36} = 0{,}69487 - 0{,}00173 = 0{,}69314.$$

(Compare the example of the last article.)

ART. 32. To *rectify* a curve, or from its equation $y = f(x)$ between the co-ordinates $AM = x$ and $MP = y$, Fig. 43, to deduce an equation between the arc $AP = s$ and one or other of the co-ordinates, we determine the differential of the arc AP of the curve, and then we seek its integral. If x be increased by a quantity $MN = PR = dx$, y is increased by $RQ = dy$, and s by the element $PQ = ds$, and according to the Theorem of Pythagoras we have

FIG. 43.



$$\overline{PQ}^2 = \overline{PR}^2 + \overline{QR}^2,$$

I.E.,

$$ds^2 = dx^2 + dy^2,$$

$$ds = \sqrt{dx^2 + dy^2},$$

hence the arc of the curve itself is

$$s = \int \sqrt{dx^2 + dy^2}.$$

E.G., for Neil's parabola (see Art. 9, Fig. 17), whose equation is $a y^2 = x^3$, we have $2 a y d y = 3 x^2 d x$, whence

$$d y = \frac{3 x^2 d x}{2 a y} \text{ and } d y^2 = \frac{9 x^4 d x^2}{4 a^2 y^2} = \frac{9 x d x^2}{4 a},$$

and $$d s^2 = \left(1 + \frac{9 x}{4 a}\right) d x^2, \text{ hence}$$

$$s = \int \sqrt{1 + \frac{9 x}{4 a}} d x = \frac{4 a}{9} \int \left(1 + \frac{9 x}{4 a}\right)^{\frac{1}{2}} d \left(\frac{9 x}{4 a}\right)$$

$$= \frac{4 a}{9} \int u^{\frac{1}{2}} d u = \frac{4 a}{9} \tfrac{2}{3} u^{\frac{3}{2}} = \tfrac{8}{27} a \sqrt{\left(1 + \frac{9 x}{4 a}\right)^3}.$$

In order to find the necessary constant, we make s begin with x and y, and we obtain

$$0 = \tfrac{8}{27} a \sqrt{1^3} + Con., \text{ or } Con. = - \tfrac{8}{27} a$$

and $$s = \tfrac{8}{27} a \left[\sqrt{\left(1 + \frac{9 x}{4 a}\right)^3} - 1\right],$$

E.G., for the piece $A P_1$ whose abscissa $x = a$, we have

$$s = \tfrac{8}{27} a \left[\sqrt{(\tfrac{13}{4})^3} - 1\right] = 1{,}736\, a.$$

Introducing the tangential angle $Q P R = P T M = a$ (Fig. 43) we have

$$Q R = P Q \,.\, sin.\, Q P R \text{ and } P R = P Q\, cos.\, Q P R,$$

I.E., $$d y = d s\, sin.\, a \text{ and } d x = d s\, cos.\, a,$$

and besides, $tang.\, a = \frac{d y}{d x}$ (see Art. 6),

also, $$sin.\, a = \frac{d y}{d s} \text{ and } cos.\, a = \frac{d x}{d s}; \text{ and finally,}$$

$$s = \int \sqrt{1 + tang.^2 a} \,.\, d x = \int \frac{d y}{sin.\, a} = \int \frac{d x}{cos.\, a}.$$

If the equation between any two of the quantities x, y, s and a is given, we can find the equation between any two others.

If, E.G., $cos.\, a = \frac{s}{\sqrt{c^2 + s^2}}$, we have

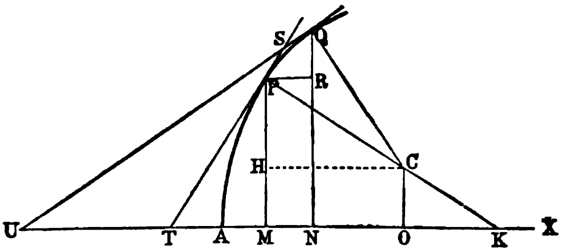
$$d x = d s\, cos.\, a = \frac{s d s}{\sqrt{c^2 + s^2}}, \text{ and}$$

$$x = \int \frac{s d s}{\sqrt{c^2 + s^2}} = \tfrac{1}{2} \int \frac{2 s d s}{\sqrt{c^2 + s^2}} = \tfrac{1}{2} \int \frac{d u}{\sqrt{u}} = \tfrac{1}{2} \int u^{-\frac{1}{2}} d u = u^{\frac{1}{2}}$$

$= \sqrt{c^2 + s^2} + Const.$, and if x and s are equal to zero at the same time, $x = \sqrt{c^2 + s^2} - c$.

Art. 33. A right line perpendicular to the tangent $P\,T$, Fig. 44, is also normal to the curve at the point of tangency, for the tangent gives the direction of the curve at this point.

Fig. 44.



The portion $P\,K$ of the line between the point of tangency P and the axis of abscissas is called simply the Normal, and the projection of the same $M\,K$ on the axis of abscissas the Subnormal. We have for the latter, since the angle $M\,P\,K$ is equal to the tangential angle $P\,T\,M = a$,

$$M\,K = M\,P \,.\, tang.\; a,$$

i.e., the subnormal $= y\; tang.\; a = y\,\frac{d\,y}{d\,x}$.

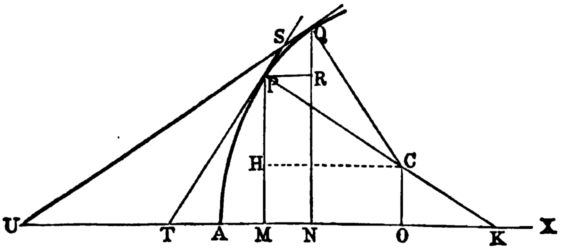
Since for the system of curves $y = x^m$, $tang.\; a = m\,x^{m-1}$, it follows that the subnormal is $= m\,x^m \,.\, x^{m-1} = m\,x^{2m-1} = \frac{m\,y^2}{x}$, and for the common parabola, whose equation is $y^2 = p\,x$, we have the subnormal $= y\,\frac{p}{2\,y} = \frac{p}{2}$, that is constant.

If to a second point Q, infinitely near the point P, we draw another normal $Q\,C$, we obtain in the point of intersection of these two lines the centre C of a circle which can be described through the points of tangency P and Q. It is called the *circle of curvature*, and the portions $C\,P$ and $C\,Q$ of the normals are radii of this circle, or, as they are styled, the *radii of curvature*. This circle is the one of all those, which can be made to pass through P and Q, which keeps closest to the element $P\,Q$ of the curve, and we can therefore assume that its arc $P\,Q$ coincides with the element $P\,Q$ of the curve. It is called the *osculatory* circle.

Denoting the radius $C\,P = C\,Q$ by r, the arc $A\,P$ of curve by s or its element $P\,Q$ by $d\,s$, and the tangential angle or arc of $P\,T\,M$ by a, and its element $S\,U\,M - S\,T\,M$, i.e., $-\,U\,S\,T = -$

$P\,C\,Q$ by $d\,a$, we have, since $P\,Q = C\,P$. arc of the angle $P\,C\,Q$, $d\,s = -r\,d\,a$, whence the radius of curvatures $r = -\frac{d\,s}{d\,a}$.

FIG. 45.



We can generally determine a from the equation of the co-ordinates by putting $tang.\,a = \frac{d\,y}{d\,x}$.

Now $d\,tang.\,a = \frac{d\,a}{cos.^2\,a}$, and $cos.\,a = \frac{d\,x}{d\,s}$, whence

$$d\,a = cos.^2\,a\,.\,d\,tang.\,a = \frac{d\,x^2}{d\,s^2.}\,d\,tang.\,a \text{ and}$$

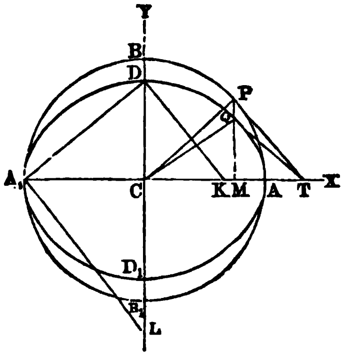
$$r = -\frac{d\,s}{cos.^2\,a\,d\,tang.\,a} = -\frac{d\,s^3}{d\,x^2\,d\,tang.\,a}.$$

For a convex curve $r = +\frac{d\,s}{d\,a} = +\frac{d\,s^3}{d\,x^2\,d\,tang.\,a}$, and for a point of inflexion $r = \infty$.

For the co-ordinates $A\,O = u$ and $O\,C = v$ of the centre C of curvature, we have

$u = A\,M + H\,C = x + C\,P\,sin.\,C\,P\,H$, I.E., $u = x + r\,sin.\,a$, and
$v = O\,C = M\,P - H\,P = y - C\,P\,cos.\,C\,P\,H$, I.E., $v = y - r\,cos.\,a$.

FIG. 46



The continuous line formed by the centres of curvature forms a curve, which is called *evolute* of $A\,P$, and whose course is determined by the co-ordinates u and v.

If the ellipse $A\,D\,A_1\,D_1$, Fig. 46, is laid upon the circle $A\,B\,A_1\,B_1$, its co-ordinates $C\,M = x$ and $M\,Q = y$ can be expressed by means of the central angle $P\,C\,B = \phi$ of the circle. We have here

$x = C\,P\, sin.\ C\,P\,M = C\,P\, sin.\ B\,C\,P = a\, sin.\ \phi$, and

$y = M\,Q = \frac{b}{a}\,M\,P = \frac{b}{a}\,C\,P\, cos.\ C\,P\,M = b\, cos.\ \phi$.

From the latter we obtain $d\,x = a\, cos.\ \phi\, d\,\phi$ and $d\,y = -\,b\, sin.\ \phi\, d\,\phi$, and consequently for the tangential angle of the ellipse $Q\,T\,X = a$

$tang.\ a = \frac{d\,y}{d\,x} = -\,\frac{b\, sin.\ \phi}{a\, cos.\ \phi} = -\,\frac{b}{a}\, tang.\ \phi$, and for its complementary angle $Q\,T\,C = a_1 = 180° - a$,

$$tang.\ a_1 = \frac{b}{a}\, tang.\ \phi \text{ and } cotg.\ a_1 = \frac{a}{b}\, cotg.\ \phi.$$

Hence the subtangent of the ellipse is

$$M\,T = M\,Q\, cotg.\ M\,T\,Q$$

$$= y\, cotg.\ a_1 = \frac{a\,y}{b}\, cotg.\ \phi = y_1\, cotg.\ \phi,$$

when y_1 designates the ordinate $M\,P$ of the circle. Since the tangent $P\,T$ to the latter is perpendicular to the radius $C\,P$, we have also $P\,T\,M = P\,C\,B = \phi$, and therefore the subtangent $M\,T$ of the same is also $= M\,P\, cotg.\ M\,T\,P = y_1\, cotg.\ \phi$.

Therefore the two points of the ellipse and circle which have the same ordinate, have one and the same subtangent.

Farther, for an elementary arc of the ellipse

$$d\,s^2 = d\,x^2 + d\,y^2 = (a^2\, cos.^2\,\phi + b^2\, sin.^2\,\phi)\, d\,\phi^2,$$

and the differential of $tang.\ a$,

$$d\, tang.\ a = -\,\frac{b}{a}\, d\, tang.\ \phi = -\,\frac{b}{a}\,\frac{d\,\phi}{cos.^2\,\phi},$$

whence it follows that the radius of curvature of the ellipse is

$$r = -\,\frac{d\,s^2}{d\,x^2\, d\, tang.\ a} = \frac{(a^2\, cos.^2\,\phi + b^2\, sin.^2\,\phi)^{\frac{3}{2}}}{a^2\, cos.^2\,\phi \cdot \frac{b}{a\, cos.^2\,\phi}}$$

$$= \frac{(a^2\, cos.^2\,\phi + b^2\, sin.^2\,\phi)^{\frac{3}{2}}}{a\,b}.$$

E.G., for $y = 0$, I.E., for $sin.\ \phi = 0$, and $cos.\ \phi = 1$, we have the maximum radius of curvature

$$r_m = \frac{a^3}{a\,b} = \frac{a^2}{b},$$

and, on the contrary, for $\phi = 90°$, I.E., for $sin.\ \phi = 1$ and $cos.\ \phi = 0$, the minimum radius of curvature

$$r_n = \frac{b^3}{a\,b} = \frac{b^2}{a}.$$

The first value of r corresponds to the point D, and the last to the point A, and both are determined by the portions of the axes $C\,L$ and $C\,K$, which are cut off by the perpendiculars erected upon the chord $A_1\,D$ at its ends A_1 and D.

ART. **34.** Many functions, which occur in practice, are composed of the various functions which we have already studied, such as

$$y = x^m,\ y = e^x, \text{ and } y = sin.\ x,\ y = cos.\ x, \text{ etc.};$$

and it is easy, with the assistance of the foregoing rules, to determine their properties, such as the position of their tangents, their quadrature, their radius of curvature, etc., as well as to construct the curves, as is shown by the following examples:

For the curve, whose equation is $y = x^2\left(1 - \frac{x}{3}\right) = x^2 - \frac{1}{3}\,x^3$,

we have $\qquad d\,y = 2\,x\,d\,x - x^2\,d\,x,$

whence $\qquad tang.\ \alpha = 2\,x - x^2 = x\,(2 - x).$

Since this tangent becomes $= 0$ for $x = 0$ and $x = 2$, its direction at these two points is parallel to that of the axis of abscissas.

Farther, $d\ tang.\ \alpha = 2\,d\,x - 2\,x\,d\,x = 2\,(1 - x)\,d\,x,$

whence for $\qquad x = 0,\ d\ tang.\ \alpha = +\ 2\,d\,x,$

and for $\qquad x = 2,\ d\ tang.\ \alpha = -\ 2\,d\,x,$

and therefore the ordinate of the first point is a minimum, and that of the second point a maximum. If we put $d\ tang.\ \alpha = 0$, we obtain $x = 1$ and $y = \frac{2}{3}$, the co-ordinates of a point of inflexion in which the concave portion of the curve joins the convex.

Farther, for an element $d\,s$ of the curve we have

$d\,s^2 = d\,x^2 + d\,y^2 = d\,x^2 + x^2\,(2 - x)^2\,d\,x^2 = [1 + x^2\,(2-x)^2]\,d\,x^2,$

whence the radius of curvature is

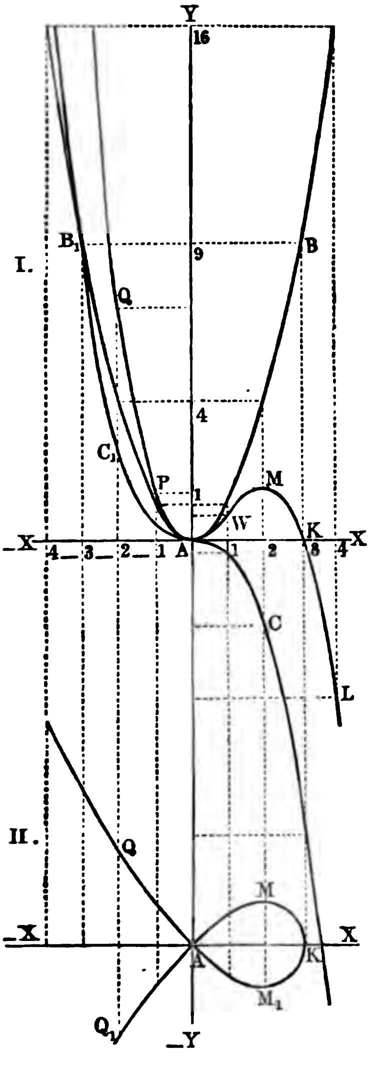
$$r = -\frac{d\,s^3}{d\,x^2\ d\ tang\ \alpha} = -\frac{[1 + x^2\,(2 - x)^2]^{\frac{3}{2}}}{2\,(1 - x)}.$$

E.G., for $x = 0$ we have $r = \frac{-1}{2} = -\ \frac{1}{2}$, for $x = 1$, $r = -\ \frac{2^{\frac{3}{2}}}{0} = \infty$,

for $x = 2$, $r = \frac{-1}{-2} = +\ \frac{1}{2}$, and for $x=3$, $r = \frac{1}{4}\,.\,10^{\frac{3}{2}} = +7{,}906.$

The corresponding curve is shown in Fig. 47, in which A is the origin and $X\overline{X}$, $Y\overline{Y}$ the axes of co-ordinates. The parabola $B\,A\,B_1$, which extends symmetrically upon both sides of the axis of $A\,Y$, represents the first part $y_1 = x^2$ of the equation, and, on the contrary, the curve $C\,A\,C_1$, which upon the right-hand side of $Y\overline{Y}$ descends below $X\overline{X}$, and on the left-hand side rises above it, and thus diverges more and more from the axis $X\overline{X}$, as it increases its distance from $Y\overline{Y}$, corresponds to the second part $y_2 = -\frac{1}{3}x^3$.

FIG. 47.



In order to find for a given abscissa x, the corresponding point of the curve $y = x^2 - \frac{1}{3}x^3$, we have but to add algebraically the corresponding ordinates of the first two curves; E.G., since for $x = 1$ we have $y_1 = 1$ and $y_2 = -\frac{1}{3}$, it follows that the corresponding ordinate of the point W is $y = y_1 + y_2 = 1 - \frac{1}{3} = \frac{2}{3}$; farther, for $x = 2$ we have $y_1 = 4$, and $y_2 = -\frac{8}{3}$, and hence the co-ordinate of the point M is $y = 4 - \frac{8}{3} = \frac{4}{3}$. In the same way $x = 3$ gives $y = y_1 + y_2 = 9 - 9 = 0$; $x = 4$, $y = 16 - \frac{64}{3} = -\frac{16}{3}$; $x = -1$, $y = 1 + \frac{1}{3} = \frac{4}{3}$; $x = -2$, $y = 4 + \frac{8}{3} = \frac{20}{3}$, etc., and we perceive that the curve from A towards the right has the form $A\,W\,M\,K\,L$, and that in the beginning it runs above the abscissa $A\,K = 3$, but from that point it extends to infinity below the axis

$X\overline{X}$, and that from A towards the left it forms but one branch $A\ P\ Q\ldots$, which rises to infinity. From what precedes we see that W is a point of inflexion, and M a point of the curve where the ordinate is a maximum. While the curve has in A and M the direction of $X\overline{X}$, in W it rises at an angle of 45°, for we have for the latter $tang.\ a = x\,(2 - x) = 1$; on the contrary, the angle of inclination at K, is $tang.\ a = -3$, consequently a is $= 71°\ 34'$, etc. The quadrature of the curve is given by the integral

$$F = \int y\,d\,x = \int (x^2 - \tfrac{1}{3}\,x^3)\,dx = \int x^2\,d\,x - \tfrac{1}{3}\int x^3\,d\,x$$
$$= \frac{x^3}{3} - \frac{x^4}{12} = \frac{x^3}{3}\left(1 - \frac{x}{4}\right).$$

Hence, E.G., we have for the area of the portion of surface $A\ W\ M\ K$ above $A\ K = 3$

$$F = \frac{3^3}{3}(1 - \tfrac{3}{4}) = \tfrac{9}{4},$$

and on the contrary the area of the portion of surface $\overline{3\ L\ 4}$ below the abscissa $\overline{3\ 4}$ is

$$F_1 = \frac{4^3}{3}(1 - \tfrac{4}{4}) - \frac{3^3}{3}(1 - \tfrac{3}{4}) = 0 - \tfrac{9}{4} = -\tfrac{9}{4}.$$

Finally, to find the length of a portion of the curve, E.G., $A\ W\ M$, we put

$$s = \int \sqrt{1 + x^2\,(2 - x)^2}\,d\,x = \int_c^{c_1} \phi\,(x)\,d\,x,$$

and employ the method of integration explained in Art. 30. Here c is $= 0$, and $c_1 = 2$, and taking $n = 4$ we have $d\,x = \dfrac{c_1 - c}{n}$ $= \dfrac{2-0}{4} = \tfrac{1}{2}$, then substituting successively the values $0, \tfrac{1}{2}, 1, \tfrac{3}{2}$ and 2 for x in the function $\phi\,(x) = \sqrt{1 + x^2\,(2 - x)^2}$, we obtain the values

$\phi\,(0) = \sqrt{1} = 1$, $\phi\,(\tfrac{1}{2}) = \sqrt{1 + \tfrac{9}{16}} = \tfrac{5}{4}$, $\phi\,(1) = \sqrt{1+1} = \sqrt{2} = 1{,}414\ldots.$
$\phi\,(\tfrac{3}{2}) = \sqrt{1 + \tfrac{9}{16}} = \tfrac{5}{4}$ and $\phi\,(2) = \sqrt{1} = 1$,
and therefore the length of the arc $A\ W\ M$ is

$$s = \left(\phi\,(0) + 4\,\phi\,(\tfrac{1}{2}) + 2\,\phi\,(1) + 4\,\phi\,(\tfrac{3}{2}) + \phi\,(2)\right)\frac{c_1 - c}{3\,.\,4}$$
$$= (1 + 5 + 2{,}828 + 5 + 1)\,.\,\tfrac{1}{6} = 2{,}471.$$

By means of the curve $y=x^2\left(1-\frac{x}{3}\right)$ we can easily determine the course of the curve $y=x\sqrt{1-\frac{x}{3}}$ by extracting the square roots of the values of the co-ordinates of the first, which give the corresponding co-ordinates of the latter. But since the square root of negative quantities are imaginary, this curve does not continue beyond the point K to the right; and since every square root of a positive number gives two values, equal and with opposite signs, the new curve ($I\,I$) runs in two symmetrical branches $Q\,A\,M\,K$ and $Q_1\,A\,M_1\,K$ on both sides of the axis of abscissas.

ART. 35. When the quotient $y=\frac{\phi(x)}{\psi(x)}$ of two functions $\phi(x)$ and $\psi(x)$ takes the indeterminate form of $\frac{0}{0}$ for a certain value a of x, which always occurs when, as E.G., in $y=\frac{x^2-a^2}{x-a}$, the numerator and denominator of a fraction have a common factor $x-a$, we can find the real value of the same by differentiating the numerator and denominator.

If x is increased by $d\,x$, and y by the corresponding element $d\,y$, we have

$$y+d\,y=\frac{\phi(x)+d\,\phi(x)}{\psi(x)+d\,\psi(x)}, \text{ but for } x=a$$

$$\phi(x)=0 \text{ and } \psi(x)=0, \text{ whence}$$

$$y+d\,y=\frac{d\,\phi(x)}{d\,\psi(x)};$$

but since $d\,y$ is infinitely small in comparison to y, we have

$$y=\frac{\phi(x)}{\psi(x)}=\frac{d\,\phi(x)}{d\,\psi(x)}=\frac{\phi_1(x)}{\psi_1(x)},$$

in which $\phi_1(x)$ and $\psi_1(x)$ designate the differential quotients of $\phi(x)$ and $\psi(x)$.

If $y=\frac{\phi_1(x)}{\psi_1(x)}$, is also $=\frac{0}{0}$, we can differentiate it anew, and put

$$y=\frac{d\,\phi_1(x)}{d\,\psi_1(x)}=\frac{\phi_2(x)}{\psi_2(x)}.$$

In the same way the indeterminate expressions $y=\frac{\infty}{\infty}$ and

$0 \times \infty$, etc., can be treated, for $\infty = \frac{1}{0}$, whence $\frac{\infty}{\infty}$ and $0 \times \infty$ can be put $= \frac{0}{0}$:

E.G., $y = \dfrac{3x^3 - 7x^2 - 8x + 20}{5x^3 - 21x^2 + 24x - 4}$ becomes for $x = 2$, $y = \frac{0}{0}$.

For this we can put

$$y = \frac{d\,(3x^3 - 7x^2 - 8x + 20)}{d\,(5x^3 - 21x^2 + 24x - 4)} = \frac{9x^2 - 14x - 8}{15x^2 - 42x + 24},$$

which for $x = 2$ gives again $y = \frac{0}{0}$, and we can again put

$$y = \frac{d\,(9x^2 - 14x - 8)}{d\,(15x^2 - 42x + 24)} = \frac{18x - 14}{30x - 42} = \frac{9x - 7}{15x - 21} = \frac{11}{9}.$$

The factor $(x - 2)$ is really contained twice in the numerator, and twice in the denominator. If we divide both by $x - 2$, we obtain

$$y = \frac{3x^2 - x - 10}{5x^2 - 11x + 2},$$

and dividing the last again by $(x - 2)$

$$y = \frac{3x + 5}{5x - 1},$$

which for $x = 2$ gives $y = \frac{11}{9}$.

We have also for $y = \dfrac{a - \sqrt{a^2 - x}}{x}$ when $x = 0$, $\frac{0}{0}$,

but since $d\,(a - \sqrt{a^2 - x}) = -\,d\,(a^2 - x)^{\frac{1}{2}} = \dfrac{\frac{1}{2}\,dx}{\sqrt{a^2 - x}}$,

in this case $\qquad y = \dfrac{\frac{1}{2}}{\sqrt{a^2 - x}} = \dfrac{1}{2a}$;

further $\qquad y = \dfrac{l\,x}{\sqrt{1 - x}}$, for $x = 1$, gives $y = \frac{0}{0}$,

but $\qquad d\,l\,x = \dfrac{dx}{x}$ and $d\sqrt{1 - x} = \dfrac{-\,dx}{2\sqrt{1 - x}}$,

hence it follows that $y = -\dfrac{2\sqrt{1 - x}}{x} = \dfrac{2.0}{1} = 0$.

Finally, $y = \frac{1 - sin.\,x + cos.\,x}{-1 + sin.\,x + cos.\,x}$ gives for $x = \frac{\pi}{2}$ (90°)

$y = \frac{1 - 1 + 0}{-1 + 1 + 0} = \frac{0}{0}$, we have therefore

$$y = \frac{d\,(1 - sin.\,x + cos.\,x)}{d\,(-1 + sin\,x + cos.\,x)} = \frac{-\,cos.\,x - sin.\,x}{cos.\,x - sin.\,x}$$

$$= \frac{-\,0 - 1}{0 - 1} = 1.$$

ART. 36. When, for a function $y = a\,u + \beta\,v$, a series of corresponding values of the variables u, v and y has been determined by observation or measurement, we can require the values of the constants a and β which are the freest from accidental or irregular errors of observation and measurement, and which express most exactly the relation between the quantities u, v and y, of which u and v are known functions of one and the same variable x. Of all the methods that can be employed for the resolution of this problem, I.E., for the determination of the most possible, or the most probably correct, values of the constants, the method of the least squares is the most general, and rests upon the most scientific basis.

If the results of the observations corresponding to the function $y = a\,u + \beta\,v$ are,

$$\left\{\begin{matrix} u_1, v_1, y_1 \\ u_2, v_2, y_2 \\ u_3, v_3, y_3 \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ u_n, v_n, y_n \end{matrix}\right\}$$

we have the following values for the errors of observation, and for their corresponding squares.

$$\left\{\begin{matrix} z_1 = y_1 - (a\,u_1 + \beta\,v_1) \\ z_2 = y_2 - (a\,u_2 + \beta\,v_2) \\ z_3 = y_3 - (a\,u_3 + \beta\,v_3) \\ \cdot\quad \cdot\quad \cdot\quad \cdot \\ \cdot\quad \cdot\quad \cdot\quad \cdot \\ z_n = y_n - (a\,u_n + \beta\,v_n) \end{matrix}\right\}$$

$$\left\{\begin{array}{l}
z_1^2=y_1^2-2\,a\,u_1\,y_1-2\,\beta\,v_1\,y_1+a^2\,u_1^2+2\,a\,\beta\,u_1\,v_1+\beta^2\,v_1^2 \\
z_2^2=y_2^2-2\,a\,u_2\,y_2-2\,\beta\,v_2\,y_2+a^2\,u_2^2+2\,a\,\beta\,u_2\,v_2+\beta^2\,v_2^2 \\
z_3^2=y_3^2-2\,a\,u_3\,y_3-2\,\beta\,v_3\,y_3+a^2\,u_3^2+2\,a\,\beta\,u_3\,v_3+\beta^2\,v_3^2 \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
z_n^2=y_n^2-2\,a\,u_n\,y_n-2\,\beta\,v_n\,y_n+a^2\,u_n^2+2\,a\,\beta\,u_n\,v_n+\beta^2\,v_n^2
\end{array}\right\}$$

Employing the sign of summation Σ to denote the sum of quantities of the same kind, $y_1^2 + y_2^2 + y_3^2 + \ldots + y_n^2 = \Sigma\,(y^2)$, $v_1\,y_1 + v_2\,y_2 + v_3\,y_3 + \ldots + v_n\,y_n = \Sigma\,(v\,y)$, etc., we have for the sum of the squares of the errors

$$\Sigma\,(z^2) = \Sigma\,(y^2) - 2\,a\,\Sigma\,(u\,y) - 2\,\beta\,\Sigma\,(v\,y) + a^2\,\Sigma\,(u^2) + 2\,a\,\beta\,\Sigma\,(u\,v) + \beta^2\,\Sigma\,(v^2).$$

In this equation, besides the sum of the squares of the errors $\Sigma\,(z^2)$, which is to be considered as the dependent variable, only a and β are unknown. The method of the smallest squares requires us to choose such values for a and β as shall cause $\Sigma\,(z^2)$ to be a minimum; and therefore we must differentiate the function $\Sigma\,(z^2)$, which we have obtained, once in reference to a and once in reference to β, and put each differential quotient of $\Sigma\,(z^2)$ thus obtained by itself equal to zero. In this way we obtain the following equations of condition for a and β,

$$-\,\Sigma\,(u\,y) + a\,\Sigma\,(u^2) + \beta\,\Sigma\,(u\,v) = 0,$$
$$-\,\Sigma\,(v\,y) + \beta\,\Sigma\,(v^2) + a\,\Sigma\,(u\,v) = 0,$$

and resolving these we have

$$a = \frac{\Sigma\,(v^2)\,\Sigma\,(u\,y) - \Sigma\,(u\,v)\,\Sigma\,(v\,y)}{\Sigma\,(u^2)\,\Sigma\,(v^2) - \Sigma\,(u\,v)\,\Sigma\,(u\,v)}, \text{ and}$$

$$\beta = \frac{\Sigma\,(u^2)\,\Sigma\,(v\,y) - \Sigma\,(u\,v)\,\Sigma\,(u\,y)}{\Sigma\,(u^2)\,\Sigma\,(v^2) - \Sigma\,(u\,v)\,\Sigma\,(u\,v)}. \quad \text{(See Ingenieur, page 77.)}$$

These formulas give for a function $y = a + \beta\,v$, since here $u = 1$, and $\Sigma\,(u\,v) = \Sigma\,(v)$, $\Sigma\,(u\,y) = \Sigma\,(y)$, and $\Sigma\,(u^2) = 1 + 1 + 1 + \ldots = n$, I.E., the number of equations or observations,

$$a = \frac{\Sigma\,(v^2)\,\Sigma\,(y) - \Sigma\,(v)\,\Sigma\,(v\,y)}{n\,\Sigma\,(v^2) - \Sigma\,(v)\,\Sigma\,(v)},$$

$$\beta = \frac{n\,\Sigma\,(v\,y) - \Sigma\,(v)\,\Sigma\,(y)}{n\,\Sigma\,(v^2) - \Sigma\,(v)\,\Sigma\,(v)}.$$

For the still simpler function $y = \beta\,v$, in which $a = 0$, we have

$$\beta = \frac{\Sigma\,(v\,y)}{\Sigma\,(v^2)},$$

and, finally, for the most simple case $y = a$, where we have to determine the most probable value of a single quantity,

$$a = \frac{\Sigma(y)}{n},$$

that is the arithmetical mean of all the values found by measurement or by observation.

EXAMPLE.—In order to discover the law of a uniformly accelerated motion, I.E., the initial velocity c and the acceleration p, we have measured the different times t_1, t_2, t_3, etc., and the corresponding spaces s_1, s_2, s_3, etc., described, and have found the following results,

Times . . .	0	1	3	5	7	10 sec.
Spaces . . .	0	5	20	38	$58\frac{1}{2}$	101 feet.

Now if $s = c\,t + \frac{p\,t^2}{2}$ is the fundamental law of this motion, we are required to determine the constants c and p. Putting in the foregoing formulas $u = t$, and $v = t^2$, and also $\alpha = c$, $\beta = \frac{p}{2}$ and $y = s$, we obtain for the calculation of c and p the following formulas:

$$c = \frac{\Sigma(t^4)\,\Sigma(s\,t) - \Sigma(t^3)\,\Sigma(s\,t^2)}{\Sigma(t^2)\,\Sigma(t^4) - \Sigma(t^3)\,\Sigma(t^3)} \text{ and}$$

$$\frac{p}{2} = \frac{\Sigma(t^2)\,\Sigma(s\,t^2) - \Sigma(t^3)\,\Sigma(s\,t)}{\Sigma(t^2)\,\Sigma(t^4) - \Sigma(t^3)\,\Sigma(t^3)},$$

from which the following calculations can be made,

t	t^2	t^3	t^4	s	$s\,t$	$s\,t^2$
1	1	1	1	5	5	5
3	9	27	81	20	60	180
5	25	125	625	38	190	950
7	49	343	2401	58.5	409.5	2866.5
10	100	1000	10000	101	1010	10100
Sum	184 $=\Sigma(t^2)$	1496 $=\Sigma(t^3)$	13108 $=\Sigma(t^4)$	222.5 $=\Sigma(s)$	1674.5 $=\Sigma(s\,t)$	14101.5 $=\Sigma(s\,t^2)$.

from which we obtain

$$c = \frac{13108 \,.\, 1674{,}5 - 1496 \,.\, 14101{,}5}{184 \,.\, 13108 - 1496 \,.\, 1496} = \frac{85340}{17386} = 4{,}908 \text{ feet, and}$$

$$\tfrac{1}{2}\,p = \frac{184 \,.\, 14101{,}5 - 1496 \,.\, 1674{,}5}{184 \,.\, 13108 - 1496 \,.\, 1496} = \frac{89624}{173860} = 0{,}5155 \text{ feet.}$$

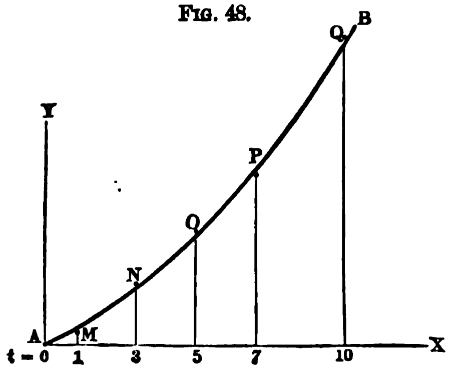
Whence the formula for the observed movement is

$$s = 4{,}908\, t + 0{,}5155\, t^2,$$

and from this formula we have

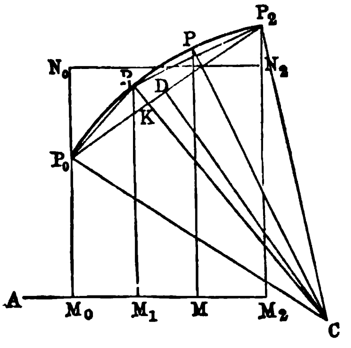
For the times .	0	1	3	5	7	10 sec.
For the spaces	0	5.43	19.36	37.43	59.62	100.63 feet.

FIG. 48.



If we consider the times (t) as abscissas, and lay off the calculated as well as the observed spaces (s) as ordinates, we can draw a curve through the extremities of the calculated ordinates, which will pass between the points M, N, O, P, Q, determined by the observed co-ordinates, so that the sum of the squares of the deviation of the curve from these points shall be as small as possible.

FIG. 49.



ART. 37. If we have no formula for the successive values of a quantity y, or for its dependence upon another quantity x, and we wish to determine its value for a given value of x, determined by experiment, or taken from a table, we employ the so-called *method of interpolation*, of which only the most important part will be given here.

If the abscissas $A\,M_0 = x_0$, $A\,M_1 = x_1$ and $A\,M_2 = x_2$, Fig. 49, and the corresponding ordinates $M_0\,P_0 = y_0$, $M_1\,P_1 = y_1$, $M_2\,P_2 = y_2$ are given, we can express the ordinate $MP = y$, corresponding to the new abscissa $AM = x$, by the formula $y = \alpha + \beta x + \gamma x^2$, provided three given points P_0, P_1, P_2, lie nearly in a straight line or in a slightly curved arc. If we change the origin of co-ordinates from A to M_0, the generality of the expression will not be affected, and we obtain for $x = 0$ simply $y = \alpha$, and consequently the constant member $\alpha = y_0$. Substituting in the supposed equation, in the first place x_1 and y_1, and then in the second place x_2 and y_2, we obtain the two following equations of condition,

$$y_1 - y_0 = \beta x_1 + \gamma x_1^2, \text{ and}$$

$$y_2 - y_0 = \beta x_2 + \gamma x_2^2, \text{ hence}$$

$$\beta = \frac{(y_1 - y_0) x_2^2 - (y_2 - y_0) x_1^2}{x_1 x_2^2 - x_2 x_1^2}, \text{ and}$$

$$\gamma = \frac{(y_1 - y_0) x_2 - (y_2 - y_0) x_1}{x_1^2 x_2 - x_2^2 x_1}$$

from which we have

$$y = y_0 + \left(\frac{(y_1 - y_0) x_2^2 - (y_2 - y_0) x_1^2}{x_1 x_2^2 - x_2 x_1^2}\right) x + \left(\frac{(y_1 - y_0) x_2 - (y_2 - y_0) x_1}{x_1^2 x_2 - x_2^2 x_1}\right) x^2.$$

If the ordinate y_1 lies midway between y_0 and y_2, we have $x_2 = 2 x_1$, and therefore more simply

$$y = y_0 - \left(\frac{3 y_0 - 4 y_1 + y_2}{2 x_1}\right) x + \left(\frac{y_0 - 2 y_1 + y_2}{2 x_1^2}\right) x^2$$

If but two pair of co-ordinates x_0, y_0, and x_1, y_1 are given, we must regard the limiting line $P_0 P_1$ as a straight line, and consequently put $\qquad y = y_0 + \beta x$

and $\qquad y_1 = y_0 + \beta x_1$,

whence we have $\qquad \beta = \frac{y_1 - y_0}{x_1}$, and

$$y = y_0 + \left(\frac{y_1 - y_0}{x_1}\right) x.$$

When it is required to interpolate by construction between three ordinates y_0, y_1, y_2 a fourth ordinate y, we draw, through the extremities P_0, P_1, P_2 of these ordinates a circle, and take $y =$ to the ordinate of the same. The centre C of the circle is determined in the usual way by joining the points P_0 P_1 P_2 by straight lines and erecting perpendiculars at the middle points of the chords. The point of intersection C of the perpendiculars is the required centre.

If the distances of the middle point P_1 from the two others P_0

and P_2, are s_0 and s_2, and the distance $P_1 K$ of the point P from the chord $P_0 P_2 = s_1 = h$, we have for the angle at the periphery $a = P_1 P_0 P_2 = \frac{1}{2}$ the angle at the centre $P_1 C P_2$

$$sin.\, a = \frac{h}{s_0},$$

and consequently the radius of curvature $C P = C P_0 = C P_1 = C P_2$ is

$$r = \frac{s_2}{2\, sin.\, a} = \frac{s_0\, s_2}{2\, h};$$

consequently we find the centre C of the circle passing through the points P_0, P_1, P_2, by describing from P_0 or P_1 or P_2 with a radius equal to the value of r, calculated by means of this formula, an arc whose intersection with the perpendicular to the chord $P_0 P_2$ erected at its centre D is the required point.

ART. **38.** The mean of all the ordinates upon the line $M_0 M_2$ is the altitude of a rectangle $M_0 M_2 N_2 N_0$ with the same base $M_0 M_2$, and having the same area as the surface $M_0 M_2 P_2 P_1 P_0$, and can therefore easily be determined from this surface. According to Art. 29 we have

$$\begin{aligned}
F &= \int_0^{x_2} y\, d x = \int_0^{x_2} (y_0 + \beta x + \gamma x^2)\, d x \\
&= y_0 x_2 + \frac{\beta x_2^2}{2} + \frac{\gamma x_2^3}{3} \\
&= y_0 x_2 + \left(\frac{(y_1 - y_0)\, x_2^2 - (y_2 - y_0)\, x_1^2}{x_1 x_2^2 - x_2 x_1^2}\right) \frac{x_2^2}{2} \\
&\quad + \left(\frac{(y_1 - y_0)\, x_2 - (y_2 - y_0)\, x_1}{x_1^2 x_2 - x_2^2 x_1}\right) \frac{x_2^3}{3} \\
&= \left(y_0 + \frac{(y_1 - y_0)\, x_2^2}{6\, x_1\, (x_2 - x_1)} - \frac{(y_2 - y_0)\,(3\, x_1 - 2\, x_2)}{6\, (x_2 - x_1)}\right) x_2 \\
&= \left(\frac{y_0 + y_2}{2}\right) x_2 + \left(\frac{(y_1 - y_0)\, x_2 - (y_2 - y_0)\, x_1}{6\, x_1\, (x_2 - x_1)}\right) x_2^2,
\end{aligned}$$

and consequently the mean ordinate is

$$y_m = \frac{F}{x_2} = \frac{(y_0 + y_2)}{2} + \left(\frac{(y_1 - y_0)\, x_2 - (y_2 - y_0)\, x_1}{6\, x_1\, (x_2 - x_1)}\right) x_2.$$

If $\frac{y_2 - y_0}{y_1 - y_0}$ were $= \frac{x_2}{x_1}$, the boundary would be a right line, and we would have simply

$$F = \left(\frac{y_0 + y_2}{2}\right) x_2$$

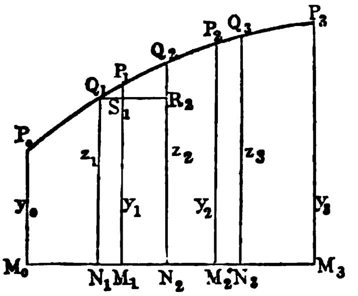
and

$$y_m = \frac{(y_0 + y_2)}{2}.$$

If also $x_2 = 2\,x_1$, that is, if y_1 is equidistant between y_0 and y_2, we have

$$F = (y_0 + 4\,y_1 + y_2)\,\frac{x_2}{6} \text{ (see Art. 30), and } y_m = \frac{y_0 + 4\,y_1 + y_2}{6}.$$

FIG. 50.



If a surface $M_0\,M_3\,P_3\,P_0$, Fig. 50, is determined by four co-ordinates $M_0\,P_0 = y_0$, $M_1\,P_1 = y_1$, $M_2\,P_2 = y_2$, $M_3\,P_3 = y_3$, which are equidistant from one another, we can determine approximately the area of the same in the following simple manner:

Let us denote by x_3 the base $M_0\,M_3$, by $z_0\;z_1\;z_3$, three ordinates intercalated between y_0 and y_3, and equidistant from each other, we can then put approximatively the surface

$$M_0\,M_3\,P_3\,P_0 = F = (\tfrac{1}{2}\,y_0 + z_1 + z_2 + z_3 + \tfrac{1}{2}\,y_3)\,\frac{x_3}{4}; \text{ but}$$

$$\frac{z_1 + z_2 + z_3}{3} = \frac{2\,z_1 + 2\,z_2 + 2\,z_3}{6} = \frac{2\,z_1 + z_2}{6} + \frac{2\,z_3 + z_2}{6} \text{ and}$$

$$y_1 = z_1 + \tfrac{1}{3}\,(z_2 - z_1) = \frac{2\,z_1 + z_2}{3}, \text{ as well as } y_2 = \frac{2\,z_3 + z_2}{3}$$

whence it follows that $\frac{z_1 + z_2 + z_3}{3} = \frac{y_1 + y_2}{2}$, and

$$F = [\tfrac{1}{2}\,y_0 + \tfrac{3}{2}\,(y_1 + y_2) + \tfrac{1}{2}\,y_3]\,\frac{x_3}{4}$$

$$= [y_0 + 3\,(y_1 + y_2) + y_3]\,\frac{x_3}{8}, \text{ and also}$$

$$y_m = \frac{y_0 + 3\,(y_1 + y_2) + y_3}{8}.$$

While the former formula for y_m is employed when the surface is divided into an even number of strips, the latter is employed when the number of these divisions is uneven.

Hence we can write approximately

$$\int_c^{c_1} y\,d\,x = \int_c^{c_1} \phi\,(x)\,d\,x = [y_0 + 3\,(y_1 + y_2) + y_3]\,\frac{c_1 - c}{8}, \text{ if}$$

$y_0 = \phi\ (c), y_1 = \phi\left(\frac{2\,c + c_1}{3}\right), y_2 = \phi\left(\frac{c + 2\,c_1}{3}\right)$ and $y_3 = \phi\ (c_1)$ are four known values of $y = \phi\,(x)$. E.G., for $\int_1^2 \frac{d\,x}{x}$ (see example, Art. 30) we have $c = 1, c_1 = 2$ and $\phi\ (x) = \frac{1}{x}$, whence it follows that

$y_0 = \frac{1}{1} = 1, y_1 = \frac{3}{2 + 2} = \frac{3}{4}, y_2 = \frac{3}{1 + 4} = \frac{3}{5}$ and $y\ = \frac{1}{2}$, and that the approximate value of this integral is

$$\int_1^2 \frac{d\,x}{x} = [1 + 3\,(\tfrac{3}{4} + \tfrac{3}{5}) + \tfrac{1}{2}]\cdot\tfrac{1}{8} = \frac{111}{160} = 0{,}694.$$

PART FIRST.

GENERAL PRINCIPLES OF MECHANICS.

FIRST SECTION.

PHORONOMICS OR THE PURELY MATHEMATICAL THEORY OF MOTION.

CHAPTER I.

SIMPLE MOTION.

§ 1. **Rest and Motion.**—Everybody occupies a certain position in space, and a body is said to be at *rest*, (Fr. repos, Ger. Ruhe), when it does not change that position, and, on the contrary, a body is said to be in *motion*, (Fr. mouvement, Ger. Bewegung), when it passes continually from one position to another.

The rest and motion of a body are either absolute or relative, according as its position is referred to a point which is itself at rest or in motion.

On the earth there is no rest, for all bodies upon it participate in its motion about its axis and around the sun. If we suppose the earth at rest, all the terrestrial bodies which do not change their position in regard to the earth are at rest.

§ 2 **Kinds of Motion.**—The uninterrupted succession of positions which a body occupies in its motion forms a space, that is called the *path* or *trajectory* (Fr. Chemin, trajectoire, Ger. Weg) of the moving body. The path of a point is a line. The path of a geometrical body is, it is true, a figure, but we generally understand by it the path of a certain point of the moving body, as, E.G., its centre. Motion is *rectilinear* (Fr. rectiligne, Ger. geradlinig) when the path is a right line, and *curvilinear* (Fr. curviligne, Ger. krummlinig) when the path of the moving body is a curved line.

§ **3**. In reference to time (Fr. temps, Ger. Zeit) motion is either *uniform* or *variable.* Motion is uniform (Fr. uniforme, G. gleichförmig) when equal spaces are passed through in equal arbitrary portions of time. It is variable (Fr. varié, Ger. ungleichförmig) when this equality does not exist. When the spaces described in equal times become greater and greater as the time during which the body is in motion increases, the variable motion is said to be *accelerated* (Fr. accéléré, Ger. beschleunigt); but if they decrease more and more with the increase of time, this motion is said to be *retarded* (Fr. retardé, Ger. verzögert). *Periodic* (Fr. périodique, Ger. periodisch) motion differs from uniform motion in this, that equal spaces are described only within certain finite spaces of time, which are called periods. The best example of uniform motion is given by the apparent revolution of the fixed stars, or by the motion of the hands of a clock. Examples of variable motion are furnished by falling bodies, by bodies thrown upwards, by the sinking of the surface of water in a vessel which is emptying itself, etc. The play of the piston of a steam engine, and the oscillations of a pendulum, afford good examples of periodic motion.

§ **4. Uniform Motion.**—*Velocity* (Fr. vitesse, Ger. Geschwindigkeit) is the rate or measure of a motion. The larger the space that a body passes through in a given time, the greater is its motion or its velocity. In uniform motion the velocity is constant, and in variable motion it changes at each instant. The measure of the velocity at a given moment of time is the space that this body either really describes, or which it would describe, if at that instant the motion became uniform or the velocity remained constant. We generally call this measure simply the velocity.

§ **5**. If a body in each instant of time describes the space σ, and if a second of time is made up of n (very many) such instants, then the space described within a second is the velocity, or rather the measure of the velocity, and it is

$$c = n \,.\, \sigma.$$

During a time t (seconds) $n \,.\, t$ instants elapse, and in each instant the body passes through the space σ, and therefore the total space, (Fr. l'espace, Ger. Weg), which corresponds to the time t, is

$$s = n \,.\, t \,.\, \sigma = n \,.\, \sigma \,.\, t, \text{ I.E.}$$

$$\text{I.)} \quad s = c\,t.$$

In uniform motion the space (s) *is a product of the velocity* (c) *and the time* (t).

$$\text{Inversely II.)} \quad c = \frac{s}{t}.$$

$$\text{III.)} \quad t = \frac{s}{c}.$$

Example.—1. A locomotive advancing with a velocity of 30 feet passes in two hours = 120 minutes = 7200 seconds, over the space $s = 30 \,.\, 7200 = 216000$ feet.

2. If we require $4\frac{1}{2}$ minutes = 270 seconds to raise a bucket out of a pit, which is 1200 feet deep, we have its mean velocity $(c) = \frac{1200}{270} = \frac{40}{9} = 4\frac{4}{9} = 4{,}444 \ldots$ feet.

3. A horse advancing with a velocity of 6 feet requires, to pass over five miles, or 26400 feet, the time $t = \frac{26400}{6} = 4400$ seconds, or 1 hour 13 minutes and 20 seconds.

§ **6.** If we compare two different uniform motions, we obtain the following result:

As the spaces are $s = c\,t$ and $s_1 = c_1\,t_1$ their ratio is $\frac{s}{s_1} = \frac{c\,t}{c_1\,t_1}$.

If we put $t = t_1$ we have $\frac{s}{s_1} = \frac{c}{c_1}$; if we take $c = c_1$ we obtain $\frac{s}{s_1} = \frac{t}{t_1}$; and finally, if $s = s_1$ it follows that $\frac{c}{c_1} = \frac{t_1}{t}$.

The spaces described in the same time in different uniform motions are to each other as the velocities; the spaces described with equal velocities are to each other as the times; and the velocities corresponding to equal spaces are inversely as the times.

§ **7. Uniformly Variable Motion.**—A motion is *uniformly variable,* (Fr. uniformément varié, Ger. gleichförmig verändert), when the increase or diminution of the velocity within equal, arbitrarily small, portions of time is always the same. It is either *uniformly accelerated* (Fr. uniformément accéléré, Ger. gleichförmig beschleunigt) or *uniformly retarded* (Fr. uniformément retardé, Ger. gleichförmig verzögert). In the first case a gradual augmentation, and in the second a gradual diminution of velocity takes place.

A body falling *in vacuo* is uniformly accelerated, and a body projected vertically upwards would be uniformly retarded, if the air exerted no influence upon it.

§ 8. The amount of the change in the velocity of a body is called the *acceleration* (Fr. accélération, Ger. Beschleunigung and Acceleration). It is either positive (acceleration) or negative (retardation), the former when there is an increase, and the latter when there is a diminution of velocity. In uniformly variable motion the acceleration is constant. We can therefore measure it by the increase or decrease of velocity which takes place in a second. For any other motion, the acceleration is the increase or decrease of velocity, which a body would undergo if, from the instant for which we wish to give the acceleration, the acceleration became constant, and the motion was changed to a uniformly varied one.

This measure is generally called simply the acceleration.

§ 9. If the velocity of an uniformly accelerated motion in a very small (infinitely small) instant of time is increased by a quantity κ, and if the second of time is composed of n (an infinite number of) such instants, the increase of velocity in a second, or the so-called acceleration, is

$$p = n\,\kappa,$$

and the increase after t seconds is $= n\,t\,.\,\kappa = n\,\kappa\,.\,t = p\,t$.

If the *initial velocity* (at the moment from which we begin to count t) is $= c$, we have for the *final velocity*, I.E., for the velocity at the end of the time t,

$$v = c + p\,t.$$

For a motion starting from rest c is $= 0$, whence $v = p\,t$; and when the motion is uniformly retarded, in which case the acceleration $(-p)$ is negative, we have

$$v = c - p\,t.$$

EXAMPLE.—1. The acceleration of a body falling freely in vacuo is $= 32{,}20$ feet. It acquires therefore after 3 seconds the velocity $v = p\,t = 32{,}20\,.\,3 = 96{,}60$ feet.

2. A ball rolling down an inclined plane has in the beginning a velocity

of 25 feet, and the acceleration is 5 feet per second. Its velocity after $2\frac{1}{2}$ seconds is therefore $v = 25 + 5 \,.\, 2{,}5 = 37{,}5$ feet; I.E., if from the last moment it moved forward uniformly, it would pass over 37,5 feet in every second.

3. A locomotive moving with a velocity of 30 feet loses, in consequence of the action of the brake, 3,5 feet of its velocity every second; its acceleration is therefore $-\,3{,}5$ feet and its velocity after 6 seconds is $v = 30 - 3{,}5 \,.\, 6 = 30 - 21 = 9$ feet.

§ **10. Uniformly Accelerated Motion.**—Within an infinitely small instant of time τ we can consider the velocity of every motion as constant, and put the space passed through in this instant

$$\sigma = v \,.\, \tau,$$

and we obtain the space passed through in the finite time t by summing these small spaces. But the time in which all these small spaces were described is one and the same τ, and we can put their sum equal to the product of this instant of time and the sum of the velocities corresponding to the different equal instants.

For uniformly accelerated motion the sum $(0 + v)$ of the velocities in the first and last instant is just as great as the sum $p\,\tau + (v - p\,\tau)$ of those in the second and last but one instants, and equal to the sum $2\,p\,\tau + (v - 2\,p\,\tau)$ of those in the third and last but two instants, etc., and this sum is in general equal to v; the sum of all these velocities is therefore equal to $\left(v \,.\, \frac{n}{2}\right)$ the product of the final velocity and half the number of the elements of the time, and the space described is equal to the product $\left(v \,.\, \frac{n}{2} \,.\, \tau\right)$ of the final velocity v and half the number of the elements of the time and one of these elements. Now the magnitude (τ) of an element of the time multiplied by their number gives the whole time t, whence the space described in the time t with an uniformly accelerated motion is $s = \frac{v\,t}{2}$.

The space described with uniformly accelerated motion is the same as that described with uniform motion when the velocity of the latter is half the final velocity of the former.

EXAMPLE.—1. If a body in uniformly varied motion has acquired in 10 seconds a velocity $v = 26$ feet, the space described in the same time is $s = \frac{26 \,.\, 10}{2} = 130$ feet.

2. A wagon whose motion is uniformly accelerated and which describes 25 feet in $2\frac{1}{4}$ seconds, possesses at the end of that time the velocity

$$v = \frac{2 \,.\, 25}{2{,}25} = \frac{50 \,.\, 4}{9} = 22{,}22 \ldots \text{ feet.}$$

§ **11.** The two fundamental formulas of uniformly accelerated motion

$$\text{I.)}\ v = p\,t \text{ and}$$

$$\text{II.)}\ s = \frac{v\,t}{2},$$

which show that the velocity is a product of the acceleration and the time, and that the space is the product of half the terminal velocity and the time, furnish two other equations, when we eliminate in the first place v and in the second t. By this operation we obtain

$$\text{III.)}\ s = \frac{pt^2}{2} \text{ and}$$

$$\text{IV.)}\ s = \frac{v^2}{2\,p}.$$

Hence, *in uniformly accelerated motion, the space described is equal to the product of half the acceleration and the square of the time, and also to the square of the terminal velocity divided by double the acceleration.*

From these four principal formulas we deduce by inversion, and by the elimination of one or other of the quantities contained in them, eight other formulas, which are collected together in a table in the "Ingenieur," page 325.

EXAMPLE.—1. A body moving with the acceleration 15,625 feet, describes in 1,5 seconds the space $\frac{15{,}625 \,.\, (1{,}5)^2}{2} = 15{,}625 \,.\, \frac{9}{8} = 17{,}578$ feet.

2. A body, which acquires a velocity $v = 16{,}5$ in consequence of an acceleration $p = 4{,}5$ feet, has described in so doing the space $s = \frac{(16{,}5)^2}{2 \,.\, 4{,}5} = 30{,}25$ feet.

§ **12.** On comparing two different uniformly accelerated motions, we arrive at the following conclusions.

The velocities are $v = p\ t$ and $v_1 = p_1\ t_1$. The spaces, on the contrary, are $s = \frac{p\,t^2}{2}$ and $s_1 = \frac{p_1\,t_1^2}{2}$, whence we have

$$\frac{v}{v_1} = \frac{p\,t}{p_1\,t_1} \text{ and } \frac{s}{s_1} = \frac{p\,t^2}{p_1\,t_1^2} = \frac{v\,t}{v_1\,t_1} = \frac{v^2\,p_1}{v_1^2\,p}.$$

Putting $t_1 = t$ we obtain:

$\frac{s}{s_1} = \frac{v}{v_1} = \frac{p}{p_1}$; the times being equal, the ratio of the spaces described is equal to that of the final velocities or of the accelerations.

If we put $p_1 = p$ we have

$$\frac{v}{v_1} = \frac{t}{t_1} \text{ and } \frac{s}{s_1} = \frac{t^2}{t_1^2} = \frac{v^2}{v_1^2}.$$

The acceleration being the same, I.E., when we have the same uniformly accelerated motion, the final velocities are to each other as the times, the spaces described as the squares of the times, and also as the squares of the final velocities.

Farther, if we take $v_1 = v$ it gives $\frac{p}{p_1} = \frac{t_1}{t}$ and $\frac{s}{s_1} = \frac{t}{t_1}$; for the same final velocities the accelerations are to each other inversely, and the spaces directly as the times.

Finally, for $s_1 = s$ we have $\frac{p}{p_1} = \frac{t_1^2}{t^2} = \frac{v^2}{v_1^2}$; for equal spaces described the accelerations are to each other inversely as the squares of the times and directly as the squares of the velocities.

§ **13.** For a uniformly accelerated motion with the initial velocity c we have from § 9

$$\text{I.)} \quad v = c + p\,t,$$

and since the space $c\,t$ belongs to the constant velocity c, and the space $\frac{p\,t^2}{2}$ to the acceleration p

$$\text{II.)} \quad s = c\,t + \frac{p\,t^2}{2}.$$

Eliminating p from the two equations, we obtain

$$\text{III.)} \quad s = \frac{c + v}{2}\,t,$$

or eliminating t, we find

$$\text{IV.)} \quad s = \frac{v^2 - c^2}{2\,p}.$$

EXAMPLE.—1. A body moving with the initial velocity $c = 3$ feet and with the acceleration $p = 5$ feet describes in 7 seconds the space

$$s = 3 \cdot 7 + 5 \cdot \frac{7^2}{2} = 21 + 122{,}5 = 143{,}5 \text{ feet.}$$

2. Another body, which in 3 minutes = 180 seconds changes its velocity from 2½ feet to 7½ feet, describes during this time the space $\frac{2{,}5 + 7{,}5}{2} \cdot 180 = 900$ feet.

§ 14. **Uniformly Retarded Motion.**—For uniformly retarded motion with the initial velocity c we have the following formulas, which are deduced from those of the foregoing paragraph by making p negative.

$$\text{I.)}\ v = c - p\,t,$$

$$\text{II.)}\ s = c\,t - \frac{p\,t^2}{2},$$

$$\text{III.)}\ s = \frac{c + v}{2} \cdot t,$$

$$\text{IV.)}\ s = \frac{c^2 - v^2}{2\,p}.$$

While in uniformly accelerated motion the velocity increases without limit, in uniformly retarded motion the velocity decreases up to a certain time, when it is $= 0$, and afterwards it becomes negative, I.E., the motion continues in the opposite direction.

If we put $v = 0$ in the first formula, we obtain $p\,t = c$, whence the time in which the velocity becomes $= 0$ is $t = \frac{c}{p}$; substituting this value of t in the second equation, we obtain the space described by the body during this time, $s = \frac{c^2}{2p}$.

If the time is greater than $\frac{c}{p}$, the space is smaller than $\frac{c^2}{2p}$; and if the time is $= \frac{2c}{p}$ the space becomes $= 0$, the body having returned to its point of departure; finally, if the time is greater than $\frac{2c}{p}$, s is negative, I.E., the body is on the opposite side of the point of departure.

EXAMPLE.—A body which is rolled up an inclined plane with an initial velocity of 40 feet, and which suffers a retardation of 8 feet per second, rises only during $\frac{40}{8} = 5$ seconds and reaches a height of $\frac{40^2}{2 \cdot 8} = 100$ feet, after which it rolls back and arrives after 10 seconds with a velocity of 40 feet at the point from whence it started, and after 12 seconds is already $40 \cdot 12 - 4 \cdot 12^2$ or $-(40 \cdot 2 + 4 \cdot 2^2) = 96$ feet below its point of departure, if the plane continues beneath it.

§ **15. The Free Fall of Bodies.**—The *free or vertical fall of bodies in vacuo* (Fr. mouvement vertical des corps pesants, Ger. der freie oder senkrechte Fall der Körper) furnishes the most important example of uniformly accelerated motion. The acceleration of this motion produced by gravity (Fr. gravité, Ger. Schwerkraft) is designated by g, and its mean value is

9,81 meters.
30,20 Paris feet.
32,20 English feet.
31,03 Vienna feet.
$31\frac{1}{4}$ = 31,25 Prussian feet.
32,7 Bavarian or meter feet.

If any of these values of g be substituted in the formulas $v = g\,t$, $s = \frac{g\,t^2}{2}$ and $s = \frac{v^2}{2\,g}$, $v = \sqrt{2\,g\,s}$, all possible questions in relation to the free fall of bodies can be answered.

For the metrical system of measures we have

$$v = 9{,}81\,.\,t = 4{,}429\sqrt{s},$$
$$s = 4{,}905\,t^2 = 0{,}0510\,v^2,$$
$$t = 0{,}1019\,v^2 = 0{,}4515\sqrt{s};$$

and for English measures

$$v = 32{,}2\,t = 8{,}025\sqrt{s},$$
$$s = 16{,}1\,t^2 = 0{,}0155\,v^2,$$
$$t = 0{,}031\,v = 0{,}249\sqrt{s}.$$

EXAMPLE.—1.) A body attains when it falls unhindered in 4 seconds a velocity $v = 32{,}2\,.\,4 = 128{,}8$ feet, and describes in this time the space $s = 16{,}1\,.\,4^2 = 257{,}6$ feet. 2.) A body which has fallen from the height $s = 9$ feet, has the velocity $v = 8{,}025\sqrt{9} = 24{,}075$. 3) A body projected vertically upwards with a velocity of 10 feet rises to the height $s = 0{,}0155\,.\,10^2 = 1{,}55$ feet, in the time

$$t = 0{,}031\,.\,10 = 0{,}31,$$

or nearly $\frac{1}{3}$ of a second.

§ **16.** The following Table shows how the motion takes place as the time elapses,

Time in seconds	0	1	2	3	4	5	6	7	8	9	10
Velocity.	0	$1g$	$2g$	$3g$	$4g$	$5g$	$6g$	$7g$	$8g$	$9g$	$10g$
Space .	0	$1\frac{g}{2}$	$4\frac{g}{2}$	$9\frac{g}{2}$	$16\frac{g}{2}$	$25\frac{g}{2}$	$36\frac{g}{2}$	$49\frac{g}{2}$	$64\frac{g}{2}$	$81\frac{g}{2}$	$100\frac{g}{2}$
Difference	0	$1\frac{g}{2}$	$3\frac{g}{2}$	$5\frac{g}{2}$	$7\frac{g}{2}$	$9\frac{g}{2}$	$11\frac{g}{2}$	$13\frac{g}{2}$	$15\frac{g}{2}$	$17\frac{g}{2}$	$19\frac{g}{2}$

The last horizontal column of this table gives the spaces described by a body falling freely in each single second. We see that these spaces are to each other as the uneven numbers 1, 3, 5, 7, etc., while the times and the velocities are to each other as the regular series of numbers 1, 2, 3, 4, 5, etc., and the distances fallen through as their squares 1, 4, 9, 16, etc. Whence, E.G., the velocity after 6 seconds is $= 6\,g = 193{,}2$ feet, I.E., the body, if from this moment it continued to move uniformly as on a horizontal plane which offered no resistance, would describe in every second the space $6\,g = 193{,}2$ feet. It does not really describe this space in the following or seventh second, but from the last column we see that it describes exactly $13\,\frac{g}{2} = 13 \,.\, 16{,}1 = 209{,}3$ feet, and in the eighth second $15\,\frac{g}{2} = 15 \,.\, 16{,}1 = 241{,}5$ feet.

REMARK.—Older German writers designate the space 16,1 feet, described by a body falling freely in the first second, by g, and call it also the acceleration of gravity. They employ for the free fall of bodies the formulas

$$v = 2\,g\,t = 2\,\sqrt{g\,s},$$

$$s = g\,t^2 = \frac{v^2}{4g},$$

$$t = \frac{v}{2g} = \sqrt{\frac{s}{g}}.$$

This usage, known only in Germany, is tending gradually to disappear, which, on account of the frequent misapprehensions and errors resulting from it, is much to be desired.

§ 17. **Free Fall with an Initial Velocity.**—If the free fall of a body takes place with an initial velocity (Fr. vitesse initiale, Ger.

Anfangsgeschwindigkeit) c, the formulas assume the following form:

$$v = c + g\,t = c + 32{,}2\,t \text{ feet} = c + 9{,}81\,t \text{ meters},$$
$$v = \sqrt{c^2 + 2\,g\,s} = \sqrt{c^2 + 64{,}4\,s} \text{ feet} = \sqrt{c^2 + 19{,}62\,s} \text{ meters},$$
$$s = c\,t + \frac{g}{2}\,t^2 = c\,t + 16{,}1\,t^2 \text{ feet} = c\,t + 4{,}905\,t^2 \text{ meters},$$

and $s = \dfrac{v^2 - c^2}{2\,g} = 0{,}0155\,(v^2 - c^2) \text{ feet} = 0{,}0510\,(v^2 - c^2) \text{ meters}.$

If, on the contrary, the body is projected vertically upwards, we have

$$v = c - g\,t = c - 32{,}2\,t \text{ feet} = c - 9{,}81\,t \text{ meters},$$
$$v = \sqrt{c^2 - 2\,g\,s} = \sqrt{c^2 - 64{,}4\,s} \text{ feet} = \sqrt{c^2 - 19{,}62\,s} \text{ meters},$$
$$s = c\,t - \frac{g}{2}\,t^2 = c\,t - 16{,}1\,t^2 \text{ feet} = c\,t - 4{,}905\,t^2 \text{ meters},$$

and $s = \dfrac{c^2 - v^2}{2\,g} = 0{,}0155\,(c^2 - v^2) \text{ feet} = 0{,}0510\,(c^2 - v^2) \text{ meters}.$

If we consider a given velocity c as a velocity acquired by a free fall, we call the space fallen through

$$\frac{c^2}{2\,g} = 0{,}0155\,c^2 \text{ feet} = 0{,}0510\,c^2 \text{ meters},$$

"*the height due to the velocity*" (F. hauteur due à la vitesse, Ger. Geschwindigkeitshöhe). By the substitution of the above, several of the foregoing formulas may be expressed more simply. If we denote the height $\left(\dfrac{c^2}{2\,g}\right)$ due to the initial velocity by k, and that $\left(\dfrac{v^2}{2\,g}\right)$ due to the final velocity by h, we have for falling bodies,

$$h = k + s \text{ and } s = h - k,$$

and for ascending bodies,

$$h = k - s \text{ and } s = k - h.$$

The space described in falling or ascending is therefore equal to the difference of the heights due to the velocities.

EXAMPLE.—If for a uniformly varied motion the velocities are 5 feet and 11 feet, and the heights due to the velocities are $0{,}0155 \,.\, 5^2 = 0{,}3875$, and $0{,}0155 \,.\, 11^2 = 1{,}8755$, the space described in passing from one velocity to the other is $s = 1{,}8755 - 0{,}3875 = 1{,}4880$ feet.

§ 18. Vertical Ascension.—If in the formula $s = \frac{c^2 - v^2}{2g}$ for the vertical ascension of bodies we put the final velocity $v = 0$, we obtain the maximum height of ascension,

$$s = \frac{c^2}{2g};$$

consequently the maximum height of ascension, corresponding to the velocity c, is equal to the height of fall k due to the final velocity c, and therefore $c = \sqrt{2gk}$ is not only the final velocity for the height k of free fall, but also the initial velocity for the maximum height of ascension k. Hence it follows that a body projected vertically upwards has at any point the same velocity, which it would have, in the opposite direction, if it fell from a height equal to the remaining height of ascension to that point, and which it really possesses afterwards, when it reaches it upon falling back.

Example.—A body projected vertically upwards, with a velocity of 15 feet, after ascending 2 feet meets an elastic obstruction, which throws it back instantaneously with the same velocity with which it struck. How great is this velocity, and how much time does the body require to ascend and fall back again? The height due to the initial velocity 15 feet is $k =$ 3,49 feet, and the height due to the velocity at the instant of collision is $h = 3{,}49 - 2{,}00 = 1{,}49$, and, consequently, the velocity itself is $= 8{,}025 \sqrt{1{,}49} = 9{,}8$ feet. The time necessary to ascend the entire height (3,49 feet) would be $t = 0{,}031\, c = 0{,}031 \,.\, 15 = 0{,}465$ seconds, while the time necessary to ascend the height 1,49 is $t_1 = 0{,}031 \,.\, 9{,}8 = 0{,}3038$ seconds, whence the time necessary to ascend the 2 feet is $t - t_1 = 0{,}465 - 0{,}3038 = 0{,}1612$ seconds, and finally the whole time employed in ascending and falling is $= 2 \,.\, 0{,}1612 = 0{,}3224$ seconds. This, therefore, is but $\frac{3224}{9300}$, or about ⅓ of the time, which would be employed by the body in rising and falling if it met with no obstacle. This case occurs in practice in forging red-hot iron, for we are obliged to give as many strokes of the hammer as possible in a short space of time, on account of the gradual cooling of the iron. If by means of an elastic spring we cause the hammer to be thrown back, it can, under the circumstances supposed in the example, make three times as many blows as when its rise was unimpeded.

Remark 1.—In practical mechanics, particularly in hydraulics, we are often obliged to convert velocity into height due to velocity, or the latter into the former. A table, by means of which this operation can be performed at once, is of the greatest service to the practical man. Such a one, calculated for the Prussian foot, is to be found in the "Ingenieur," page 326 to 329.

REMARK 2.—The formulas deduced in the foregoing paragraphs are strictly correct only for bodies falling freely *in vacuo;* they are, however, sufficiently accurate for practical purposes, when the weight of the body is great compared to its volume, and when the velocities are not very great. They are, besides, employed in many other cases, as will be shown hereafter.

§ 19. Variable Motion in General.—The formula $s = c\,t$ (§ 5) for uniform motion holds good also for every *variable motion,* if instead of t we substitute an element or an infinitely small instant of the time τ, and instead of s the space σ described in this instant, for we can assume that during the instant τ the velocity c, which we here denote by v, remains constant, and that the motion itself is uniform.

Hence, we have for every variable motion

$$\text{I.)} \quad \sigma = v\,\tau, \text{ and } v = \frac{\sigma}{\tau} \text{ (compare § 10).}$$

The velocity (v) *for every instant is given by the quotient of the element of the space divided by that of the time.*

In like manner the formula $v = p\,t$ (§ 11) for uniformly accelerated motion holds good also for every variable motion, if instead of t and v we substitute the element of time τ and the infinitely small increase of velocity κ during that time, for the acceleration p does not vary sensibly in an instant τ, and the motion can be regarded as uniformly accelerated during this instant. Consequently we have for all motions

$$\text{II.)} \quad \kappa = p\,\tau, \text{ and } p = \frac{\kappa}{\tau}.$$

The acceleration (p) *is, therefore, equal to the element of the velocity divided by the element of the time.*

If we put the total duration of the motion $t = n\,\tau$, and the velocities in the successive instants τ are $v_1, v_2, v_3 \ldots v_n$, the corresponding elements of the space are $\sigma_1 = v_1\,\tau$, $\sigma_2 = v_2\,\tau$, $\sigma_3 = v_3\,\tau \ldots$, $\sigma_n = v_n\,\tau$, and the total space described is

$$s = (v_1 + v_2 + v_3 \ldots v_n)\,\tau = \left(\frac{(v_1 + v_2 + \ldots + v_n)}{n}\right) n\,\tau, \text{ I.E.,}$$

$$\text{I*)} \quad s = \left(\frac{v_1 + v_2 + \ldots + v_n}{n}\right) t = v\,t, \text{ when}$$

$v = \frac{v_1 + v_2 + v_3 \ldots v_n}{n}$ denotes the mean velocity of the body while describing the space s.

In like manner if c denotes the initial and v the final velocity, and if $p_1, p_2 \ldots p_n$ denote the accelerations in the equal successive instants τ, we have

$$v - c = (p_1 + p_2 + \ldots p_n)\,\tau = \left(\frac{p_1 + p_2 + \ldots + p_n}{n}\right) n\,\tau, \text{ I.E.,}$$

$$\text{II*)} \quad v - c = \left(\frac{p_1 + p_2 + \ldots p_n}{n}\right) t = p\,t, \text{ when}$$

$p = \frac{p_1 + p_2 + \ldots + p_n}{n}$ denotes the mean acceleration.

By combining the formulas I. and II. we obtain the following not less important equation:

$$\text{III.)} \quad v\,\kappa = p\,\sigma.$$

If, while the space $s = n\,\sigma$ is described, the acceleration assumes successively the values $p_1, p_2 \ldots p_n$, the sum of the products $p\,\sigma$ is

$$(p_1 + p_2 \ldots + p_n)\,\sigma = \left(\frac{p_1 + p_2 + \ldots + p_n}{n}\right) n\,\sigma$$

$$= \left(\frac{p_1 + p_2 + \ldots + p_n}{n}\right) s = p\,s.$$

If the initial velocity c is transformed by successive increases of $\kappa = \frac{v - c}{n}$ into the final velocity v, the sum of the products $v\,\kappa$ is

$$c\kappa + (c + \kappa)\,\kappa + \ldots\ldots + (v - \kappa)\,\kappa + v\,\kappa = [c + (c + \kappa) + \ldots + (v - \kappa) + v]\,\kappa$$

$$= (v + c)\,\frac{n\,\kappa}{2} = \frac{(v + c)\,(v - c)}{2} = \frac{v^2 - c^2}{2},$$

and therefore we can write

$$\text{III*)} \quad \frac{v^2 - c^2}{2} = p\,s, \text{ or } s = \frac{v^2 - c^2}{2\,p} \text{ (compare IV., § 13).}$$

With the aid of the foregoing formulas we can solve the most varied problems of phoronomics and mechanics.

The time, in which the space $s = n\,\sigma$ is described with the variable velocities $v_1, v_2, \ldots v_n$, is

$$\text{IV.)} \quad t = \sigma\left(\frac{1}{v_1} + \frac{1}{v_2} + \ldots \frac{1}{v_n}\right) = \frac{s}{n}\left(\frac{1}{v_1} + \frac{1}{v_2} + \ldots + \frac{1}{v_n}\right) = \frac{s}{v},$$

when we put the value $\frac{1}{n}\left(\frac{1}{v_1} + \frac{1}{v_2} + \ldots + \frac{1}{v_n}\right) = \frac{1}{v}$, whose reciprocal v can be considered as the mean velocity.

EXAMPLE.—When a body moves according to the law $v = a\,t^2$, we have $v + \kappa = a\,(t + \tau)^2 = a\,(t^2 + 2\,t\,\tau + \tau^2)$, and $\kappa = a\,\tau\,(2\,t + \tau)$, consequently $p = \frac{\kappa}{\tau} = 2\,a\,t$.

The velocities of the body at the end of the times

$$\tau,\ 2\tau,\ 3\,\tau \ldots n\,\tau \text{ are } a\,\tau^2,\ a\,(2\,\tau)^2,\ a\,(3\,\tau)^2 \,..\, a\,(n\,\tau)^2,$$

whence it follows that the space described in $t = n\,\tau$ seconds is

$$s = [a\,\tau^2 + a\,(2\,\tau)^2 + \,..\, a\,(n\,\tau)^2]\,\tau = (1^2 + 2^2 + 3^2 + \,..\, + n^2)\,a\,\tau^3,$$

but from Article 15, IV., of the Introduction to the Calculus we have

$$1^2 + 2^2 + 3^2 + \,..\, + n^2 = \frac{n^3}{3}, \text{ hence}$$

$$s = \frac{n^3}{3}\,a\,\tau^3 = \frac{a}{3}\,(n\,\tau)^3 = \frac{a\,t^3}{3}$$

(§ 20.) Differential and Integral Formulas of Phoronomics.—The general formulas of motion found in the foregoing paragraphs assume, when the notations of the calculus are employed, I.E., when the element of time τ is designated by $d\,t$, the element of space σ by $d\,s$, and the element of velocity κ by $d\,v$, the following form:

I.) $v = \dfrac{d\,s}{d\,t}$, or $d\,s = v\,d\,t$, whence $s = \displaystyle\int v\,d\,t$, and $t = \displaystyle\int \frac{d\,s}{v}$.

II.) $p = \dfrac{d\,v}{d\,t}$, or $d\,v = p\,d\,t$, whence $v = \displaystyle\int p\,d\,t$, and $t = \displaystyle\int \frac{d\,v}{p}$.

III.) $v\,d\,v = p\,d\,s$, or $s = \displaystyle\int \frac{v\,d\,v}{p}$, and $\dfrac{v^2 - c^2}{2} = \displaystyle\int p\,d\,s$,

in which c denotes the initial and v the final velocity, while the space s is being described.

We see from the above *that the difference of the squares of the velocities is equal to twice the integral of the product of the acceleration and the differential $d\,s$, or equal to the product of the mean acceleration and the space described by the body in passing from the velocity c to the velocity v.*

According to the theory of maxima and minima the space is a maximum, and the motion attains the greatest extension, when we have

$$\frac{d\,s}{d\,t} = v = 0,$$

and the velocity is a maximum or minimum when

$$\frac{d\,v}{d\,t} = p = 0.$$

The foregoing are the fundamental formulas of the higher Phoronomics and Mechanics.

EXAMPLE.—1. From the equation for the space $s = 2 + 3\,t + t^2$, we deduce by differentiation the equation for the velocity $v = 3 + 2\,t$, and that for the acceleration $p = 2$; the latter is constant and the motion is uniformly accelerated,

For $t = 0, 1, 2, 3 \dots$ seconds, we have
$v = 3, 5, 7, 9 \dots$ (Feet), and
$s = 2, 6, 12, 20 \dots$ (Feet).

2. From the formula for the velocity
$v = 10 + 3t - t^2$, we obtain by integration

$$s = \int 10\,dt + \int 3t\,dt - \int t^2\,dt = 10t + \tfrac{3}{2}t^2 - \frac{t^3}{3},$$

and on the contrary by differentiation $p = 3 - 2t$.

Consequently, for $3 - 2t = 0$, I.E., for $t = \frac{3}{2}$ seconds, the acceleration is 0 and the velocity is a maximum ($v = 12\frac{1}{4}$), and for $10 + 3t - t^2 = 0$, I.E., for $t = \frac{3}{2} + \sqrt{10 + \frac{9}{4}} = \frac{3 + 7}{2} = 5$ the velocity is $= 0$ and the space is a maximum.

For $t = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6$ seconds we have
$p = 3,\ 1,\ -1,\ -3,\ -5,\ -7,\ -9$ feet,
$v = 10,\ 12,\ 12,\ 10,\ 6,\ 0,\ -8$ feet,
$s = 0,\ 11\frac{1}{6},\ 23\frac{1}{3},\ 34\frac{1}{2},\ 42\frac{2}{3},\ 45\frac{5}{6},\ 42$ feet.

3. For the motion expressed by the formula $p = -\mu s$, in which μ designates a constant coefficient, we have

$$\frac{v^2 - c^2}{2} = \int p\,ds = -\mu \int s\,ds = -\frac{\mu s^2}{2}, \text{ or } v^2 = c^2 - \mu s^2;$$

whence $\quad v = \sqrt{c^2 - \mu s^2}$ and $s = \sqrt{\dfrac{c^2 - v^2}{\mu}}$.

We have also $\quad dt = \dfrac{ds}{v} = \dfrac{ds}{\sqrt{c^2 - \mu s^2}} = \dfrac{1}{c}\,\dfrac{ds}{\sqrt{1 - \left(\dfrac{s\sqrt{\mu}}{c}\right)^2}}$

$$= \frac{d\left(\dfrac{s\sqrt{\mu}}{c}\right)}{\sqrt{\mu}\sqrt{1 - \left(\dfrac{s\sqrt{\mu}}{c}\right)^2}} = \frac{du}{\sqrt{\mu}\sqrt{1 - u^2}},$$

when we put $\dfrac{s\sqrt{\mu}}{c} = u$; and it follows that (see Art. 26, V.; of the Introduction to the Calculus).

$t = \dfrac{1}{\sqrt{\mu}}\, \sin.^{-1} u = \dfrac{1}{\sqrt{\mu}} \sin.^{-1} \dfrac{s\sqrt{\mu}}{c}$, and

$s = \dfrac{c}{\sqrt{\mu}} \sin. (t\sqrt{\mu})$, as well as

$v = \dfrac{ds}{dt} = c \cos. (t\sqrt{\mu})$ and

$p = \dfrac{dv}{dt} = -c\sqrt{\mu} \sin. (t\sqrt{\mu})$.

When the motion begins we have, for $t = 0$, $s = 0$, $v = c$ and $p = 0$, and afterwards for

$t\sqrt{\mu} = \frac{\pi}{2}$, or $t = \frac{\pi}{2\sqrt{\mu}}$, $s = \frac{c}{\sqrt{\mu}}$, $v = 0$ and $p = -c\sqrt{\mu}$, for

$t\sqrt{\mu} = \pi$, or $t = \frac{\pi}{\sqrt{\mu}}$, $s = 0$, $v = -c$ and $p = 0$, for

$t\sqrt{\mu} = \frac{3}{2}\pi$, or $t = \frac{3\pi}{2\sqrt{\mu}}$, $s = -\frac{c}{\sqrt{\mu}}$, $v = 0$ and $p = c\sqrt{\mu}$, and for

$t\sqrt{\mu} = 2\pi$, or $t = \frac{2\pi}{\sqrt{\mu}}$, $s = 0$, $v = c$ and $p = 0$.

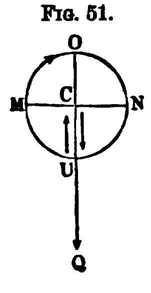
The moving point has therefore a vibratory motion upon both sides of the fixed point of beginning, to which it returns every time that it has described, with a velocity which gradually increases from 0 to $v = \pm c$, the space $s = \pm \frac{c}{\sqrt{\mu}}$.

§ 21. Mean Velocity.—The velocity $c_1 = \frac{s}{t}$, which we find when we divide the space described during a certain time, E.G., during the period of a periodic motion, by the time itself, differs from the velocity $v = \frac{\sigma}{\tau}\left(\frac{d\,s}{d\,t}\right)$ for an instant or during the element of time τ $(d\,t)$. We call the former the *mean velocity* (Fr. vitesse moyenne, Ger. mittlere Geschwindigkeit), and we can consider it as the velocity that a body must have, to describe uniformly in a certain time (t) the space (s) which it really does describe with a variable motion in the same time. When the motion is uniformly variable the mean velocity is equal to the half sum of the initial and of the final velocity, for according to § 13 the space is equal to this sum $\left(\frac{c + v}{2}\right)$ multiplied by the time (t).

In general, the mean velocity is (according to § 19) $c_1 = \frac{v_1 + v_2 + \ldots v_n}{n}$, in which $v_1, v_2, \ldots v_n$ denote the velocities corresponding to equal and very small intervals of time.

EXAMPLE.—While a crank is turned uniformly in a circle $U\,M\,O\,N$, Fig. 51, the load Q attached to it, E.G., the piston of an air or water pump, etc., moves with a variable motion up and down; the velocity of this load is at the highest and lowest points U and O a minimum, and equal to zero, and at half the height at M and N a maximum, and equal to the velocity of the crank. Within a half revolution the mean velocity is equal to the whole height of ascent, I.E., the diameter $U\,O$ of the circle in which the crank revolves, divided by the time of a half revolution. If we put the radius of the circle in which the crank revolves, $C\,U = C\,O = r$, that

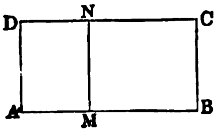
FIG. 51.



is, its diameter $= 2\,r$, and the time equal to t, it follows that the mean velocity $c_1 = \frac{2\,r}{t}$. The crank in the same time describes a half circle $\pi\,r$, and its velocity is $c = \frac{\pi\,r}{t}$, and therefore the mean velocity of the load $c_1 = \frac{2}{\pi}\,c = \frac{2\,t}{3{,}141}\,c$ is 0,6366 times as great as the constant velocity c of the crank.

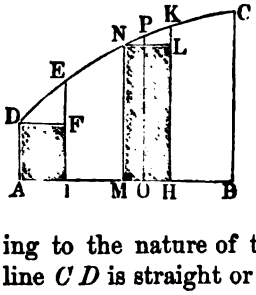
§ 22. Graphical Representation of the Formulas of Motion. The laws of motion which have been found in the foregoing paragraphs can be expressed by geometrical figures, or, as we say, graphically represented. Graphical representations, as they render the conception of the formula more easy, assist the memory, protect us from many errors, and serve also directly for the determination of quantities which may be required, are of the greatest use in mechanics. In uniform motion, the space (s) is the product ($c\,t$) of the velocity and the time, and in Geometry the area of a rectangle is equal to the product of the base by the altitude; we can therefore represent the space described (s) by a rectangle $A\,B\,C\,D$, Fig. 52, whose base $A\,B$ is the time t, and whose altitude $A\,D = B\,C$ is the velocity c, provided the time and the velocity are expressed by similar units of length, that is, if the second and the foot are represented by one and the same line.

FIG. 52.



§ 23. While in uniform motion the velocity ($M\,N$) at any moment ($A\,M$) is the same, in variable motion it is different for each instant; therefore this motion can only be represented by a four-sided figure, $A\,B\,C\,D$, Fig. 53, the base of which $A\,B$, denotes the time (t), the other boundaries being the three lines, $A\,D$, $B\,C$, and $C\,D$. The first two of these lines denote the initial and final velocities, and the last one is determined by the extremities (N) of the different lines representing the velocities corresponding to the intermediate times (M). According to the nature of the variable motion in question, the fourth line $C\,D$ is straight or curved, rises or sinks from its origin, and is

FIG. 53.



concave or convex towards the base. In every case, however, the area of this figure is equal to the space (s) described; for every surface $A\ B\ C\ D$, Fig. 53, can be divided into a series of small strips $M\ O\ P\ N$, which may be considered as rectangles, and the area of each of which is a product of the base ($M\ O$) and the corresponding altitude ($M\ N$) or ($O\ P$), and in like manner the space described in a certain time is composed of small portions, each one of which is a product of an element of time and the velocity of the body during that instant. The figure also shows the difference between the measure of the velocity and the space actually described in the following unit of time. The rectangle $M\ L$, above the base $M\ H =$ unity $(1) = v\ .\ 1$ is the measure of the velocity M, and on the contrary, the surface $M\ K$ above the same base represents the space actually described. In the same way the rectangle $A\ F$ over $A\ I =$ unity is the measure of the initial velocity $A\ D = c$, and the surface $A\ E$ that of the space actually described in the first second.

§ **24.** In *uniformly variable motion* the increase or decrease $v-c$ of the velocity ($= p\ t$, § 13) is proportional to the time (t). If in Fig. 54 and Fig. 55 we draw the line $D\ E$ parallel to the base $A\ B$, we cut off from the lines $B\ C$ and $M\ N$, which represent the velocities, the equal portions $B\ E$ and $M\ O$, which are equal to the line $A\ D$ representing the initial velocity, there remain the pieces $C\ E$ and $N\ O$, which represent the increase or decrease in velocity; for these we have from what precedes the proportion

FIG. 54.

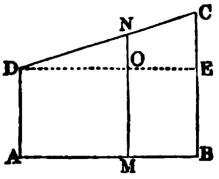
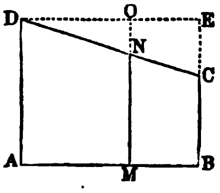


FIG. 55.



$$N\ O : C\ E = D\ O : D\ E.$$

Such a proportion requires that N, as well as every point of the line $C\ D$, shall be upon the straight line uniting C and D, or that the line $C\ D$, which limits the velocities $M\ N$, shall be straight. Consequently the space described in uniformly accelerated or retarded motion can be represented by the area of a Trapezoid $A\ B\ C\ D$, whose altitude *A B* is the time (*t*) and whose two parallel bases *A D* and *B C* are the initial and final velocity. The formula found in § 13 $s = \frac{c + v}{2} . t$ corresponds exactly to this figure. For uniformly accelerated motion the fourth side *D C* rises from the point of origin, and for uniformly retarded motion this line descends from the same point. When the uniformly accelerated motion begins with a velocity equal to zero, the trapezoid becomes a triangle, whose area is $\frac{1}{2}\, B\, C . A\, B = \frac{1}{2}\, v\, t$.

§ **25.** The mean velocity of a variable motion is the quotient of the space divided by the time; it gives, when multiplied by the time, the space, and can be considered as the altitude *A F* = *B E* of the rectangle *A B E F*, Fig. 56, the base of which *A B* is equal to the time *t*, and the area of which is equal to that of the four-sided figure *A B C N D*, which measures the space described. The mean velocity is found by changing the four-sided figure *A B C N D* into an equally long rectangle *A B E F*. Its determination is especially important for periodic motion, which occurs in almost all machines. The law of this motion is represented by the serpentine line *C D E F G*, Fig. 57. If the right line *L M*, drawn parallel to *A B*, cuts off the same space as the serpentine line, then *L M* is also the axis of *C D E F G*, and the distance *A L* = *B M* between the two parallels *A B* and *L M* is the mean velocity of the periodic motion, and, on the contrary, *A C*, *O E*, *B G*, etc., are the maximum, and *N D* and *P F* the minimum velocities of a period *A O*, *O B*, etc.

FIG. 56.

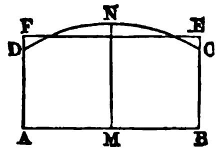
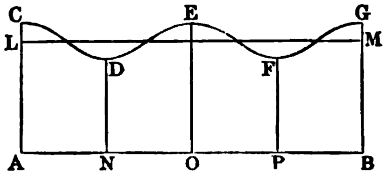


FIG. 57.



§ **26.** The *acceleration* or the continuous increase of velocity in a second can easily be determined from the figure. In uniformly accelerated motion it is constant, and is therefore the difference *P Q*, Fig. 58 and Fig. 59, between the two velocities *O P* and *M N*, one of which corresponds to a time ($M\ O$) one second greater than the other. If the motion is variable, but not uniformly, and the line

FIG. 58.

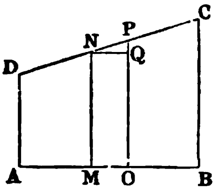
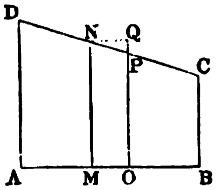


FIG. 59.



of velocity $C\ D$ therefore a curve, the acceleration at every instant is different, and consequently it is not really the difference $P\ Q$ of the velocities $O\ P$ and $M\ N = O\ Q$, Figs. 60 and 61, which are those at times differing one second $M\ O$ from each other, but it

FIG. 60.

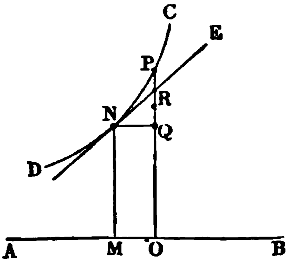
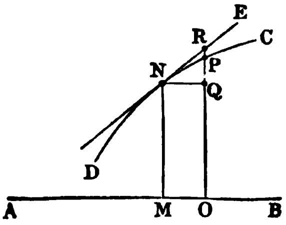


FIG. 61.



is the increase $R\ Q$ of the velocity $M\ N$, which would take place, if from the instant M the motion became a uniformly accelerated one, that is if the curve $N\ P\ C$ became a straight line $N\ E$. But the tangent $N\ E$ is the line in which a curve $D\ N$ would prolong itself, if from a certain point (N), its direction remained unchanged; the new line of velocity coincides with the tangent, and the perpendicular $O\ R$ which reaches to this line is the velocity which would have existed at the end of a second, if at the beginning of the same the motion had become a uniformly accelerated one, and therefore the difference $R\ Q$ between this velocity and the initial one ($M\ N$) is the acceleration for the instant which corresponds to the point M in the time line $A\ B$. We can also of course consider the time and the accelerations as the co-ordinates of a curve, in which case the velocities are represented by surfaces.

CHAPTER II.

COMPOUND MOTION.

§ 27. **Composition of Motion.**—The same body can possess, at the same time, two or more motions; every (relative) motion is composed of the motion within a certain space, and of the motion of this space within or in relation to another space. Every point on the earth possesses already two motions; for it revolves once every day around the earth's axis, and with the earth once a year around the sun. A person moving on a ship has two motions in relation to the shore, his own motion proper and that of the ship; the water which flows out of an opening in the side or in the bottom of a vessel carried upon a wagon has two motions, that from the vessel, and that with the vessel, etc.

Hence we distinguish *simple* and *compound motion.* The rectilinear motions of which other rectilinear or curvilinear motions are composed (Fr. composés, Ger. zusammengesetzt), or of which we can imagine them to be composed, are *simple motions* (Fr. simple, Ger. einfach). How several simple motions can be united so as to form a compound one, and how the decomposition of a compound motion into several simple ones is accomplished, will be shown in what follows.

§ 28. If the simple motions take place in the same straight line, their sum or difference gives the resulting compound motion, the former when the motions are in the same direction, and the latter when the motions are in opposite directions. The correctness of this proposition becomes evident, when we combine the spaces described in the same time by virtue of the simple motions. The spaces $c_1 t$ and $c_2 t$ described in the same time correspond to uniform motions whose velocities are c_1 and c_2, and if these motions are in the same direction the space described in t seconds is

$$s = c_1 t + c_2 t = (c_1 + c_2) t,$$

and consequently the resulting velocity of the compound motion is the sum of the velocities of the simple motions. When the motions are in contrary directions, we have

$$s = c_1 t - c_2 t = (c_1 - c_2) t,$$

and the resulting velocity is equal to the difference of the simple velocities.

EXAMPLE.—1. A person, walking upon the deck of a ship with a velocity of 4 feet in the direction of the motion of the latter, appears to people on shore, when the ship moves with a velocity of 6 feet, to pass by with a velocity of 4 + 6 = 10 feet.

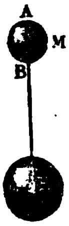
2. The water discharged from an opening in the side of a vessel with a velocity of 25 feet, while it is moved simultaneously with the vessel in the opposite direction with a velocity of 10 feet, has in reference to the other objects which are at rest a velocity of only 25 — 10 = 15 feet.

§ **29.** The same relations also obtain for variable motion. If the same body has, besides the initial velocities c_1 and c_2, the constant accelerations p_1 and p_2, the corresponding spaces are $c_1 t$, $c_2 t$, $\frac{1}{2} p_1 t^2$, $\frac{1}{2} p_2 t^2$, and if the velocities and the accelerations have the same directions, the total space described in virtue of the component motions is

$$s = (c_1 + c_2) t + (p_1 + p_2) \frac{t^2}{2}.$$

If we put $c_1 + c_2 = c$ and $p_1 + p_2 = p$, we obtain $s = c t + p \frac{t^2}{2}$, whence it follows that not only the sum of the component velocities gives the velocity of the resulting or compound motion, but also that the sum of the accelerations of the simple motions gives its acceleration.

FIG. 62.

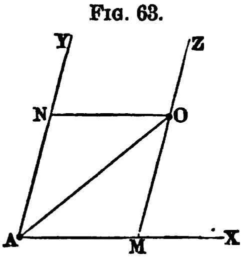


EXAMPLE.—A body upon the moon has imparted to it by the moon an acceleration $p_1 = 5{,}15$ feet, and from the earth an acceleration $p_2 = 0{,}01$ feet. Therefore, a body *A*, Fig. 62, beyond the moon *M* and the earth *E*, falls towards the centre of the moon with an acceleration of 5,16 feet, and a body *B* between *M* and *E* with an acceleration of 5,14 feet.

§ **30. Parallelogram of Motions.**—If a body possesses at the same time two motions which differ from each other in direction, it takes a direction which lies between those of the two motions, and if these motions are of different kinds, E.G., if one is uniform and the other variable, the direction changes at every point, and the motion is curvilinear.

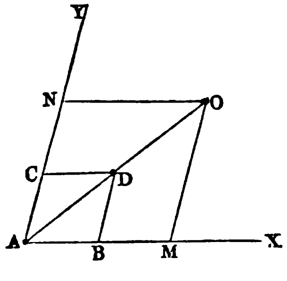
We find the point *O*, Fig. 63, which a body moving at the same time in the direction *A X* and *A Y*, occupies at the end of a certain time (t) by seeking the fourth corner O of the parallelogram $A\ M\ O\ N$, determined by the spaces $A\ M = x$ and $A\ N = y$, described simultaneously, and by the angle $X\ A\ Y$ which the directions of motion form with one another. We can convince ourselves of the correctness of this proceeding by supposing the spaces x and y described not simultaneously, but one after the other. By virtue of one motion the body describes the space $A\ M = x$, and by virtue of the other from M in the direction $A\ Y$, that is on a line $M\ O$ parallel to $A\ Y$, the space $A\ N = y$. If we make $M\ O = A\ N$, we obtain in O the position of the body which corresponds to the two motions x and y, and which, according to this construction, is the fourth corner of the parallelogram. We can also imagine the space $A\ M = x$ to be described in a line $A\ X$, which with all its points moves forward in the direction $A\ Y$, and therefore carries M parallel to $A\ Y$ and causes this point to describe the path $M\ O = A\ N = y$.

FIG. 63.



§ **31. Parallelogram of Velocities.**—If the two motions in the directions $A\ X$ and $A\ Y$ take place uniformly with the velocities c_1 and c_2, the spaces described in a certain time t are $x = c_1\ t$ and $y = c_2\ t$, and their ratio $\frac{y}{x} = \frac{c_2}{c_1}$ is the same for all times, a peculiarity which is possessed only by the right line $A\ O$, Fig. 64. It follows therefore that the direction of the compound motion is always a straight line. If we construct with the velocities $A\ B = c_1$ and $A\ C = c_2$ the parallelogram $A\ B\ C\ D$, its fourth corner D gives the point where the body is at the end of the first second, but since the resulting motion is rectilinear, it follows that it takes place in the direction of the diagonal of the parallelogram constructed with the velocities. If we designate by s the space $A\ O$ really described in the time t, we have from the similarity of the triangles $A\ M\ O$ and $A\ B\ D$

FIG. 64.



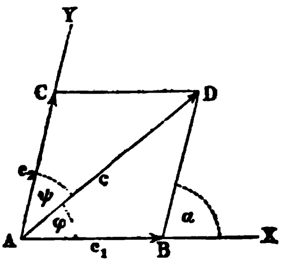
$\frac{s}{x} = \frac{A\,D}{A\,B}$, whence it follows that this space

$$s = \frac{x \,.\, A D}{A B} = \frac{c_1\, t \,.\, \overline{A\,D}}{c_1} = \overline{A\,D} \,.\, t.$$

According to the last equation the space described in the diagonal is proportional to the time (t), and therefore the compound motion is itself uniform and its velocity c equal to $A\,D$.

Therefore the diagonal of a parallelogram, constructed with two velocities and with the angle inclosed by them, gives the direction and magnitude of the velocity, with which the resulting motion actually takes place. This parallelogram is called the *parallelogram of velocities* (Fr. parallelogramme de vitesse, Ger. Parallelogram der Geschwindigkeiten); the simple velocities are called *components* (Fr. composantes, Ger. Seitengeschwindigkeiten), and the compound velocity the *resultant* (Fr. resultante, Ger. die resultirende or mittlere).

FIG. 65.



§ **32.** By employing trigonometrical formulas, the direction and magnitude of the resulting velocity can be found by calculating one of the equal triangles, E.G., $A\;B\;D$, of which the parallelogram of velocities is composed, by which we obtain the resulting velocity $A\,D = c$ in terms of the components $A\,B = c_1$ and $A\;C = c_2$ and of the angle included between them $B\,A\,C = a$.

For we obtain c by the formula

$$c = \sqrt{c_1^2 + c_2^2 + 2\, c_1\, c_2 \cos. a},$$

and the angle $B\,A\,D = \phi$, which the resultant makes with the velocity c_1, by the formula $\sin. \phi = \frac{c_2 \sin. a}{c}$, or

$$\tang. \phi = \frac{c_2 \,.\, \sin.\, a}{c_1 + c_2 \cos.\, a}, \text{ or } \cotang. \phi = \cotang. a + \frac{c_1}{c_2 \sin. a}.$$

We have also

$$\tang. \left(\frac{a}{2} - \phi\right) = \frac{c_1 - c_2}{c_1 + c_2} \tang. \frac{a}{2}.$$

If the velocities c_1 and c_2 are equal to each other, the parallelogram is a Rhombus, and in consequence of the diagonals being at right angles to each other, we have more simply

$$c = 2\, c_1 \cos. \tfrac{1}{2}\, a \text{ and } \phi = \tfrac{1}{2}\, a.$$

If the velocities are at right angles, we have also more simply

$$c = \sqrt{c_1^2 + c_2^2} \text{ and } tang.\ \phi = \frac{c_2}{c_1}.$$

Example.—1. The water discharged from a vessel or from a machine has a velocity $c_1 = 25$ feet, while the vessel itself is moved with a velocity $c_2 = 19$ feet in a direction, which forms with that of the water an angle $a^0 = 130°$. What is the direction of the resultant or absolute velocity of the water?

$$c = \sqrt{25^2 + 19^2 + 2\,.\,25\,.\,19\, cos.\, 130^0} = \sqrt{625 + 361 - 50\,.\,19\,.\,cos.\, 50^0}.$$

$$= \sqrt{986 - 950\ cos.\, 50^0} = \sqrt{986 - 610{,}7} = \sqrt{375{,}3} = 19{,}37 \text{ feet}$$

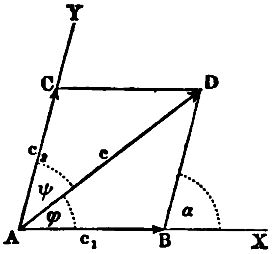
is the required resulting velocity.

Further, $sin.\ \phi = \frac{19\, sin.\, 130^0}{19{,}37\,.} = 0{,}9808\ sin.\ 50^0 = 0{,}7513$, hence the angle formed by the direction of the resultant with that of the velocity c_1 is $\phi = 48^0\, 42^1$, and the angle formed by it with the direction of the motion of the vessel is $a - \phi = 81^0\, 18^1$.

2. If the foregoing velocities were at right angles to each other, we would have $cos.\ a = cos.\ 90^0 = 0$, and therefore the resulting velocity $c = \sqrt{986} = 31{,}40$ feet, and also $tang.\ \phi = \frac{19}{25} = 0{,}76$, hence the angle formed by it with the first velocity is $\phi = 37^0\, 14^1$.

§ 33. We can also consider every velocity to be composed of two components, and therefore under certain conditions can decompose it into such components. If, for example, the angles $D\,A\,X = \phi$, and $D\,A\,Y = \psi$, Fig. 66, which the required velocities form with the resultant $A\,D = c$, are given, we draw through the extremity D of the line representing c other lines parallel to the directions $A\,X$ and $A\,Y$: the points of intersection B and D cut off the velocities sought, and we have

FIG. 66.



$$A\,B = c_1 \text{ and } A\,C = c_2.$$

Trigonometry gives these velocities by the formulas

$$c_1 = \frac{c\ sin.\ \psi}{sin.\ (\phi + \psi)},\ c_2 = \frac{c\ sin.\ \phi}{sin.\ (\phi + \psi)}.$$

Generally, in the application of these formulas, the two velocities are at right angles to each other, and

$$\phi + \psi = 90^0,\ sin.\ (\phi + \psi) = 1, \text{ whence}$$
$$c_1 = c\ cos.\ \phi \text{ and } c_2 = c\ sin.\ \phi.$$

We can also determine, when one component (c_1) and its angle of direction (ϕ) are given, the magnitude and direction of the other. Finally, if the three velocities c, c_1 and c_2 are given, we can determine their angles of direction by the same method that we employ to find the angles of a triangle, when three sides are given.

Example.—If the velocity $c = 10$ feet is to be decomposed into two components whose directions form with that of c the angles $\phi = 65°$ and $\psi = 70°$, we have

$$c_1 = \frac{10 \sin. 70°}{\sin. 135°} = \frac{9,397}{\sin. 45°} = 13,29 \text{ feet and } c_2 = \frac{10 \sin. 65°}{\sin. 135°} = \frac{9,068}{0,7071} = 12,81 \text{ feet.}$$

§ 34. Composition and Decomposition of Velocities.—By repeated use of the parallelogram of velocities, any number of velocities can be combined so as to give a single resultant. The construction of the parallelogram $A\,B\,D\,C$ (Fig. 67) gives the resultant $A\,D$ of c_1 and c_2, the construction of the parallelogram $A\,D\,F\,E$ gives the resultant of $A\,D$ and $A\,E = c_3$, and from the construction of the parallelogram $A\,F\,H\,G$ we obtain the resultant $A\,H = c$ of $A\,F$ and $A\,G = c_4$, or that of c_1, c_2, c_3 and c_4.

The most simple manner of resolving this problem is by the construction of a polygon $A\,B\,D\,F\,H$, whose sides $A\,B$, $B\,D$, $D\,F$ and $F\,H$ are parallel and equal to the given velocities c_1, c_2, c_3 and c_4, and whose last side is always equal to the resulting velocity.

Fig. 67.

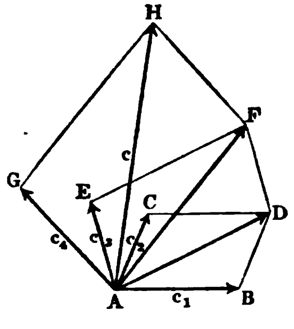
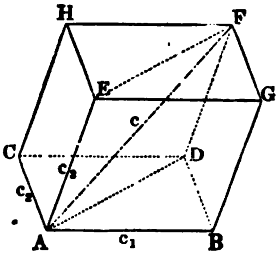


Fig. 68.



In case the velocities do not lie in the same plane, the resultant can also be found by repeated application of the parallelogram of velocities. The resultant $A\,F = c$ (Fig. 68) of three velocities $A\,B = c_1$, $A\,C = c_2$ and $A\,E = c_3$, not in the same plane, is the diagonal of a parallelopipedon whose sides are equal to the velocities. We often employ for this reason the term *parallelopipedon of velocities.*

§ **35. Composition of Accelerations.**—By the composition of two uniformly accelerated motions, beginning with a velocity = 0, we obtain also a uniformly accelerated motion in a straight line. If we designate the accelerations of the motions in the directions $A\,X$ and $A\,Y$ (Fig. 69) by p_1 and p_2, the spaces described during the time t are

$$A\,M = x = \frac{p_1\,t^2}{2} \text{ and}$$

$$A\,N = y = \frac{p_2\,t^2}{2},$$

and their ratio is

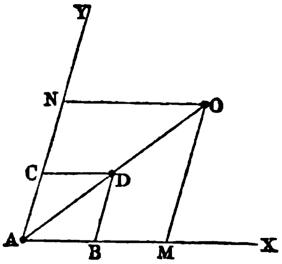
$$\frac{x}{y} = \frac{p_1\,t^2}{p_2\,t^2} = \frac{p_1}{p_2},$$

which is entirely independent of the time, therefore the path $A\ O$ is a straight line. If we make $A\,B = p_1$, and $B\,D = A\ C = p_2$, we obtain a parallelogram $A\ B\ D\ C$, and we have

$$\frac{A\ O}{A\ D} = \frac{A\ M}{A\ B} = \frac{\frac{1}{2}p_1\,t^2}{p_1} = \tfrac{1}{2}\,t^2, \text{ whence } A\ O = \tfrac{1}{2}\,\overline{A\,D}\,.\,t^2$$

According to this equation the space $A\ O$ of the compound motion is proportional to the square of the time; the motion itself is therefore *uniformly accelerated,* and its acceleration is the diagonal $A\ D$ of the parallelogram constructed with the two simple accelerations.

FIG. 69.



We see, therefore, that we can combine several accelerations so as to form a single one, or decompose a single one into several others by means of the *parallelogram of accelerations* (Fr. parallélogramme des accélérations, Ger. Parallelogram der Accelerationen) according to exactly the same rules as we perform the composition and decomposition of velocities by means of the parallelogram of velocities.

§ **36. Composition of Velocities and Accelerations.**—By the combination of a uniform motion with a uniformly accelerated one we obtain, when the directions of the two motions do not coincide, a motion which is completely irregular. If during a certain time t, by virtue of the velocity c, the space

$$A\,N = y = c\,t$$

is described in the direction $A\ Y$, Fig. 70, and if during the same time, by virtue of a constant acceleration, the space

$$A\ M = x = \frac{p\ t^2}{2}$$

is described in the direction $A\ X$ at right angles to the former, then the body will be in the corner O of the parallelogram constructed with $y = c\ t$ and $x = \frac{p\ t^2}{2}$. By the aid of these formulas, it is true, we can find the position of the body for any given time, but these positions do not lie in the same straight line; for if we substitute the value of $t = \frac{y}{c}$ taken from the first equation, in the second we obtain the equation of the path

$$x = \frac{p\ y^2}{2\ c^2}.$$

According to this formula the space (x) described in one direction varies, not as the space, but as the square (y^2) of the space described

FIG. 70.

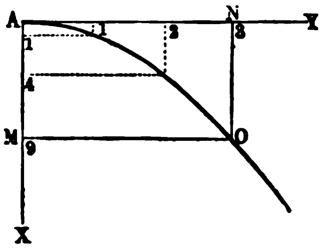
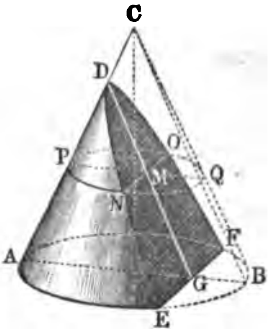


FIG. 71.



in the other direction, and the path of the body is therefore not a straight line, but a certain curve known in Geometry as the parabola (Fr. parabole, Ger. Parabel).

REMARK.—Let $A\ B\ C$, Fig. 71, be a cone with a circular base $A\ E\ B\ F$, and $D\ E\ F$ a section of the same parallel to the side $B\ C$ and at right angles to the section $A\ B\ C$, and let $O\ P\ N\ Q$ be a second section parallel to the base and therefore circular. Further, let $E\ F$ be the line of intersection between the base and the first section, and finally, let us suppose the parallel diameters $A\ B$ and $P\ Q$ to be drawn in the triangular section $A\ B\ C$ and the axis $D\ G$ in the section $D\ E\ F$. Then for the half chord $M\ N = M\ O$ we have the equation $\overline{M\ N}^2 = P\ M\ .\ M\ Q$; but $M\ Q = G\ B$ and for

$P M$ we have the proportion $P M : D M = A G : D G$, whence

$$M N^2 = B G \,.\, \frac{D M \,.\, A G}{D G}.$$

But we have also $G E^2 = B G \,.\, A G$; whence, dividing the first equation by the second,

$$\frac{\overline{D M}}{D G} = \frac{\overline{M N^2}}{\overline{G E^2}}.$$

The portions cut off from the axis (*abscissas*) *are as the squares of the corresponding perpendiculars* (*Ordinates*). This law coincides exactly with the law of motion just found; the motion takes place then in a curved line $D N E$, which is one of the conic sections. For the construction, position of the tangent, and other properties of the parabola, see the Ingenieur, page 175, etc.

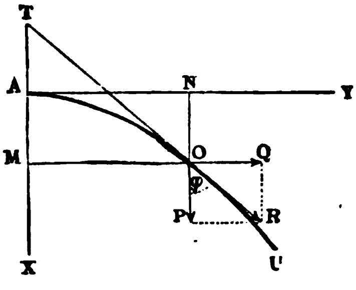
§ 37. **Parabolic Motion.**—In order thoroughly to understand the motion produced by the combination of velocity and acceleration, we must be able to give for any time (t) the *direction*, *velocity*, and the *space described*. The velocity parallel to $A Y$ is constant and $= c$, and that parallel to $A X$ is variable and $= p t$; if we construct with these velocities $O Q = c$ and $O P = p t$ the parallelogram $O P R Q$, Fig. 72, we obtain in the diagonal $O R$ the mean velocity, or that with which the body in O describes the parabolic path $A O U$. This velocity itself is

$$v = \sqrt{c^2 + (p t)^2}.$$

$O R$ gives also the tangent or the direction in which the body moves for an instant; consequently, for the angle $P O R = X T O = \phi$, which the same makes with the direction (axis) $A X$ of the second motion, we have the following formula

$$tang. \phi = \frac{O Q}{O P} = \frac{c}{p t}.$$

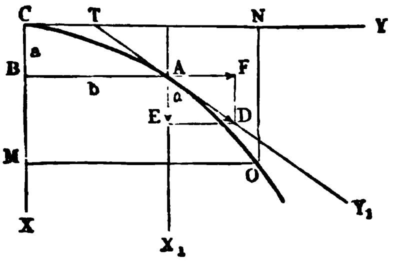
FIG. 72.



Finally, to obtain the space described or the arc of the curve $A O = s$, we can employ the formula $\sigma = v \tau$ (§ 19), by the aid of which we can calculate the small portions which we can consider as elements. The calculus also gives a complicated formula for the computation of an arc of a parabola.

§ **38.** We have previously supposed that the primitive directions of motion were at right angles to each other, and we must now consider the case, when the direction of the acceleration makes any arbitrary angle with that of the velocity. If the velocity of the body in the direction $A\ Y_1$ (Fig. 73) is c, and if, in the direction $A\ X_1$ which forms an angle $X_1\ A\ Y_1 = a$ with the former, the acceleration is p, A is no longer the vertex, and $A\ X_1$ no longer the axis, but only the direction of the axis of the parabola. The vertex of the parabola is situated at a point whose co-ordinates, in reference to the point of beginning of the motion, are $C B = a$ and $B A = b$, of which the former lies in the axis of the parabola and the latter is at right angles to it. The velocity $A\ D = c$ is composed of the two components $A\ F = c$ *sin.* a and $A\ E = c$ *cos.* a. The first of these is constant, and the latter is variable, and always equal to the variable velocity $p\ t$, provided that the body requires the time t to pass from the vertex C to the real point of beginning.

FIG. 73.



Hence we have

$$c \, cos.\, a = p\, t, \text{ whence } t = \frac{c\, .\, cos.\, a}{p}, \text{ and therefore}$$

$$1)\quad C B = a = \frac{p\, t^2}{2} = \frac{c^2\, cos.^2\, a}{2 p}, \text{ and}$$

$$2)\quad B A = b = c\, sin.\, a\, .\, t = \frac{c^2\, sin.\, a\, cos.\, a}{p} = \frac{c^2\, sin.\, 2\, a}{2\, p}.$$

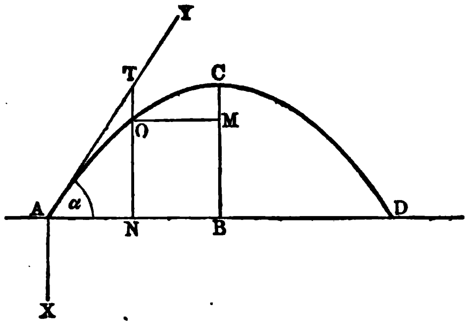
If we have determined by these distances the vertex C of the parabola, starting from this point we can, for any given time, determine the position O of the body. Besides, if we put $C M = x$ and $M\ O = y$, the general formula

$$x = \frac{p\, y^2}{2\, c^2\, sin.^2\, a}, \text{ or } y = c\, sin.\, a \sqrt{\frac{2\, x}{p}} \text{ holds good.}$$

REMARK.—One of the most important applications of the theory of parabolic motion, just discussed, is to the motion of projectiles. A body projected in an inclined direction either upward or downward would describe, in virtue of its initial velocity c and of the acceleration of gravity ($g = 32 . 2$ feet), an arc of a parabola, if the resistance of the air were done away with, or if its motion took place *in vacuo*. If the velocity of projection is not very great and if the body is very heavy compared with its volume, the divergence of the body from a parabolic path is small enough to be neglected. The most perfect parabolic trajectories are those described by jets of water issuing from vessels, fire-engines, etc. Bodies shot from guns, etc., E.G., musket balls, describe, in consequence of the great resistance of the air, paths which differ very sensibly from a parabola.

§ 39. Motion of Projectiles.—A body projected in the direction $A\ Y$ at an angle of elevation $Y A D = a$, Fig. 74, ascends to a certain height $B\ C$, which is called the ***height of projection*** (Fr. hauteur du jet, Ger. Wurfhöhe), and it reaches the horizontal plane from which it started in A, at a distance $A\ D$ from it, which is called the ***range of projection*** (Fr. amplitude du jet, Ger. Wurfweite).

FIG. 74.



From the velocity c, the acceleration g and the angle of elevation, we obtain, according to § 38, when we replace p by g and a° by $90^\circ + a^\circ$, or *cos. a* by *sin. a*, etc.

$$\text{the height of projection } C B = a = \frac{c^2 \sin.^2 a}{2 g} \text{ and}$$

$$\text{half the range of projection } A B = b = \frac{c^2 \sin. 2 a}{2 g}.$$

From the last formula we see that the range of projection is a maximum for $\sin. 2 a = 1$, or $2 a = 90^\circ$, that is for $a = 45^\circ$. A body projected at an angle of elevation of 45° attains the greatest range of projection.

We have also

$$a = \frac{g b^2}{2 c^2 \cos.^2 a},$$

and for a point O in the path of the projectile for which $C M = x$ and $M O = y$,

$$x = \frac{g y^2}{2 c^2 \cos.^2 a},$$

or when its position is given by the co-ordinates $A N = x_1$ and $N O = y_1$, since in that case

$$x = CM = BC - NO = a - y_1 \text{ and}$$
$$y = MO = AB - AN = b - x_1, \text{ we have}$$
$$a - y_1 = \frac{g\,(b - x_1)^2}{2\,c^2\,cos.^2\,a},$$

whence

$$y_1 = a - \frac{g\,(b - x_1)^2}{2\,c^2\,cos.^2\,a}, \text{ or since } a - \frac{g\,b^2}{2\,c^2\,cos.^2\,a} = 0$$

$$y_1 = x_1\,tang.\,a - \frac{g\,x_1^2}{2\,c^2\,cos.^2\,a}.$$

Substituting in the equation $y_1 = x_1\,tang.\,a - \frac{g\,x_1^2}{2\,c^2\,cos.^2\,a}$, for $\frac{1}{cos.^2\,a}$, the value $1 + tang.^2\,a$, and resolving the same in reference to $tang.\,a$, we obtain the following expression for the angle of elevation (a), required to reach a point given by the co-ordinates x_1 and y_1,

$$tang.\,a = \frac{c^2}{g\,x_1} \pm \sqrt{\left(\frac{c^2}{g\,x_1}\right)^2 - \left(1 + \frac{2\,c^2\,y_1}{g\,x_1^2}\right)}.$$

If $\left(\frac{c^2}{g\,x_1}\right)^2 = 1 + \frac{2\,c^2\,y_1}{g\,x_1^2}$, or $c^4 - 2\,g\,y_1\,c^2 = g^2\,x_1^2$, then we have

$$c = \sqrt{g\,(y_1 + \sqrt{x_1^2 + y_1^2})} \text{ and}$$
$$tang.\,a = \frac{c^2}{g\,x_1}.$$

Smaller values of c make $tang.\,a$ imaginary, and larger values of c give two values for $tang.\,a$; in the first case the point cannot be attained, and in the second case it would be attained either in the rise or in the fall of the projectile.

EXAMPLE.—1. A jet of water rises with a velocity of 20 feet at an angle of 66°. The height due to the velocity is $h = 0{,}0155\,.\,20^2 = 6{,}2$ feet, and the jet ascends to a height $a = h\,sin.^2\,a = 6{,}2\,.\,(sin.\,66°)^2 = 5{,}17$ feet, the range of the jet is $2\,b = 2\,.\,6{,}2\;sin.\,132° = 2\,.\,6{,}2\;sin.\,48 = 9{,}21$ feet. The time, which each particle of water requires to describe the entire arc $A\,C\,D$ of the parabola, is $t = \frac{2\,c\,sin.\,a}{g} = \frac{2\,.\,20\;sin.\,66°}{32{,}2} = 1{,}14$ seconds. The height corresponding to the horizontal distance $A\,N = x_1 = 3$ feet is

$$y_1 = 3\,.\,tang.\,66° - \frac{32{,}2\,.\,9}{2\,.\,400\,.\,cos.\,(66°)^2} = 6{,}738 - \frac{0{,}36225}{0{,}16543}$$
$$= 6{,}738 - 2{,}189 = 4{,}549 \text{ feet.}$$

2. A jet of water discharged from a horizontal tube has, for a height 1¾ feet, a range of 5¼ feet; how great is its velocity?

From the formula $x = \frac{g y^2}{2 c^2} = \frac{y^2}{4 h}$, we deduce $h = \frac{y^2}{4 x}$, in which we must substitute $x = 1{,}75$ and $y = 5{,}25$, and thus we obtain $h = \frac{5{,}25^2}{4 \,.\, 1{,}75} = 3{,}937$ feet and the corresponding velocity $c = 15{,}92$ feet.

§ **40. Jets of Water.**—The peculiarities of the motion of jets of water are explained and shown in what follows. From what precedes we have

$$y = x \, tang. \, a - \frac{g x^2 \, [1 + (tang. \, a)^2]}{2 c^2} \text{ and}$$

$$y_1 = x_1 \, tang. \, a_1 - \frac{g x_1^2 \, [1 + (tang. \, a_1)^2]}{2 c^2}$$

for the equations of the parabolas formed by the paths of two ascending jets of water whose velocities c are the same, and whose angles of elevation a and a_1 are different. If we put $x_1 = x$ and subtract these equations from one another, we obtain

$$y - y_1 = x \, (tang. \, a - tang. \, a_1) - \frac{g x^2}{2 c^2} \, [(tang. \, a)^2 - (tang. \, a_1)^2]$$

$$= x \, (tang. \, a - tang. \, a_1) \left(1 - \frac{g x}{2 c^2} \, (tang. \, a + tang. \, a_1)\right).$$

If we assume that the two streams have nearly the same angle of elevation and require the two parabolas to have a point in common, we must put $y_1 = y$ and consequently we have

$$x \, (tang. \, a - tang. \, a_1) \left(1 - \frac{g x}{2 c^2} \, (tang. \, a + tang. \, a_1)\right) = 0, \text{ or}$$

$$\frac{g x}{2 c^2} \, (tang. \, a + tang. \, a_1) = 1,$$

or, since we can put $a_1 = a$ we have simply

$$\frac{g \, x \, tang. \, a}{c^2} = 1, \text{ whence } tang. \, a = \frac{c^2}{g x}.$$

Substituting this value in the equation

$$y = x \, tang. \, a - \frac{g x^2}{2 c^2} \, [1 + (tang. \, a)^2],$$

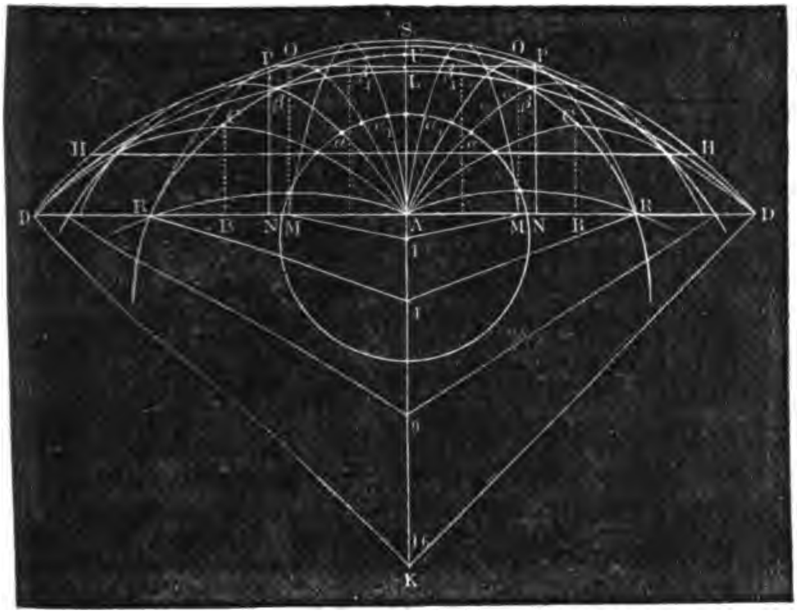
we obtain the equation

$$y = \frac{c^2}{g} - \frac{g x^2}{2 c^2} \left(1 + \frac{c^4}{g^2 x^2}\right) = \frac{c^2}{2 g} - \frac{g x^2}{2 c^2}$$

of the curve *D P S P D*, Fig. 75, which passes through the neighboring points, in which every two parabolas starting from the same point *A* at different angles cut each other, and which, therefore, touches or envelops the whole system of parabolas *A C D*, *A O R*, etc.

The height to which a vertical jet of water rises is $A\,S = \frac{c^2}{2g}$, and the range of projection of a jet $A\,C\,D$ rising at an angle of

FIG. 75



45° is $A\,D = 2\,.\,\frac{c^2\,sin.\,2\,a}{2\,g} = 2\,.\,\frac{c^2}{2g} = 2\,\overline{A\,S}.$

If we transfer the origin of co-ordinates from A to S, replacing the co-ordinates $A\,N = x$ and $N\,P = y$ by the co-ordinates $S\,U = u$ and $U\,P = v$, we have

$$y = A\,S - S\,U = \frac{c^2}{2g} - u \text{ and } x = A\,N = U\,P = v,$$

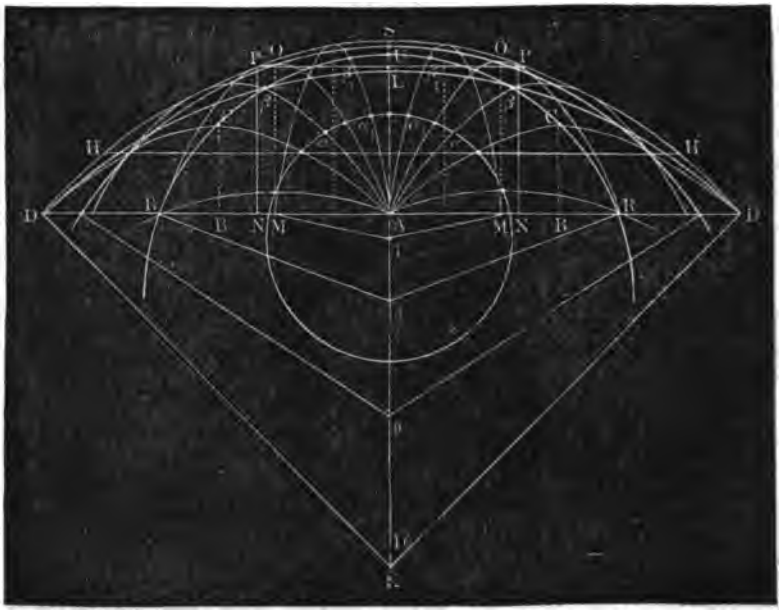
and the equation

$$y = \frac{c^2}{2\,g} - \frac{g\,x^2}{2\,c^2} \text{ is thus transformed into}$$

$$u = \frac{g\,v^2}{2\,c^2} \text{ or } v^2 = \frac{2\,c^2}{g}\,u.$$

This equation is that of the common parabola whose parameter is $p = \frac{2\,c^2}{g} = 4\,\overline{A\,S}$, and therefore the *envelope* $D\,P\,S\,P\,D$ of all the jets of water rising from the point *A* is a common parabola, whose vertex is *S* and whose axis is *S A*.

FIG. 76.



A bunch of jets rising from *A* in all directions would be enveloped by the paraboloid generated by the revolution of the envelope *D P S P D* around *A S*. If t is the time in which a body rising in a parabola describes the arc *A O*, Fig. 76, the co-ordinates of which are $A M = x$ and $M O = y$, we have

$$x = c\, t\, cos.\, a \text{ and } y = c\, t\, sin.\, a - \frac{g\, t^2}{2}, \text{whence}$$

$$cos.\, a = \frac{x}{c\, t} \text{ and } sin.\, a = \frac{y + \frac{1}{2} g\, t^2}{c\, t}.$$

Substituting these values for $cos.\, a$ and $sin.\, a$ in the well-known trigonometrical formula $(cos.\, a)^2 + (sin.\, a)^2 = 1$, we obtain the following formula

$$\frac{x^2}{(c\, t)^2} + \frac{(y + \frac{1}{2} g\, t^2)^2}{(c\, t)^2} = 1, \text{ or } x^2 + (y + \tfrac{1}{2} g\, t^2)^2 = c^2\, t^2.$$

If from a point *A*, Fig. 76, bodies be projected at the same moment and in the same vertical plane at different angles of elevation, the positions that they occupy after the lapse of a certain time (t) are determined by the last equation, which is that of a *circle* whose radius is $r = c\,t$ and whose centre is situated vertically below A at a distance $a = \frac{1}{2}\, g\, t^2$, and which can therefore be written in the following form,

$$x^2 + (y + a)^2 = r^2.$$

The circumference of this circle would therefore be reached at the same moment by all the elementary jets $A\ C\ D$, $A\ O\ P$, $A\ L\ S$.... rising at the same moment from the point A.

If in the formula $t_1 = \frac{x}{c\ cos.\ \alpha}$ we substitute $\alpha = 45°$, and $x = A\ B = \frac{c^2}{2\,g}$, we obtain $t_1 = \frac{c}{2\ g\ cos.\ 45°} = \frac{c}{g}\ \sqrt{\tfrac{1}{2}}$, hence the time required to describe the whole arc of the parabola $A\ C\ D$ is $t = 2\ t_1 = \frac{c}{g}\ \sqrt{2}$, and the radius of the circle $D\ L\ D$, which is reached simultaneously by the different elements of the water, is

$$KD = r = c\,t = \frac{c^2}{g}\ \sqrt{2} = \frac{c^2}{2\,g}\ \sqrt{8} = 2{,}828\ \frac{c^2}{2g} = 2{,}828\ .\ \overline{A\ S},$$ and

the distance of the centre K from A is

$$A\ K = a = \tfrac{1}{2}\,g\,t^2 = \frac{c^2}{g} = 2\,\frac{c^2}{2\,g} = 2\ \overline{A\ S}.$$

If we divide $D\ K$ in 4, and $A\ K$ in 16 equal parts, we can, since r is proportional to t and a to t^2, from the points of division 1, 4, 9 in $A\ K$, describe other circles with the radii $\frac{1}{4}\ D\ K$, $\frac{2}{4}\ D\ K$, and $\frac{3}{4}\ D\ K$, which cut off the parabolic arcs described in the same time, E.G., the circle described from 1 with $1\ a = \frac{1}{4}\ D\ K$, cuts off in the points a, a_1,...., the parabolic paths $A\ a$, $A\ a_1$...., described simultaneously, and the circle described from 4 with $4\ \beta = \frac{1}{2}\ D\ K$ cuts off in the points β, β_1.... the parabolic arcs $A\ \beta$, $A\ \beta_1$, etc., which are also simultaneously described.

If these circles be revolved about the vertical axis $K\ L$, they describe spherical surfaces which bound the parabolic paths described simultaneously, when the jets are projected all around A at all angles of elevation.

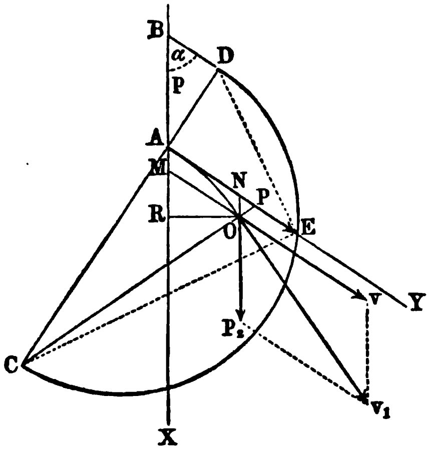
§ **41. Curvilinear Motion in General.**—By the combination of several velocities and several constant accelerations, we obtain also a parabolic motion, for not only the velocities but also the accelerations can be combined so as to form a single resultant; the problem is then the same as if there were one velocity and one acceleration, I.E., as if there were but one uniform and one uniformly accelerated motion.

If the accelerations are variable, they can be combined so as to give a resultant, as well as if they were constant, for we can consider them as constant during an infinitely small period of time (τ), and the motion as uniformly accelerated during this time. The resulting acceleration is, it is true, like its components themselves, variable. If we combine this resulting acceleration with the given velocity, we obtain the small parabolic arc, in which the motion takes place during this instant. If we determine also for the following instant the velocity and the acceleration, we obtain another portion of an arc belonging to another parabola, and proceeding in the same manner, we obtain approximately the entire curve of the path.

§ 42 We can consider every small arc of a curve as an arc of a circle. The circle to which this arc belongs is called the *circle of curvature or osculatory circle* (Fr. cercle osculateur, Ger. Krümmungskreis), and its radius is the *radius of curvature* (Fr. rayon de courbure, Ger. Krümmungshalbmesser). The path of a body in motion can be considered as composed of such arcs of circles, and we can therefore deduce a formula for its radii. Let $A\ M$ (Fig. 77) $= x = \frac{pt^2}{2}$ be a very small space described in the direction $A\ X$ with uniformly accelerated motion, $A\ N = y = v\tau$ a very small space described uniformly and O the fourth corner of the parallelogram constructed with x and y, that is, the position that the body starting from A occupies at the end of the instant (τ).

Let us draw $A\ C$ perpendicular to $A\ Y$, and let us see from what point C in this line an arc of a circle can be described through A and O. In consequence of the smallness of $A\ O$ we can consider not only $C A$, but also $C\ O\ P$ as perpendicular to

FIG. 77.



$A\ Y$, so that in the triangle $N\ O\ P$ the angle $N\ P\ O$ can be treated as a right angle. The resolution of this triangle gives

$$O\ P = O\ N \sin.\ O\ N\ P = A\ M \sin.\ X\ A\ Y = \frac{p\,\tau^2}{2} \sin.\ a,$$

and the tangent

$$A\ P = A\ N + N\ P = v\,\tau + \frac{p\,\tau^2}{2} \cos.\ a = \left(v + \frac{p\,\tau}{2} \cos.\ a\right) \tau,$$

can be put $= v\,\tau$, for $\frac{p\,\tau}{2} \cos.\ a$ can be neglected in the presence of v, in consequence of the infinitely small factor τ. Now, from the properties of the circle we know that $\overline{A\ P}^2 = P\ O\,.\,(P\ O + 2\ \overline{C\ O})$, or since $P\ O$ can be neglected in the presence of $2\ C\ O$, $\overline{A\ P}^2 = P\ O\,.\,2\ C\ O$; whence it follows that the radius of curvature is

$$C\ A = C\ O = r = \frac{\overline{A\ P}^2}{2\ P\ O} = \frac{v^2\,\tau^2}{p\,\tau^2 \sin.\ a} = \frac{v^2}{p \sin.\ a}$$

In order to determine by construction the radius of curvature, we lay off upon the normal to the original direction of the motion $A\ Y$ the normal acceleration, I.E., its normal component $p \sin.\ a = A\ D$, and join the extremity E of the velocity $A\ E = v$ to D by the right line $D\ E$, then we erect upon $D\ E$ a perpendicular $E\ C$; the point of its intersection with the first normal is the centre of the osculatory circle of the point A.

By inverting the last formula we obtain the *normal acceleration* $n = p \sin.\ a = \frac{v^2}{r}$; from which we see that it increases directly as the square of the velocity, and inversely as the radius of curvature, or directly as the greatness of the curvature.

EXAMPLE.—The radius of curvature of the parabolic trajectory produced by the acceleration of gravity is $r = 0{,}031 \frac{c^2}{\sin.\ a}$, and for the vertex of this curve where $a = 90°$, and therefore $\sin.\ a = 1$, it becomes $r = 0{,}031\ c^2$ feet. For a velocity $c = 20$ feet we obtain $r = 12{,}4$ feet; the farther the body is distant from the vertex the smaller a becomes, and consequently the greater is the radius of curvature.

§ **43.** If the point A has described the elementary space $A\ O = \sigma$, its velocity has changed; for the initial velocity v in the direction $A\ Y$ is now combined with the velocity $p\ \tau$ acquired in the direction $A\ X$, and consequently from the parallelogram of velocities we have for the velocity v_1

$$v_1^2 = v^2 + 2\ v\ p\ \tau \cos.\ a + p^2\,\tau^2 = v^2 + p\ \tau\,(2\ v \cos.\ a + p\ \tau),$$

but $p\ \tau$ vanishes in the presence of $2\ v \cos.\ a$, and we have

$$v_1^2 = v^2 + 2\,p\,v\,\tau\,cos.\,\alpha.$$

But $v\,\tau$ is the elementary space $A\,N = A\,O = \sigma$, and $p\,cos.\,\alpha$ is the *tangential acceleration*, I.E., the component k of the acceleration p in the direction of the tangent or of the motion, whence we have

$$\frac{v_1^2 - v^2}{2} = k\,\sigma.$$

Here $\sigma\,cos.\,\alpha$ is the projection $A\,R = \xi_1$ of the space upon the direction of the acceleration, and consequently we have

$$\frac{v_1^2 - v^2}{2} = p\,\xi_1.$$

As the motion progresses v_1 changes successively into v_2, $v_3 \ldots$ v_n, and the projections of the elementary spaces are increased by the quantities ξ_2, $\xi_3 \ldots \xi_n$, therefore we have

$$\frac{v_2^2 - v_1^2}{2} = p\,\xi_2, \frac{v_3^2 - v_2^2}{2} = p\,\xi_3, \ldots \frac{v_n^2 - v_{n-1}^2}{2} = p\,\xi_n,$$

and by addition

$$\frac{v_n^2 - v^2}{2} = p\,(\xi_1 + \xi_2 + \ldots \xi_n) = p\,x,$$

in which x denotes the total projection of the acceleration upon $A\,X$. We can also put

$$\frac{v_n^2 - v^2}{2} = \left(\frac{p_1 + p_2 + \ldots + p_n}{n}\right) x,$$

when the acceleration is variable and assumes successively the values $p_1, p_2 \ldots p_n$.

We see from the above that the *variation of the velocity* does not in the least depend upon the form or length of the path described, but only on its *projection* x upon the direction of the acceleration. For this reason all the jets of water, Fig. 76, have one and the same velocity on reaching the same horizontal plane $H\,H$. If c is the initial velocity or velocity of efflux, v the velocity at $H\,H$, and b the height of the line $H\,H$ above A, we have

$$\frac{v^2 - c^2}{2} = -\,g\,b, \text{ whence}$$

$$v = \sqrt{c^2 - 2\,g\,b}.$$

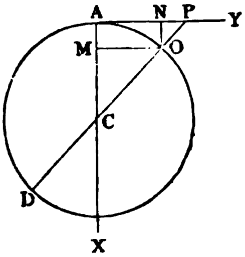
If at a certain point of the motion we have $\alpha = 90°$, the tangential acceleration $k = p\,cos.\,\alpha$ becomes $= 0$, and the normal acceleration $n = p\,sin.\,\alpha$ is equal to the mean acceleration p. In this case the variation of the squares of the velocities while the element σ of the space is being described, is $v_1^2 - v^2 = 0$, and we have $v_1 = v$; and if the motion continues in a curve, the direction of the acceleration changing in such a manner as always to remain normal to the direction of the motion (I.E., if there is no tangential acceleration), $v_1^2 - v^2 = 0$, or $v_1 = v$ remains constant while the point is describing any finite space, and the final velocity is equal to the initial velocity c.

The normal acceleration, for which the velocity remains constant, is

$$p = \frac{c^2}{r},$$

an example of which is afforded by motion in a circle, for then the radius of curvature $C A = C O = C D = r$ is constant. Inversely a constant acceleration, which always acts at right angles to the direction in which the body is moving, causes uniform motion in a circle.

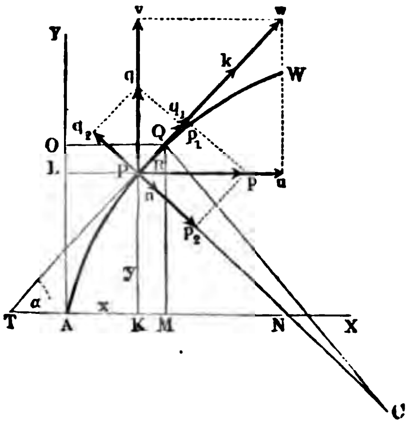
FIG. 78



EXAMPLE.—A body, revolving in a circle 5 feet in diameter in such a manner as to make each revolution in 5 seconds, has a velocity $c = \frac{2\pi r}{t} = \frac{2\pi . 5}{5} = 2 . \pi = 6{,}288$ feet, and a normal acceleration $p = \frac{(6{,}288)^2}{5} = 7{,}896$ feet, I.E., in every second it would be diverted from the straight line a distance $\frac{1}{2}\, p = \frac{1}{2} . 7{,}896 = 3{,}948$ feet.

(§ **44.**) **Curvilinear Motion in General.**—If a point P, Fig. 79, moves in two directions $A\ X$ and $A\ Y$ at the same time, we can consider the spaces described $A\ K = L\ P = x$ and $A\ L = K P = y$ as the co-ordinates of the curve $A\ P\ W$ formed by the path, and if $d\ t$ is the element of time, in which the body describes the elementary spaces $P\ R = d\ x$ and $R\ Q = d\ y$, we have (from § 20) the velocity along the abscissa

FIG. 79.



$$1)\quad u = \frac{d\,x}{d\,t},$$

and that along the ordinate

$$2)\quad v = \frac{d\,y}{d\,t},$$

and therefore the resulting *tangential velocity*, or that along the curve, when the directions $A\,X$ and $A\,Y$ of the motions are at right angles to each other,

$$3)\qquad w = \sqrt{u^2 + v^2} = \sqrt{\left(\frac{d\,x}{d\,t}\right)^2 + \left(\frac{d\,y}{d\,t}\right)^2} = \sqrt{\frac{d\,x^2 + d\,y^2}{d\,t^2}} = \frac{d\,s}{d\,t}, =$$

in which formula $d\,s$ denotes the element $P\,Q$ of the curve which, according to Art. 32 of the Introduction to the Calculus, is equal to

$$\sqrt{d\,x^2 + d\,y^2}.$$

The acceleration along the abscissa is, according to § 20,

$$4)\quad p = \frac{d\,u}{d\,t}, \qquad a_x$$

and that along the ordinate

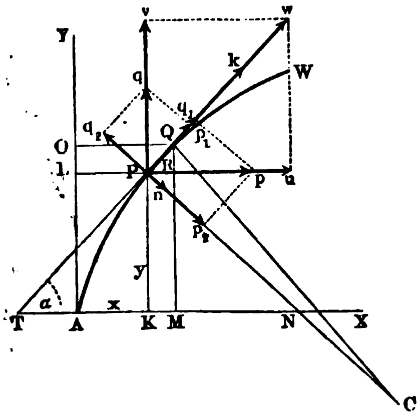
$$5)\quad q = \frac{d\,v}{d\,t}. \qquad a_y$$

For the tangential angle $P\,T\,X = Q\,P\,R = a$, formed by the direction of motion $\overline{P\,w}$ with the direction of the abscissas, we have,

$$tang.\,a = \frac{v}{u} = \frac{d\,y}{d\,x},$$

and also

FIG 80.



$$sin.\,a = \frac{v}{w} = \frac{d\,y}{d\,s} \text{ and}$$

$$cos.\,a = \frac{u}{w} = \frac{d\,x}{d\,s}.$$

The accelerations p and q can be decomposed into the following components in the directions of the tangent $P\,T$ and of the normal $P\,N$,

$$p_1 = p\,cos.\,a \text{ and } p_2 = p\,sin.\,a,$$

$$q_1 = q\,sin.\,a \text{ and } q_2 = q\,cos.\,a.$$

Consequently the tangential acceleration is

$$a_t \qquad k = p_1 + q_1 = p\,cos.\,a + q\,sin.\,a$$

$$= \frac{d\,u}{d\,t} \cdot \frac{u}{w} + \frac{d\,v}{d\,t} \cdot \frac{v}{w} = \frac{u\,d\,u + v\,d\,v}{w\,d\,t},$$

and the normal acceleration is

$$n = p_1 - q_1 = p \sin. a - q \cos. a$$
$$= \frac{du}{dt} \cdot \frac{v}{w} - \frac{dv}{dt} \cdot \frac{u}{w} = \frac{v\,du - u\,dv}{w\,dt}$$

But by differentiating $u^2 + v^2 = w^2$ we obtain

$$u\,du + v\,dv = w\,dw,$$

and therefore we have more simply for the tangential acceleration

$$6) \quad k = \frac{w\,dw}{w\,dt} = \frac{dw}{dt} = a_t$$

From $tang.\, a = \frac{v}{u}$ we obtain $d\, tang.\, a = \frac{u\,dv - v\,du}{u^2}$, (Introduction to the Calculus, Art. 8) and the radius of the curvature $CP = CQ$ of the elementary arc PQ (according to Art. 33 of the Introduction to the Calculus) is

$$r = -\frac{ds^3}{dx^2\, d\, tang.\, a},$$

whence it follows that

$$v\,du - u\,dv = -u^2\, d\, tang.\, a = \frac{u^2\,ds^3}{r\,dx^2} = \frac{ds^3}{2\,dt^2} = \frac{ds}{r}\left(\frac{ds}{dt}\right)^2 = \frac{w^2\,ds}{r^2}$$

and that the *normal acceleration* is simply

$$7) \quad n = \frac{w^2\,ds}{r\,w\,dt} = \frac{w}{r} \cdot \frac{ds}{dt} = \frac{w^2}{r}.$$

Finally we have

$$k\,ds = \frac{dw}{dt} \cdot ds = \frac{ds}{dt}\,dw = w\,dw;$$

from which we obtain (as in § 20),

$$8) \quad \frac{w^2 - c^2}{2} = \int k\,ds,$$

when we suppose that while describing the space s the velocity changes from c to w. Therefore, *in curvilinear motion half the difference of the squares of the velocities is equal to the product of the mean acceleration* (k) *and the space* s. In like manner

$$p\,dx + q\,dy = u\,du + v\,dv = w\,dw, \text{ and therefore}$$

$$9) \quad \frac{w^2 - c^2}{2} = \int (p\,dx + q\,dy) = \int p\,dx + \int q\,dy, \text{ and}$$

$$10) \quad \int k\,ds = \int p\,dx + \int q\,dy, \text{ or}$$
$$k\,ds = p\,dx + q\,dy.$$

The product of the tangential acceleration and the element of the curve is equal to the sum of the products of the accelerations along the co-ordinates and the corresponding elements of co-ordinates.

EXAMPLE.—A body moves on one axis $A\,X$ with the velocity $u = 12\,t$, and on the other $A\,Y$ with the velocity $v = 4\,t^2 - 9$; required the other conditions of the resulting motion. The corresponding accelerations along the co-ordinates are

$$p = \frac{d\,u}{d\,t} = 12, \text{ and } q = \frac{d\,v}{d\,t} = 8\,t,$$

and the co-ordinates, or spaces described along the axes, are

$$x = \int u\,d\,t = \int 12\,t\,d\,t = 6\,t^2, \text{ and}$$

$$y = \int v\,d\,t = \int (4\,t^2 - 9)\,d\,t = \frac{4}{3}\,t^3 - 9\,t,$$

in which equations the spaces count from the time $t = 0$. The tangential velocity, or that along the curve, is

$$w = \sqrt{u^2 + v^2} = \sqrt{144\,t^2 + (4\,t^2 - 9)^2} = \sqrt{16\,t^4 + 72\,t^2 + 81} = 4\,t^2 + 9,$$

consequently the tangential acceleration is

$$k = \frac{d\,w}{d\,t} = 8\,t = \text{the acceleration } q \text{ along the ordinate.}$$

We have also for the space described along the curve

$$s = \int w\,d\,t = \int (4\,t^2 + 9)\,d\,t = \frac{4}{3}\,t^3 + 9\,t.$$

When the direction of the motion is given by the formula,

$$tang.\ a = \frac{v}{u} = \frac{4\,t^2 - 9}{12\,t} = \frac{\frac{2}{3}\,x - 9}{2\,\sqrt{6\,x}},$$

we have

$$d\,tang.\,a = \frac{4\,t^2 + 9}{12\,t^2}\,d\,t,$$

and therefore the radius of curvature of the trajectory is

$$r = -\frac{d\,s^3}{d\,x^2\,d\,tang.\,a} = -\frac{(4\,t^2 + 9)^3 \,.\, 12\,t^2}{144\,t^2\,(4\,t^2 + 9)} = -\frac{(4\,t^2 + 9)^2}{12},$$

$$\text{or, } r = -\frac{w^2}{12}.$$

Consequently the normal acceleration, which produces a constant change of direction of the motion of the body, is

$$n = \frac{w^2}{r} = -12, \text{ or constant.}$$

The equation of the curve of the trajectory of the body is found by substituting $t = \sqrt{\frac{x}{6}}$ in the foregoing equation, and it is

$$y = \frac{4}{3}\sqrt{\left(\frac{x}{6}\right)^3} - 9\,\sqrt{\frac{x}{6}} = \left(\frac{2}{9}\,x - 9\right)\sqrt{\frac{x}{6}}.$$

The ordinate y is a (negative) maximum for $v = 0$, I.E., for $t^2 = \frac{9}{4}$, or $t = \frac{3}{2}$, and $x = 6\,.\,t^2 = 6\,.\,\frac{9}{4} = \frac{27}{2}$, and then

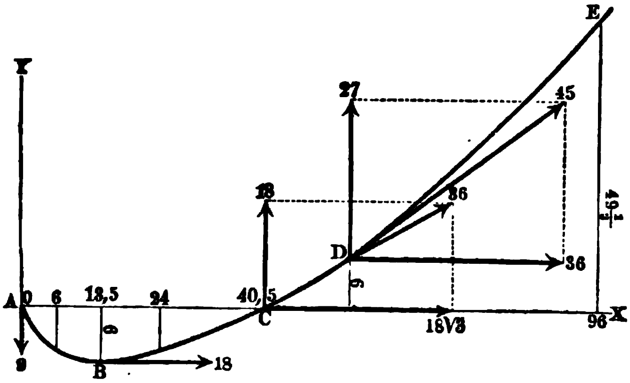
$$y = \frac{4}{3}\,.\,\frac{9}{4}\,.\,\frac{3}{2} - 9\,.\,\frac{3}{2} = -9;$$

and on the contrary, it is $= 0$, for $t^2 = \frac{27}{4}$ or $t = \frac{3}{2}\sqrt{3}$, and $x = \frac{81}{2}$.

The curve which forms the path of the body runs at first below the axis of abscissas, and after the time $t = \sqrt{\frac{27}{4}}$ it cuts it at a point whose abscissa is $x = \frac{81}{2}$, and from that time it remains above the axis.

The following table contains a collection of the corresponding values of t, u, v, w, x, y, *tang.* a, r and s, from which the curve $A\ B\ C\ D\ E$, Fig. 81, is constructed.

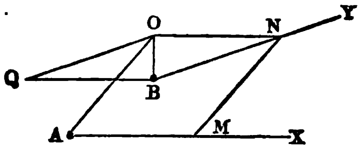
FIG. 81.



t	u	v	w	x	y	*tang.* a	r	s
0	0	-9	9	0	0	∞	$-\frac{27}{4}$	0
1	12	-5	13	6	$-\frac{23}{3}$	$-\frac{5}{12}$	$-\frac{169}{12}$	$\frac{31}{3}$
$1\frac{1}{2}$	18	0	18	$\frac{27}{2}$	-9	0	-27	18
2	24	7	25	24	$-\frac{22}{3}$	$\frac{7}{24}$	$-\frac{625}{12}$	$\frac{86}{3}$
$\frac{3}{2}\sqrt{3}$	$18\sqrt{3}$	18	36	$\frac{81}{2}$	0	$\sqrt{\frac{1}{3}}$	-108	$27\sqrt{3}$
3	36	27	45	54	$+9$	$\frac{3}{4}$	$-\frac{675}{4}$	63
4	48	55	75	96	$+\frac{148}{3}$	$\frac{55}{48}$	$-\frac{1875}{4}$	$\frac{364}{3}$

§ 45. Relative Motion.—If two bodies are moving simultaneously, a continual change in their relative positions, distances apart, etc., takes place, the value of which may be determined for any instant by the aid of what precedes. Let A, Fig. 82, be the point where one and B that where the other motion begins; the first

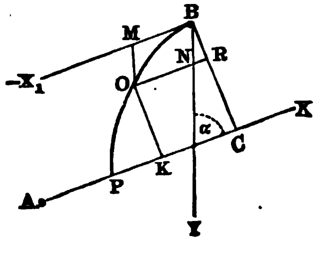
FIG. 82.



body passes in a given time (t) in the direction $A\ X$ to the position M, and the other body in the same time in the direction $B\ Y$ to the point N. Now if we draw $M\ N$, this line will give us the relative position and distance from each other of the bodies A and B at the end of this time. Drawing $A\ O$ parallel to $M\ N$, and making $A\ O = M\ N$, the line $A\ O$ will also give the relative position of the bodies A and B. If we now draw $O\ N$, we obtain a parallelogram, in which $O\ N$ is $= A\ M$. If, finally, we make $B\ Q$ equal and parallel to $N\ O$ and draw $O\ Q$, we obtain a new parallelogram $B\ N\ O\ Q$, in which the one side $B\ N$ is the absolute space (y) described by the second body, the other side $B\ Q$ is the space (x) described by the other body in the opposite direction, and the fourth corner O is the relative position of the second body, that is, in reference to the position of the first body, which we consider to be fixed. Hence we can determine the relative position O of a moving body (B) by giving to this body besides its motion ($B\ N$) another, equal to but in the opposite direction from that $A\ M$ of the body (A), to which its position is referred, and then by combining in the ordinary way, as, E.G., by the aid of a parallelogram, these two motions.

§ **46.** If the motions of the bodies A and B are uniform, we can substitute for $A\ M$ and $B\ N$ the velocities c and c_1, that is the spaces described in one second. In this way we obtain the relative velocity of one body when we give to it besides its own absolute velocity, that of the body to which we refer the first velocity, but in the opposite direction.

FIG. 83.



The same relation holds good for the accelerations. If, E.G., a body A, Fig. 83, moves uniformly in the direction $A\ C$ with the velocity c, and a body B moves in the direction $B\ Y$, which makes an angle α with $B\ X_1$, with an initial velocity $= 0$ and with the constant acceleration p, we can also suppose that A stands still and that B possesses, besides the acceleration p, also the velocity $(-c)$ in the direction $B X_1$ parallel to $A X$; the body will then describe the parabolic path $B O P$.

The spaces described in the time t in the directions $B Y$ and $B X_1$ are $B N = \frac{p t^2}{2}$ and $B M = c t$, the first of which can be decomposed into the components $N R = \frac{p t^2}{2} \cos. a$ and $B R = \frac{p t^2}{2} \sin. a$, which are parallel and at right angles to $A X$.

Now if $A C = a$ and $C B = b$ are the original co-ordinates of the point B in reference to A, and $A K = x$ and $K O = y$ the co-ordinates of the same after the time t, we have, since $A K = A C - O N - N R$ and $K O = C B - B R$,

$$x = a - c t - \frac{p t^2}{2} \cos. a \text{ and } y = b - \frac{p t^2}{2} \sin. a,$$

and consequently the corresponding relative velocities

$$u = - c - p t \cos. a \text{ and } v = - p t \sin. a.$$

From the abscissa x we determine the time by the formula

$$t = \sqrt{\frac{2 (a - x)}{p \cos. a} + \left(\frac{c}{p \cos. a}\right)^2} - \frac{c}{p \cos. a},$$

and, on the contrary, from the ordinate y by the formula

$$t = \sqrt{\frac{2 (b - y)}{p \sin. a}}.$$

If the body B moves in the line $A X$ towards A, we have $b = 0$ and also $a = 0$, and therefore

$$t = \sqrt{\frac{2 (a - x)}{p} + \left(\frac{c}{p}\right)^2} - \frac{c}{p},$$

putting $x = 0$, we obtain for the time, when two bodies will meet,

$$t = \sqrt{\frac{2 a}{p} + \left(\frac{c}{p}\right)^2} - \frac{c}{p} = \frac{\sqrt{2 a p + c^2} - c}{p}.$$

If, on the contrary, the body B moves in the line $A X$ ahead of the body A, then $a = 180°$, and the distance of the former from the latter body is $x = a - c t + \frac{p t^2}{2}$, and, inversely, the time, at the end of which the bodies are at a distance x from each other, is

$$t = \pm \sqrt{- \frac{2 (a - x)}{p} + \left(\frac{c}{p}\right)^2} + \frac{c}{p}.$$

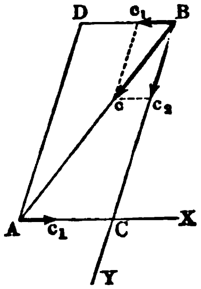
The corresponding velocity $u = - c + p t$ is $= 0$, and the distance x is a minimum for $t = \frac{c}{p}$, and its value is $x = a - \frac{c^2}{2 p}$

For every other value of x we have two values for the time, one of which is greater and the other less than $\frac{c}{p}$.

Remark.—The foregoing theory of relative motion is often applied, not only in celestial mechanics, but also in the mechanics of machines. Let us consider the following case.

A body A, Fig. 84, moves in the direction $A\,X$ with the velocity c_1, and should be met by another body B which has the velocity c_2; what direction must we give the latter? If we draw $A\,B$, lay off from B, c_1 in the opposite direction and complete with c_1 and c_2 a parallelogram $B\,c_1\,c\,c_2$, whose diagonal c coincides with $A\,B$, we obtain in the direction $B\,c_2 = c_2$ of its side, not only the direction $B\,Y$ in which the body B must move, but also in the point of intersection C of the two directions $A\,X$ and $B\,Y$, the point where the two bodies will meet. If α is the angle $B\,A\,X$ formed by $A\,X$, and β the angle $A\,B\,Y$ formed by $B\,Y$ with $A\,B$, we have

FIG. 84.



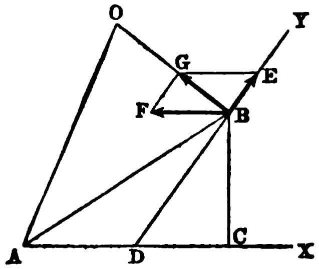
$$\frac{\sin.\beta}{\sin.\alpha} = \frac{c_1}{c_2}.$$

This formula is applicable to the *aberration of the light of the stars* which is caused by the composition of the velocity c_1 of the earth A around the sun with the velocity c_2 of the light of the star B. Here c_1 is about 19 miles, and c_2 about 192,000 miles, consequently

$$\sin.\beta = \frac{c_1}{c_2}\sin.\alpha = \frac{19\,\sin.\alpha}{192000} = \frac{\sin.\alpha}{10105},$$

hence the aberration or the angle $A\,B\,C = \beta$, formed by the apparent direction $A\,B$ of the star (which can be supposed to be infinitely distant) with the true direction $B\,C$ or $A\,D$, is $\beta = 20''\,\sin.\alpha$; and for $\alpha = 90°$, that is, for a star, which is vertically above the path of the earth (in the so-called pole of the ecliptic), we have $\beta = 20''$. In consequence of this divergence we always see a star 20″ in the direction of the motion of the earth behind its true position, and consequently a star in the neighborhood of the pole of the ecliptic describes apparently in the course of a year a small circle of 20″ radius around its true position. For stars in the plane of the earth's path this apparent motion takes place in a straight line, and for the other stars in an apparent ellipse.

FIG. 85.



Example.—A locomotive moves from A upon the railroad track $A\,X$, Fig. 85, with 35 feet velocity, and another at the same time from B with 20 feet velocity upon the track $B\,Y$, which forms an angle $B\,D\,X = 56°$ with the first. Now if the initial distances are $A\,C = 30000$ feet, and $C\,B = 24000$ feet, how great is the distance $A\,O$ after a quarter of an hour? From the absolute velocity $B\,E = c_1 = 20$ feet of the second train, the inverse velocity $B\,F = c = 35$ feet of the first, and the included angle $E\,B\,F = a = 180° - B\,D\,C = 180° - 56° = 124°$, we obtain the relative velocity of the second train

$$B\,G = \sqrt{c^2 + c_1{}^2 + 2\,c\,c_1\,\cos.\,a} = \sqrt{35^2 + 20^2 - 2\,.\,35\,.\,20\,.\,\cos.\,56°}$$
$$= \sqrt{1225 + 400 - 1400\,\cos.\,56°} = \sqrt{1625 - 782{,}9} = \sqrt{842{,}1} = 29{,}02 \text{ feet.}$$

For the angle $G\,B\,F = \phi$, included between the direction of the relative motion and the direction of the first motion, we have

$$\sin.\,\phi = \frac{c_1\,\sin.\,56°}{29{,}02} = \frac{20\,.\,0{,}8290}{29{,}02};\ \log \sin.\,\phi = 0{,}75690 - 1, \text{ whence } \phi = 34°{,}50'.$$

The relative space described in 15 minutes $= 900$ seconds is $B\,O = 29{,}02\,.\,900 = 26118$ feet, the distance $A\,B$ is $= \sqrt{(30000)^2 + (24000)^2} = 38419$ feet, the value of the angle $B\,A\,C = A\,B\,F$, whose tangent is $\frac{24000}{30000} = 0{,}8$, is $\psi = 38°\ 40'$, and therefore the angle

$$A\,B\,O = \phi + \psi = 34°\ 50' + 38°\ 40' = 73°\ 30',$$

and the distance apart of the two trains after 15 minutes is

$$A\,O = \sqrt{\overline{A\,B}^2 + \overline{B\,O}^2 - 2\,A\,B\,.\,B\,O\,\cos.\,A\,B\,O}$$
$$= \sqrt{38419^2 + 26118^2 - 2\,.\,38419\,.\,26118\,\cos.\,73°\ 30'}$$
$$= \sqrt{1588190000} = 39850 \text{ feet.}$$

SECOND SECTION.

MECHANICS, OR THE PHYSICAL SCIENCE OF MOTION IN GENERAL.

CHAPTER I.

FUNDAMENTAL PRINCIPLES AND LAWS OF MECHANICS.

§ 47. **Mechanics.**—*Mechanics* (Fr. mécanique, Ger. Mechanik) is the science which treats of the laws of the motion of *material bodies.* It is an application to the bodies of the exterior world of that part of Phoronomics or Cinematics which deals with the motions of geometrical bodies without considering the cause. Mechanics is a part of *Natural Philosophy* (Fr. physique générale, Ger. Naturlehre) or of the science of the laws, in accordance with which the changes in the material world take place, viz., that part of it, which treats of the changes in the material world arising from measurable motions.

§ 48. **Force.**—*Force* (Fr. force, Ger. Kraft) is the cause of the motion, or of the change in the motion of material bodies. Every change in motion, E.G., every change of velocity, must be regarded as the effect of a force. For this reason we attribute to a body falling freely a force, which we call *gravity;* for the velocity of the body changes continually. But, on the other hand, we cannot infer from the fact that a body is at rest or moving uniformly that it is free from the action of any force; for forces may balance each other without causing any visible effect. Gravity, which causes a body to fall, acts as strongly upon it when it lies upon a table, but its effect is here destroyed by the resistance of the table or other support.

§ 49. Equilibrium.—A body is in *equilibrium* (Fr. équilibre, Ger. Gleichgewicht), or the forces acting on a body hold each other in equilibrium, or balance each other, when they counterbalance or neutralize each other without leaving any resulting action, or without causing any motion or change of motion. E.G. When a body is suspended by a string, gravity is in equilibrium with the cohesion of the string. The equilibrium of several forces is destroyed and motion produced when one of the forces is removed or neutralized in any way. Thus a steel spring, which is bent by a weight, begins to move as soon as the weight is removed, for then the force of the spring, which is called its elasticity, comes into action.

Statics (Fr. statique, Ger. Statik) is that part of mechanics which treats of the laws of equilibrium. *Dynamics* (Fr. dynamique, Ger. Dynamik), on the contrary, treats of forces as producers of motion.

§ 50. Classification of the Forces.—According to their action, we can divide forces into *motive forces* (Fr. forces motrices puissance, Ger. bewegende Kräfte), and *resistances* (Fr. résistances, Ger. Widerstände). The former produce, or can produce, motion, the latter can only prevent or diminish it. Gravity, the elasticity of a steel spring, etc., belong to the moving forces, friction, resistance of bodies, etc., to the resistances; for although they can hinder or diminish motion or neutralize moving forces, they are in no way capable of producing motion. The moving forces are either *accelerating* (Fr. accéleratrices, Ger. beschleunigende) or *retarding* (Fr. retardatrices, Ger. verzögernde). The former cause a positive, the latter a negative, acceleration, producing in the first case an accelerated, and in the second a retarded motion. The resistances are always retarding forces, but all retarding forces are not necessarily resistances. When a body is projected vertically upward, gravity acts as a retarding force, but gravity is not on this account a resistance, for when the body falls it becomes an accelerating force. We distinguish also *uniform* (Fr. constantes, Ger. beständige, constante) and *variable forces* (Fr. variable, Ger. veranderliche). While uniform forces act always in the same way, and therefore in the equal instants of time produce the same effect, I.E., the same increase or decrease of velocity, the effects of variable forces are different at different times; hence the former forces produce uniformly variable motions, and the latter variably accelerated or retarded motions.

§ **51. Pressure.**—*Pressure* (Fr. pression, Ger. Druck), and *traction* (Fr. traction, Ger. Zug), are the first effects of force upon a material body. In consequence of the action of a force bodies are either compressed or extended, or, in general, a change of form is caused.

The pressure or traction, produced by gravity acting vertically downwards and to which the support of a heavy body or the string, to which it is suspended, is subjected, is called the *weight* (Fr. poids, Ger. Gewicht) of the body.

Pressure and traction, and also weight, are quantities of a peculiar kind, and can be compared only with themselves; but since they are effects of force they may be employed as measures of the latter.

The most simple and therefore the most common way of measuring forces is by means of weights.

§ **52. Equality of Forces.**—Two weights, two pressures, two tractions, or the two forces corresponding to them are equal, when we can replace one by the other without producing a different action. When, E.G., a steel spring is bent in exactly the same manner by a weight G suspended to it as by another weight G_1 hung upon it in exactly the same manner, the two weights, and therefore the forces of gravity of the two bodies are equal. If in the same way a loaded scale (Fr. balance, Ger. Waage) is made to balance as well by the weight G as by another G_1, with which we have replaced G, then these weights are equal, although the arms of the balance may be unequal, and the other weight be greater or less.

A pressure or weight (force) is 2, 3, 4, etc., or in general n times as great as another pressure, etc., when it produces the same effect as 2, 3, 4 .... n pressures of the second kind acting together. If a scale loaded in any arbitrary manner is caused to balance by the weight (G) as well as by 2, 3, 4, etc., equal weights (G_1), then is the weight (G) 2, 3, 4, etc. times as great as the weight (G_1).

§ **53. Matter.**—*Matter* (~~Fr. Matière~~, Ger. Materie) is that, by which the bodies of the exterior world (which in contradistinction to geometrical bodies are called material bodies) act upon our senses. *Mass* (Fr. masse, Ger. Masse) is the quantity of matter which makes up a body.

Bodies of equal volume (~~Fr. volume~~, Ger. Volumen) or of equal geometrical contents generally have different weights. Therefore we can not determine from the volume of a body its weight; it is necessary for that purpose to know the weight of the unit of volume, E.G., of a cubic foot, cubic meter, etc.

§ **54. Unit of Weight.**—The measurement of weights or forces consists in comparing them to some given unchangeable weight, which is assumed as the unit. We can, it is true, choose this unit of weight or force arbitrarily, but practically it is advantageous to choose for this purpose the weight of a certain volume of some body, which is universally distributed. This volume is generally one of the common units of space. One of the units of weight is the gram, which is determined by the weight of a cubic centimetre of pure water at its maximum density (at a temperature of about 4° *C.*). The old Prussian pound is also a unit referred to the weight of water. A Prussian cubic foot of distilled water weighs at 15° *R. in vacuo* 66 Prussian pounds. Now a Prussian foot is $=$ 139,13 Paris lines $=$ 0,3137946 meter; whence it follows that a Prussian pound $=$ 467,711 grams. The Prussian new or custom-house pound weighs exactly $\frac{1}{2}$ kilogramm. The English pound is determined by the weight of a cubic foot of water at a temperature of 39°, 1 F. The pound is equal to 453,5926 grams. A cubic foot of water weighs 62,425 lbs.

§ **55. Inertia** (Fr. inertie, Ger. Trägheit) is that property of matter, in virtue of which matter cannot move of itself nor change the motion, that has been imparted to it. Every material body remains at rest as long as no force is applied to it, and if it has been put in motion continues to move uniformly in a straight line, as long as it is free from the action of any force. If, therefore, changes in the state of motion of a material body occur, if a body changes the direction of its motion, or if its velocity becomes greater or less, this result must not be attributed to the body as a certain quantity of matter, but to some exterior cause, I.E., to a force.

Since, whenever there is a change in the state of motion of a body, a force is developed, we can in this sense count inertia as one of the forces. If a moving body could be removed from the influence of all the forces which act upon it, it would move forward uniformly for ever; but such a uniform motion is nowhere to be found, since it is impossible for us to remove a body from the influence of every force. If a mass moves upon a horizontal table the action of gravity is counterbalanced by the table, and therefore does not act directly upon the body, but in consequence of the pressure of the body on the table a resistance is developed, which will be treated hereafter under the name of *friction.* This resistance continually diminishes the velocity of the moving body, and the body therefore assumes a uniformly retarded motion and finally comes to rest. The air also opposes a resistance to its motion, and even if the friction of the body could be completely put aside, a continual decrease of velocity would be caused by the former. But we find that the loss of velocity becomes less and less, and that the motion approximates more and more to a uniform one, the more we diminish the number and magnitude of these resistances, and we can therefore conclude, that if all moving forces and resistances were removed, a perfectly uniform motion would ensue.

§ 56. **Measure of Forces.**—The force (P) which accelerates an inert mass (M) is proportional to the acceleration (p) and to the mass (M) itself. When the masses are the same, it increases with the infinitely small increments of velocity produced in the infinitely small spaces of time, and when the velocities are equal it increases in the same ratio as the masses themselves. In order to produce an m fold acceleration of the same mass, or of equal masses, we require an m fold force, and an n fold mass requires an n fold force to produce the same acceleration.

Since we have not as yet adopted a measure for the masses, we can assume

$$P = Mp,$$

or that *the force is equal to the product of mass and the acceleration,* and at the same time we can substitute instead of the force its effect, I.E., the pressure produced by it.

The correctness of this general law of motion can be proved by direct experiment, when we, E.G., drive along upon a horizontal table by means of bent steel springs similar or different movable masses; but the important proof lies in this, that all the results and rules for compound motion, deduced from the law, correspond exactly with our observations and with natural phenomena.

§ 57. **Mass.**—All bodies at the same point on the earth fall *in vacuo* equally quickly, namely, with the constant acceleration $g = 9{,}81$ meter $= 32{,}2$ feet (§ 15). If the mass of a body is $= M$

and the weight which measures the force of gravity $= G$, we have from the last formula

$$G = Mg,$$

I.E., the *weight of a body is a product of its mass and the acceleration of gravity,* and inversely

$$M = \frac{G}{g},$$

I.E., the *mass of a body is the weight of the same divided by the acceleration of gravity,* or the mass is that weight which a body would have if the acceleration of gravity were $= 1$, E.G., a meter, a foot, etc. For that point upon or in the neighborhood of the earth or of any other celestial body, where the bodies fall with a velocity (at the end of the first second) of 1 meter instead of 9,81 meters, the mass, or rather the measure of the same, is given directly by the weight of the body.

According as the acceleration of gravity is expressed in meters or feet we have for the masses

$$M = \frac{G}{9{,}81} = 0{,}1019\ G, \text{ or}$$

$$M = \frac{G}{32{,}2} = 0{,}031\ G.$$

Hence the mass of a body, whose weight is 20 pounds, is $M = 0{,}031 \times 20 = 0{,}62$ pounds, and inversely the weight of a mass of 20 pounds is $G = 32{,}2 \times 20 = 644$ pounds.

§ **58.**—If we suppose the acceleration (g) of gravity to be constant, it follows that the mass of a body is exactly proportional to its weight, and that, when the masses of two bodies are M and M_1 and their weights G and G_1, we have

$$\frac{M}{M_1} = \frac{G}{G_1}.$$

Therefore, the weight of a body can be employed as a measure of its mass, so that the greater the mass a body is the greater is its weight.

However the acceleration of gravity is variable, becoming greater as we approach the poles and diminishing as we approach the equator; it is a maximum at the poles and a minimum at the equator. It also decreases when a body is elevated above the level of the sea. Now since a mass, so long as we take nothing from it nor add anything to it, is a constant quantity and remains the same for all points on the earth, and even on the moon, it follows that the weight of a body must be variable and depend upon the position of the body, and that in general it must be proportional to the acceleration of gravity, or that $\frac{G}{G_1}$ must be $= \frac{g}{g_1}$.

The same steel spring would therefore be differently deflected by the same weight at different points on the earth—at the equator and on high mountains the least, and at the poles at the level of the sea the most.

§ 59. **Heaviness** (Fr. densité, Ger. Dichtigkeit) is the intensity with which matter fills space. The heavier a body is, the more matter is contained in the space it occupies. The natural measure of the heaviness is that quantity of matter (the mass) which fills the unity of volume; but since matter can only be measured by weight, the weight of a unit of volume, E.G., of a cubic meter or of a cubic foot of another matter, must be employed as the measure of its heaviness. Hence, the heaviness of water at 39°.1 F. is = 62,425 pounds, and that of cast iron at 32° F. is = 452 pounds, I.E., a cubic foot of water weighs 62,425, and a cubic foot of cast iron 452. In ordinary calculations we assume that of water to be 62½ pounds. From the volume V of a body and its heaviness γ we have its weight $G = V\gamma$.

The product of the volume and the heaviness is the weight.

The heaviness of a body is uniform (Fr. homogène, uniforme, Ger. gleichförmig) or variable, (Fr. variable, hétérogène, Ger. ungleichförmig), according as equal portions of the volume have equal or different weights, E.G., the heaviness of the simple metals is uniform, since equal parts of them, however small, weigh the same. Granite, on the contrary, is a body of variable heaviness, since it is composed of parts of different density.

EXAMPLE.—1. If the heaviness of lead is 712 pounds, then 3,2 cubic feet of lead weigh $G = V\gamma = 2278,4$ lbs. If the weight of a cubic foot of bar iron be 480 pounds, the volume of a piece, whose weight is 205 pounds, is

$$V = \frac{G}{\gamma} = \frac{205}{480} = 0,4271 \text{ cubic feet} = 0,4271 \times 1728 = 738 \text{ cubic inches.}$$

Note.—In German and French the word "*density*" is employed to express the weight of a cubic foot, a cubic meter, etc., of any material. In English, unfortunately, it is employed as a synonym of specific gravity.—TR.

If 10,4 cubic feet of hemlock, thoroughly saturated with water, weighs 577, then its heaviness is

$$\gamma = \frac{G}{V} = \frac{577}{10,4} = 55,5 \text{ pounds.}$$

§ 60. **Specific Gravity.**—*Specific weight*, or *specific gravity*, (Fr. poids spécifique, Ger. specifisches or eigenthümliches Gewicht) is the ratio of the heaviness of one body to that of another body, generally water, which is assumed as the unit. But the heaviness is equal to the weight of the unit of volume; therefore the specific gravity is also the ratio of the weight of one body to that of another, E.G., water, of equal volume.

In order to distinguish the specific gravity or specific weight from the weight of a body of a given volume, the latter is called the *absolute weight* (Fr. poids absolu, Ger. absolutes Gewicht).

If γ is the heaviness of the matter (water), to which the others are referred, and γ_1 the heaviness of any matter whose specific gravity is denoted by ε, we have the following formula:

$$\varepsilon = \frac{\gamma_1}{\gamma} \text{ and } \gamma_1 = \varepsilon\, \gamma,$$

therefore the heaviness of any matter is equal to the specific gravity of the same multiplied by the heaviness of water.

The absolute weight G of a mass of whose volume is V, and whose specific gravity is ε, is:

$$G = V\gamma_1 = V\, \varepsilon\, \gamma.$$

EXAMPLE.—1. The heaviness of pure silver is 655 pounds, and that of water 62,425 pounds; consequently the specific gravity of the former (in relation to water) is $= \frac{655}{62,425} = 10,50$, I.E., every mass of silver is 10½ times as heavy as a mass of water that occupies the same space. 2) If we take 13,598 for the specific gravity of mercury, and the heaviness of water as 62,425, then we have for the heaviness of mercury,

$$\gamma = 13,598 \,.\, 62,425 = 848,86 \text{ pounds.}$$

A mass of 35 cubic inches of the same weighs, since 1,728 cubic inches are a cubic foot,

$$G = 848,86\, V = \frac{848,86 \,.\, 35}{1728} = 17,19 \text{ pounds.}$$

REMARK.—The use of the French weights and measures possesses the advantage that we can perform the multiplication by ε and γ by simply changing the position of the decimal point, for a cubic centimeter weighs a gram, and a cubic meter a million grams, or 1,000 kilograms. The heaviness of mercury is therefore, when we employ the French measure, $\gamma_1 = 13,598 \,.\, 1000 = 13598$ kilograms; that is, a cubic meter of mercury weighs 13598 kilograms.

§ **61.** The following table contains the specific gravities of those substances, which are met with the oftenest in practical mechanics. A complete table of specific gravities is to be found in the Ingenieur, page 310.

Mean specific gravity of the wood of deciduous trees, dry	0,659
saturated with water	1,110
Mean specific gravity of the wood of evergreen trees, dry	0,453
saturated with water	0,889[*]
Mercury	13,56
Lead	11,33
Copper, cast and dense	8,75
" hammered	8,97
Brass	8,55
Iron, cast, white	7,50
" " grey	7,10
" " medium	7,06
" in rods	7,60
Zinc, cast	7,05
" rolled	7,54
Granite	2,50 to 3,05
Gneiss	2,39 to 2,71
Limestone	2,40 to 2,86
Sandstone	1,90 to 2,70
Brick	1,40 to 2,22
Masonry with mortar made of lime and quarry stone:	
Fresh	2,46
Dry	2,40
Masonry with mortar made of lime and sandstone:	
Fresh	2,12
Dry	2,05
Brickwork with mortar made of lime:	
Fresh	1,55 to 1,70
Dry	1,47 to 1,59
Earth, clayey, stamped:	
Fresh	2,06
Dry	1,93
Garden earth:	
Fresh	2,05
Dry	1,63
Dry poor earth	1,34

§ **62. State of Aggregation.**—Bodies present themselves to us in three different states, depending upon the manner in which their parts are held together. This is called their *state of aggregation.* They are either *solid* (Fr. solides, Ger. fest) or *fluid* (Fr. fluides, Ger. flüssig), and the latter are either *liquid* (Fr. liquides, Ger. tropfbar flüssig) or *gaseous* ((Fr. gazeux, aériformes, Ger. elastisch flüssig). Solid bodies are those, whose parts are held together so firmly, that a certain force is necessary to change their forms or to produce a separation of their parts. Fluids are bodies, the position of whose parts in reference to each other is changed by the smallest force. Elastic fluids, the representative of which is the air, are distinguished from liquids, the representative of which is water, by the fact that they tend continually to expand more and more, which tendency is not possessed by water, etc.

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[*] See the absorption of water by wood, polytechnische Mittheilungen, Vol. II, 1845.

While every solid body possesses a peculiar form of its own and a definite volume, liquids have only a determined volume, but no peculiar form. Gases or aeriform fluids possess neither one nor the other.

§ 63. **Classification of the Forces.**—Forces are very different in their nature; we give here only the most important ones:

1) *Gravity*, by virtue of which all bodies tend to approach the centre of the earth.
2) *The Force of Inertia*, which manifests itself when a change in the velocity or in the direction of the moving body takes place.
3) *The Muscular Force* of living beings, or the force produced by means of the muscles of men and animals.
4) *The Elastic Force*, or that of springs, which bodies exhibit when a change of form or of volume occurs.
5) *The Force of Heat*, by virtue of which bodies expand and contract, when a change of temperature takes place.
6) *The Force of Cohesion*, or the force by which the parts of a body hold together, and with which they resist separation.
7) *The Force of Adhesion*, or the force with which bodies brought into close contact attract each other.
8) *The Magnetic Force*, or the attractive and repulsive force of the magnet.

Then we have the *electric* and the *electro-magnetic forces*, etc.

The resistances due to friction, rigidity, resistance of bodies, etc., are due principally to the force of cohesion, which, like the elasticity, etc., is due to the so-called molecular force, or the force with which the molecules, or the smallest parts of a body, act upon one another.

§ 64. **Forces, how Determined.**—For every force, we must distinguish:

1) The *point of application* (Fr. point d'application; Ger. Angriffspunkt), the point of the body to which the force is directly applied.
2) The *direction of the force* (Fr. direction, Ger. Richtung), the right line, in which a force moves the point of applica-

tion, or tends to move it or hinder its motion. The direction of a force has, like every direction of motion, two senses. It can take place from left to right, or from right to left, from above downwards, or from below upwards. One is considered as positive, and the other as negative. As we read and write from left to right, and from above downwards, it is natural to consider these motions as positive, and the opposite motions as negative.

3) The *absolute magnitude or intensity* (Fr. grandeur absolue, intensité, Ger. absolute Grösse) of the force, which we have seen is measured by weights, E.G. pounds, kilograms, etc.

Forces are graphically represented by straight lines, whose direction and length indicate the direction and magnitude of the forces, and one of whose extremities can be considered as the point of application of the forces.

§ 65. **Action and Reaction.**—The first effect produced by a force upon a body is an extension or compression, combined with a change of form or of volume, which commences at the point of application, and from there gradually spreads itself farther and farther into the body. By this inward change in the body the elasticity inherent in it comes into action and sets itself in equilibrium with the force, and is, therefore, equal to it, but acts in the opposite direction. *Hence, action and reaction are equal and opposite.* This law is true, not only for the effects of forces acting by contact, but also for those acting by attraction and repulsion, among which the magnetic forces, and also that of gravity, must be counted. A bar of iron attracts a magnet exactly as much as it is attracted itself by the magnet. The force, with which the moon is attracted towards the earth (by gravity), is equal to the force with which the moon reacts upon the earth.

The force with which a weight presses upon its support is returned by the latter in the opposite direction. The force, with which a workman pulls, pushes, etc., a machine, reacts upon the workman, and tends to move him in the opposite direction. When one body impinges upon another, the first presses upon the second exactly as much, as the second does upon the first.

§ 66. **Division of Mechanics.**—General mechanics are divided into two principal divisions, according to the state of aggregation of the bodies:

1) Into the mechanics of solid or rigid bodies (Fr. mécanique des corps solides, Ger. Mechanik der festen oder starren Körper).
2) Into the mechanics of fluids (Fr. mécanique des fluides, Ger. Mechanik der flüssigen Körper). The latter can again be divided:
a) Into the mechanics of water and other liquids or hydraulics (Fr. hydraulique, Ger. Hydraulik, Hydromechanik); and
b) Into the mechanics of air and other aëriform bodies (Fr. mécanique des fluides aëriformes, Ger. Mechanik der luftförmigen Körper).

If we take into consideration the division of mechanics into statics and dynamics, we can again divide it into:

1) Statics of rigid bodies.
2) Dynamics of rigid bodies.
3) Statics of water, etc., or hydrostatics.
4) Dynamics of water, etc., or hydrodynamics.
5) Statics of air (of gases and vapor) or aërostatics.
6) Dynamics of air (of gases and vapors) or aërodynamics or pneumatics.

CHAPTER II.

MECHANICS OF A MATERIAL POINT.

§ 67. A *material point* (Fr. point matérial, Ger. materieller Punkt) is a material body whose dimensions in all directions are infinitely small compared with the space described by it. In order to simplify the discussion, we will now consider the motion and equilibrium of a material point alone. A (finite) body is a continuous union of an infinite number of material points or molecules. If the different points or elements of a body move in exactly the same manner, I.E., with same velocity in parallel straight lines, the theory of the motion of material point is applicable to the whole body; for in this case we can suppose that equal portions of the mass are impelled by equal portions of the force.

§ **68. Simple Constant Force.**—If p is the acceleration with which a mass M is impelled by a force P, we have from § 56

$$P = Mp, \text{ or inversely the acceleration } p = \frac{P}{M}.$$

Putting the mass $M = \frac{G}{g}$, G denoting the weight of the body and g the acceleration of gravity, we obtain the force

$$1)\ P = \frac{p}{g} G,$$

and the acceleration

$$2)\ p = \frac{P}{G} g.$$

We find then the force (P) which moves a body with the acceleration (p) by multiplying the weight (G) of the body by the ratio $\left(\frac{p}{g}\right)$ of its acceleration to that of gravity.

Inversely we obtain the acceleration (p), with which a force (P) will move a mass M, by multiplying the acceleration (g) of gravity by the ratio $\left(\frac{P}{G}\right)$ of the force to the weight of the body.

EXAMPLE.—Let us imagine a body placed upon a very smooth horizontal table, which opposes no resistance to its motion, but which counteracts the effect of gravity. If this body be subjected to the action of a horizontal force, the body yields and moves forward in the direction of the force. If the weight of the body is $G = 50$ pounds and the force which acts uninterruptedly upon it is $P = 10$ pounds, it will assume a uniformly accelerated motion, the acceleration of which is $p = \frac{P}{G} g = \frac{10}{50}\ 32{,}2 = 6{,}44$ feet. If, on the contrary, the acceleration produced in a body weighing 42 pounds by a force P is $p = 9$ feet, then the force is $P = \frac{p}{g} G = \frac{9}{32{,}20}$ $.42 = 0{,}031 . 378 = 11{,}7$ pounds.

§ **69.** If the force acting upon a body is *constant*, a uniformly variable motion is the result, and it is uniformly accelerated, when the direction of the force coincides with the original direction of motion, and uniformly retarded, when the force acts in the opposite direction. If we substitute in the formulas of § 13 and § 14, instead of p, its value $\frac{P}{M} = \frac{P}{G} g$, we obtain the following:

I. For uniformly accelerated motion:

$$1)\ v = c + \frac{P}{G} g\, t = c + 32{,}2\, \frac{P}{G} t \text{ feet} = c + 9{,}81\, \frac{P}{G} t \text{ metres},$$

$$2)\ s = c\, t + \frac{P}{G} \frac{g\, t^2}{2} = c\, t + 16{,}1\, \frac{P}{G} t^2 \text{ feet} = c\, t + 4{,}905\, \frac{P}{G} t^2 \text{ metres.}$$

II. For uniformly retarded motion:

$$1)\ v = c - \frac{P}{G} g\, t = c - 32{,}2\, \frac{P}{G} t \text{ feet} = c - 9{,}81\, \frac{P}{G} t \text{ metres.}$$

$$2)\ s = c\, t - \frac{P}{G} \frac{g\, t^2}{2} = c\, t - 16{,}1 \frac{P}{G} t^2 \text{ feet} = c\, t - 4{,}905\, \frac{P}{G}\, t^2 \text{ metres.}$$

By means of the above formulas all questions, which can arise in reference to the rectilinear motions produced by a constant force, can be answered.

EXAMPLE.—1) A wagon weighing 2,000 pounds moves upon a horizontal road, which opposes no resistance to it, with a velocity of 4 feet, and is impelled during 15 seconds by a constant force of twenty-five pounds; with what velocity will it proceed after the action of this force? The required velocity is $v = c + 32{,}2\, \frac{P}{G}\, t$, but here we have $c = 4$, $P = 25$, $G = 2{,}000$ and $t = 15$, whence $v = 4 + 32{,}2 \, . \, \frac{25}{2000} \, . \, 15 = 4 + 6{,}037 = 10{,}037$ feet. 2) Under the same circumstances a wagon weighing 5,500 pounds, which in the three previous minutes had described uniformly 950 feet, was impelled during 30 seconds by a constant force, so that afterwards it described 1650 feet uniformly in three minutes. What was this force? The initial velocity is $c = \frac{950}{3\, .\, 60} = 5{,}277$ feet, and the final velocity is $v = \frac{1650}{3\, .\, 60} = 9{,}166$ feet, whence $\frac{P}{G}\, g\, t = v - c = 3{,}889$, and the force $P = \frac{3{,}889\, .\, G}{g\, t} = 0{,}031\, .\, 3{,}889\, .\, \frac{5500}{30} = 0{,}120559\, .\, \frac{550}{3} = 22{,}10$ pounds. 3) A sled weighing 1500 pounds and sliding on a horizontal support with a velocity of 15 feet loses, in consequence of the friction, in 25 seconds, the whole of its velocity. What is the amount of the friction? The motion is here uniformly retarded and the final velocity is $v = 0$, hence $c = 32{,}2\, .\, \frac{P\, t}{G}$, and $P = 0{,}031\, \frac{G\, c}{t} = 0{,}031\, \frac{1500\, .\, 15}{25} = 0{,}031\, .\, 900 = 27{,}9$ pounds, which is the friction in question. 4) Another sled, weighing 1200 pounds and moving with an initial velocity of 12 feet, is obliged to overcome a friction of 45 pounds when in motion. What is its velocity after 8 seconds, and what is the space described?

The final velocity is

$$v = 12 - 32,2 \frac{45 \, . \, 8}{1200} = 12 - 9,66 = 2,34 \text{ feet,}$$

and the space described is

$$s = \left(\frac{c + v}{2}\right) t = \left(\frac{12 + 2,34}{2}\right) . \, 8 = 57,36 \text{ feet.}$$

§ **70. Mechanical Effect.***—*Mechanical effect* or *work done* (Fr. travail mecanique, Ger. Leistung or Arbeit der Kraft) is that effect which a force accomplishes in overcoming a resistance, as, E.G., gravity, friction, inertia, etc. Work is done when we elevate a weight, when a greater velocity is communicated to a body, when the forms of bodies are changed, when they are divided, etc. The work done depends not only upon the force, but also on the space during which it is in action, or during which it overcomes a resistance. If we raise a body slowly enough to be able to disregard the inertia, the work done is proportional to its weight and to the height which it is lifted for 1) the effect is the same if a body of the m (3) fold weight is lifted a certain height, or if m (3) bodies of the weight (G) are lifted the same height; it is m times as great as that necessary to raise the simple weight the same height; and in like manner 2) the work done is the same, if one and the same weight be raised the n (5) fold height ($n \, h$) or if it is raised n (5) times to the simple height, and in general n (5) times so great, as when it is raised to the simple height. In like manner, the work done by a weight sinking slowly is proportional to the weight and to the distance it sinks. This proportion is, however, true for every other kind of work done; in order to make a saw cut of twice the length and of the same depth as another we are obliged to separate twice as many particles, and the work done is therefore double; the double length requires the force to describe double the distance, and consequently the work is proportional to the space described. In like manner the work done by a run of millstones increases evidently with the number of grains of a certain kind of corn which it grinds to a certain fineness. This quantity is, however, under the same circumstances proportional to the number of revolutions, or rather to the space described by the upper millstone while this quantity of corn is being ground. The work done increases, therefore, directly with the space described.

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* *Energy* is the capacity of a body to perform work. *Energy* is said to be stored when this capacity is increased, and to be *restored* when it is diminished. The unit of energy is the same as that of work.—TR.

§ **71.** As the work done by a force depends upon the intensity of the force and the space described by it, we can assume as the *unit of work* or *dynamical unit* (Fr. unité dynamique, Ger. Einheit der mechanischen Arbeit oder Leistung) the work done in overcoming a resistance, whose intensity is the unit of weight (pound, kilogram) over a space equal to the unit of length (foot, metre), and we can also put this measure equal to the product of the force or resistance into the space described by it in its direction while overcoming the resistance.

If we put the amount of the resistance itself $= P$ and the space described by the force, or rather by its point of application, while overcoming it $= s$, then the work done in overcoming this resistance is

$$A = P\,s \text{ units of work.}$$

In order better to define the units of work (which we can style simply *dynam*) the units of both factors P and s are generally given, and instead of units of work we say kilogram-meters and pound-feet, or inversely meterkilograms, foot-pounds, etc., according as the weight and the space are expressed in kilograms and meters, or in pounds and feet. For simplicity we write instead of meterkilogram, *m k* or *k m*; and instead of foot pound, *lb. ft.*, or *ft. lb.*

EXAMPLE.—1. In order to raise a stamp weighing 210 pounds, 15 inches high, the work to be done is $A = 210 \,.\, \frac{15}{12} = 262{,}5$ *ft. lbs.* 2. By a mechanical effect of 1500 foot pounds a sled, which when moving must overcome a friction of 75 pounds, will be drawn forward a distance

$$s = \frac{A}{P} = \frac{1500}{75} = 20 \text{ feet.}$$

§ **72.** Not only when the force is invariable, or the resistance is constant, but also when the resistance varies while the force is overcoming it, can the work done be expressed by the product of the force and the space described, provided we assume for the value of the force the mean value of the continuous succession of forces. The relation between the time, velocity and space is therefore the same here; for we can regard the latter as the product of the time and of the mean of the velocities. We can also employ here the same graphical representations. The work done can be regarded as the area of a rectangle $A\ B\ C\ D$, Fig. 86, whose base $A\ B$ is the space (s) described and whose height is either the constant force P or the mean value of the different forces. In general, however, the work done can be represented by the area of a figure $A\ B\ C\ N\ D$, Fig. 87, the base of which is the space s described, and the height of which above each point of the base is equal to the force corresponding to that point of the path. If we transform the figure $A\ B\ C\ N\ D$ in a rectangle $A\ B\ E\ F$ with the same base and the same area, its altitude $A\ F = B\ E$ gives the mean value of the force.

FIG. 86.

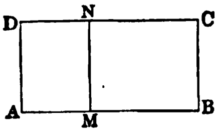
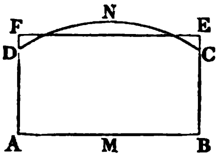
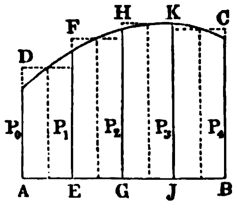


FIG. 87.



§ **73.** Arithmetic and Geometry give several different methods for finding the mean value of a continuous succession of quantities, the most important of which are to be found in the *Ingenieur*. The method known as *Simpson's Rule* is, however, the one most generally employed in practice, because in many cases it unites great simplicity with a high degree of accuracy.

FIG. 88.



In every case it is necessary to divide the space $A\ B = s$ (Fig. 88), in n (as many as possible) equal parts, such as $A\ E = E\ G = G\ J$, etc., and to determine the forces $E\ F = P_1$, $G\ H = P_2$, $J\ K = P_3$, etc., at the ends of these portions of the path. If we put the initial force $A\ D = P_0$ and the final one $B\ C = P_n$ we have the *mean force* $P = (\frac{1}{2}\ P_0 + P_1 + P_2 + P_3 + \ldots + P_{n-1} + \frac{1}{2}\ P_n) : n$, and consequently its work

$$P s = (\tfrac{1}{2} P_0 + P_1 + P_2 + \ldots + P_{n-1} + \tfrac{1}{2} P_n) \frac{s}{n}.$$

If the number of parts (n) be even, I.E. . 2, 4, 6, 8, etc., Simpson's Rule gives more exactly the mean force

$$P = (P_0 + 4\ P_1 + 2\ P_2 + 4\ P_3 + \ldots + 4\ P_{n-1} + P_n) : 3n,$$

whence the corresponding work done is

$$P\,s = (P_0 + 4\ P_1 + 2\ P_2 + 4\ P_3 + \ldots + 4\ P_{n-1} + P_n)\frac{s}{3\,n}.$$

If n is an uneven number, we can put

$$P\,s = [\tfrac{3}{8}(P_0 + 3\ P_1 + 3\ P_2 + P_3) + \tfrac{1}{3}(P_3 + 4\ P_4 + 2\ P_5 + \ldots . + 4\ P_{n-1} + P_n)]\ \frac{s}{n}.$$

(See Art. 38 of the Introduction to the Calculus.)

EXAMPLE.—In order to determine the work done by a horse, in drawing a wagon along a road, we employ a dynamometer (or force measurer), one side of which is attached to the wagon and the other to the horse, and we observe from time to time the intensity of the force. If the initial force is $P = 110$ pounds, that after moving 25 feet 122 pounds, that after 50 feet 127 pounds, that after 75 feet 120 pounds, and that at the end of the whole distance, 100 feet, 114 pounds, we have for the mean value of the force according to the first formula

$$P = (\tfrac{1}{2} . 110 + 122 + 127 + 120 + \tfrac{1}{2} \times 114) : 4 = 120{,}25 \text{ pounds},$$

and for the mechanical effect

$$P\,s = 120{,}25 \times 100 = 12025 \text{ foot-pounds.}$$

According to the second formula we have

$$P = (110 + 4 . 122 + 2 . 127 + 4 . 120 + 114) : (3 . 4) = \frac{1446}{12} = 120{,}5 \text{ pounds},$$

and the mechanical effect

$$P\,s = 120{,}5 . 100 = 12050 \text{ foot-pounds.}$$

§ 74. Principle of the Vis Viva or Living Forces.—If in the formula $s = \frac{v^2 - c^2}{2\,p}$ or $p\,s = \frac{v^2 - c^2}{2}$, found in § 14, we substitute for p its value $\frac{P}{G}\,g$, we obtain the mechanical effect $A = P\,s = \left(\frac{v^2 - c^2}{2\,g}\right)G$, or designating the heights due to the velocities $\frac{v^2}{2\,g}$ and $\frac{c^2}{2\,g}$ by h and k,

$$P\,s = (h - k)\ G.$$

This equation, so important in practical mechanics, means that the mechanical effect ($P\,s$), which a mass absorbs when its velocity changes from a lesser to a greater, or that which it gives out, when its velocity is forced to change from a greater to a less, is always equal to the product of the weight of the mass into the difference of the heights due to the different velocities $\left(\frac{v^2}{2g} - \frac{c^2}{2g}\right)$.

EXAMPLE.—1. In order to impart, upon a perfectly smooth railroad, a velocity of 30 feet to a wagon weighing 4000 pounds, the work to be done is $Ps = \frac{v^2}{2g}G = 0{,}0155\, v^2\, G = 0{,}0155 \times 900 \times 4000 = 55800$ pounds, and this wagon will perform the same amount of work if a resistance be opposed to it, so as to cause it gradually to come to rest. 2. Another wagon, weighing 6000 pounds and moving with a velocity of 15 feet, acquires in consequence of the action of a force a velocity of 24 feet; how much mechanical effect is stored by the wagon, or how much work is performed by the force?

The heights due to the velocities 15 and 24 feet are $k = \frac{c^2}{2g} = 3{,}487$ and $h = \frac{v^2}{2g} = 8{,}928$ feet. Consequently the work done $P\, s = (h - k)\, G = (8{,}928 - 3{,}487) \times 6000 = 5{,}441 \times 6000 = 32646$ foot-pounds.

If the space described is known the force can be found, and if the force is known the space can be found. Let us suppose, E.G., in the last case, that the space described by the wagon, while the velocity changes from 11 to 24 feet, is but 100 feet, we have then the force $P = (h - k)\frac{G}{s} = \frac{32646}{100} = 326{,}46$ pounds. If, however, the force was 2000 pounds, the space would be $s = (h - k)\frac{G}{P} = \frac{32646}{2000} = 16{,}323$ feet. 3. If a sled weighing 500 pounds, and moving with a velocity of 16 feet, loses in consequence of the friction the whole of its velocity while describing 100 feet, the resistance of the friction is

$$P = \frac{h \times G}{s} = 0{,}0155 \times 16^2 \times \frac{500}{100} = 0{,}0155 \times 256 \times 5 = 19{,}84 \text{ pounds.}$$

§ **75.** The formula for the work done, found in the preceding paragraph,

$$P\, s = \left(\frac{v^2 - c^2}{2g}\right) G = (h - k)\, G,$$

holds good not only when the forces are constant, but also when they are variable, if we substitute (according to § 73) instead of P the mean value of the force; for according to III*), in § 19, we have, in general, for every continuous motion

$$\frac{v^2 - c^2}{2} = p\, s,$$

in which $p = \frac{p_1 + p_2 + \ldots + p_n}{n}$ denotes the mean acceleration with which the space s is described, and we have also

$$p = \frac{P_1 + P_2 + \,.\,.\, P_n}{n\,M}, \text{ whence}$$

$$\left(\frac{v^2 - c^2}{2}\right) M = \left(\frac{P_1 + P_2 + \ldots + P_n}{M}\right) s \text{ and}$$

$$P\,s = \left(\frac{v^2 - c^2}{2}\right) M = \frac{v^2 - c^2}{2\,g}\, G = (h - k)\, G,$$

in which $P = \frac{P_1 + \ldots + P_n}{n}$ denotes the mean of all the forces measured after the spaces $\frac{s}{n}, \frac{2s}{n}, \frac{3s}{n} \ldots \frac{n\,s}{n}$ are described.

The force P can also be calculated by means of one of the formulas of § 73, when the number n of the parts is not assumed to be very great.

We are very often required to calculate the change of velocity that a given mass M undergoes, when a given amount of mechanical effect $P\,s$ is imparted to it. The principal equation which we have found is then to be employed in the following form

$$h = k + \frac{P.s}{G} \text{ or } v = \sqrt{c^2 + 2\,g\,\frac{P\,s}{G}}.$$

If we have calculated by means of this formula the velocities $v_1, v_2 \ldots v_n$ which correspond to the spaces $\frac{s}{n}, \frac{2s}{n}, \frac{3s}{n} \ldots s$, we can calculate by means of the formula

$$t = \frac{s}{n}\left(\frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3} + \ldots + \frac{1}{v_n}\right)$$

the time in which the space s is described.

In the form $G = M g = \frac{2\,P\,s}{v^2 - c^2} = \frac{P\,s}{\frac{1}{2}\,(v + c)\,(v - c)}$ the principal formula we have found serves to determine the mass M, which in consequence of the mechanical effect $P\,s$ imparted to it will undergo a change of velocity $v - c$.

When the motion of a body is continuous and the final velocity v is equal to the initial one c, then the work done becomes $= 0$, I.E, the accelerated part of the motion absorbs exactly as much work as the retarding portion gives out.

EXAMPLE.—If a wagon weighing 2500 pounds, moving without friction with an initial velocity of 10 feet, has imparted to it a mechanical effect of 8000 foot-pounds, what is its final velocity?

Here $v = \sqrt{10^2 + 64,4 \cdot \frac{8000}{2500}} = \sqrt{100 + 206,08} = 17,49$ feet,

REMARK.—We call, without attaching any particular idea to the term, the product of the mass $M = \frac{G}{g}$ into the square of the velocity (v^2), that is $M v^2$, the *vis viva* (Fr. force vive, Ger. lebendige Kraft) of the moving mass, and we can therefore put the mechanical effect, which a mass which is moved absorbs, equal to the *half* of its *vis viva*. If an inert mass passes from a velocity c to another v, then the work gained or lost is equal to the half difference of the vis viva at the beginning and of that at the end of the change of velocity. This law of the mechanical effect bodies produce by virtue of their inertia is called the principle of vis viva (Fr. principe des forces vives, Ger. Princip der lebendigen Kräfte).

§ 76. Composition of Forces.—If two forces P_1 and P_2 act upon the same body 1) in the same or 2) in opposite directions, then their effect is the same as when a single force equal to 1) the sum or 2) the difference of these forces acted upon the body; for these forces impart to the mass the accelerations

$$p_1 = \frac{P_1}{M} \text{ and } p_2 = \frac{P_2}{M};$$

whence, according to § 28, the resulting acceleration is

$$p = p_1 \pm p_2 = \frac{P_1 \pm P_2}{M},$$

and consequently the corresponding force is

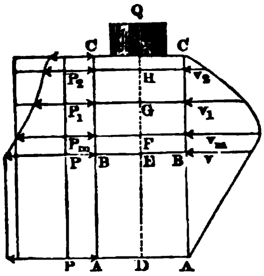
$$P = M p = P_1 \pm P_2.$$

We call the force P derived from the two forces and capable of producing the same effect (equipollent) their *resultant* (Fr. résultante, Ger. Resultirende), and its constituents P_1 and P_2 its *components* (Fr. composantes, Ger. Componenten).

EXAMPLE.—1) A body lying upon the flat of the hand presses with its absolute weight on it only so long as the hand is at rest, or is moved with the body uniformly up or down; but if we lift the hand with an accelerated motion, it experiences a heavier pressure; and if, on the contrary, we allow it to sink with an accelerated motion, then the pressure becomes less than the weight, and even $= 0$ when the hand is lowered with an acceleration equal to that of gravity. If the pressure on the hand is P, then the body falls with the force $G - P$ only, if its mass is $M = \frac{G}{g}$; if we put the acceleration with which the hand descends $= p$ we have $G - P = \frac{G}{g} p$, and therefore the pressure $P = G - \frac{p}{g} G = \left(1 - \frac{p}{g}\right) G$. If, on the contrary, we raise the body upon the hand with an acceleration p, then the acceleration p is opposite to the acceleration g, and the pressure becomes $P = \left(1 + \frac{p}{g}\right) G$. According as we lower or raise a body with an acceleration of 20 feet, the pressure upon the hand is $\left(1 - \frac{20}{32,2}\right) G = (1 - 0,62)\, G =$ 0,38 times the weight of the body, or $1 + 0,62 = 1,62$ times the same weight. 2) If with the flat of the hand I throw a body weighing 3 pounds 14 feet vertically upward, by urging it on continuously during the first two feet, then the work done is $P s = G h = 3 . 14 = 42$ pounds, and the pressure of the body on the hand is $P = \frac{42}{2} = 21$ pounds. Hence the body when at rest presses with a weight of three pounds upon the hand, and, on the contrary, during the act of throwing it, it reacts with a force of 21 pounds upon the hand.

3) What load Q can a piston movable in a cylinder $A\, A\, C\, C$, Fig. 89, raise to the height $D\, K = s = 6$ feet, if during the first half of its course the air which flows in from a very large reservoir acts upon it with a force of 6000 pounds, and if during the second half of its course this air enclosed in the cylinder expands according to the law of Mariotte, while the exterior air acts with a constant pressure of 2000 pounds in the opposite direction. Since the air shut in the cylinder at the end of the second half of the course of the piston has expanded to double its volume, the pressure of the same upon the piston at the end of the course is only $\frac{1}{2} . P = 3000$ pounds. The air inclosed in the cylinder, when the piston has traveled 3 feet, presses with a force of 6000 pounds upon it, on the contrary at the end of four feet with a force of $\frac{3}{4} . 6000 = 4500$ pounds, at the end of 5 feet with $\frac{3}{5} . 6000 = 3600$ pounds, and at the end of the entire course with a force of $\frac{1}{2} . 6000 = 3000$ pounds. Hence the mean force during the expansion $= \frac{1}{8}\, [6000 + 3\,(4500 + 3600) + 3000] = \frac{33300}{8} = 4162$ pounds, and consequently the mean force during the whole of the course of the piston is $= \frac{6000 + 4162}{2} = 5081$ pounds. If we subtract the constant opposing force of 2000 pounds from this, it follows that the weight to be raised by the piston is

FIG. 89.



$$Q = 5081 - 2000 = 3081 \text{ pounds.}$$

The motive force for the first half of the course is then $P - (Q + 2,000) = 6000 - 5081 = 919$ pounds, and consequently the acceleration of the motion is $p = \left(\frac{P - (Q + 2000)}{Q}\right) g = \frac{919}{3081} . 32,2 = 9,6$ feet, and the velocity at the end of the first half of the course of the piston $s_1 = \frac{s}{2}$ $= 3$ feet is $v = \sqrt{2ps_1} = \sqrt{6.9,6} = \sqrt{57,6} = 7,589$ feet, and the time in which this space is described by the piston is $t_1 = \frac{2s_1}{v} = \frac{6}{7,589} = 0,790$ seconds. The distance, which has been traveled by the piston when the force and the load balance each other, that is, when the motive force and consequently the acceleration is $= 0$, and the velocity of the piston is a maximum, is

$$x = \left(\frac{P}{Q + 2000}\right)\frac{s}{2} = \frac{6000 \,.\, 3}{5081} = 3,543 \text{ feet.}$$

When the distance $\frac{6,543}{2} = 3,2715$ feet has been described, the force acting on the inside piston is $\frac{6000 \,.\, 3}{3,2715} = 5502$, and consequently the motive force is $= 5502 - 5081 = 421$ pounds, and the mean value of the same while the piston passes from 3 to 3,543 feet is $\frac{919 + 4 \,.\, 421 + 0}{6} = 434$ pounds. The corresponding mean acceleration is $= \frac{434}{3081}\, g = \frac{434 \,.\, 32,2}{3081}$ $= 4,535$ feet, and consequently the maximum velocity of the piston at the end of the space $x = s_1 + s_2 = 3,543$ feet is

$v_m = \sqrt{v^2 + 2ps_2} = \sqrt{57,6 + 2 \times 4,535 \times 0,543} = \sqrt{62,525} = 7,907$ feet.

The time required to describe the space $s_2 = 0,543$ can be put

$$= t_2 = \frac{s_2}{2}\left(\frac{1}{v} + \frac{1}{v_m}\right) = 0,2715\left(\frac{1}{7,589} + \frac{1}{7,907}\right) = 0,070 \text{ seconds.}$$

If the piston has described the space 5,5 the motive force is $\frac{18000}{5,500} -$ $5081 = -1808$ pounds, and if the piston is midway between this point and the point of maximum velocity, this force is then $= \frac{18000}{4,5215} - 5081$ $= -1100$ pounds, and the corresponding accelerations are $= -\frac{1808 \times 32,2}{3081}$ $= -18,89$ feet, and $= -\frac{1100 \times 32,2}{3081} = -11,49$ feet.

The mean acceleration while the piston describes the portion of the space $5,500 - 3,543 = 1,957$ feet is consequently $= -\frac{0 + 4 \times 11,49 + 18,89}{6}$ $= -10,81$ feet, and therefore the velocity acquired at the end of this space is $= \sqrt{62,525 - 2 \times 10,81 \times 1,957} = \sqrt{20,215} = 4,496$ feet. On the contrary, during the first half of the last portion of the course, the mean acceleration is $= -\frac{0 + 11,49}{2} = -5,745$ feet, and therefore the velocity at the end of the space 4,5215 feet $v_1 = \sqrt{62,525 - 2 \times 5,745 \times 0,9785} = \sqrt{51,282} =$ 7,161 feet, and we have for the time required to describe the space $s_3 =$

1,957, $t_3 = \frac{s_3}{6}\left(\frac{1}{v_m} + \frac{4}{v_1} + \frac{1}{v_2}\right) = 0,326\left(\frac{1}{7,907} + \frac{4}{7,161} + \frac{1}{4,496}\right) = 0,326$ $\times$ $0,9075 = 0,296$ seconds. Finally, we can put the time during which the last portion $s_4 = 0,5$ of the whole course is described $t_4 = \frac{2\,s_4}{v_2} = \frac{1}{4,496}$ $= 0,2224$ seconds, and the time required by the piston to describe its entire course $t = t_1 + t_2 + t_3 + t_4 = 0,790 + 0,070 + 0,296 + 0,2224 = 1,378$ *seconds.*

§ 77. **Parallelogram of Forces.**—If a mass (a material point) M, Fig. 90, is acted upon by two forces, P_1 and P_2, whose direction, $M\,X$ and $M\,Y$, form an angle $X\,M\,Y = a$ with each other, the forces cause in these directions the accelerations

$$p_1 = \frac{P_1}{M} \text{ and } p_2 = \frac{P_2}{M},$$

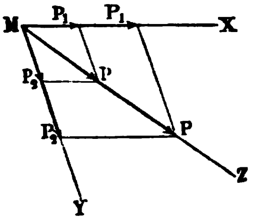
and by combining them, a resulting acceleration (§ 35) in the direction $M\,Z$, which is determined by the diagonal of a parallelogram constructed with p_1, p_2, and a, is obtained; this resulting acceleration is

$$p = \sqrt{p_1^2 + p_2^2 + 2\,p_1\,p_2\,cos.\,a},$$

and we have for the angle ϕ, which its direction makes with the direction $M\,X$ of the acceleration p_1

$$sin.\,\phi = \frac{p_2\,sin.\,a}{p}.$$

FIG. 90.



Substituting in these two formulas the given values of p_1 and p_2, we obtain

$$p = \sqrt{\left(\frac{P_1}{M}\right)^2 + \left(\frac{P_2}{M}\right)^2 + 2\left(\frac{P_1}{M}\right)\left(\frac{P_2}{M}\right) cos.\,a} \text{ and}$$

$$sin.\,\phi = \left(\frac{P_2}{M}\right)\frac{sin.\,a}{p},$$

and multiplying the first equation by M, we have

$$M\,p = \sqrt{P_1^2 + P_2^2 + 2\,P_1\,P_2\,cos.\,a},$$

or since $M\,p$ is the force P corresponding to the acceleration p, we find

$$1)\ P = \sqrt{P_1^2 + P_2^2 + 2\,P_1\,P_2\,cos.\,a},$$

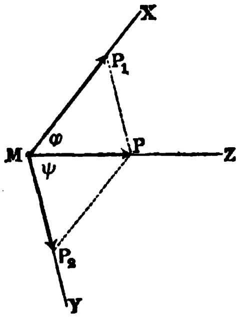
$$2)\ sin.\,\phi = \frac{P_2\,sin.\,a}{P}$$

The resultant or mean force is determined in magnitude and direction from the component forces in exactly the same manner, as the resulting acceleration is determined from the component accelerations.

If we represent the forces by right lines, making the ratio of their length the same as that of the weights (E.G. pounds) to each other, the resultant can then be represented by the diagonal of the parallelogram whose sides are formed by the component forces, and one angle of which is equal to the angle formed by the component forces with each other. The parallelogram thus constructed with the component forces, the diagonal of which represents the resultant, is called the *parallelogram of forces.*

EXAMPLE.—If a body, Fig. 91, weighing 150 pounds and resting on a perfectly smooth table, is acted on by two forces $P_1 = 30$ pounds, and $P_2 = 24$ pounds, which form with each other an angle $P_1 M P_2 = a = 105°$, in what direction and with what acceleration will the motion take place? Since $cos.\ a = cos.\ 105° = - cos.\ 75°$, we have the resultant

FIG. 91.



$$P = \sqrt{30^2 + 24^2 - 2 \times 24 \times 30 \times cos.\ 75°}$$
$$= \sqrt{900 + 576 - 1440\ cos.\ 75°}$$
$$= \sqrt{1476 - 372{,}7} = 33{,}22 \text{ pounds;}$$

and the corresponding acceleration

$$p = \frac{P}{M} = \frac{Pg}{G} = \frac{33{,}22 \times 32{,}2}{150} = 7{,}13 \text{ feet.}$$

The direction of the motion forms an angle ϕ with the direction of the first force, which is determined by the following formula

$$sin.\ \phi = \frac{24}{33{,}22}\ sin.\ 105° = 0{,}7224\ sin.\ 75° = 0{,}6978;$$

and ϕ is $= 44°\ 15'$.

REMARK.—The resultant (P) depends (according to the formula just found) upon the components alone, and not upon the mass (M) of the body upon which the forces act. For this reason we find in many works upon mechanics the correctness of the parallelogram of forces demonstrated without reference to the mass, but with the assumption of some one of the fundamental laws of statics. Such pure statical demonstrations are numerous. In each of the following works we find a different one: "Eytelwein's Handbuch der Statik fester Körper;" "Gerstner's Handbuch der Mechanik;" "Kayser's Handbuch der Statik;" "Mobius' Lehrbuch der Statik;" "Rühlman's Technische Mechanik." The demonstration in Gerstner's "Mechanik" is based upon the theory of the lever; it is really very simple, and is to be found in old, and also in later works, E.G., in those of Kästner, Monge, Whewell, etc. Kayser's demonstration is that of Poisson in elementary shape. Mobius' discussion of it is based upon a particular theory of couples (des couples) introduced by Poisson (Elements de Statique). A peculiar demonstration is given by Duchayla in the Correspondence sur l'école polytechnique No. 4, which is reproduced by Brix in his Lehrbuch der Statik fester Körper. It is also given in many other works, E.G., in Moseley's Mechanical Principles, etc. The demonstration of the parallelogram of forces given by Navier in his "Leçons de Mécanique" (German by Meier, 1851) is also to be found in Rühlmann's "Grundzüge der Mechanik," Leipzig, 1860. A theory of this parallelogram, founded on the laws of motion, is to be found in Newton's "Principia." It is also employed in many later works, I.E., by Venturoli, Poncelet, Burg, etc. See "Elementi di Mecanica e d'Idraulica di Venturoli," "Mecanique industrielle par Poncelet," "Compendium der populären Mechanik and Machinenlehre von Burg." A new demonstration by Mobius is to be found in the Berichten der Gesellshaft der Wissenshaften zu Leipzig (1850), another by Ettingshausen in the papers of the Academy of Vienna (1851), and a third, by Schlömich in his "Zeitschrift für Mathematik and Physik" (1857).

§ 78. Decomposition of Forces.—With the aid of the parallelogram of forces we can not only combine two or more forces so as to find a single resultant, but also decompose a given force, under given circumstances, into two or more forces. If the angles ϕ and ψ, which the components $M P_1 = P_1$ and $M P_2 = P_2$, Fig. 91, make with the given force $M P = P$ are given, then the components are determined by the following formulas

$$P_1 = \frac{P \sin. \psi}{\sin. (\phi + \psi)}, \; P_2 = \frac{P \sin. \phi}{\sin. (\phi + \psi)}.$$

If the components are at right angles, then $\phi + \psi = 90°$ and $\sin. (\phi + \psi) = 1$, and we have

$$P_1 = P \cos. \phi \text{ and } P_2 = P \sin. \phi.$$

and if, finally, ψ and ϕ are equal, we have

$$P_2 = P_1 = \frac{P. \sin. \phi}{\sin. 2 \phi} = \frac{P}{2 \cos. \phi}$$

EXAMPLE.—1) How heavily will a table $A B$, Fig. 92, be pressed by a body M whose weight is $G = 70$ pounds, and which acted on by a force $P = 50$ pounds, which is inclined to the horizon at an angle $P M P_1 = \phi = 40°$? The horizontal component is

$$P_1 = P \cos. \phi = 50 \cos. 40° = 38{,}30 \text{ pounds},$$

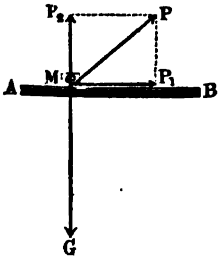
and the vertical component

$$P_2 = P \sin. \phi = 50 \sin. 40° = 32{,}14 \text{ pounds}.$$

The latter tends to raise the body from the table, and consequently the pressure on the table is

$$G - P_2 = 70 - 32{,}14 = 37{,}86 \text{ pounds}.$$

FIG. 92.



2) If a body M, Fig. 91, weighing 110 pounds, is moved upon a horizontal support by two forces, so that in the first second it describes a distance of 6,5 feet in a direction, which forms with the two directions of the forces

the angles $\phi = 52^\circ$ and $\psi = 77^\circ$, the forces can be found as follows: The acceleration is double the space described in the first second, or in this case $p = 2 \,.\, 6{,}5 = 13$ feet, and the resultant is

$$P = \frac{p\,G}{g} = 0{,}031 \,.\, 13 \,.\, 110 = 44{,}33 \text{ pounds.}$$

Hence one of the components is

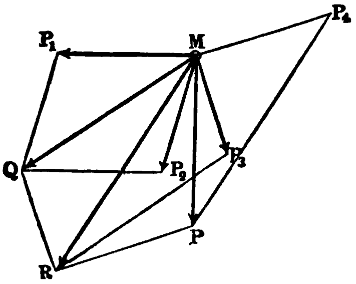
$$P_1 = \frac{P \, sin.\, 77^\circ}{sin.\, (52^\circ + 77^\circ)} = \frac{44{,}33 \, sin.\, 77^\circ}{sin.\, 51^\circ} = 55{,}58 \text{ pounds,}$$

and the other is

$$P_2 = \frac{44{,}33 \, sin.\, 52^\circ}{sin.\, 51^\circ} = 44{,}95 \text{ pounds.}$$

§ 79. Composition of Forces in a Plane.—In order to find the resultant P of a number of component forces P_1, P_2, P_3, etc. (Fig. 93), we can pursue exactly the same method that we employed in the composition of velocities. We can, by employing repeatedly the parallelogram of forces, combine the forces two by two so as to form one, until but one is left. The force P_1 and P_2 give, E.G., by means of the parallelogram $M\,P_1\,Q\,P_2$, the resultant $M\,Q = Q$; and if we combine this with P_3 we obtain, by means of the parallelogram $M\,Q\,R\,P_3$ the resultant $M\,R = R$, and the latter, combined with P_4, gives, by means of the diagonal $M\,P = P$, the resultant of all four forces P_1, P_2, P_3, and P_4. It is not necessary, when combining these forces, to complete the parallelograms and to find their diagonals. We have but to construct a polygon $M\,P_1\,Q\,R\,P$ by drawing its sides $M\,P_1$, $P_1\,Q$, $Q\,R$, $R\,P$, equal and parallel to the given components P_1, P_2, P_3, P_4. The last side $M\,P$, which closes the parallelogram, is the resultant required, or rather the measure of the same.

FIG. 93.



REMARK.—The solution of mechanical problems by construction is very useful. Although the results are not as accurate as those obtained by calculation, yet they are of great value as checks against gross errors, and can therefore always be employed as proofs of calculations. In Fig. 93 we have drawn the forces as meeting each other and forming the given angles $P_1\,M\,P_2 = 72^\circ\,30'$, $P_2\,M\,P_3 = 33^\circ\,20'$, and $P_3\,M\,P_4 = 92^\circ\,40'$; and their length is such, that a pound is represented by a line or $\frac{1}{12}$ of a

Prussian inch. The forces $P_1 = 11{,}5$ pounds, $P_2 = 10{,}8$ pounds, $P_3 = 8{,}5$ pounds, and $P_4 = 12{,}2$, are therefore expressed by sides 11,5 lines, 10,8 lines, 8,5 lines, and 12,2 lines long. A careful construction of the polygon of forces gives the value of the resultant $P = 14{,}6$ pounds, and the angle formed by the direction $M\,P$ with the direction $M\,P_1$ of the first force $a = 86\frac{1}{2}°$.

§ **80.** We can determine the resultant P more simply by decomposing each of the components P_1, P_2, P_3, etc., into two components Q_1 and R_1, Q_2 and R_2, Q_3 and R_3, etc., in the direction of the rectangular axes $X\,X$ and $Y\,Y$, Fig. 94, by then adding algebraically the forces which lie in the same axis, and by seeking the intensity and direction of the resultant of the two forces which have been thus obtained, and whose directions are at right angles to each other. If the angles $P_1\,M\,X$, $P_2\,M\,X$, $P_3\,M\,X$, etc., P_1, P_2, P_3, etc., form with the axis of X are $= a_1$, a_2, a_3, etc., we have the components $Q_1 = P_1 \cos. a_1$, $R_1 = P_1 \sin. a_1$; $Q_2 = P_2 \cos. a_2$, $R_2 = P_2 \sin. a_2$, etc.; whence it follows from the equation

$$Q = Q_1 + Q_2 + Q_3 + \ldots, \text{ that}$$

1) $Q = P_1 \cos. a_1 + P_2 \cos. a_2 + P_3 \cos. a_3 + \ldots,$

and also from $R = R_1 + R_2 + R_3 \ldots$, that

2) $R = P_1 \sin. a_1 + P_2 \sin. a_2 + P_3 \sin. a_3 + \ldots$

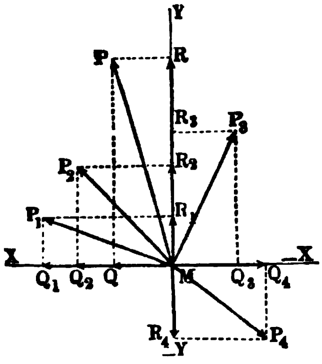
We find the value of the resultant of the two components Q and R, just obtained, by the aid of the formula

3) $P = \sqrt{Q^2 + R^2},$

and that of the angle $P\,M\,X = a$, formed by its direction with the axis $X\,\overline{X}$, by means of the formula

4) $tang.\ a = \dfrac{R}{Q}.$

FIG. 94.



In adding algebraically the forces we must pay particular attention to their signs; for if they are different for two different forces, I.E. if these forces act in opposite directions from the point of application, then this addition becomes an arithmetical subtraction. The angle a is acute as long as R and Q are positive; it is between 90°—180°, when Q is negative and R positive; it is between 180°—270°, when Q and R are both negative, and is finally between 270°—360°, when R alone is negative.

EXAMPLE.—What is the direction and intensity of the resultant of the forces $P_1 = 30$ pounds, $P_2 = 70$ pounds, and $P_3 = 50$ pounds, whose directions lie in the same plane and form the angles $P_1 M P_2 = 56°$ and $P_2 M P_3 = 104°$ with each other? If we lay the axis $X X$, Fig. 94, in the direction of the first force, we obtain $a_1 = 0°$, $a_2 = 56°$, and $a_3 = 56° + 104° = 160°$; hence

1) $Q = 30 \,.\, cos.\, 0° + 70 \,.\, cos.\, 56° + 50\, cos.\, 160° = 30 + 39,14 - 46,98$
$= 22,16$ pounds,

2) $R = 30 \,.\, sin.\, 0° + 70 \,.\, sin.\, 56° + 50 \,.\, sin.\, 160° = 0 + 58,03 + 17,10$
$= 75,13$ pounds, and

3) $tang.\, a = \dfrac{75,13}{22,16} = 3,3903,$

and therefore the angle formed by the resultant with the positive portion of the axis $M X$ is $a = 73° 34'$, and the resultant itself is $P = \sqrt{Q^2 + R^2} = \dfrac{Q}{cos.\, a} = \dfrac{R}{sin.\, a} = \dfrac{75,13}{sin.\, 73° 34'} = \dfrac{75,13}{0,9591} = 78,33$ pounds.

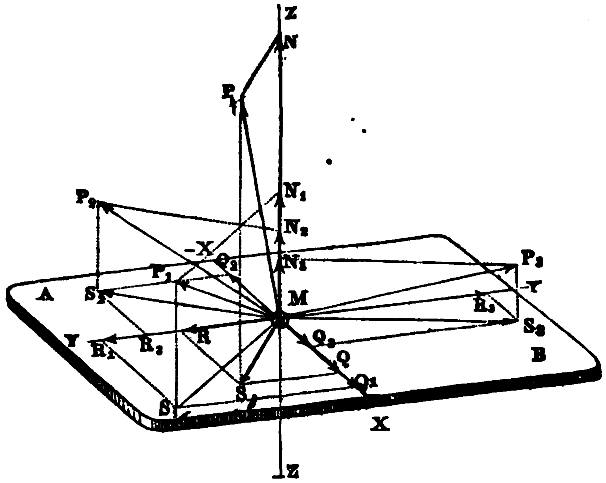
§ 81. Forces in Space.—If the direction of the forces do not lie in the same plane, we pass a plane through the point of application and decompose the forces into two others, one of which lies in the plane, and the other at right angles to it. The components thus obtained, which lie in the plane, are combined according to the rule given in the last paragraph, so as to give a single resultant, and those at right angles to the plane give, by simple addition, another resultant. From these two components, which are at right angles to each other, we find the resultant according to the well-known rule (§ 77).

This method of proceeding is graphically represented in Fig. 95. $\overline{M P_1} = P_1$, $\overline{M P_2} = P_2$, $\overline{M P_3} = P_3$ are the simple forces, $A\, B$ is the plane (plane of projection) and $Z Z$ is the axis at right angles to it. From the decomposition of the forces $P_1\, P_2$, etc., we obtain the forces $S_1\, S_2$, etc., in the plane, and the forces $N_1\, N_2$, etc., along the normal $Z\, Z$. The former are again decomposed into the components Q_1, Q_2, etc., R_1, R_2, etc., which, by addition, give the resultants Q and R, from which, as components, we determine the resultant S, which, combined with the sum of all the normal forces $N_1\, N_2$, etc., gives the required resultant P.

If we put the angles of inclination of the directions of the forces to the plane equal to β_1, β_2, etc., we obtain for the forces in the plane $S_1 = P_1\, cos.\, \beta_1$, $S_2 = P_2\, cos.\, \beta_2$, etc., and for the normal forces $N_1 = P_1\, sin.\, \beta_1$, $N_2 = P_2\, sin.\, \beta_2$, etc. Designating the angles which the projections of the directions of the forces in the plane

A B form with the axis *X X* by a_1, a_2, etc., that is, putting $S_1 M X = a_1, S_2 M X = a_2$, etc., we obtain the following three forces, which

FIG. 95.



form the edges of a rectangular parallelopipedon (parallelopipedon of forces):

$Q = S_1 cos.\ a_1 + S_2\ cos.\ a_2 + \ldots$, or

1) $Q = P_1\ cos.\ \beta_1\ cos.\ a_1 + P_2\ cos.\ \beta_2\ cos.\ a_2 + \ldots$,

2) $R = P_1\ cos.\ \beta_1\ sin.\ a_1 + P_2\ cos.\ \beta_2\ sin.\ a_2 \ldots$ and

3) $N = P_1\ sin.\ \beta_1 + P_2\ sin.\ \beta_2 + \ldots$

From these three forces we obtain the final resultant

4) $P = \sqrt{Q^2 + R^2 + N^2}$,

and its angle $P\ M\ S = \beta$ of inclination to the plane of projection by the aid of the formula

5) $tang.\ \beta = \dfrac{N}{S} = \dfrac{N}{\sqrt{Q^2 + R^2}}$.

Finally, the angle $S\ M\ X = a$, which the projection of the resultant in the plane *A B* forms with first axis *X X*, is given by the formula

6) $tang.\ a = \dfrac{R}{Q}$.

If λ_1, λ_2, etc., are the angles formed by the forces P_1, P_2 with the axis *M X*, μ_1, μ_2 . . ., the angles formed by them with the axis *M Y* and ν_1, ν_2, etc., the angles formed by them with the axis *M Z*, we have also

1*) $Q = P_1 \cos. \lambda_1 + P_2 \cos. \lambda_2 + \ldots$,

2*) $R = P_1 \cos. \mu_1 + P_2 \cos. \mu_2 + \ldots$ and

3*) $N = P_1 \cos. \nu_1 + P_2 \cos. \nu_2 + \ldots$

The value of the resultant is given by the formula

4*) $P = \sqrt{Q^2 + R^2 + N^2}$,

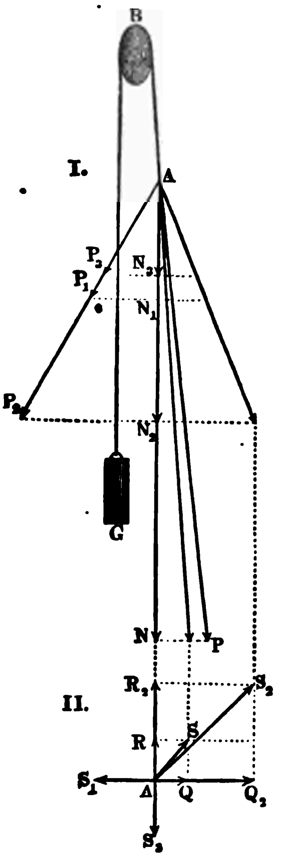
and the direction of the same by the formulas

5*) $\cos. \lambda = \frac{Q}{P}, \cos. \mu = \frac{R}{P}, \cos. \nu = \frac{N}{P}$,

in which λ, μ and ν denote the angles formed by P with the axes MX, MY, MZ.

We have also $\cos. \lambda = \cos. a \cos. \beta$, $\cos. \mu = \sin. a \cos. \beta$, and $\nu = 90° - \beta$, or $\cos. \nu = \sin. \beta$.

FIG. 96.



EXAMPLE.—In order to raise vertically a weight G, Fig. 96, I and II, by means of a rope passing over a fixed pully, three workmen pull at the end of the rope A with the forces $P_1 = 50$ pounds, $P_2 = 100$ pounds and $P_3 = 40$ pounds; the directions of these forces are inclined at an angle of 60° to the horizon, and form the horizontal angles $S_1 A S_2 = S_2 A S_3 = 135°$ and $S_3 A S_1 = 90°$ with each other. What is the intensity and direction of the resultant which we can put equal to the weight G, and how great could this weight be made, if the forces had the same direction?

The vertical components of the forces are $N_1 = P_1 \sin. \beta_1 = 50 \sin. 60° = 43,30$ pounds, $N_2 = P_2 \sin. \beta_2 = 100 \sin. 60° = 86,60$ pounds and $N_3 = P_3 \sin. \beta_3 = 40 \sin. 60° = 34,64$ pounds; consequently, the vertical force is $N = N_1 + N_2 + N_3 = 164,54$ pounds.

The horizontal components are

$S_1 = P_1 \cos. \beta_1 = 50 \cos. 60° = 25$ pounds,
$S_2 = P_2 \cos. \beta_2 = 100 \cos. 60° = 50$ pounds
and $S_3 = P_3 \cos. \beta_3 = 40 \cos. 60° = 20$ pounds.

If we pass an axis $X\overline{X}$ in the direction of the force S_1, we have for the component forces in this direction

$Q = Q_1 + Q_2 + Q_3 = S_1 \cos. a_1 + S_2 \cos. a_2 + S_3 \cos. a_3 = 25 \cos. 0° + 50 \cos. 135° + 20 \cos. 270° \doteq 25 . 1 - 50 . 0,7071 - 20 . 0 = 25 - 35,355 = - 10,355$ pounds, and for the component in the direction $Y\overline{Y}$

$R = R_1 + R_2 + R_3 = S_1 \; sin. \; a_1 + S_2 \; sin. \; a_2 + S_3 \; sin. \; a_3 = 25 \; sin. \; 0° +$ $50 \; sin. \; 135° + 20 \; sin. \; 270° = 50 \,.\, 0,7071 - 20 = 15,355$ pounds, and for the horizontal resultant

$$S = \sqrt{Q^2 + R^2} = \sqrt{10,355^2 + 15,355^2} = 18,520 \text{ pounds.}$$

The angle a, formed by this resultant with the axis $X \overline{X}$, is determined by the formula

$$tang. \; a = \frac{R}{Q} = - \frac{15,355}{10,355} = - 1,4828, \text{ whence } a = 180° - 56° = 124°.$$

The final resultant is

$$P = \sqrt{N^2 + S^2} = \sqrt{164,54^2 + 18,520^2} = 165,58 \text{ pounds.}$$

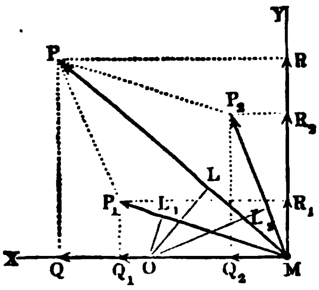
The angle of inclination of this force to the horizon is determined by the formula

$$tang. \; \beta = \frac{N}{S} = \frac{164,54}{18,520} = 8,8848, \text{ whence we have } \beta = 83° \; 35'.$$

If all the forces acted in the same direction, the resultant would be = $50 + 100 + 40 = 190$ pounds, or $190 - 165,58 = 24,42$ pounds greater than the one just found.

§ 82. Principle of Virtual Velocities.—From the foregoing rules for the composition of forces, two others can be deduced, which are of great importance in their practical applications. Let M, Fig. 97, be a material point, $\overline{MP_1} = P_1$ and $\overline{MP_2} = P_2$ the forces acting upon it, and $\overline{MP} = P$ the resultant of the forces P_1 and P_2. If we pass through M two axes $M X$ and $M Y$ at right angles to each other, and decompose the forces P_1 and P_2, as well as their resultant P, into their components in the direction of these axes, I.E., P_1 into Q_1, and R_1, P_2 in Q_2 and R_2 and P into Q and R, we obtain the forces in the direction of one axis Q_1, Q_2 and Q, and those in the direction of the other R, R_1 and R_2, and we have $Q = Q_1 + Q_2$ and $R = R_1 + R_2$. If from any point O in the axis $M X$ we let fall the perpendiculars $O L_1$, $O L_2$ and $O L$ upon the directions of the forces P_1, P_2 and P, we obtain the right-angled triangles $M O L_1$ $M O L_2$, and $M O L$, which are similar to the triangles formed by the three forces, viz.,

FIG. 97.



$$\Delta \; M O L_1 \backsim \Delta \; M P_1 \, Q_1,$$
$$\Delta \; M O L_2 \backsim \Delta \; M P_2 \, Q_2,$$
$$\Delta \; M O L \backsim \Delta \; M P \, Q.$$

In consequence of this similarity we have $\frac{M Q_1}{M P_1}$ I.E., $\frac{Q_1}{P_1} = \frac{M L_1}{M O}$, $\frac{Q_2}{P_2} = \frac{M L_2}{M O}$ and $\frac{Q}{P} = \frac{M L}{M O}$; substituting these values of Q_1, Q_2 and Q in the formula $Q = Q_1 + Q_2$, we obtain

$$P.\overline{ML} = P_1.\overline{ML_1} + P_2.\overline{ML_2}.$$

In like manner we have

$$\frac{R_1}{P_1} = \frac{O L_1}{M O}, \frac{R_2}{P_2} = \frac{O L_2}{M O} \text{ and } \frac{R}{P} = \frac{O L}{M O},$$

whence

$$P.\overline{OL} = P_1.\overline{OL_1} + P_2.\overline{OL_2}.$$

The formulas hold good, when P is the resultant of three or more forces P_1, P_2, P_3, etc., since we have, in general,

$$Q = Q_1 + Q_2 + Q_3 + \ldots$$
$$R = R_1 + R_2 + R_3 + \ldots$$

We can, therefore, put, in general,

1) $P.\overline{ML} = P_1.\overline{ML_1} + P_2.\overline{ML_2} + P_3.\overline{ML_3} + \ldots$,

2) $P.\overline{OL} = P_1.\overline{OL_1} + P_2.\overline{OL_2} + P_3.\overline{OL_3} + \ldots$

The resultant P of the forces P_1, P_2, P_3, etc., must correspond to both these equations, and they can therefore be employed to determine P.

The first of these two formulas can also be employed for a system of forces in space, N, Q, R, Fig. 95, since here we have also $N = N_1 + N_2 + N_3 + \ldots$, or

$P \cos. \nu = P_1 \cos. \nu_1 + P_2 \cos. \nu_2 + P_3 \cos. \nu_3 + \ldots$, and also

$P.\overline{MO} \cos. \nu = P_1.\overline{MO} \cos. \nu_1 + P_2 \overline{MO} \cos. \nu_2 + P_3 \overline{MO} \cos. \nu_3 + \ldots$

§ **83.** If the point of application M, Fig. 98 and Fig. 99, moves to O, or if we imagine the point of application moved forward

FIG. 98.

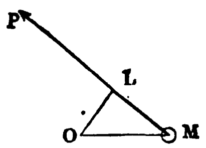
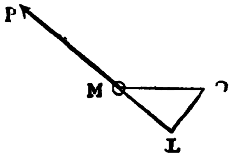


FIG. 99.



through the space $M O = x$, we call the projection $M L = s$ of this space x upon the direction of the force $M P$ the *space described* by the force P, and the product $P s$ of the force by the space *is the* *work done by the force.* If we substitute these quantities in the equation (1) of the last paragraph we obtain

$$P\,s = P_1\,s_1 + P_2\,s_2 + P_3\,s_3 + \ldots,$$

hence *the work done by the resultant is equal to the sum of the work done by the component forces.*

In adding the mechanical effects we must, as in adding the forces, pay attention to the signs of the same. If one of the forces Q_1, Q_2, of the foregoing paragraph, acts in the opposite direction to the others, then it must be introduced as negative quantity; this force, as for example, Q_3 in Fig. 94, § 80, is, however, a component of a force P_3 which, under the circumstances supposed in the foregoing paragraph, opposes the motion $M\,L_3$ of its point of application; we are, therefore, obliged to treat the force P, Fig. 99, which acts in opposition to the motion $M\,L$, as negative, if we consider the force P, Fig. 98, which acts in the direction of the motion $M\,L$, to be positive.

If the forces are variable, either in magnitude or in direction, then the formula

$$P\,s = P_1\,s_1 + P_2\,s_2 + P_3\,s_3 + \ldots$$

is correct only for an infinitely small space s, s_1, s_2, etc.

We call the infinitely small spaces σ_1, σ_2, σ_3, etc., described by the forces corresponding to the infinitely small space described by the material point, the *virtual velocities* (Fr. vitesses virtuelles, Ger. virtuelle Geschwindigkeiten) of the same, and the law corresponding to the formula $P\,\sigma = P_1\,\sigma_1 + P_2\,\sigma_2 + P_3\,\sigma_3$ is known as the *principle of virtual velocities.*

§ 84. Transmission of Mechanical Effect.—According to the principle of vis viva for a rectilinear motion the work ($P\,s$) done by a force (P), when the velocity c of a mass M is changed into a velocity v, is

$$P\,s = \left(\frac{v^2 - c^2}{2}\right) M.$$

Now if P is the resultant of the forces P_1, P_2, etc., which act on the mass M, and if the spaces described by them are s_1, s_2, etc., while the mass M describes the space s, we have, from the foregoing paragraph,

$$P\,s = P_1\,s_1 + P_2\,s_2 + \ldots,$$

from which we deduce the following general formula,

$$P_1\,s_1 + P_2\,s_2 + \ldots = \left(\frac{v^2 - c^2}{2}\right) M;$$

therefore the sum of the work done by the single forces is equal to half the increase of the vis viva of the mass.

If the velocity during the motion is constant, I.E., if $v = c$ and the motion itself is uniform, we have $v^2 - c^2 = 0$, and therefore there is neither gain nor loss of vis viva, whence

$$P_1 s_1 + P_2 s_2 + P_3 s_3 + \ldots = 0;$$

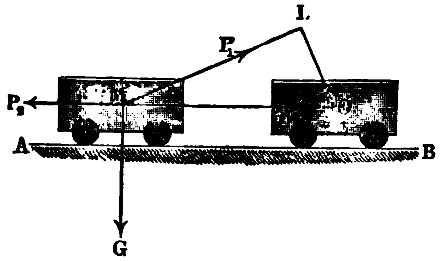
and the sum of the mechanical effects of the single forces is null.

If, on the contrary, the sum of the mechanical effects is null, then the forces do not change the motion of the body in the given direction; if the body has no motion in the given direction, it will not have any imparted to it in this direction by the action of the forces; if it had before a certain velocity in a given direction, it will retain the same.

If the forces are variable, the variable velocity v can, after a certain time, become the initial. This phenomena occurs in all periodic motions, which are very common in machinery. But $v = c$ gives the work done $\left(\frac{v^2 - c^2}{2}\right) M = 0$, and therefore the gain or loss of mechanical effect during a period of the motion is $= 0$.

EXAMPLE.—A wagon, Fig. 100, weighing $G = 5000$ pounds is moved forward on a horizontal road by a force $P_1 = 660$ pounds, inclined at an angle $a = 24°$ to the horizon, and is obliged to overcome a horizontal resistance $P_2 = 450$ produced by the friction, what work must the force P_1 do, in order to change the initial velocity of 2 feet of the wagon into a velocity of 5 feet?

FIG. 100,



If we put the space described by the wagon $MO = s$, we have the work done by the force P_1

$$= P_1 . \overline{ML} = P_1\, s \cos. a = 660 . s \cos. 24° = 602{,}94 . s,$$

and the work done by the force P_2 acting as a resistance is

$$= (-P_2) . s = - 450 . s,$$

consequently the work done by the motive force is

$$P s = P_1\, s \cos. a - P_2\, s \cos. 0 = (602{,}94 - 450)\, s = 152{,}94\, s \text{ foot pounds.}$$

The mass, however, absorbed during the change of velocity the mechanical effect

$\left(\frac{v^2-c^2}{2g}\right) G = \left(\frac{5^2-2^2}{2g}\right) . 5000 = 0{,}0155 . (25-4) . 5000 = 1627{,}5$ foot-pounds; putting the two effects equal to each other we obtain $152{,}94 . s = 1627{,}5$, whence the space described by the wagon is

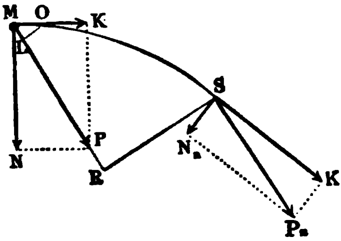
$$M\,O = s = \frac{1627{,}5}{152{,}94} = 10{,}64 \text{ feet},$$

and finally the mechanical effect of the force P_1 is

$$P_1\, s \cos. a = 602{,}94 . 10{,}64 = 6415 \text{ foot-pounds.}$$

§ **85. Curvilinear Motion.**—If we suppose the spaces (σ, σ_1, etc.,) infinitely small, we can apply the foregoing formulas to curvilinear motion. Let $M\,O\,S$, Fig. 101, be the trajectory of the material point, and $\overline{M\,P} = P$ the resultant of all the forces acting upon it. If we decompose this force into two others, the one of which $\overline{M\,K} = K$ is tangent and the other $\overline{M\,N} = N$ normal to the curve, we call the former the *tangential* and the latter the *normal force.*

Fig. 101.



While the material point describes the element $M\,O = \sigma$ of its curved path $M\,O\,S$, and its velocity changes from c to v_1, the mass M absorbs the mechanical effect $\left(\frac{v_1^2 - c^2}{2}\right) M$, during the same time the tangential force K performs the work $K\,\sigma$, and the normal force the work $N . 0 = 0$, and consequently we have

$$K\,\sigma = \left(\frac{v_1^2 - c^2}{2}\right) M.$$

If, while the point describes the space $M\,O\,S = s = n\,\sigma$, the tangential velocity changes from c to v, and at the same time the tangential force assumes successively the values K_1, K_2, .. K_n, then

$$(K_1 + K_2 + \ldots + K_n)\,\sigma = \left(\frac{K_1 + K_2 + \ldots + K_n}{n}\right) s = \left(\frac{v^2 - c^2}{2}\right) M,$$

and the work done is

$$A = K\,s = \left(\frac{v^2 - c^2}{2}\right) M, \text{ when } K = \frac{K_1 + K_2 + \ldots + K_n}{n}$$

denotes the mean value of the variable tangential force.

If we put the projection of the elementary space $\overline{M\,O} = \sigma$ upon the direction $\overline{ML}$ of the force $= \xi$, we have also $P\,\xi = K\,\sigma$; if, therefore, while the point describes the space $M\,O\,S = s = n\,\sigma$ the resultant P assumes successively the values $P_1, P_2 \ldots P_n$, the projections of the elementary spaces are successively $\xi_1, \xi_2 \ldots \xi_n$, and we have also

$$P_1\,\xi_1 + P_2\,\xi_2 + \ldots + P_n\,\xi_n = (K_1 + K_2 + \ldots + K_n)\,\sigma,$$

and therefore

$$A = P_1\,\xi_1 + P_2\,\xi_2 + \ldots + P_n\,\xi_n = \left(\frac{v^2 - c^2}{2}\right) M.$$

When the direction of the force P remains constant, the projections $\xi_1, \xi_2 \ldots \xi_n$ of the portions $\sigma, \sigma \ldots$ of the space or that of the whole space $s = n\,\sigma$ form a straight line

$$M\,R = x = \xi_1 + \xi_2 + \ldots \xi_n.$$

If we put $x = m\,\xi$, we can also write

$$A = (P_1 + P_2 + \ldots + P_m)\,\xi = (P_1 + P_2 + \ldots + P_m)\,\frac{x}{m} = P\,x,$$

where P denotes the mean $\frac{P_1 + P_2 + \ldots + P_m}{m}$ of the forces, which correspond to the equal portions $\xi = \frac{x}{m}$ of the projections of the path on the direction of the force.

We have, therefore, also

$$P\,x = \left(\frac{v^2 - c^2}{2}\right) M = (h - k)\,G,$$

in which k denotes the height due to the initial velocity c and h that due to the final velocity v, and G the weight $M\,g$ of the moving body.

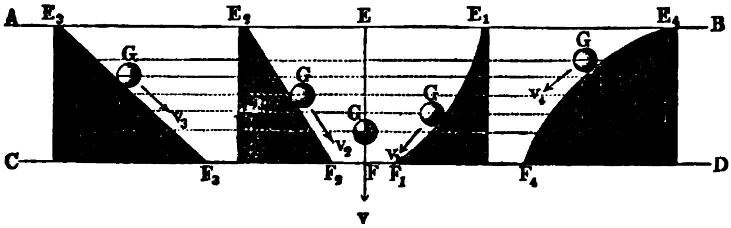
Therefore, *in curvilinear motion, the entire work done is equal to the product of the weight of the body moved and the difference of the heights due to the velocities.*

REMARK.—The formulas, thus obtained by the combination of the principle of vis viva with that of virtual velocities, are particularly applicable to the cases of bodies, which are compelled to describe a given path, either because there is a support placed under them, or because they are suspended by a string, etc. If such a body is impelled by gravity alone, then the work performed by its weight G in descending a distance, whose vertical projection is s, is $= G\,s$, whence

$$G\,s = (h - k)\,G, \text{ i.e. } s = h - k.$$

Whatever may be the path on which a body descends from one horizontal plane $A B$, Fig. 102, to another horizontal one $C D$, the difference of the heights due to the velocities is always equal to the vertical height of descent. Bodies, which begin to describe the paths $E F$, $E_1 F_1$, $E_2 F_2$, etc., with equal velocities (c), arrive at the end of these paths with the same velocity, although they require different times to acquire it.

FIG. 102.



If, for example, the initial velocity is $c = 10$ feet, and the vertical height of fall $= s = 20$ feet, or $h = s + k = 20 + 0,0155 . 10^2 = 21,55$ feet, we have for the final velocity

$$v = \sqrt{2 g h} = 8,025 \sqrt{21,55} = 37,24 \text{ feet},$$

whatever may be the straight or curved line in which the descent takes place.

THIRD SECTION.

STATICS OF RIGID BODIES.

CHAPTER I.

GENERAL PRINCIPLES OF THE STATICS OF RIGID BODIES.

§ 86. **Transference of the Point of Application.**—Although the form of every rigid body is changed by the forces which act upon it, that is, it is compressed, extended, bent, etc., yet in many cases we can consider the body as perfectly rigid, not only because this change of form or displacement of its parts is often very small, but also because it takes place during a very short space of time. For the sake of simplicity we will therefore consider, when nothing to the contrary is stated, a rigid body to be a system of points rigidly united to each other.

FIG. 103.

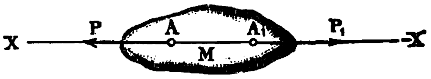
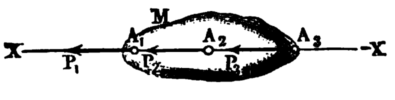


FIG. 104.



A force P, Fig. 103, which acts upon a rigid body at a point A, transmits itself unchanged in its own direction $X\bar{X}$ through the whole body, and an equal opposite force P_1 will balance it, when its point of application A_1 lies in the direction $X\ \bar{X}$. The distance of these points of application A and A_1 from each other has no influence upon the state of equilibrium; the two opposite forces balance each other, whatever the distance may be, if the points are rigidly connected. We can therefore assert, that the action of a force P_1 (Fig. 104) remains the same, no matter in what point A_1, A_2, A_3, etc., of its direction it may be applied or act upon the body M.

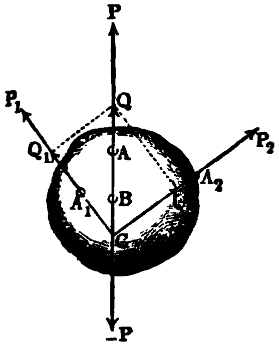
§ **87.** If two forces P_1 and P_2, Fig. 105, acting in the same plane are applied at different points A_1 and A_2 to a body, their action upon the body is the same as if the point C at which the two directions intersect were the common point of application C of these forces; for, according to the law just laid down, both points of application can be transferred to C without producing any change in the action of the forces. If, therefore, we make

$$\overline{C\,Q_1} = \overline{A_1\,P_1} = P_1 \text{ and}$$

$$\overline{C\,Q_2} = \overline{A_2\,P_2} = P_2,$$

and complete the parallelogram $C\ Q_1\ Q\ Q_2$, its diagonal will give us the resultant $\overline{C\,Q} = P$ of $\overline{C\,Q_1}$ and $\overline{C\,Q_2}$ and also of the forces P_1 and P_2. The point of application of this resultant can be any other point A in the direction of the diagonal.

FIG. 105.



If at a point B on the diagonal we apply a force $\overline{B\,P} = -P$ equal and opposite to the resultant $\overline{A\,P} = P$, the forces P_1, P_2 and $-P$ will balance each other.

§ **88. Statical Moment.**—If from any point O, Fig. 106, in the plane of the forces we let fall the perpendiculars $O\,L_1$, $O\,L_2$ and $O\,L$ upon the directions of the component forces P_1 and P_2 and of the resultant P, we have, according to § 82,

$$P\,.\,\overline{O\,L} = P_1\,.\,\overline{O\,L_1} + P_2\,.\,\overline{O\,L_2},$$

and, therefore, from the perpendiculars or distances $O\,L_1$ and $O\,L_2$ of the components we can find that of the resultant by putting

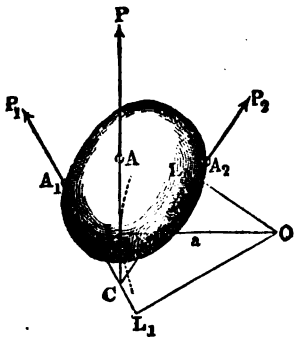
$$O\,L = \frac{P_1\,.\,\overline{O\,L_1} + P_2\,.\,\overline{O\,L_2}}{P}.$$

While the intensity and direction of the resultant is found by means of the parallelogram of forces, the position L of the point of application is obtained by means of the last formula.

If the directions of the forces, when sufficiently prolonged, form an angle $P_1 \, C \, P_2 = \alpha$, the value of the resultant is

$$1) \quad P = \sqrt{P_1^2 + P_2^2 + 2 \, P_1 \, P_2 \cos. \alpha}.$$

FIG. 106.



If the direction of the resultant forms an angle $P \, C \, P_1 = \alpha_1$ with the direction of the component P_1, we have

$$2) \quad \sin. \alpha_1 = \frac{P_2 \sin. \alpha}{P}.$$

If, finally, the distances from any point to the directions $C \, P_1$ and $C \, P_2$ of the given forces are $O \, L_1 = a_1$ and $O \, L_2 = a_2$, then the distance $O \, L = a$ from this point to the direction $C \, P$ of the resultant is

$$3) \quad a = \frac{P_1 \, a_1 + P_2 \, a_2}{P}$$

By the aid of the last distance a we can determine the position of the resultant without reference to any auxiliary point C by describing from O with the radius a a circle, and by drawing a tangent $L \, P$ to it, the direction of which is given by the angle α_1.

EXAMPLE.—A body is acted upon by the forces $P_1 = 20$ pounds and $P_2 = 34$ pounds, whose directions form an angle $P_1 \, C \, P_2 = \alpha = 70°$ with each other, and their distances from a certain point are $O \, L_1 = a_1 = 4$ feet and $O \, L_2 = a_2 = 1$ foot; what is the intensity, direction and position of the resultant? The value of the resultant is

$$P = \sqrt{20^2 + 34^2 + 2 \, . \, 20 \, . \, 34 \cos. 70°} = \sqrt{400 + 1156 + 1360 \, . \, 0{,}34202}$$
$$= \sqrt{2021{,}15} = 44{,}96 \text{ feet};$$

and its direction is determined by the angle α_1, whose sine is

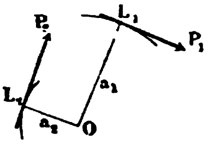
$$\sin. \alpha_1 = \frac{34 \, . \sin. 70°}{44{,}96}, \text{ hence } \log \sin. \alpha_1 = 0{,}85163 - 1,$$

and the angle formed by the direction of the resultant with that of the force P_1 is $\alpha_1 = 45° \, 17'$. The position or line of application of the resultant is finally determined by its distance $O \, L$ from O, which is

$$a = \frac{20 \, . \, 4 + 34 \, . \, 1}{44{,}96} = \frac{114}{44{,}96} = 2{,}536 \text{ feet.}$$

§ **89.**—We call the normal distances $O\ L_1 = a_1$ and $O\ L_2 = a_2$ of the directions of the forces from an arbitrary point O, Fig. 107, the *arms of the lever*, or simply the *arms* (Fr. bras du levier, Ger. Hebelarme) of the forces, because they form an important element in the theory of the lever, which will be discussed hereafter. The product $P\ a$ of the force and the arm of the lever is called the *statical moment* of the force (Fr. moment des forces, Ger. statisches or Kraftmoment). Since $P\ a = P_1\ a_1 + P_2\ a_2$, the statical moment of the resultant is equal to the sum of the statical moments of the two components.

FIG. 107.



In adding the moments, we must pay attention to the positive and negative signs. If the forces P_1 and P_2 act in the same direction around O, as in Fig. 107, if, E.G., the direction of the forces coincide with the direction of motion of the hands of a watch, they and their moments are said to have the same sign, and if one of them is taken as positive, the other must also be considered as positive. If, on the contrary, the two forces, as in Fig. 108, act in opposite directions around the point O, they and their statical moments are said to be opposite to each other, and when one is assumed to be positive, the other must be taken as negative.

FIG. 108.

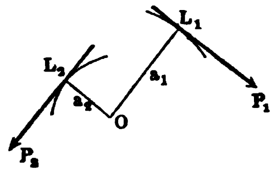
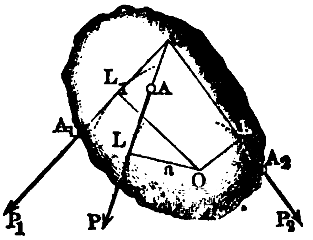


FIG. 109.

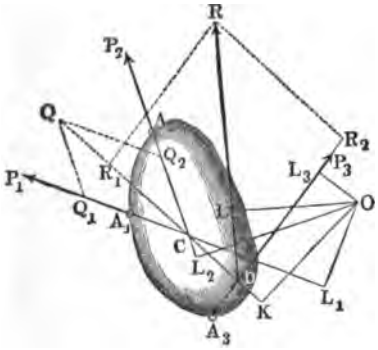


In the combination represented in Fig. 109 we have $P\ a = P_1\ a_1 - P_2\ a_2$, since P_2 is opposite to the force P_1, or its moment $P_2\ a_2$ is negative, while in the combination in Fig. 106 $P\ a = P_1\ a_1 + P_2\ a_2$.

§ **90. Composition of Forces in the Same Plane.**—If three forces P_1, P_2, P_3, Fig. 110, are applied to a body at three different points A_1, A_2, A_3 in the same plane, we first combine two (P_1, P_2) of these forces so as to obtain a resultant $\overline{C\ Q} = Q$, and then combine the latter with the third force (P_3) according to the same rule, constructing with $D R_1 = C Q$ and $D R_2 = A_2 P_2$ the parallelogram $D R_1 R R_2$. The diagonal $D R$ is the required resultant P of P_1, P_2, and P_3. It is easy to see how we must proceed, when a fourth force P_4 is added.

Here the intensity and direction of the resultant is found in exactly the same manner as when the forces are applied at the same point (see § 80); the rules given in § 80 can be employed to calculate the first two elements of the resultant, but the third element, viz., the position of the resultant or its line of application, must be determined by means of the formula for the statical moments. If $O L_1 = a_1$, $O L_2 = a_2$, $O L_3 = a_3$ and $O L = a$ are the arms of the three component forces P_1, P_2, P_3 and of their resultant P in reference to an arbitrary point O, and if Q is the resultant of P_1 and P_2 and $O K$ its arm, we have

Fig. 110.



$$P a = Q . \overline{O K} + P_3 a_3 \text{ and } Q . \overline{O K} = P_1 a_1 + P_2 a_2.$$

Combining these two equations, we obtain

$$P a = P_1 a_1 + P_2 a_2 + P_3 a_3,$$

and in like manner when there are several forces

$$P a = P_1 a_1 + P_2 a_2 + P_3 a_3 + \ldots$$

I.E., *the (statical) moment of the resultant is always equal to the algebraical sum of the (statical) moments of the components.*

§ **91.** If P_1, P_2, P_3, etc., Fig. 111, are the individual forces of a system, α_1, α_2, α_3, etc., the angles $P_1 D_1 X$, $P_2 D_2 X$, $P_3 D_3 X$, etc., formed by the directions of these forces with any arbitrary axis $X \overline{X}$ and a_1, a_2, a_3, etc., their arms $O L_1$, $O L_2$, $O L_3$, etc., in reference to the point of intersection O of the two axes $X \overline{X}$ and $Y \overline{Y}$, we have, according to §§ 80 and 90,

1) the component parallel to the axis $X \overline{X}$

$$Q = P_1 \cos. \alpha_1 + P_2 \cos. \alpha_2 + \ldots,$$

2) the component parallel to the axis $Y \overline{Y}$

$$R = P_1 \sin. \alpha_1 + P_2 \sin. \alpha_2 + \ldots$$

3) the resultant of the whole system

$$P = \sqrt{Q^2 + R^2},$$

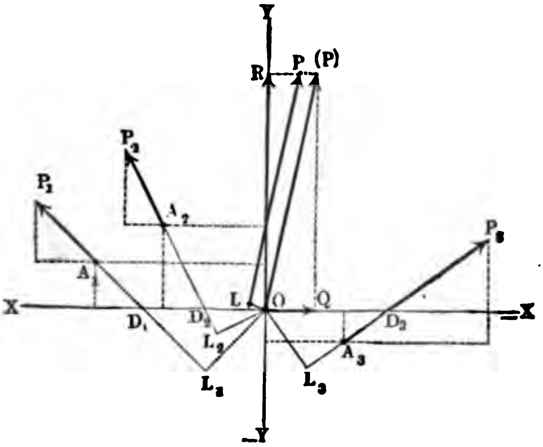
4) the angle α formed by the resultant with the axis for which

$$tang.\ \alpha = \frac{R}{Q},$$

5) and the arm of the resultant or the radius of the circle to which the direction of the resultant is tangent

$$a = \frac{P_1 a_1 + P_2 a_2 + \ldots}{P_1 + P_2 + \ldots}.$$

FIG. 111.



If b, b_1, b_2 etc., denote the distances $O\,D$, $O\,D_1$, $O\,D_2$, etc., cut off from the axis $X\,\overline{X}$, we have

$$a = b\ sin.\ \alpha,\ a_1 = b_1\ sin.\ \alpha_1,\ a_2 = b_2\ sin.\ \alpha_2,\ \text{etc.},$$

and therefore also

$$b = \frac{P_1\,b_1\ sin.\ \alpha_1 + P_2\,b_2\ sin.\ \alpha_2 + \ldots}{P\ sin.\ \alpha} = \frac{R_1\,b_1 + R_2\,b_2 + \ldots}{R}.$$

If we replace the resultant (P) by an equal opposite force $(-P)$, the forces $P_1, P_2, P_3 \ldots (-P)$ will balance each other.

If $x_1, x_2 \ldots$ and $y_1, y_2 \ldots$ denote the co-ordinates of the points of application $A_1, A_2 \ldots$ of the given forces $P_1, P_2 \ldots$, the moments of the components of the latter are $R_1\,x_1, R_2\,x_2 \ldots$ and $Q_1\,y_1, Q_2\,y_2 \ldots$, and the moment of the resultant is

$$P\,a = (R_1\,x_1 + R_2\,x_2 + \ldots) - (Q_1\,y_1 + Q_2\,y_2 + \ldots),$$

and its arm is

$$a = \frac{(R_1\, x_1 + R_2\, x_2 + \ldots) - (Q_1\, y_1 + Q_2\, y_2 + \ldots)}{\sqrt{(R_1 + R_2 + \ldots)^2 + (Q_1 + Q_2 + \ldots)^2}}.$$

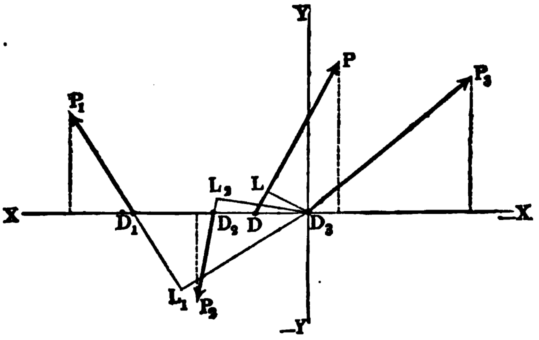
EXAMPLE.—The forces $P_1 = 40$ pounds, $P_2 = 30$ pounds, $P_3 = 70$ pounds, Fig. 112, form with the axis $X\bar{X}$ the angles $a_1 = 60°$, $a_2 = -80°$, $a_3 = 140°$, and the distances between the points of intersection D_1, D_2, D_3 of the directions of the forces with the axis are $D_1 D_2 = 4$ feet, and $D_2 D_3 = 5$ feet. Required the elements of the resultant. The sum of the components parallel to the axis $X\bar{X}$ is

$$\begin{aligned} Q &= 40 \; cos.\; 60° + 30 \; cos.\; (-80°) + 70 \; cos.\; 142° \\ &= 40 \; cos.\; 60° + 30 \; cos.\; 80° - 70 \; cos.\; 38° \\ &= 20 + 5{,}209 - 55{,}161 = -29{,}952 \text{ pounds.} \end{aligned}$$

The sum of those parallel to the axis $Y\bar{Y}$ is

$$\begin{aligned} R &= 40 \; sin.\; 60° + 30 \; sin.\; (-80°) + 70 \; sin.\; 142° \\ &= 40 \; sin.\; 60° - 30 \; sin.\; 80° + 70 \; sin.\; 38° \\ &= 34{,}641 - 29{,}544 + 43{,}096 = 48{,}193. \end{aligned}$$

FIG. 112.



Hence it follows that the resultant

$$P = \sqrt{Q^2 + R^2} = \sqrt{29{,}952^2 + 48{,}193^2} = \sqrt{3219{,}68} = 56{,}742 \text{ pounds.}$$

The angle a formed by the latter with the axis is determined by the formula

$$tang.\; a = \frac{R}{Q} = -\frac{48{,}193}{29{,}952} = -1{,}6090, \text{ from which we obtain}$$

$$a = 180° - 58°\, 8' = 121°\, 52'.$$

If we transfer the origin of the co-ordinates O to D_3, we have the arm of the force

$$\overline{O\,L} = a = \frac{P_1 \; sin.\; a_1\, b_1 + P_2 \; sin.\; a_2\, b_2 + \ldots}{P} = \frac{R_1\, b_1 + R_2\, b_2 + \ldots}{P}$$

$$= \frac{34{,}641 \,.\, (4 + 5) - 29{,}544 \,.\, 5 + 0}{56{,}742} = \frac{164{,}049}{56{,}742} = 2{,}891 \text{ feet,}$$

and, on the contrary, the distance cut off on the axis $X\overline{X}$

$$O\,D = b = \frac{164{,}049}{48{,}193} = 3{,}404 \text{ feet.}$$

§ **92. Parallel Forces.**—If the forces P_1, P_2, P_3, etc., Fig. 113, of a rigid system of forces are parallel, their arms $O\,L_1$, $O\,L_2$, $O\,L_3$, etc., coincide with each other; if through the origin O we draw an arbitrary line $X\overline{X}$, the directions of the forces will cut off from it the portions $O\,D_1$, $O\,D_2$, $O\,D_3$, etc., which are proportional to the arms $O\,L_1$, $O\,L_2$, $O\,L_3$, etc., for we have $\triangle\ O\,D_1\,L_1 \backsim \triangle\ O\,D_2\,L_2 \backsim \triangle\ O\,D_3\,L_3$, etc. Designating the angle $D_1\,O\,L_1 = D_2\,O\,L_2$, etc., by a, the arms $O\,L_1$, $O\,L_2$, etc., by a_1, a_2, etc., and the distances cut off $O\,D_1$, $O\,D_2$, etc., by b_1, b_2, etc., we have

$$a_1 = b_1 \cos.\, a,\ a_2 = b_2 \cos.\, a, \text{ etc.}$$

Finally, substituting these values in the formula

$$P\,a = P_1\,a_1 + P_2\,a_2 + \ldots,$$

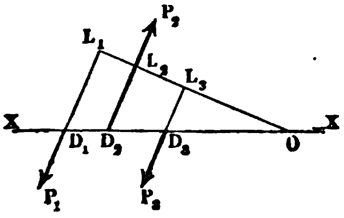
we obtain

$$P\,b \cos.\, a = P_1\,b_1 \cos.\, a + P_2\,b_2 \cos.\, a + \ldots,$$

or, omitting the common factor $\cos.\, a$, we have

$$P\,b = P_1\,b_1 + P_2\,b_2 + \ldots$$

FIG. 113.



In every system of parallel forces we can substitute for the arms the distances $O\,D_1$, $O\,D_2$, etc., cut off from any oblique line by the directions of the forces. Since the intensity and direction of the resultant of a system of forces with different points of application is the same as that of a system of forces applied in one point, the resultant of the system of parallel forces has the same direction as the components, and is equal to their algebraical sum; hence we have

$$1)\qquad P = P_1 + P_2 + P_3 + \ldots \text{ and}$$

$$2)\qquad a = \frac{P_1\,a_1 + P_2\,a_2 + \ldots}{P_1 + P_2 + \ldots}, \text{ or}$$

$$3)\qquad b = \frac{P_1\,b_1 + P_2\,b_2 + \ldots}{P_1 + P_2 + \ldots}.$$

EXAMPLE.—The directions of the three forces $P_1 = 12$ pounds, $P_2 = -32$ pounds and $P_3 = 25$ pounds cut a straight line in the points D_1, D_2 and D_3, Fig. 113, whose distances from each other are $D_1 D_2 = 21$ inches, and $D_2 D_3 = 30$ inches; required the resultant. The intensity of this force is

$$P = 12 - 32 + 25 = 5 \text{ pounds},$$

and the distance $D_1 D$ of its point of application D in the axis $X\overline{X}$ from the point D_1 is

$$b = \frac{12 \,.\, 0 - 32 \,.\, 21 + 25 \,.\, (21 + 30)}{5} = \frac{0 - 672 + 1275}{5} = 120{,}6 \text{ inches.}$$

§ **93. Couples.**—The resultant of two equal and opposite forces P_1 and $-P_1$ is

$$P = P_1 + (-P_1) = P_1 - P_1 = 0,$$

and its arm is

$$a = \frac{P_1 a_1 + P_2 a_2}{0} = \infty \text{ (infinitely great).}$$

FIG. 114.

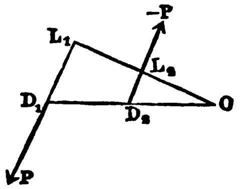
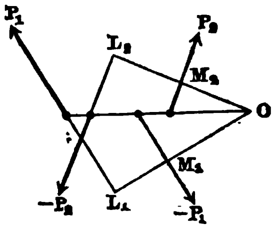


FIG. 115.



No finite force acting at a finite distance can balance a couple, but two such couples can balance each other. Let P_1 and $-P_1$ and $-P_2$ and P_2, Fig. 115, be two such couples, and $O L_1 = a_1$, $O M_1 = O L_1 - L_1 M_1 = a_1 - b_1$, $O L_2 = a_2$ and $O M_2 = O L_2 - L_2 M_2 = a_2 - b_2$ their arms measured from a certain point O, then, when equilibrium exists, we have

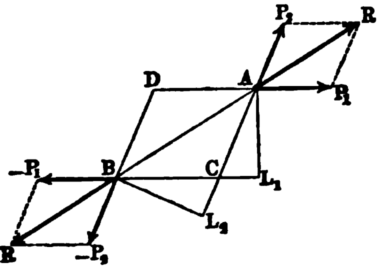
$$P_1 a_1 - P_1 (a_1 - b_1) - P_2 a_2 + P_2 (a_2 - b_2) = 0, \text{ I.E.}$$
$$P_1 b_1 = P_2 b_2.$$

Two such couples balance each other when the product of one force by its distance from the opposite one is the same for both couples.

A pair of equal opposite forces is called simply a *couple* (Fr. couple, Ger. Kräftepaar), and the product of one of its forces by their normal distance apart is called the *moment of the couple*.

From the foregoing we see that two couples acting in opposite directions balance each other, when their moments are equal. That this rule is correct can be proved in the following manner. If we transfer the points of application of the forces P_1, P_2 and $-P_1, -P_2$ of the couples $(P_1, -P_1)$ and $(P_2, -P_2)$, Fig. 116, to the points of intersection A and B of their lines of application, we can combine P_1 and P_2 as well as $-P_1$ and $-P_2$ by means of the parallelogram of forces and obtain the resultants. If the directions of these resultants lie in the prolongation of the line $A\ B$, then these forces, and consequently the corresponding couples $(P_1, -P_1)$, and $(P_2, -P_2)$, balance each other. If equilibrium exists, the triangle $A\ B\ C$ formed by $A\ B$ and by the directions of the forces $-P$ and P_2 must be similar to the triangles $R\ A\ P_1$ and $B\ \overline{R\ P}_1$, and consequently we have the proportion

FIG. 116.



$$\frac{C B}{C A} = \frac{P_1}{P_2} \text{ or the equation } P_1 . \overline{C A} = P_2 . \overline{C B}.$$

But the perpendiculars $A\ L_1 = b_1$ and $B\ L_2 = b_2$ to the directions of the couples are proportional to the hypothenuses $C\ A$ and $C\ B$ of the similar triangles $A\ C\ L_1$ and $B\ C\ L_2$, and we can therefore put

$$P_1\, b_1 = P_2\, b_2.$$

The moments of two couples which balance each other are consequently equal to each other.

If in the formula (§ 91) for the arm a of the resultant

$$a = \frac{P_1\, a_1 + P_2\, a_2 + \ldots}{P}$$

we substitute $P = 0$, while the sum of the statical moments has a finite value, we obtain $a = \infty$, a proof that in this case there can be no other resultant than a couple.

If the forces of a system shall balance each other, it is necessary not only that the resultant $P = \sqrt{Q^2 + R^2}$ of the components Q and R, but also that its moment

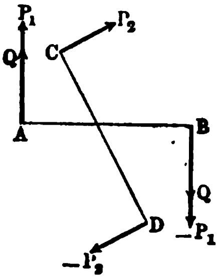
$$P\, a = P_1\, a_1 + P_2\, a_2 + \ldots \text{shall be} = 0.$$

Example.—If one couple consists of the forces $P_1 = 25$ pounds and $-P_1 = -25$ pounds and the other of the forces $P_2 = 18$ pounds and $-P_2 = -18$ pounds, and if the normal distance between the first couple is $b = 3$ feet, then to produce equilibrium it is necessary that the normal distance or arm of the second couple shall be

$$b_2 = \frac{25 . 3}{18} = 4\tfrac{1}{6} \text{ feet.}$$

§ 94. **Composition and Decomposition of Couples.**—The composition and decomposition of couples acting in the same plane is accomplished by a mere algebraical addition, and is therefore much simpler than the composition and decomposition of single forces. Since two opposite couples balance each other, when their moments are equal, the action of two couples is the same and the couples are said to be equivalent, when the moment of one couple is equal to that of the other. If, therefore, the two couples $(P_1, - P_1)$ and $(P_2, - P_2)$, Fig. 117, are to be combined, we can replace the one $(P_2, - P_2)$ by another which has the same arm $A\ B = b_1$ as the former couple $(P_1, - P_1)$, and can then add the forces thus obtained to the others, and thus obtain a single couple. If b_2 is the arm $C\ D$ of the one couple and $(Q, - Q)$ the reduced couple, we have $Q\ b_1 = P_2\ b_2$, and consequently $Q = \frac{P_2 b_2}{b_1}$, hence one component of the resulting couple is

Fig. 117.



$$P_1 + Q = P_1 + \frac{P_2 b_2}{b_1}$$

and the required moment of the resulting couple is

$$(P_1 + Q)\ b_1 = P_1\ b_1 + P_2\ b_2.$$

In same manner the resultant of three couples may be found. If $P_1\ b_1$, $P_2\ b_2$, and $P_3\ b_3$ be the moments of these couples, we can put

$$P_2\ b_2 = Q\ b_1 \text{ and } P_3\ b_3 = R\ b_1, \text{ or}$$

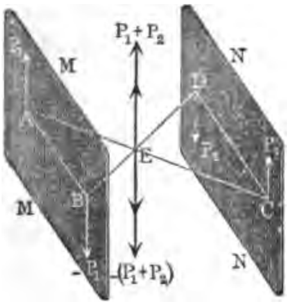
$$Q = \frac{P_2\ b_2}{b_1} \text{ and } R = \frac{P_3\ b_3}{b_1},$$

from which we obtain the resultant

$$(P_1 + Q + R)\ b_1 = P_1\ b_1 + P_2\ b_2 + P_3\ b_3.$$

In combining these couples to obtain a single resultant we must pay attention to the signs, since the moments of the couples tending to turn the body in one direction are positive, and the moments of those tending to turn it in the other are negative. We can now adopt the following principle for indicating the direction of rotation of a couple. Let us assume arbitrarily a centre of rotation between the lines of application of the forces of a couple; then if the couple tends to turn in the direction of the hands of a watch, the couple is to be considered as positive, and if in the other direction, as negative.

FIG. 118.



The foregoing rule for the composition of couples is also applicable, when the forces act in parallel planes. If the parallel couples (P_1, $-P_1$) and (P_2, $-P_2$), Fig. 118, in the parallel planes $M\,M$ and $N\,N$ have equal moments $P_1\,b_1$ and $P_2\,b_2$ and act in opposite directions to each other, they will also balance each other; for they give rise to two resultants $P_1 + P_2$ and $-(P_1 + P_2)$, which balance each other, as they are applied in the same point E, which is determined by the equations

$$\overline{E\,A}\,.\,P_1 = \overline{E\,C}\,.\,P_2, \overline{E\,B}\,.\,P_1 = \overline{E\,D}\,.\,P_2 \text{ and}$$

$$P_1\,b_1 = P_2\,b_2, \text{ I.E. } \overline{A\,B}\,.\,P_1 = \overline{C\,D}\,.\,P_2, \text{ whence}$$

$$E\,A : E\,B : A\,B = E\,C : E\,D : C\,D;$$

hence this point coincides with the point of intersection of the two transverse lines $A\,C$ and $B\,D$.

Since the couple (P_2, $-P_2$) balances every other couple acting in a parallel plane with an equal and opposite moment, it follows that every couple can be replaced by another which has the same moment, and which acts in a plane parallel to that of the first.

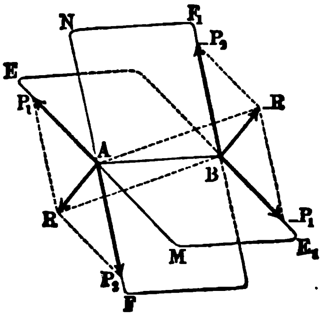
If, therefore, several couples whose planes of action are parallel are applied to a body, they can be replaced by a single couple whose moment is the algebraical sum of their moments, and whose plane, which in other respects is arbitrary, is parallel to the planes of the given system.

§ **95.** If two couples (P_1, $-P_1$) and (P_2, $-P_2$) act in two different planes $E\,M\,E_1$ and $F\,N\,F_1$, Fig. 119, whose line of intersection is the straight line $A\ B$, and which form with each other a given angle

$$E\ A\ F = E_1\ B\ F_1 = a$$

we can, after having reduced them to the same arm $A\ B$, combine them by means of the parallelogram of forces. We obtain thus from P_1 and P_2 the resultant R, and from $- P_1$ and $- P_2$ the resultant $- R$. These two resultants being equal and opposite, form another couple, whose plane is given by the direction of R and $- R$.

FIG. 119.



The resultant R can be found according to § 77 by means of the formulas

$$R = \sqrt{P_1^2 + P_2^2 + 2\ P_1\ P_2 \cos. a} \text{ and}$$

$$\sin. \beta = \frac{P_2 \sin. a}{R},$$

in which β denotes the angle $E\ A\ R = E_1\ B\ \bar{R}$ formed by the direction of the resultant with that of the component P_1. If the arm is $A\ B = c$, and if we put the moment $P_1\ c = P\ a$ and the moment $P_2\ c = Q\ b$ or $P_1 = \frac{P\ a}{c}$ and $P_2 = \frac{Q\ b}{c}$ we obtain

$$R = \sqrt{\left(\frac{P\ a}{c}\right)^2 + \left(\frac{Q\ b}{c}\right)^2 + 2\ \frac{P\ a}{c} . \frac{Q\ b}{c} \cos. a},$$

or the moment of the resultant of the couples $(P, - P)$ and $(Q, - Q)$

$$R\ c = \sqrt{(P\ a)^2 + (Q\ b)^2 + 2\ P\ a . Q\ b . \cos. a},$$

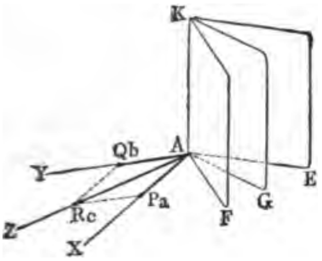
and in like manner for the angle formed by its plane with that of the first couple $(P, - P)$ we have

$$\sin. \beta = \frac{Q\ b}{R\ c} \sin. a.$$

We can therefore combine and decompose couples acting in the different planes in exactly the same manner as forces applied at the same point, by substituting instead of the latter the moments of the former, and instead of the angles, which the directions of the former make with each other, those formed by their planes of action.

The referring back of the theory of couples to the principle of the decomposition of simple forces can be greatly simplified by introducing the axis of rotation instead of the plane of rotation of the couple. We understand by the *axis of rotation* or *axis of a couple*, any perpendicular to its plane. Since every couple can be arbitrarily displaced in its plane without changing its action upon the body, we can pass the axis of the couple through any given point.

FIG. 120.



Since the plane and the axis of a couple are at right angles to each other, the axes $A X$, $A Y$ and $A Z$, Fig. 120, form the same angles with each other as the planes $A E K$, $A F K$ and $A G K$ themselves. If one of the couples is the resultant of the other two, we see from what precedes, that the diagonal of the parallelogram constructed with the moments $P a$ and $Q b$ will give the moment of the resultant; if therefore we lay off upon the axes $A X$ and $A Y$ the moments $P a$ and $Q b$, and then complete the parallelogram, we obtain in its diagonal not only the axis $A Z$ of the resulting couple, but also its moment $R c$. We see, therefore, *that couples are combined and decomposed in exactly the same way as simple forces*, provided we substitute for the directions of the forces the axes of the couples and the moments of the latter for the forces themselves. All the rules for the composition and decomposition of forces given in § 76 and § 77, etc., are in this sense applicable to the composition and decomposition of couples.

§ 96. Centre of Parallel Forces.—If the parallel forces lie in different planes, their composition must be effected in the following manner. Prolonging the straight line $A_1 A_2$, Fig. 121, which joins the points of application of two parallel forces P_1 and P_2, until it meets the plane which contains the axes $M X$ and $M Y$, which are at right angles to each other, and taking the point of intersection K as the origin, we have for the point of application A of the resultant $P_1 + P_2$ of these forces

$$(P_1 + P_2) . \overline{K A} = P_1 . \overline{K A_1} + P_2 . \overline{K A_2}.$$

Now since B, B_1 and B_2 are the projections of the points of application A, A_1 and A_2 upon the plane $X\,Y$, we have

$$A\,B : A_1\,B_1 : A_2\,B_2 = K\,A : K\,A_1 : K\,A_2,$$

and therefore also

$$(P_1 + P_2)\,.\,\overline{A\,B} = P_1\,.\,\overline{A_1\,B_1} + P_2\,.\,\overline{A_2\,B_2}.$$

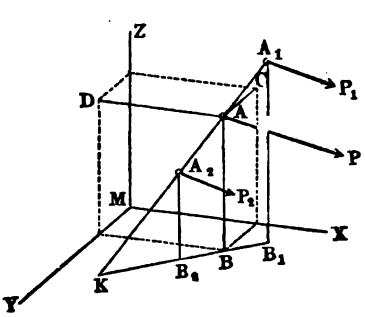
If we designate the normal distances $A_1\,B_1$, $A_2\,B_2$, $A_3\,B_3$, etc., of the points of application from the plane $X\,\overline{X}$ by z_1, z_2, z_3, etc., and the normal distance of the point of application A from this plane by z, we have for two forces

$$(P_1 + P_2)\,z = P_1\,z_1 + P_2\,z_2;$$

and for three forces, since $(P_1 + P_2)$ can be considered as one force with the moment $P_1\,z_1 + P_2\,z_2$,

$$(P_1 + P_2 + P_3)\,z = P_1\,z_1 + P_2\,z_2 + P_3\,z_3, \text{ etc.}$$

FIG. 121.



Consequently we have in general

$$(P_1 + P_2 + P_3 + \ldots)\,z = P_1\,z_1 + P_2\,z_2 + P_3\,z_3 \ldots,$$

and therefore

$$1) \qquad z = \frac{P_1\,z_1 + P_2\,z_2 + \ldots}{P_1 + P_2 + \ldots}.$$

If, in like manner, we denote the distances $A\,C$ and $A\,D$ of the point of application A of the resultant from the planes $X\,Z$ and $Y\,Z$ by y and x, and the distances of the points of application A_1, $A_2 \ldots$ from the same planes by y_1, $y_2 \ldots$ and x_1, $x_2 \ldots$, we obtain

$$2) \qquad y = \frac{P_1\,y_1 + P_2\,y_2 + \ldots}{P_1 + P_2 + \ldots} \text{ and}$$

$$3) \qquad x = \frac{P_1\,x_1 + P_2\,x_2 + \ldots}{P_1 + P_2 + \ldots}.$$

The distances, x, y and z, from three fixed planes, as, E.G., from the floor and two sides of a room, determine completely the point A; for it is the eighth corner of the parallelopipedon constructed with x, y and z; hence there is but one point of application of the resultant of such a system of forces.

Since the three formulas for x, y and z do not contain the angles formed by the forces with the fixed planes, the point of application is not dependent upon them or upon the direction of the forces;

the whole system can therefore be turned about this point without its ceasing to be the point of application, as long as the forces remain parallel.

In a system of parallel forces we call the product of a force by the distance of its point of application from a plane or line the *moment* of this force in reference to the plane or line, and it is also customary to call the point of application of the resultant the *centre of parallel forces* (Fr. centre des forces parallèles, Ger. Mittelpunkt des ganzen Systems). We obtain the distance of *the centre of a system of parallel forces from any plane or line* (the latter, when the forces are in the same plane) by dividing the sum of the statical moments by the sum of the forces themselves.

EXAMPLE.—If the forces are	P_n	5	− 7	10	4 pounds.
and their distances or the co-ordinates of their points of application are	x_n	1	2	0	9 feet.
	y_n	2	4	5	3 "
	z_n	8	3	7	10 "
we will have the moments	$P_n\,x_n$	5	− 14	0	36 foot pounds.
	$P_n\,y_n$	10	− 28	50	12 "
	$P_n\,z_n$	40	− 21	70	40 "

Now the sum of the forces is $= 19 - 7 = 12$ pounds, and therefore the distances of the centre of parallel forces from the three co-ordinate planes are

$$x = \frac{5 + 36 - 14}{12} = \frac{27}{12} = \frac{9}{4} = 2{,}25 \text{ feet,}$$

$$y = \frac{10 + 50 + 12 - 28}{12} = \frac{44}{12} = \frac{11}{3} = 3{,}66 \text{ feet, and}$$

$$z = \frac{40 + 70 + 40 - 21}{12} = \frac{129}{12} = \frac{43}{4} = 10{,}75 \text{ feet.}$$

§ 97. Forces in Space.—If we wish to combine a system of forces directed in different directions, we pass a plane through them and transfer all their points of application to this plane, and then decompose each force into two components, one perpendicular to and the other in the plane. If $\beta_1, \beta_2 \ldots$ are the angles formed by the directions of the forces with the plane, the components normal to the plane are $P_1 \sin. \beta_1$, $P_2 \sin. \beta_2 \ldots$ and those in the plane are $P_1 \cos. \beta_1$, $P_2 \cos. \beta_2$, etc. The resultant of the latter can be obtained as indicated in § 91, and that of the former as indicated in the last paragraph. Generally the directions of the two resultants do not cut each other at all, and the composition of the forces so as to form a single resultant is not possible. If, however, the resultant of the parallel forces passes through a point K, Fig. 122, in the direction $A\,B$ of the resultant P of the forces lying in the plane (that of the paper), a composition is possible. Putting the ordinates of the points of application K of the first resultant $O\,C = D\,K = u$ and $O\,D = C\,K = v$, the arm of the other $O\,L = a$ and the angle $B\,A\,O$ formed by the latter with the axis $X\,\overline{X_1} = \alpha$, then the condition for the possibility of the composition is

$$u \sin. \alpha + v \cos. \alpha = a.$$

If this equation is not satisfied, if, E.G., the resultant of the normal forces passes through K_1, it is not possible to refer the whole system of forces to a single resultant, but they can be replaced by

FIG. 122.

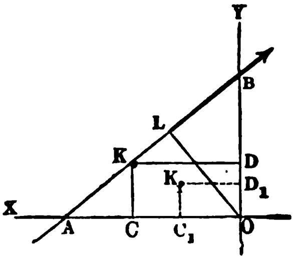
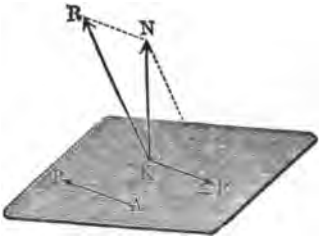


FIG. 123.

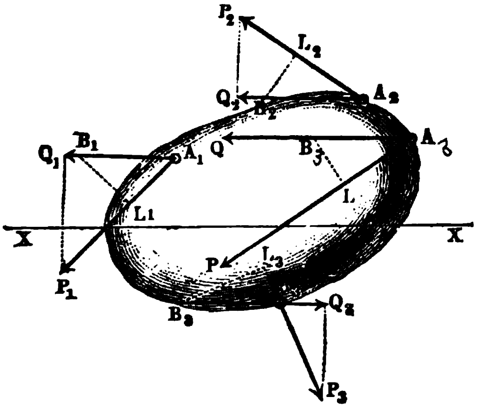


a resultant R, Fig. 123, and a couple $(P, -P)$ by decomposing the resultant N of the parallel forces into the forces $-P$ and R, one of which is equal, parallel and opposite to the resultant P of the forces in the plane.

We can accomplish directly this referring of a system of forces to a single force and to a couple by imagining a system of couples, whose positive components are exactly equal in amount and direction to the given forces, to be applied to the body at any arbitrary point. These couples naturally do not change the state of equilibrium, for being applied at the same point they counteract themselves. On the contrary, the positive components can be combined according to known rules (§ 81) so as to give one resultant, while the negative components form with the given forces couples, whose resultant (according to § 95) is a single couple. After these operations have been performed, we have only one force and one couple.

§ 98. Principle of Virtual Velocities.—If a system of forces P_1, P_2, P_3, Fig. 124, which act in a plane, have a motion of translation, that is, if all the points of application A_1, A_2, A_3 describe equal parallel spaces $A_1\,B_1$, $A_2\,B_2$, $A_3\,B_3$, then (according to the meaning of § 81) the work done by the resultant is equal to the sum of the work done by the components, and consequently, when the forces balance each other, this sum is $= 0$. If the projections of the common space $A_1\,B_1 = A_2\,B_2$, etc., upon the directions of the forces are $A_1\,L_1$, $A_2\,L_2$, etc., $= s_1$, s_2, etc., the work done by the resultant is

FIG. 124.



$$P\,s = P_1\,s_1 + P_2\,s_2 + .$$

This law is a consequence of one of the formulas in § 91. According to it, the component Q of the resultant parallel to the axis $X\,X$ is equal to the sum

$$Q_1 + Q_2 + Q_3 + \dots$$

of the components of the forces P_1, P_2, etc., which are parallel to it. Now from the similarity of the triangles $A_1\,B_1\,L_1$ and $A_1\,P_1\,Q_1$ we know that

$$\frac{Q_1}{P_1} = \frac{A_1\,L_1}{A_1\,B_1} = \frac{s_1}{A\,B},$$

and therefore we have

$$Q_1 = \frac{P_1\,s_1}{A\,B},\; Q_2 = \frac{P_2\,s_2}{A\,B}, \text{ etc. and } Q = \frac{P\,s}{A\,B}.$$

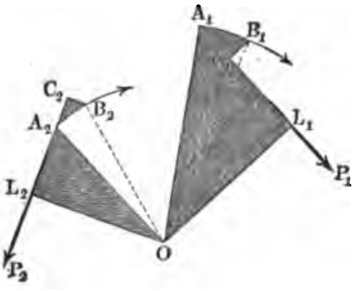
Hence, instead of

$$Q = Q_1 + Q_2 + \dots$$

we can put

$$P\,s = P_1\,s_1 + P_2\,s_2 + \dots$$

FIG. 125.



§ **99. Equilibrium in a Rotary Motion.**—If a system of forces P_1, P_2, etc., Fig. 125, acting in the same plane, is caused to turn a very small distance about a point O, the principle of virtual velocities announced in § 83 and § 98 is applicable here also, as can be demonstrated in the following manner. According to § 89 the moment of the resultant $P \cdot \overline{OL} = P\, a$ is equal to the sum of the moments of the components, or

$$P\, a = P_1\, a_1 + P_2\, a_2 + \ldots$$

The space $A_1 B_1$, corresponding to a rotation through a small angle $A_1\, O\, B_1 = \beta^\circ$ or a small arc $\beta = \frac{\beta^\circ}{180^\circ} \cdot \pi$, is situated at right angles to the radius $O\, A_1$, therefore the triangle $A_1\, B_1\, C_1$ formed by letting fall the perpendicular $B_1\, C_1$ upon the direction of the force, is similar to the triangle $O\, A_1\, L_1$ formed by the arm $O\, L_1 = a_1$, and we have

$$\frac{O\, L_1}{O\, A_1} = \frac{A_1\, C_1}{A_1\, B_1}.$$

If we put the virtual velocity $\overline{A_1\, C_1} = \sigma_1$ and the arc $\overline{A_1\, B_1} = \overline{O\, A_1} \cdot \beta$, we obtain

$$a_1 = \frac{O\, A_1 \cdot \sigma_1}{O\, A_1 \cdot \beta} = \frac{\sigma_1}{\beta}, \text{ and in like manner } a_2 = \frac{\sigma_2}{\beta}, \text{ etc.}$$

Substituting these values of a_1, a_2, etc., in the above equation, we obtain

$$\frac{P\, \sigma}{\beta} = \frac{P_1\, \sigma_1}{\beta} + \frac{P_2\, \sigma_2}{\beta} + \ldots, \text{ etc.,}$$

or since β is a common divisor,

$$P\, \sigma = P_1\, \sigma_1 + P_2\, \sigma_2 + \ldots,$$

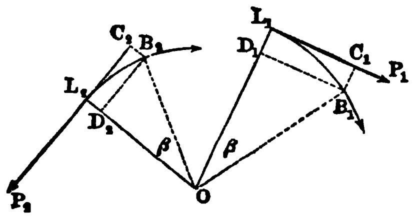
as we found in § 83.

Therefore, for a small rotation, the work ($P\, \sigma$) done by the resultant is equal to the sum of the work done by the components.

§ **100.**—The principle of virtual velocities holds good for any arbitrarily great rotation, when, instead of the virtual velocities of the points of application, we substitute the projections

$L_1\,C_1$, $L_2\,C_2$, Fig. 126, of the spaces described by the ends L_1, L_2,

FIG. 126.



etc., of the perpendiculars; for multiplying the well-known equation for the statical moment

$$P\,a = P_1\,a_1 + P_2\,a_2 + \ldots$$

by *sin.* β and substituting in the new equation

$$P\,a\,sin.\,\beta = P_1\,a_1\,sin.\,\beta + P_2\,a_2\,sin.\,\beta,$$

instead of $a_1\,sin.\,\beta$, $a_2\,sin.\,\beta \ldots$ the spaces

$$O\,B_1\,sin.\,L_1\,O\,B_1 = D_1\,B_1 = L_1\,C_1 = s_1,$$

$$O\,B_2\,sin.\,L_2\,O\,B_2 = D_2\,B_2 = L_2\,C_2 = s_2, \text{ etc.},$$

we obtain

$$P\,s = P_1\,s_1 + P_2\,s_2 + \ldots$$

This principle remains correct for finite rotations, when the directions of the forces revolve with the system, or when the point of application or end of the perpendicular changes continually so that the arms $O\,L_1 = O\,B_1$, etc., remain constant; for from

$$P\,a = P_1\,a_1 + P_2\,a_2 + \ldots,$$

by multiplying it by β we obtain

$$P\,a\,\beta = P_1\,a_1\,\beta + P_2\,a_2\,\beta + \ldots, \text{ I.E.},$$

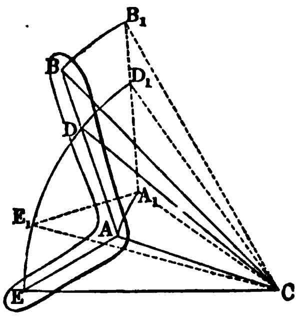
$$P\,s = P_1\,s_1 + P_2\,s_2 + \ldots,$$

when s_1 s_2, etc., denote the arcs $L_1\,B_1$, $L_2\,B_2$, etc., described by the points of application L_1, L_2, etc.

§ 101. A Small Displacement Referred to a Rotation.—Every small motion or displacement of a body in a plane can be considered as a small rotation about a movable centre as we will now proceed to show. Let A and B, Fig. 127, two points of the body (surface or line), be subjected to a small displacement, in consequence of which they now occupy the positions A_1 and B_1, $A_1\,B_1$ being $= A\,B$. If we erect at these points perpendiculars to the paths $A\,A_1$, and $B\,B_1$, they will cut each other at a point C, about which we can imagine the spaces $A\,A_1$ and $B\,B_1$, considered as arcs of circles, to be described. But since $A\,B = A_1\,B_1$, $A\,C =$

A_1 C and $B\ C = B_1\ C$, the two triangles $A\ B\ C$ and $A_1\ B_1\ C$ are similar; the angle $B_1\ C\ A_1$ is therefore equal to the angle $B\ C\ A$, and the angle of rotation $A\ C\ A_1$ equal to the angle of rotation $B\ C\ B_1$. If we make $A_1\ D_1 = A\ D$ we obtain, since the angles $D_1\ A_1 C$ and $D\ A\ C$ and the sides $C\ A_1$ and $C\ A$ are equal to each other, two equal, similar triangles $C\ A_1 D_1$ and $C\ A\ D$, in which $C\ D_1 = C\ D$ and $\angle\ A_1\ C\ D_1 = \angle\ A\ C\ D$. Consequently, $\angle\ A\ C\ A_1$ is also $= \angle\ D\ C\ D_1$, and when the displacement of the line $A\ B$ is small, every other point D of it will describe an arc of a circle. Finally, if E is a point lying without the line $A\ B$ but rigidly connected with it, the small space $E\ E_1$ described by it can also be regarded as a small arc of a circle, whose centre is at C; for if we make the angle $E_1\ A_1\ B_1 = E\ A\ B$ and the distance $A_1\ E_1 = A\ E$, we obtain again two equal and similar triangles $A_1\ C\ E_1$ and $A\ C\ E$, whose sides $C\ E_1$ and $C\ E$ and whose angles $A_1\ C\ E_1$ and $A\ C\ E$ are equal to each other, and the same thing can be proved for every other point rigidly connected with $A\ B$. We can, therefore, consider any small motion of a surface or of a solid body rigidly connected with $A\ B$ as a small rotation about a centre, which is determined by the point of intersection C of the perpendiculars to the spaces $A\ A_1$ and $B\ B_1$ described by two points of the body.

FIG. 127.



§ 102. Generality of the Principle of Virtual Velocities.—According to a foregoing paragraph (99) the mechanical effect of the resultant is equal to the mechanical effect of its components for a small revolution of the system, and according to the last paragraph (101) any small motion can be considered as a revolution; the principle of virtual velocities is therefore applicable to any small motion of a body or of a system of forces.

If, therefore, a system of forces is in equilibrium, I.E., if the resultant is null, then after a small arbitrary motion the sum of the mechanical effects must be equal to 0. If, on the contrary, for a small motion of the body the sum of all the mechanical effects is equal to zero, it does not necessarily follow that the system is in equilibrium, for then this sum must be = 0 for all possible small motions. Since the formula expressing the principle of virtual velocities fulfils but one of the conditions of equilibrium, in order that equilibrium shall exist it is necessary that this formula shall be true for as many independent motions as there are conditions, E.G., for a system of forces in a plane for three independent motions.

CHAPTER II.

THE THEORY OF THE CENTRE OF GRAVITY.

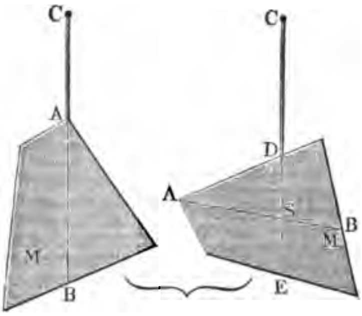
§ **103. Centre of Gravity.**—The weights of the different parts of a heavy body form a system of parallel forces, whose resultant is the weight of the whole body and whose centre can be determined by the three formulas of paragraph 96. We call this centre of the forces of gravity of a body or system of bodies the *centre of gravity* (Fr. centre de gravité, Ger. Schwerpunkt), and also the centre of the mass of the body or system of bodies. If a body be caused to rotate about its centre of gravity, that point will never cease to be the centre of gravity, for if we suppose the fixed planes, to which the points of application of the single weights are referred, to rotate with the body, during this rotation the position of the directions of the forces in regard to these planes change, and on the contrary the distances of the points of application from these planes remain constant. Therefore the centre of gravity is that point at which the weight of a body acts as a force vertically downwards, and at which it must be supported in order to keep the body at rest.

§ **104. Line and Plane of Gravity.**—Every straight line, which contains the centre of gravity, is called a *line of gravity*, and every plane passing through the centre of gravity a *plane of gravity*. The centre of gravity is determined by the intersection of two lines of gravity, or by that of a line of gravity and a plane of gravity, or by the point where three planes of gravity cut each other.

Since the point of application of a force can be transferred arbitrarily in the direction of the force without affecting the action of the latter, a body is in equilibrium whenever any point of the vertical line passing through the centre of gravity is held fast.

If a body *M*, Fig. 128, be suspended at the end of a string *C A*, we obtain in the prolongation *A B* of this string a line of gravity, and if it be suspended in another way we find a second line of gravity *D E*. The point of intersection *S* of the two lines is the centre of gravity of the whole body. If we suspend a body by means of an axis, or if we balance it upon a sharp edge (knife edge), the vertical plane passing through the axis or knife edge is a plane of gravity.

FIG. 128.



Empirical determinations of the centre of gravity, such as we have just given, are seldom applicable; we generally employ some of the geometrical methods, given in the following pages, to determine with accuracy the centre of gravity. In many bodies, such as rings, etc., the centre of gravity is without the body. If such a body is to be suspended by its centre of gravity, it is necessary to fasten to it a second body in such a manner that the centres of gravity of the two bodies shall coincide.

§ 105. Determination of the Centre of Gravity.—Let x_1, x_2, x_3, etc., be the distances of the parts of a heavy body from one co-ordinate plane, y_1, y_2, y_3, etc., those from the second, and z_1, z_2, z_3, etc., those from the third, and let P_1, P_2, P_3, etc., be the weights of these parts, we have, from § 96, for the distances of the centre of gravity of the body from the three planes

$$x = \frac{P_1 x_1 + P_2 x_2 + P_3 x_3 + \ldots}{P_1 + P_2 + P_3 + \ldots},$$

$$y = \frac{P_1 y_1 + P_2 y_2 + P_3 y_3 + \ldots}{P_1 + P_2 + P_3 + \ldots}, \text{ and}$$

$$z = \frac{P_1 z_1 + P_2 z_2 + P_3 z_3 + \ldots}{P_1 + P_2 + P_3}.$$

If we denote the volume of these parts of the body by V_1, V_2, V_3, etc., and the weight of their units of volume by γ_1, γ_2, γ_3, etc., we can write

$$x = \frac{V_1 \gamma_1 x_1 + V_2 \gamma_2 x_2 + V_3 \gamma_3 x_3 + \ldots}{V_1 \gamma_1 + V_2 \gamma_2 + V_3 \gamma_3 + \ldots}, \text{ etc.}$$

If the body is homogeneous, i.e., if γ is the same for all the parts, we have

$$x = \frac{(V_1 x_1 + V_2 x_2 + \ldots)\,\gamma}{(V_1 + V_2 + \ldots)\,\gamma},$$

or, cancelling the common factor γ,

$$1)\quad x = \frac{V_1 x_1 + V_2 x_2 + \ldots}{V_1 + V_2 + \ldots},$$

$$2)\quad y = \frac{V_1 y_1 + V_2 y_2 + \ldots}{V_1 + V_2 + \ldots}, \text{ and}$$

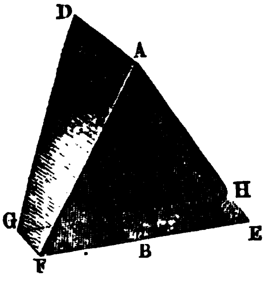
$$3)\quad z = \frac{V_1 z_1 + V_2 z_2 + \ldots}{V_1 + V_2 + \ldots}.$$

Consequently we can substitute for the weights of the different parts their volumes, and the determination of the centre of gravity becomes a question of pure geometry.

When one or two of the dimensions of a body are very small compared with the others, e.g., in the case of sheet-iron, wire, etc., we can regard them as planes or lines, and determine their centres of gravity by means of the last three formulas, substituting instead of the volumes V_1, V_2, etc., the surfaces F_1, F_2, etc., or the lengths l_1, l_2, etc.

§ **106**. In regular spaces the centre of gravity coincides with their centre, e.g., in the case of the cube, sphere, equilateral triangle, circle, etc. Symmetrical spaces have their centre of gravity in the axis or plane of symmetry. A body *A D F H*, Fig. 129, is divided by the plane of symmetry *A B C D* into two halves, which differ only in their position in regard to the plane, and the conditions are therefore the same on both sides of the plane; the moments are consequently the same on both sides, and the centre of gravity is to be found in this plane.

Fig. 129.



Since the axis of symmetry *E F* divides the plane surface *A B F C D*, Fig. 130, into two parts, one of which is the reflected image of the other, the conditions are the same on each side; consequently the moments on both sides are the same, and the centre of gravity of the whole surface lies in this line.

Finally, the axis of symmetry $K L$ of a body $A B G H$, Fig. 131, is also a line of gravity of it; for it is formed by the intersection of two planes of symmetry $A B C D$ and $E F G H$.

FIG. 130.

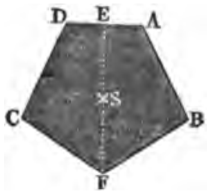
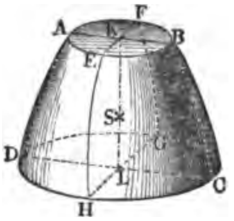


FIG. 131.

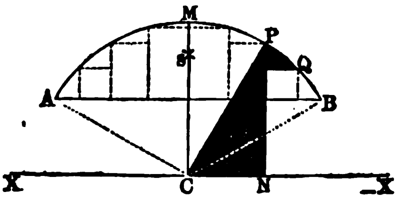


For this reason the centre of gravity of a cylinder, of a cone and of a solid of rotation, formed by the revolution of a surface, or by being turned upon a lathe, is to be found in the axis of the body.

§ **107. Centre of Gravity of Lines.**—The centre of gravity of a straight line is at its centre.

The centre of gravity of the arc of a circle $A M B = b$, Fig. 132, is to be found in the radius drawn to the middle M of the arc; for this radius is an axis of symmetry of the arc. In order to determine the distance $C S = y$ of the centre of gravity S from the centre of the circle, we divide the arc into a very great number of parts and determine their statical moment in reference to an axis $X \bar{X}$, which passes through the centre C and is parallel to the chord $A B = s$. If $P Q$ is a part of the arc and $P N$ its distance from $X \bar{X}$, its statical moment is $= P Q \, . \, P N$. Drawing the radius $P C = M C = r$ and the projection $Q R$ of $P Q$ parallel to $A B$, we obtain two similar triangles $P Q R$ and $C P N$, for which we have

FIG. 132.



$$PQ : QR = CP : PN,$$

whence we obtain for the statical moment of an element of the arc

$$PQ \, . \, PN = QR \, . \, CP = QR \, . \, r.$$

But in the statical moments of all the other elements of the arc r is a common factor, and the sum of all the projections QR of the elements of the arc is equal to the chord, which is the projection of the entire arc; consequently the moment the arc is = the chord s multiplied by the radius r. Putting this moment equal to the arc b multiplied by the distance y, or $b\,y = s\,r$, we obtain

$$\frac{y}{r} = \frac{s}{b}, \text{ or } y = \frac{s\,r}{b}.$$

The distance of the centre of gravity from the centre is to the radius as the chord is to the arc.

If the angle subtended by the arc b is $= \beta^\circ$ and the arc corresponding to the radius $1 = \beta = \frac{\beta^\circ}{180^\circ}\,\pi$, we have $b = \beta\,r$ and $s = 2\,r\,sin.\,\frac{\beta}{2}$, and consequently

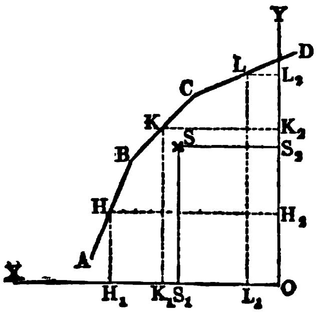
$$y = \frac{2\,sin.\,\frac{1}{2}\,\beta\,.\,r}{\beta}.$$

For a semicircle $\beta = \pi$ and $sin.\,\frac{\beta}{2} = 1$, whence

$$y = \frac{2}{\pi}\,r = 0{,}6366\ldots r, \text{ approximatively} = \frac{7}{11}\,r.$$

§ **108.** In order to find the centre of gravity of a polygon or combination of lines $A\,B\,C\,D$, Fig. 133, we first obtain the distances of the centres H, K, L of the lines $AB = l_1$, $BC = l_2$, $CD = l_3$, etc., from the two axes OX and OY, viz., $HH_1 = y_1$, $HH_2 = x_1$, $KK_1 = y_2$, $KK_2 = x_2$, etc. The distances of the centre of gravity from these axes are

FIG. 133.



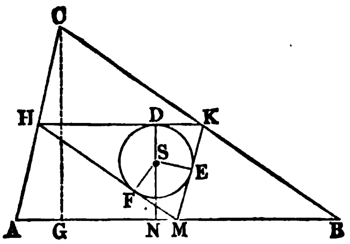
$$OS_1 = SS_2 = x = \frac{l_1 x_1 + l_2 x_2 + \ldots}{l_1 + l_2 + \ldots},$$

$$OS_2 = SS_1 = y = \frac{l_1 y_1 + l_2 y_2 + \ldots}{l_1 + l_2 + \ldots}$$

E.G., the distance of the centre of gravity S of a wire $A\ B\ C$, Fig. 134, bent in the shape of a triangle from the base $A\ B$ is

$$N S = y = \frac{\frac{1}{2} a h + \frac{1}{2} b h}{a + b + c} = \frac{a + b}{a + b + c} \cdot \frac{h}{2},$$

FIG. 134.



when the sides opposite the angles A, B, C are denoted by a, b, c and the altitude $C\ G$ by h.

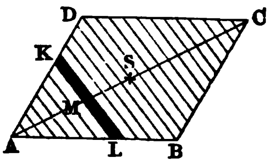
If we join the middles H, K, M of the sides of the triangle and inscribe a circle in the triangle thus obtained, its centre will coincide with the centre of gravity S; for the distance of this point from one of the sides $H\ K$ is

$$S D = N D - N S = \frac{h}{2} - \frac{a + b}{a + b + c} \cdot \frac{h}{2} = \frac{c h}{2\,(a + b + c)}$$

$$= \frac{\triangle\ A\ B\ C}{a + b + c},$$ or constant, and therefore = the distances $S\ E$ and $S\ F$ from the other sides.

§ 109. Centre of Gravity of Plane Figures.—The centre of gravity of a parallelogram $A\ B\ C\ D$, Fig. 135, is situated at the point of intersection S of its diagonals; for all strips $K\ L$, formed by drawing lines parallel to one of the diagonals $B\ D$, are divided by the other diagonal $A\ C$ into two equal parts; each of the diagonals is therefore a line of gravity.

FIG. 135.



In a triangle $A\ B\ C$, Fig. 136, every line $C\ D$ drawn from an angle to the centre D of the opposite side $A\ B$ is a line of gravity; for it bisects every element $K\ L$ of the triangle formed by drawing lines parallel to $A\ B$. If from a second angle A we draw a second line of gravity to the middle E of the opposite side $B\ C$, the point of intersection S of the two lines of gravity gives the centre of gravity of the whole triangle.

Since $B\ D = \frac{1}{2}\ B\ A$ and $B\ E = \frac{1}{2}\ B\ C$, $D\ E$ is parallel to $A\ C$ and equal to $\frac{1}{2}\ A\ C$, the triangle $D\ E\ S$ is similar to the triangle $C\ A\ S$ and $C\ S = 2\ S\ D$. Adding $S\ D$, we obtain $C\ S + S\ D$, i.e. $CD = 3\,SD$ and inversely $SD = \frac{1}{3}\,CD$. The centre of gravity S is at a distance equal to $\frac{1}{3}\,CD$ from the middle D of the base and at a distance equal to $\frac{2}{3}\,CD$ from the angle C. If we draw the perpendiculars CH and SN to the base, we have also $NS = \frac{1}{3}\,CH$; the centre of gravity S is at a distance from the base of the triangle equal to one third of the altitude.

FIG. 136.

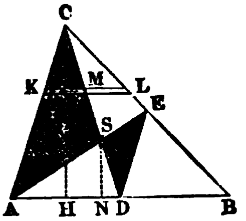
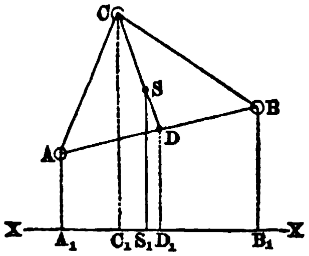


FIG. 137.



The distance of the centre of gravity of a triangle $A\,B\,C$, Fig. 137, from an axis $X\bar{X}$ is $SS_1 = DD_1 + \frac{1}{3}(CC_1 - DD_1)$, but $DD_1 = \frac{1}{2}(AA_1 + BB_1)$, and consequently we have

$$y = SS_1 = \frac{1}{3}\,CC_1 + \frac{2}{3}\cdot\frac{1}{2}(AA_1 + BB_1) = \frac{AA_1 + BB_1 + CC_1}{3},$$

i.e., the arithmetical mean of the distances of the angles from $\overline{XX}$.

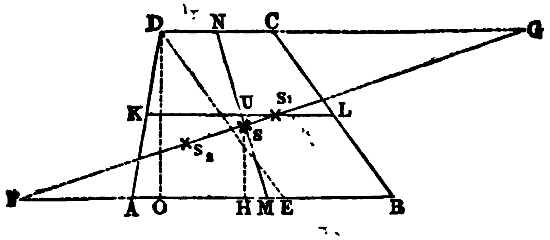
Since the distance of the centre of gravity of three equal weights, applied at the corners of a triangle, is determined in the same way, the centre of gravity of a plane triangle coincides with the centre of gravity of these three weights.

§ 110. The determination of the centre of gravity of a trapezoid $A\,B\,C\,D$, Fig. 138, can be made in the following manner. The right line $M\,N$, which joins the centres of the two bases $A\,B$ and $C\,D$, is a line of gravity of the trapezoid; for if we draw a great number of lines parallel to the bases, the figure will be divided into a number of small strips whose centres or centres of gravity lie upon the line $M\,N$. In order to determine completely the centre of gravity S, we have only to find its distance $S\,H$ from the base $A\,B$.

Let the bases $A\,B$ and $C\,D$ be denoted by b_1 and b_2 and the altitude or normal distance between the latter by h. Now if we draw $D\,E$ parallel to the side $B\,C$, we obtain a parallelogram

$B\,C\,D\,E$, whose area is $b_2\,h$ and the distance of whose centre of gravity S_1 from $A\,B$ is $= \frac{h}{2}$, and a triangle $A\,D\,E$, whose area is $\frac{(b_1 - b_2)\,h}{2}$ and the distance of whose centre of gravity from $A\,B$ is $= \frac{h}{3}$.

FIG. 188.



The statical moment of the trapezoid in reference to $A\,B$ is therefore

$$F\,y = b_2\,h\,.\,\frac{h}{2} + \frac{(b_1 - b_2)\,h}{2}\,.\,\frac{h}{3} = (b_1 + 2\,b_2)\,\frac{h^2}{6},$$

but the area of the trapezoid is $F = (b_1 + b_2)\,\frac{h}{2}$,

consequently the normal distance of the centre of gravity from the base is

$$H\,S = y = \frac{\frac{1}{6}\,(b_1 + 2\,b_2)\,h^2}{\frac{1}{2}\,(b_1 + b_2)\,h} = \frac{b_1 + 2\,b_2}{b_1 + b_2}\,.\,\frac{h}{3}.$$

The distance of this point from the middle line $K\,L = \frac{b_1 + b_2}{2}$ of the trapezoid is

$$U\,S = \frac{h}{2} - H\,S = \frac{3\,(b_1 + b_2) - 2\,(b_1 + 2\,b_2)}{b_1 + b_2}\,\frac{h}{6}, \text{ I.E., } y_1 = \frac{b_1 - b_2}{b_1 + b_2}\,.\,\frac{h}{6}.$$

In order to find the centre of gravity by construction, we have only to prolong the two bases, make the prolongation $C\,G = b_1$ and the prolongation $A\,F = b_2$, and join the extremities F and G thus obtained by a straight line; the point of intersection S with the line $M\,N$ is the required centre of gravity; for from $H\,S = \frac{b_1 + 2\,b_2}{b_1 + b_2}\,.\,\frac{h}{3}$ it follows that

$$M\,S = \frac{b_1 + 2\,b_2}{b_1 + b_2}\,.\,\frac{M\,N}{3} \text{ and } N\,S = \frac{2\,b_1 + b_2}{b_1 + b_2}\,.\,\frac{M\,N}{3}, \text{ or}$$

$$\frac{M\,S}{N\,S} = \frac{b_1 + 2\,b_2}{2\,b_1 + b_2} = \frac{\frac{1}{2}\,b_1 + b_2}{b_1 + \frac{1}{2}\,b_2} = \frac{M\,A + A\,F}{C\,G + N\,C} = \frac{M\,F}{N\,G}.$$

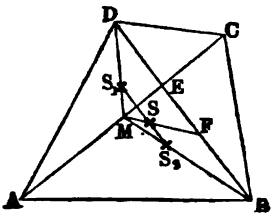
which, in consequence of the similarity of the triangles $M\,S\,F$ and $N\,S\,G$, is perfectly true.

If we denote by a the projection $A\,O$ of the side $A\,D$ upon $A\,B$, the distance of the centre of gravity from the corner A is determined by the formula

$$A\,H = x = \frac{b_1^2 + b_1\,b_2 + b_2^2 + a\,(b_1 + 2\,b_2)}{3\,(b_1 + b_2)}.$$

§ **111**. In order to find the centre of gravity of any other four-sided figure $A\;B\;C\,D$, Fig. 139, we can divide it by means of the diagonal $A\;C$ into two triangles, and then determine their centres of gravity S_1 and S_2 by means of the foregoing rules; thus we obtain a line of gravity $S_1\,S_2$. If we again divide the figure by the diagonal $B\,D$ into two other triangles, and determine their centres of gravity, we obtain a second line of gravity, whose intersection with $S_1\;S_2$ gives the centre of gravity of the whole figure.

Fig. 139.



We can proceed more simply by bisecting the diagonal $A\;C$ at M and laying off the longer portion $B\,E$ of the other diagonal upon the shorter portion, so as to have $D\,F = B\,E$. We then draw $F\,M$ and divide this line into three equal parts; the centre of gravity is at the first point of division S from M as can be proved in the following manner. We have $M\;S_1 = \frac{1}{3}\;M\;D$ and $M\,S_2 = \frac{1}{3}\,M\;B$; consequently $S_1\;S_2$ is parallel to $B\;D$, but $S\,S_1$ multiplied by $\triangle\;A\;C\,D = S\,S_2$ multiplied by $\triangle\;A\;C\,B$ or $S\,S_1\,.\,D\;E = S\,S_2\,.\,B\,E$, whence $S\,S_1 : S\,S_2 = B\;E : D\;E$. But we have $B\;E = D\;F$ and $D\;E = B\;F$, consequently also $S\,S_1 : S\,S_2 = D\;F : B\;F$. Hence the right line $M\;F$ cuts the line of gravity $S_1\,S_2$ at the centre of gravity S of the whole figure.

§ **112**. If we are required to find the centre of gravity S of a polygon $A\;B\;C\,D\,E$, Fig. 140, we divide it into triangles and find the statical moments in reference to two rectangular axes $X\,\overline{X}$ and $Y\,\overline{Y}$.

If the co-ordinates $O\;A_1 = x_1$, $O\;A_2 = y_1$, $O\;B_1 = x_2$, $O\;B_2 = y_2$, etc., of the corners are given, the statical moments of the triangles $A\;B\;O$, $B\;C\;O$, $C\,D\;O$, etc., can be determined very simply in the following manner. The area of the triangle $A\;B\;O$ is, according to the remark which follows, $= D_1 = \frac{1}{2}\;(x_1\;y_2 - x_2\,y_1)$, that of the following triangle $B\ C\ O$ is $= D_2 = \frac{1}{2}(x_2 y_3 - x_3 y_2)$, etc., the distance of the centre of gravity of $A\ B\ O$ from $Y\ \overline{Y}$ is, according to § 109,

$$u_1 = \frac{x_1 + x_2 + 0}{3} = \frac{x_1 + x_2}{3},$$

and that from $X\ \overline{X}$ is $= v_1 = \frac{y_1 + y_2}{3}$, those of the centre of gravity of the triangle $B\ C\ O$ are

$$u_2 = \frac{x_2 + x_3}{3} \text{ and } v_2 = \frac{y_2 + y_3}{3}, \text{ etc.}$$

Multiplying these distances by the areas of the triangles we obtain the statical moments of the latter, and substituting the values thus found in the formulas

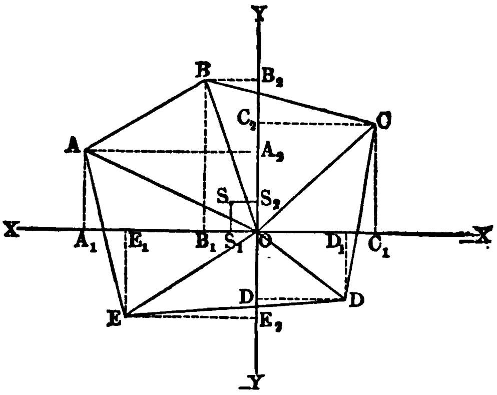
$$u = \frac{D_1 u_1 + D_2 u_2 + \ldots}{D_1 + D_2 + \ldots} \text{ and } v_1 = \frac{D_1 v_1 + D_2 v_2 + \ldots}{D_1 + D_2 + \ldots},$$

we obtain the distances $u = O\ S_1$ and $v = O\ S_2$ of the required centre of gravity S from the axes $Y\ \overline{Y}$ and $X\ \overline{X}$.

If we divide in two ways a polygon of n sides by means of a diagonal into a triangle and a polygon of $(n - 1)$ sides, and then join the centre of the former with that of the latter, we obtain in this way two lines of gravity, whose intersection gives the centre of gravity. By repeated application of this operation, we can find by construction the centre of gravity of any polygon.

EXAMPLE.—A pentagon $A\ B\ C\ D\ E$, Fig. 140, is given by the co-ordinates of its corners A, B, C, etc., and the co-ordinates of its centre of gravity are required.

FIG. 140.



Co-ordinates given.		Double area of the triangles.	The triple co-ordinate of the centre of gravity.		The sextuple statical moment.	
x	y		$3\,u_n$	$3\,v_n$	$6\,D_n\,u_n$	$6\,D_n\,v_n$
24	11	$24.21-7.11=427$	31	32	13237	13664
7	21	$7.15+21.16=441$	-9	36	-3969	15876
-16	15	$16.9+12.15=324$	-28	6	-9072	1944
-12	-9	$12.12+18.9=306$	$+6$	-21	1836	-6426
18	-12	$18.11+24.12=486$	$+42$	-1	20412	-486
		Total, 1984			22444	24572

The distance of the centre of gravity from the axis $Y\,\overline{Y}$ is therefore

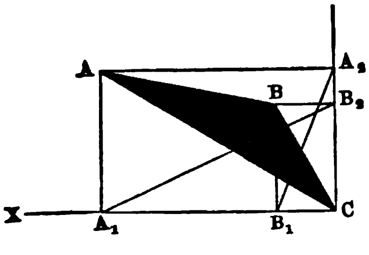
$$S\,S_2 = u = \frac{1}{3} \cdot \frac{22444}{1984} = 3{,}771$$

and from $X\,\overline{X}$ it is

$$S\,S_1 = v = \frac{1}{3} \cdot \frac{24572}{1984} = 4{,}128.$$

Remark.—If $C\,A_1 = x_1$, $C\,B_1 = x_2$, $C\,A_2 = y_1$ and $C\,B_2 = y_2$ are the co-ordinates of two corners of a triangle $A\;B\;C$, Fig. 141, the third corner C of which coincides with the origin of co-ordinates, its area is

FIG. 141.

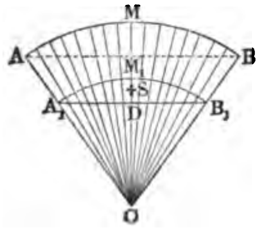


$$D = \text{trapezoid } A\,B\,B_1\,A_1 + \text{triangle } C\,B\,B_1 - \text{triangle } C\,A\,A_1$$
$$= \left(\frac{y_1 + y_2}{2}\right)(x_1 - x_2) + \frac{x_2\,y_2}{2} - \frac{x_1\,y_1}{2} = \frac{x_1\,y_2 - x_2\,y_1}{2}.$$

The area of this triangle is therefore the difference between those of two other triangles $C\,B_2\,A_1$ and $C\,A_2\,B_1$, and one co-ordinate of one point is the base of one triangle and the other co-ordinate is the altitude of the second triangle. In like manner one co-ordinate of the second point is the altitude of the first triangle and the other co-ordinate is the base of the second triangle.

§ 113. The Centre of Gravity of a Sector, $A\;C\;B$, Fig. 142, coincides with centre of gravity S of the arc $A_1\,B_1$, which has the same central angle as the former and whose radius $C\,A_1$ is two thirds of that $C\,A$ of the sector; for the latter can be divided by an infinite number of radii into small triangles, whose centres of gravity are situated at a distance from the centre C equal to two thirds of radius; the continuous succession of these centres forms the arc $A_1\ M_1\ B_1$. The centre of gravity S of the sector lies, therefore, upon the radius which bisects this surface and at the distance

FIG. 142.



$$CS = y = \frac{chord}{arc} \cdot \frac{2}{3}\,\overline{CA} = \frac{4}{3} \cdot \frac{sin. \frac{1}{2}\beta}{\beta} \cdot r$$

from the centre, when r denotes the radius of sector and β the arc which measures its central angle $A\ C\ B$.

For the semicircle $\beta = \pi$, $sin. \frac{1}{2}\,\beta = sin.\ 90° = 1$, whence

$$y = \frac{4}{3\ \pi}\,r = 0{,}4244\ r, \text{ or approximatively } \frac{14}{33}\,r.$$

For a quadrant we have

$$y = \frac{4}{3} \cdot \frac{\sqrt{\frac{1}{2}}}{\frac{1}{2}\ \pi}\,r = \frac{4\sqrt{2}}{3\ \pi}\,r = 0{,}6002\ r,$$

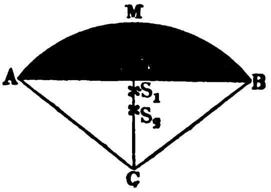
and for a sextant

$$y = \frac{4}{3} \cdot \frac{\frac{1}{2}}{\frac{1}{3}\ \pi}\,r = \frac{2}{\pi}\,r = 0{,}6366\ r.$$

§ 114. The Centre of Gravity of the Segment of a Circle,

$A\ B\ M$, Fig. 143, is found by putting its moment equal to the difference of the moments of the sector $A\ C\ B\ M$ and of the triangle $A\ C\ B$. If r is the radius $C\ A$, s the chord $A\ B$ and A the area of the segment $A\ B\ M$, we have the moment of the sector

FIG. 143.



$$= \text{sector multiplied by } \overline{CS_1} = \frac{r\,.\,arc}{2} \cdot \frac{chord}{arc} \cdot \frac{2}{3}\,r = \frac{1}{3}\,s\,r^2,$$

the moment of triangle

$$= \text{triangle multiplied by } \overline{CS_2} = \frac{s}{2}\sqrt{r^2 - \frac{s^2}{4}} \cdot \frac{2}{3}\sqrt{r^2 - \frac{s^2}{4}}$$

$$= \frac{s\,r^2}{3} - \frac{s^3}{12},$$

and consequently the moment of the segment A

$$A\,.\,\overline{CS} = A\,y = \frac{1}{3}\,s\,r^2 - \left(\frac{s\,r^2}{3} - \frac{s^3}{12}\right) = \frac{s^3}{12}.$$

Hence the required distance is $y = \dfrac{s^3}{12\,A}$.

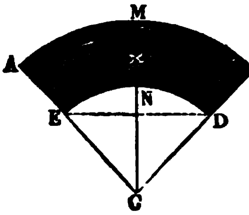
For a semicircle $s = 2\,r$ and $A = \dfrac{1}{2}\,\pi\,r^2$, and therefore

$$y = \frac{8\,r^3}{12\,.\,\dfrac{\pi\,r^2}{2}} = \frac{4\,r}{3\,\pi},$$

as we have already found.

In the same way the centre of gravity S of a section of a ring $A\;B\;D\;E$, Fig. 144, can be found; for it is the difference of two sectors $A\;C\;B$ and $D\;C\;E$. If the radii are $C\,A = r_1$ and $C\,E = r_2$ and the chords $A\;B = s_1$ and $D\;E = s_2$, we have the statical moment of the sectors $\dfrac{s_1\,r_1^2}{3}$ and $\dfrac{s_2\,r_2^2}{3}$, and consequently that of the portion of the ring

FIG. 144.



$$M = \frac{s_1\,r_1^2 - s_2\,r_2^2}{3}, \text{ or since } \frac{s_2}{s_1} = \frac{r_2}{r_1},$$

$$M = \frac{r_1^3 - r_2^3}{3}\,.\,\frac{s_1}{r_1}.$$

The area of the piece of the ring is $F = \dfrac{\beta\,r_1^2}{2} - \dfrac{\beta\,r_2^2}{2} = \beta\left(\dfrac{r_1^2 - r_2^2}{2}\right)$, in which β denotes the arc which measures the central angle $A\;C\;B$; hence the centre of gravity S of the section of the ring is determined by the formula

$$C\,S = y = \frac{M}{F} = \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\cdot\frac{2}{3}\cdot\frac{s_1}{r_1\,\beta} = \frac{2}{3}\left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right)\cdot\frac{chord}{arc}$$

$$= \frac{4}{3}\,\frac{sin.\,\frac{1}{2}\,\beta}{\beta}\cdot\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2} = \frac{sin.\,\frac{1}{2}\,\beta}{\beta}\left(1 + \tfrac{1}{12}\left(\frac{b}{r}\right)^2\right)2\,r, \text{ when } r_1 - r_2$$

$= b$ and $r_1 + r_2 = 2\,r$.

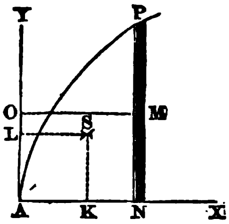
EXAMPLE.—If the radius of the extrados of an arch is $r_1 = 5$ feet, and that of the intrados is $r_2 = 3\frac{1}{2}$ feet, and if the central angle is $\beta^\circ = 130^\circ$, the distance of the centre of gravity of the front surface of the arch from its centre is

$$y = \frac{4}{3}\,\frac{sin.\;65^\circ}{arc.130^\circ}\cdot\frac{5^3 - 3{,}5^3}{5^2 - 3{,}5^2} = \frac{4\,.\,0{,}9063}{3\,.\,2{,}2689}\cdot\frac{125 - 42{,}875}{25 - 12{,}25} = \frac{3{,}6252\;.\;82{,}125}{6{,}8067\;.\;12{,}75}$$

$= 3{,}430$ feet.

(§ 115.) **Determination of the Centre of Gravity by the Aid of the Calculus.**—The determination of the centre of gravity by means of the calculus is accomplished in the following manner. Let *A N P*, Fig. 145, be the given surface, $A\,N = x$ its abscissa and $N\,P = y$ its ordinate. The area of an element of the surface is

FIG. 145.



$d\,F = y\,d\,x$ (see Introduction to the Calculus, Art. 29) and its moment in reference to the axis of ordinates *A Y* is $\overline{O\,M}\,.\,d\,F = \overline{A\,N}\,.\,d\,F = x\,y\,d\,x$; if we put the distance $L\,S = A\,K$ of the centre of gravity *S* of the whole surface *F* from the axis *A Y*, $=u$, we have

$$F\,u = \int x\,y\,d\,x,$$

and consequently 1) $u = \frac{\int x\,y\,d\,x}{F} = \frac{\int x\,y\,d\,x}{\int y\,d\,x}$.

Since the centre or centre of gravity *M* of the element *N M P* is at the distance $N\,M = \frac{1}{2}\,y$ from the axis *A X*, the moment of $d\,F$ in reference to this axis *A X* is

$$\overline{N\,M}\,.\,d\,F = \tfrac{1}{2}\,y\,d\,F = \tfrac{1}{2}\,y^2\,d\,x;$$

putting the distance $K\,S = A\,L$ of the centre of gravity *S* of the whole surface *F* from the axis *A X*, $= v$, we have

$$F\,v = \int \tfrac{1}{2}\,y^2\,d\,x, \text{ and therefore}$$

$$2)\ v = \frac{\frac{1}{2}\int y^2\,d\,x}{F} = \frac{1}{2}\,\frac{\int y^2\,d\,x}{\int y\,d\,x}.$$

E.G., for the parabola, whose equation is $y^2 = p\,x$ or $y = \sqrt{p}\,.\,x^{\frac{1}{2}}$, we have

$$u = \frac{\int \sqrt{p}\,.\,x^{\frac{1}{2}}\,x\,d\,x}{\int \sqrt{p}\,.\,x^{\frac{1}{2}}\,d\,x} = \frac{\sqrt{p}\int x^{\frac{3}{2}}\,d\,x}{\sqrt{p}\int x^{\frac{1}{2}}\,d\,x} = \frac{\int x^{\frac{3}{2}}\,d\,x}{\int x^{\frac{1}{2}}\,d\,x}$$

$$= \frac{\frac{2}{5}\,x^{\frac{5}{2}}}{\frac{2}{3}\,x^{\frac{3}{2}}} = \tfrac{3}{5}\,x,$$

or $L\,S = A\,K = \frac{3}{5}\,A\,N$, and, on the contrary,

$$v = \tfrac{1}{2}\,\frac{\int p\,x\,d\,x}{\sqrt{p}\int x^{\frac{1}{2}}\,d\,x} = \tfrac{1}{2}\,\sqrt{p}\,\frac{\int x\,d\,x}{\int x^{\frac{1}{2}}\,d\,x} = \tfrac{1}{2}\,\sqrt{p}\,\frac{\frac{1}{2}\,x^2}{\frac{2}{3}\,x^{\frac{3}{2}}}$$

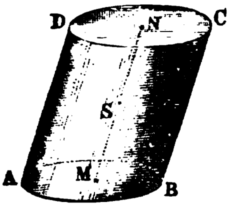
$$= \tfrac{3}{8}\,\sqrt{p\,x} = \tfrac{3}{8}\,y,$$

or

$$K\,S = A\,L = \tfrac{3}{5}\,N\,P.$$

§ 116. The Centre of Gravity of Curved Surfaces.—The centre of gravity of the curved surface (envelope) of a cylinder *A B C D*, Fig. 146, lies in the middle *S* of the axis *M N* of this body; for all the ring-shaped elements of the envelope of the cylinder, obtained by cutting the body parallel to its base, have their centres and centres of gravity upon this axis; the centres of gravity form then a homogeneous heavy line. For the same reason the centre of gravity of the envelope of a prism lies in the middle of the line, which unites the centres of gravity of its bases.

Fig. 146.



The centre of gravity *S* of the envelope of a right cone *A B C*, Fig. 147, lies in the axis of the cone one-third of its length from the base, or two-thirds from the apex; for this curved surface can be divided into an infinite number of infinitely small triangles by means of straight lines (called sides of the cone). The centre of gravity of all these triangles form a circle *H K*, which is situated at a distance equal to two-thirds of the axis from the apex *C*, and whose centre or centre of gravity *S* lies in the axis *C M*.

Fig. 147.

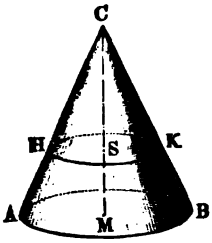
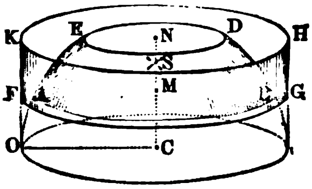


Fig. 148.

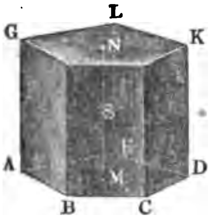


The centre of gravity of a zone *A B D E*, Fig. 148, of a sphere, and also that of spherical shell, lies in the middle *S* of its height *M N*; for, according to the teachings of geometry, the zone has the same area as the envelope *F G H K* of a cylinder, whose height is equal to that *M N* of the zone and whose radius is equal to that *C O* of the sphere, and this holds good even in the ring-shaped elements obtained by passing an infinite number of planes parallel to the base through the zone; hence the centre of gravity of the zone and of the envelope of the cylinder coincide.

Remark.—The centre of gravity of the envelope of an oblique cone or pyramid is to be found, it is true, at a distance from the base equal to one-third of the altitude, but not in the right line joining the apex to the centre of gravity of the periphery of the base, since by cutting the envelope parallel to the latter we divide it into rings of different thicknesses on different sides.

§ 117. Centre of Gravity of Bodies.—The centre of gravity S of a prism $A\ K$, Fig. 149, is the centre of the line uniting the centres of gravity M and N of the two bases $A\ D$ and $G\ K$; for by passing planes parallel to the base through the body we divide it into similar slices, whose centres lie in $M\ N$, and whose continuous succession form the homogeneous heavy line $M\ N$.

FIG. 149.



For the same reason the centre of gravity of a cylinder is to be found in the middle of its axis.

The centre of gravity of pyramid $A\ D\ F$, Fig. 150, lies in the straight line $M\ F$ joining the apex F with the centre of gravity M of the base; for all slices such as $N\ O\ P\ Q\ R$ have, in consequence of their similarity to the base $A\ B\ C\ D\ E$, their centre of gravity upon this line.

FIG. 150.

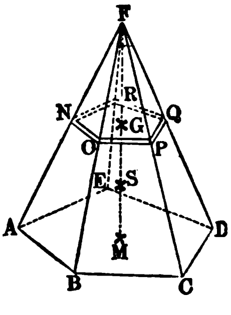
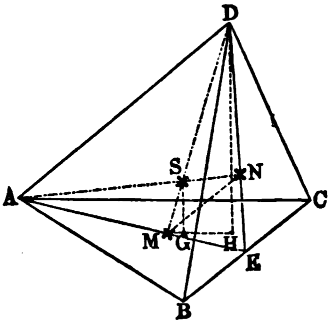


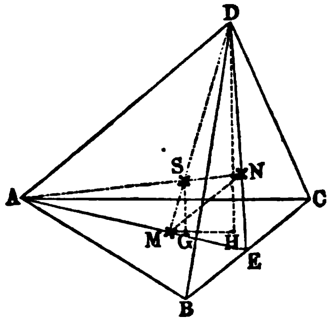
FIG. 151.



If the body is a triangular pyramid, like $A\ B\ C\ D$, Fig. 151, we can consider each of the four corners as the apex and the opposite side as the base. The centre of gravity is therefore determined by the intersection of the two straight lines drawn from the corners D and A to the centres of gravity M and N of the opposite surfaces $A\ B\ C$ and $B\ C\ D$.

If the right lines $E\ A$ and $E\ D$ are also given, we have (according to § 109) $EM = \frac{1}{3} EA$ and $EN = \frac{1}{3} ED$. MN is therefore parallel to AD and, $= \frac{1}{3} AD$, and the triangle MNS is similar to the triangle DAS. In consequence of this similarity we have also $MS = \frac{1}{3} DS$ or $DS = 3\,MS$ and $MD = MS + SD = 4\,MS$, or inversely $MS = \frac{1}{4}\,MD$. The distance of the centre of gravity of a triangular pyramid from its base along the line joining the centre of gravity M of the base to the apex D of the pyramid is equal to one-fourth of this line.

FIG. 152.



If the altitudes DH and SG are given and if we draw the line HM, we obtain the similar triangles DHM and SGM, in which, as we have just seen, $SG = \frac{1}{4}\,DH$. We can therefore assert that the distance of the centre of gravity of a triangular pyramid from its base is one-fourth and from its apex three-fourths of its altitude.

Finally, since every pyramid and every cone is composed of triangular pyramids of the same height, the centre of gravity of every pyramid and of every cone lies at a distance from the base equal to one-fourth of the altitude and at a distance from the apex equal to three-fourths of the altitude.

We determine the centre of gravity of a pyramid or of a cone by passing a plane, at a distance from the base equal to one-fourth the altitude, through the body parallel to its base and by finding the centre of gravity of this section or the point where a line drawn from the centre of gravity of the base to the apex will cut it.

§ **118.** If we know the distances AA_1, BB_1, etc., of the four corners of a triangular pyramid $ABCD$, Fig. 153, from a plane HK, the distance SS_1 of its centre of gravity S from the plane is their mean value

$$SS_1 = \frac{AA_1 + BB_1 + CC_1 + DD_1}{4},$$

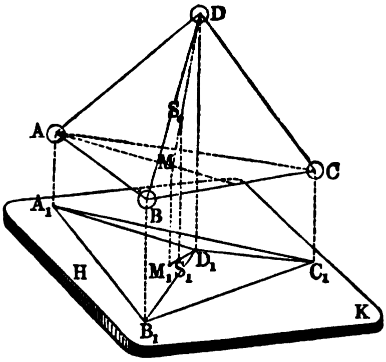
which can be proved in the following manner. The distance of the centre of gravity M of the base ABC from this plane is (§ 109)

$$MM_1 = \frac{AA_1 + BB_1 + CC_1}{3},$$

and the distance of the centre of gravity S of the pyramid is

$$S\,S_1 = M\,M_1 + \tfrac{1}{4}\,(D\,D_1 - M\,M_1),$$

FIG. 153.



in which $D\,D_1$ is the distance of the apex. Combining the last two equations, we obtain

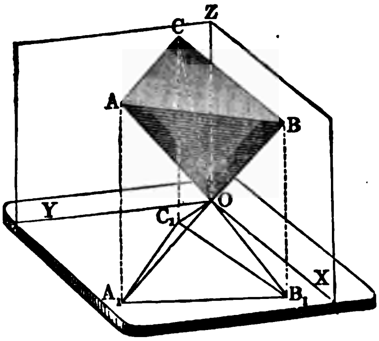
$$S\,S_1 = y = \tfrac{3}{4}\,M\,M_1 + \tfrac{1}{4}\,D\,D_1 = \frac{A\,A_1 + B\,B_1 + C\,C_1 + D\,D_1}{4}.$$

The distance of the centre of gravity of four equal weights placed at the corners of the triangular pyramid is also equal to the arithmetical mean

$$y = \frac{A\,A_1 + B\,B_1 + C\,C_1 + D\,D_1}{4};$$

consequently the centre of gravity of the pyramid coincides with that of these weights.

FIG. 154.



REMARK.—The determination of the volume of a triangular pyramid from the co-ordinates of its corners is very simple. If we pass through the apex O of such a pyramid $A\,B\,C\,O$, Fig. 154, three co-ordinate planes $X\,Y$, $X\,Z$, $Y\,Z$, and denote the distances of the corners A, B, C from these planes by z_1, z_2, z_3; y_1, y_2, y_3 and x_1, x_2, x_3, we have the volume of the pyramid

$$V = \pm\tfrac{1}{6}\,[x_1\,y_2\,z_3 + x_2\,y_3\,z_1 + x_3\,y_1\,z_2 - (x_1\,y_3\,z_2 + x_2\,y_1\,z_3 + x_3\,y_2\,z_1)],$$

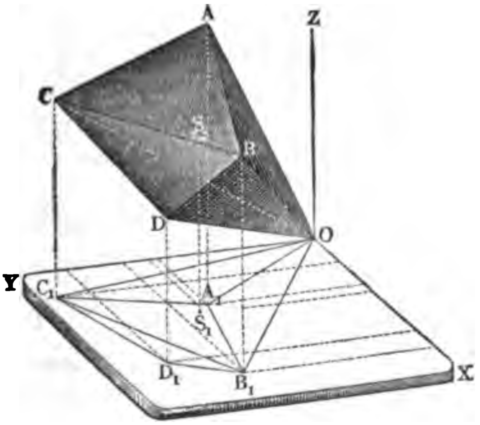
which is found by considering the pyramid as the aggregate of four obliquely truncated prisms.

The distances of the centre of gravity of this pyramid from the three co-ordinate planes YZ, XZ and XY are

$$x = \frac{x_1 + x_2 + x_3}{4},\ y = \frac{y_1 + y_2 + y_3}{4},\ \text{and } z = \frac{z_1 + z_2 + z_3}{4}$$

§ **119.** The centre of gravity S of any polyhedron, such as $A\,B\,C\,D\,O$, Fig. 155, can be found by calculating the statical moments and volumes of the triangular pyramids, such as $A\,B\,C\,O$, $B\,C\,D\,O$, into which it can be decomposed.

FIG. 155.



If the distances of the corners A, B, C, etc., from the co-ordinate planes $Y\,Z$, $X\,Z$ and $X\,Y$, passing through the common apex O of all the pyramids, are x_1, x_2, x_3, etc., y_1, y_2, y_3, etc., and z_1, z_2, z_3, etc., we have the volumes of the various pyramids

$$V_1 = \pm \tfrac{1}{6}\,(x_1\,y_2\,z_3 + x_2\,y_3\,z_1 + x_3\,y_1\,z_2 - x_1\,y_3\,z_2 - x_2\,y_1\,z_3 - x_3\,y_2\,z_1),$$
$$V_2 = \pm \tfrac{1}{6}\,(x_2\,y_3\,z_4 + x_3\,y_4\,z_2 + x_4\,y_2\,z_3 - x_2\,y_4\,z_3 - x_3\,y_2\,z_4 - x_4\,y_3\,z_2),$$

etc., and the distances of their centres of gravity from the co-ordinate planes are

$$u_1 = \frac{x_1 + x_2 + x_3}{4},\ v_1 = \frac{y_1 + y_2 + y_3}{4},\ w_1 = \frac{z_1 + z_2 + z_3}{4},$$

$$u_2 = \frac{x_2 + x_3 + x_4}{4},\ v_2 = \frac{y_2 + y_3 + y_4}{4},\ w_2 = \frac{z_2 + z_3 + z_4}{4},\ \text{etc.}$$

From these values we calculate the distances u, v, w of the centre of gravity S of the whole body by means of the formulas

$$u = \frac{V_1 u_1 + V_2 u_2 + \dots}{V_1 + V_2 + \dots},\quad v = \frac{V_1 v_1 + V_2 v_2 + \dots}{V_1 + V_2 + \dots},\ \text{and}$$

$$w = \frac{V_1 w_1 + V_2 w_2 + \dots}{V_1 + V_2 + \dots}$$

Example.—A body $A\ B\ C\ D\ O$, Fig. 155, bounded by six triangles, is determined by the following values of its co-ordinates, and we wish to find the co-ordinates of the centre of gravity.

Given Co-ordinates.			The sextuple volume of the triangular pyramids $A\ B\ C\ O$ and $B\ C\ D\ O$.	Quadruple Co-ordinates of the Centres of Gravity.			Twenty-four fold Statical Moments.		
x	y	z		$4u_n$	$4v_n$	$4w_n$	$24\ V_n u_n$	$24\ V_n v_n$	$24\ V_n w_n$
20	23	41	$6\ V_1 = \begin{Bmatrix} 20.29.28 \\ 23.30.12 \\ 41.45.40 \end{Bmatrix} - \begin{Bmatrix} 20.40.30 \\ 23.28.45 \\ 41.12.29 \end{Bmatrix} = 31072$	77	92	99	2392544	2858624	3076128
45	29	30							
12	40	28	$6\ V_2 = \begin{Bmatrix} 45.35.28 \\ 29.20.12 \\ 30.38.40 \end{Bmatrix} - \begin{Bmatrix} 45.40.20 \\ 29.28.38 \\ 30.12.35 \end{Bmatrix} = 17204$	95	104	78	1634380	1789216	1341912
38	35	20							
			Total 48276	.....			4026924	4647840	4418040

From the results of the above calculation we deduce the distances of the centre of gravity S of the whole body from the planes $Y\,Z$, $X\ Z$ and $X\ Y$,

$$u = \frac{1}{4} \cdot \frac{4026924}{48276} = 20{,}853,$$

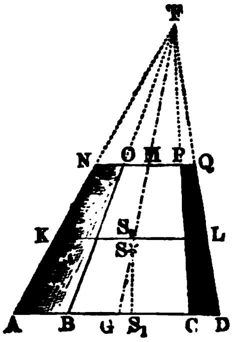
$$v = \frac{1}{4} \cdot \frac{4647840}{48276} = 24{,}069,\ \text{and},$$

$$w = \frac{1}{4} \cdot \frac{4418040}{48276} = 22{,}879.$$

Remark.—We can also determine the centre of gravity of a polyhedron by dividing it in two ways by means of a plane into two pieces and by joining the centres of gravity of each two pieces; the intersection of the two lines gives the required centre of gravity. Since both lines are lines of gravity, the intersection must be the centre of gravity of the whole body. If the body has a great number of corners, this process becomes very long, in consequence of the number of times this division must be repeated. The five-cornered body in Fig. 155, which must be divided in two ways into two triangular pyramids, has its centre of gravity at the intersection of the lines joining the centres of gravity of each two of these pyramids.

§ **120.** The centre of gravity of a *truncated pyramid* or *frustum of a pyramid A D Q N*, Fig. 156, lies in the line $G M$ joining the centres of gravity of the two (parallel) bases. In order to determine the distance of this point from one of the bases we must calculate the volumes and moments of the complete pyramid $A D F$ and of the portion $N Q F$, which has been cut away. If the areas of the bases $A D$ and $N Q$ are $= G_1$ and G_2, and if the perpendicular distance between them $= h$, the height x of the portion of the pyramid, which is wanting, is determined by the formula

FIG. 156.



$$\frac{G_1}{G_2} = \frac{(h+x)^2}{x^2},$$

whence
$$\frac{h}{x} + 1 = \sqrt{\frac{G_1}{G_2}} \text{ or } x = \frac{h\sqrt{G_2}}{\sqrt{G_1} - \sqrt{G_2}},$$

and
$$h + x = \frac{h\sqrt{G_1}}{\sqrt{G_1} - \sqrt{G_2}}.$$

The moment of the whole pyramid in reference to its base is

$$\frac{G_1(h+x)}{3} \cdot \frac{h+x}{4} = \frac{1}{12}\,\frac{h^2 G_1^2}{(\sqrt{G_1} - \sqrt{G_2})^2},$$

and that of the part of pyramid, that is wanting, is

$$\frac{G_2\,x}{3}\left(h + \frac{x}{4}\right) = \frac{1}{3}\,\frac{h^2\sqrt{G_2^3}}{\sqrt{G_1} - \sqrt{G_2}} + \frac{1}{12} \cdot \frac{h^2 G_2^2}{(\sqrt{G_1} - \sqrt{G_2})^2},$$

hence the moment of the truncated pyramid is

$$\frac{h^2}{12(\sqrt{G_1} - \sqrt{G_2})^2} \cdot [G_1^2 - 4(\sqrt{G_1 G_2^3} - G_2^2) - G_2^2] =$$

$$\frac{h^2(G_1^2 - 4\,G_2\sqrt{G_1\,G_2} + 3\,G_2^2)}{12\,(G_1 - 2\sqrt{G_1\,G_2} + G_2)} = \frac{h^2}{12} \cdot (G_1 + 2\sqrt{G_1\,G_2} + 3\,G_2).$$

Now the contents of the truncated pyramid are

$$V = (G_1 + \sqrt{G_1\,G_2} + G_2)\frac{h}{3};$$

and therefore the distance of the centre of gravity S from the base is

$$y = \frac{G_1 + 2\sqrt{G_1 G_2} + 3 G_2}{G_1 + \sqrt{G_1 G_2} + G_2} \cdot \frac{h}{4}.$$

The distance $S_0 S$ of this point from the plane $K L$, passing through the middle of the body parallel to its base and dividing its height into two equal parts, is

$$y_1 = \frac{h}{2} - y = \frac{[2(G_1 + \sqrt{G_1 G_2} + G_2) - (G_1 + 2\sqrt{G_1 G_2} + 3 G_2)]}{G_1 + \sqrt{G_1 G_2} + G_2} \frac{h}{4}$$

$$= \left(\frac{G_1 - G_2}{G_1 + \sqrt{G_1 G_2} + G_2}\right) \frac{h}{4}.$$

If the radii of the bases of a *frustum of cone* are r_1 and r_2, or $G_1 = \pi r_1^2$ and $G_2 = \pi r_2^2$, we have

$$y = \frac{r_1^2 + 2 r_1 r_2 + 3 r_2^2}{r_1^2 + r_1 r_2 + r_2^2} \cdot \frac{h}{4} \text{ and}$$

$$y_1 = \frac{r_1^2 - r_2^2}{r_1^2 + r_1 r_2 + r_2^2} \cdot \frac{h}{4}.$$

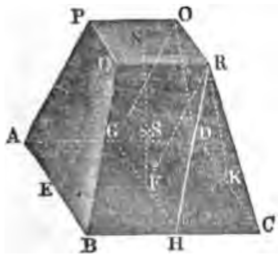
Example.—The centre of gravity of a truncated cone whose altitude is $h = 20$ inches and whose radii are $r = 12$ inches and $r_1 = 8$ inches lies, as is always the case, in the line joining the centres of the bases, and at a distance

$$y = \frac{20}{4} \cdot \frac{12^2 + 2 \,.\, 12 \,.\, 8 + 3 \,.\, 8^2}{12^2 + 12 \,.\, 8 + 8^2} = \frac{5 \,.\, 528}{304} = \frac{2640}{304} = 8{,}684 \text{ inches}$$

from the greater base.

§ **121**. An *obelisk*, I.E., a body $A\ C\ O\ Q$, Fig. 157, bounded by two dissimilar rectangular bases and by four trapezoids, can be decomposed into a parallelopipedon $A\ F\ R\ P$, into two triangular prisms $E\ H\ R\ Q$ and $G\ K\ R\ O$ and into a four-sided pyramid $H K R$. By the aid of the moments of these component parts we can find the centre of gravity of the whole body.

FIG. 157.



It is easy to see that the right line joining the middle of one base to that of the other is a line of gravity of the body; we have, therefore, but the distance of the centre of gravity from one of the bases to determine. Let us denote the length $B\ C$ and the width $A\ B$ of one base by l_1 and b_1, and the length $Q\ R$ and the width $P\ Q$ of the other base by l_2 and b_2, and the height of the body or the distance of the bases apart by h. The contents of the parallelopipedon are then $= b_2\, l_2\, h$, and its moment is $b_2\, l_2\, h \cdot \frac{h}{2} = \frac{1}{2}\, b_2\, l_2\, h^2$. The contents of the two triangular prisms are

$$= ([b_1 - b_2]\, l_2 + [l_1 - l_2]\, b_2)\, \frac{h}{2},$$

and their moments are

$$= ([b_1 - b_2]\, l_2 + [l_1 - l_2]\, b_2)\, \frac{h}{2} \cdot \frac{h}{3},$$

and finally the contents of the pyramid are

$$= (b_1 - b_2)\,(l_1 - l_2)\, \frac{h}{3},$$

and its moment is

$$= (b_1 - b_2)\,(l_1 - l_2)\, \frac{h}{3} \cdot \frac{h}{4}.$$

From the above we deduce the volume of the whole body

$$V = (6\, b_2\, l_2 + 3\, b_1\, l_2 + 3\, l_1\, b_2 - 6\, b_2\, l_2 + 2\, b_1\, l_1 + 2\, b_2\, l_2 - 2\, b_1\, l_2 - 2\, b_2\, l_1) \cdot \frac{h}{6}$$

$$= (2\, b_1\, l_1 + 2\, b_2\, l_2 + b_1\, l_2 + l_1\, b_2)\, \frac{h}{6},$$ its moment

$$Vy = (6\, b_2\, l_2 + 2\, b_1\, l_2 + 2\, l_1\, b_2 - 4\, b_2\, l_2 + b_1\, l_1 + b_2\, l_2 - b_1\, l_2 - l_1\, b_2) \cdot \frac{h^2}{12}$$

$$= (3\, b_2\, l_2 + b_1\, l_1 + b_1\, l_2 + b_2\, l_1)\, \frac{h^2}{12},$$

and the distance of its centre of gravity S from the base $b_1\, l_1$

$$y = \frac{b_1\, l_1 + 3\, b_2\, l_2 + b_1\, l_2 + b_2\, l_1}{2\, b_1\, l_1 + 2\, b_2\, l_2 + b_1\, l_2 + b_2\, l_1} \cdot \frac{h}{2}.$$

We can also put (see the "Planimetrie und Stereometrie" of C. Koppe)

$$V = \frac{b_1 + b_2}{2} \cdot \frac{l_1 + l_2}{2} \cdot h + \frac{b_1 - b_2}{2} \cdot \frac{l_1 - l_2}{2} \cdot \frac{h}{3}.$$

The distance y_1 of the centre of gravity from the cross section through the middle is determined by the formula

$$y_1 = \frac{h}{2} - y = \frac{b_1\, l_1 - b_2\, l_2}{3\,(b_1 + b_2)\,(l_1 + l_2) + (b_1 - b_2)\,(l_1 - l_2)} \cdot h$$

REMARK.—This formula is also applicable to bodies with elliptical bases. If the semi-axes of one base are a_1 and b_1 and those of the other a_2 and b_2, the volume of such a body is

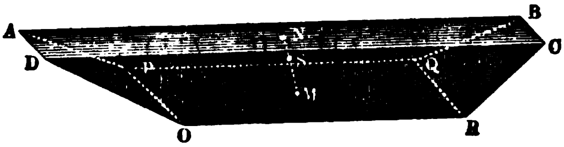
$$V = \frac{\pi h}{6}\,(2\, a_1\, b_1 + 2\, a_2\, b_2 + a_1\, b_2 + a_2\, b_1),$$

and the distance of its centre of gravity from the base $\pi\, a_1\, b_1$ is

$$y = \frac{a_1\, b_1 + 3\, a_2\, b_2 + a_1\, b_2 + a_2\, b_1}{2\, a_1\, b_1 + 2\, a_2\, b_2 + a_1\, b_2 + a_2\, b_1} \cdot \frac{h}{2}.$$

EXAMPLE.—If the embankment *A C O Q*, Fig. 158, for a dam is 20 feet high, 250 feet long and 40 wide at the bottom, and 400 feet long and 15 feet wide on top, what is the distance of its centre of gravity from its base? Here $b_1 = 40$, $l_1 = 250$, $b_2 = 15$, $l_2 = 400$, and $h = 20$, and consequently the distance is

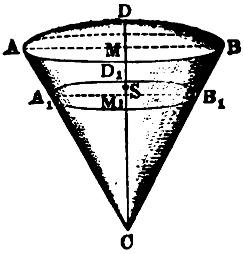
FIG. 158.



$$y = \frac{40 \,.\, 250 + 3 \,.\, 15 \,.\, 400 + 40 \,.\, 400 + 15 \,.\, 250}{2 \,.\, 40 \,.\, 250 + 2 \,.\, 15 \,.\, 400 + 40 \,.\, 400 + 15 \,.\, 250} \cdot \frac{20}{2}$$
$$= \frac{4775}{5175} \cdot 10 = \frac{1910}{207} = 9{,}227 \text{ feet.}$$

§ **122.** If the circular sector *A C D*, Fig. 159, is revolved about its radius *C D*, a *spherical sector A C B* is generated, the centre of gravity of which can be determined in the following manner. We can consider this body as the aggregate of an infinite number of infinitely thin pyramids, whose common apex is the centre *C* and whose bases form the spherical zone *A D B*. The centres of gravity of each of these pyramids are situated at a distance equal to $\frac{3}{4}$ of the radius *C D* of the sphere from its centre *C*, and they form a second spherical zone $A_1 D_1 B_1$, whose radius $C D_1 = \frac{3}{4}\, C D$. The centre of gravity of this curved surface is also that of the spherical sector; for the weights of the elementary pyramids are equally distributed over this surface, which is therefore everywhere equally heavy.

FIG. 159.



If we put the radius $C A = C D = r$ and the altitude $D M$ of the exterior zone $= h$, we have for the interior zone $C D_1 = \frac{3}{4}\, r$ and $M_1 D_1 = \frac{3}{4}\, h$, and consequently (§ 116) $S D_1 = \frac{1}{2}\, M_1 D_1 = \frac{3}{8}\, h$, and the distance of the centre of gravity of the spherical sector from the centre *C* is

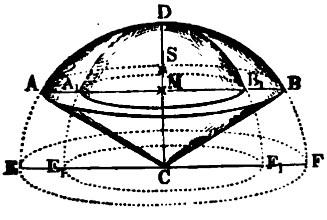
$$C S = C D_1 - S D_1 = \tfrac{3}{4}\, r - \tfrac{3}{8}\, h = \tfrac{3}{4}\left(r - \frac{h}{2}\right).$$

For a hemisphere $r = h$, and therefore the distance of its centre of gravity *S* from the centre *C* is

$$C S = \tfrac{3}{4} \cdot \frac{r}{2} = \tfrac{3}{8}\, r.$$

§ **123.** We obtain the centre of gravity S of a spherical segment $A\ B\ D$, Fig. 160, by putting the moment of the segment equal to that of the spherical sector $A\ D\ B\ C$ less that of the cone $A\ B\ C$. Denoting again the radius $C\ D$ of the sphere by r and the altitude $D\ M$ by h, we have the moment of the sector

FIG. 160.



$$= \tfrac{2}{3}\, \pi\, r^2 h \,.\, \tfrac{3}{8}\, (2\, r - h) = \tfrac{1}{4}\, \pi\, r^2 h\, (2\, r - h),$$

and that of the cone

$$= \tfrac{1}{3}\, \pi\, h (2\, r - h) \,.\, (r - h) \,.\, \tfrac{3}{4}\, (r - h) = \tfrac{1}{4}\, \pi\, h\, (2\, r - h)\, (r - h)^2;$$

hence the moment of the segment is

$$V y = \tfrac{1}{4}\, \pi\, h\, (2\, r - h)\, (r^2 - [r - h]^2) = \tfrac{1}{4}\, \pi\, h^2\, (2\, r - h)^2.$$

The contents of the segment are

$$V = \tfrac{1}{3}\, \pi\, h^2\, (3\, r - h),$$

and consequently the required distance is

$$C S = y = \frac{\tfrac{1}{4}\, \pi\, h^2\, (2\, r - h)^2}{\tfrac{1}{3}\, \pi\, h^2\, (3\, r - h)} = \tfrac{3}{4} \,.\, \frac{(2\, r - h)^2}{3\, r - h}.$$

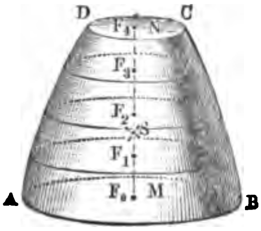
If we put again $h = r$, the segment becomes a hemisphere, and, as before, we have $C\ S = \tfrac{3}{8}\, r$.

This formula is also true for the segment $A_1\, D\, B_1$ of a spheroid generated by the revolution of the arc $D\ A_1$ of an ellipse about its major axis $C\ D = r$; for if we cut the two segments by means of planes parallel to the base $A\ B$ into thin slices, the ratio of the corresponding slices is constant and $= \dfrac{\overline{M A_1}^2}{\overline{M A}^2} = \dfrac{\overline{C E_1}^2}{\overline{C E}^2} = \dfrac{b^2}{r^2}$, when b denotes the smaller semi-axis of the ellipse. We must multiply not only the volume, but also the moment of the spherical segment by $\dfrac{b^2}{r^2}$ to obtain the volume and moment of the segment of the spheroid, and therefore the quotient $C\ S = \dfrac{\text{moment}}{\text{volume}}$ is not changed.

In general we have $C\, S = y = \tfrac{3}{4}\, \dfrac{(2\, r - h)^2}{3\, r - h}$, in which r denotes that semi-axis about which the ellipse is revolved, when generating the spheroid.

§ **124. Application of Simpson's Rule.**—In order to find the centre of gravity of an irregular body $A\ B\ C\ D$, Fig. 161, we divide it, by means of planes equally distant from each other, into thin slices and determine the area of the cross sections thus obtained and their moments in reference to the first parallel plane $A\,B$, which serves as base, and we then combine the latter by means of Simpson's rule.

FIG. 161.



If the areas of the cross-sections are F_0, F_1, F_2, F_3, F_4 and the total height or distance $M\,N$ between the two parallel planes farthest apart $= h$, we have, according to Simpson's rule, the volume of the body

$$V = (F_0 + 4\,F_1 + 2\,F_2 + 4\,F_3 + F_4)\,\frac{h}{12}.$$

Multiplying in this formula each surface by its distance from its base we obtain the moment of the body, viz.,

$$Vy = (0\,.\,F_0 + 1\,.\,4\,F_1 + 2\,.\,2\,F_2 + 3\,.\,4\,F_3 + 4\,F_4)\,\frac{h}{4}\,.\,\frac{h}{12},$$

and dividing the last equation by the first we obtain the required distance of the centre of gravity S

$$M\,S = y = \frac{(0\,.\,F_0 + 1\,.\,4\,F_1 + 2\,.\,2\,F_2 + 3\,.\,4\,F_3 + 4\,F_4)}{F_0 + 4\,F_1 + 2\,F_2 + 4\,F_3 + F_4}\,\frac{h}{4}.$$

If the number of slices $= 6$, we have

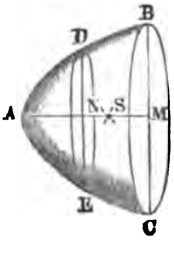
$$y = \frac{0\,.\,F_0 + 1\,.\,4\,F_1 + 2\,.\,2\,F_2 + 3\,.\,4\,F_3 + 4\,.\,2\,F_4 + 5\,.\,4\,F_5 + 6\,F_6}{F_0 + 4\,F_1 + 2\,F_2 + 4\,F_3 + 2\,F_4 + 4\,F_5 + F_6}\,.\,\frac{h}{6}.$$

It is easy to see how this formula varies, when the number of slices is changed. The rule, however, requires, that the number of slices shall be an even one, or the number of surfaces an uneven one.

In many cases we need determine but one distance, as a line of gravity is also known. Solids of rotation formed upon the turning lathe are very common examples of such bodies. Their axis of rotation is a line of gravity.

This formula is also applicable to the determination of the centre of gravity of a surface, in which case the cross sections F_0, F_1, F_2, etc., become lines.

FIG. 162.



EXAMPLE 1. For the parabolic conoid $A\,B\,C$, Fig. 162, formed by the revolution of a portion $A\,B\,M$ of a parabola about its axis $A\,M$, we obtain, when we make but one section $D\,N\,E$ through the middle, the following. Let the altitude $A\,M = h$, the radius $B\,M = r$, $A\,N = N\,M = \frac{h}{2}$ and consequently the radius $D\,N = r\,\sqrt{\tfrac{1}{2}}$. The area of the section through A is $F_0 = 0$,

that through N $F_1 = \pi \overline{D N^2} = \frac{\pi r^2}{2}$ and that through M, $F_2 = \pi r^2$. Hence it follows that the volume of this body is

$$V = \frac{h}{6} (0 + 4 F_1 + F_2) = \frac{h}{6} (2 \pi r^2 + \pi r^2) = \tfrac{1}{2} \pi r^2 h = \tfrac{1}{2} F_2 h,$$

and that its moment is

$$V y = \frac{h^2}{12} (1 . 2 \pi r^2 + 2 \pi r^2) = \tfrac{1}{3} \pi r^2 h^2 = \tfrac{1}{3} F_2 h^2.$$

Consequently the distance of the centre of gravity S from the vertex is

$$A S = y = \frac{\tfrac{1}{3} F_2 h^2}{\tfrac{1}{2} F_2 h} = \tfrac{2}{3} h.$$

FIG. 163.

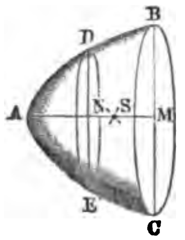
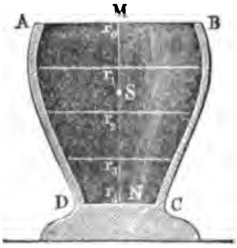


FIG. 164.



EXAMPLE 2. The mean half widths of the vessel $A B C D$, Fig. 164, are $r_0 = 1$ inch, $r_1 = 1{,}1$ inches, $r_2 = 0{,}9$ inches, $r_3 = 0.7$ inches, and $r_4 = 0{,}4$ inches, and its height $M N = 2{,}5$ inches; required the centre of gravity of the space within it. The cross sections are $F_0 = 1\ \pi$, $F_1 = 1{,}21\ \pi$, $F_2 = 0{,}81\ \pi$, $F_3 = 0{,}49\ \pi$ and $F_4 = 0{,}16\ \pi$, and therefore the distance of its centre of gravity from the horizontal plane $A B$ is

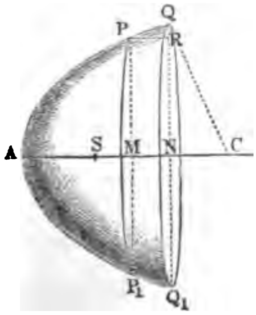
$$M S = \frac{0 . 1 \pi + 1 . 4 . 1{,}21 \pi + 2 . 2 . 0{,}81 \pi + 3 . 4 . 0{,}49 \pi + 4 . 0{,}16 . \pi}{1 \pi + 4 . 1{,}21 \pi + 2 . 0{,}81 \pi + 4 . 0{,}49 \pi + 0{,}16 . \pi} \cdot \frac{2{,}5}{4}$$

$$= \frac{14{,}60}{9{,}58} \cdot \frac{2{,}5}{4} = \frac{36{,}50}{38{,}32} = 0{,}9502 \text{ inches.}$$

The vacant space in the vessel is $V = 9{,}58\ \pi . \frac{2{,}5}{12} = 6{,}270$ cubic inches.

(§ 125.) Determination of the Centre of Gravity of Surfaces and Solids of Rotation.—The centre of gravity of curved surfaces and of bodies with curved surfaces can be determined generally by the aid of the calculus. In practice, *solids* and *surfaces of rotation* occur most frequently, and we will therefore here treat only of the determination of the centre of gravity of these forms. If the plane curve $A P$, Fig. 165, revolves about its axis $A C$, it describes a so-called surface of rotation $A P P_1$; and if the surface $A P M$ bounded by the curve $A P$ and

FIG. 165.



its co-ordinates $A\,M$ and $M\,P$ is revolved about the same axis a solid of rotation bounded by a circular surface $P\,M\,P_1$ and by a surface of rotation $A\,P\,P_1$ is produced.

If we denote the abscissa $A\,M$ by x, the corresponding ordinate by y and the corresponding arc $A\,P$ by s, and also the element $M\,N = P\,R$ of the abscissa by $d\,x$, the element $Q\,R$ of the ordinate by $d\,y$ and the element $P\,Q$ of the curve by $d\,s$, we have the area of the belt-shaped element $P\,Q\,Q_1\,P_1$ generated by the revolution of $d\,s$, when we put the surface of rotation $A\,P\,P_1 = O$,

$$d\,O = 2\,\pi\,.\,P\,M\,.\,P\,Q = 2\,\pi\,y\,d\,s,$$

and, on the contrary, the contents of the element of the solid of rotation $A\,P\,P_1 = V$, limited by this element of the surface, are

$$d\,V = \pi\,\overline{P\,M}^2\,.\,M\,N = \pi\,y^2\,d\,x.$$

Since the distance of both elements from a plane passing through A at right angles to the axis $A\,C$ is equal to the abscissa x, the moment of $d\,O$ is

$$x\,d\,O = 2\,\pi\,x\,y\,d\,s,$$

and that of $d\,V$ is

$$x\,d\,V = \pi\,x\,y^2\,d\,x.$$

Now since

$$O = \int 2\,\pi\,y\,d\,s = 2\,\pi \int y\,d\,s \text{ and}$$
$$V = \int \pi\,y^2\,d\,x = \pi \int y^2\,d\,x,$$

and since according to the above formulas the moment of O is

$$\int 2\,\pi\,x\,y\,d\,s = 2\,\pi \int x\,y\,d\,s,$$

and that of V is

$$\int \pi\,x\,y^2\,d\,x = \pi \int x\,y^2\,d\,x,$$

it follows, that the distance $A\,S = y$ of the centre of gravity S from the origin A is

1) for surfaces of rotation

$$u = \frac{2\,\pi \int x\,y\,d\,s}{2\,\pi \int y\,d\,s} = \frac{\int x\,y\,d\,s}{\int y\,d\,s},$$

and, on the contrary,

2) for solids of rotation,

$$u = \frac{\pi \int x\,y^2\,d\,x}{\pi \int y^2\,d\,x} = \frac{\int x\,y^2\,d\,x}{\int y^2\,d\,x}.$$

E.G., for a spherical zone whose radius $C\,Q = r$ we have, since

$$\frac{P\,Q}{P\,R} = \frac{C\,Q}{Q\,N} \text{ I.E. } \frac{d\,s}{d\,x} = \frac{r}{y} \text{ or } y\,d\,s = r\,d\,x,$$

$$A\,S = u = \frac{\int x\,r\,d\,x}{\int r\,d\,x} = \frac{\int x\,d\,x}{\int d\,x} = \frac{\frac{1}{2}x^2}{x} = \tfrac{1}{2}\,x = \tfrac{1}{2}\,A\,M.$$

(Compare § 116.)

For a segment of a sphere, on the contrary, we have, since we can put $y^2 = 2\,r\,x - x^2$,

$$A\,S = u = \frac{\int (2\,r\,x - x^2)\,x\,d\,x}{\int (2\,r\,x - x^2)\,d\,x} = \frac{\int 2\,r\,x^2\,d\,x - \int x^3\,d\,x}{\int 2\,r\,x\,d\,x - \int x^2\,d\,x}$$

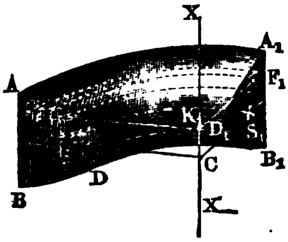
$$= \frac{\frac{2}{3}\,r\,x^3 - \frac{1}{4}\,x^4}{r\,x^2 - \frac{1}{3}\,x^3} = \frac{(\frac{2}{3}\,r - \frac{1}{4}\,x)\,x}{r - \frac{1}{3}\,x} = \left(\frac{8\,r - 3\,x}{3\,r - x}\right)\frac{x}{4},$$

and consequently

$$C\,S = r - u = \tfrac{3}{4}\,\frac{(2\,r - x)^2}{3\,r - x}. \quad \text{(Compare § 123.)}$$

§ **126. Properties of Guldinus.**—An interesting and often very useful application of the theory of the centre of gravity is the properties of Guldinus (Fr. méthode centrobarique, Ger. die Guldinische Regel). According to these *the contents of a solid of rotation* (or the area of a surface of rotation) *is equal to the product of the generating surface* (or generating line) *and the space described by its centre of gravity while generating the body* (or surface). The correctness of this rule can be proved as follows:

FIG. 166.



If a plane surface $A\ B\ D$, Fig. 166, is revolved about an axis $X\overline{X}$, every element F_1, F_2, etc. of it describes a ring; if the distances of these elements F_1, F_2, etc. from the axis of rotation $X\overline{X}$ are $F_1\,K_1$, $F_2\,K_2$, etc. $= r_1$, r_2, etc., and if the angle of rotation is $F\,K\,F_1 = S\,C\,S_1 = a^0$ or the arc corresponding to the radius 1, $= a$, the arc-shaped paths described by the elements are $r_1\,a$, $r_2\,a$, etc. The spaces described by the elements F_1, F_2, etc., can be regarded as curved prisms whose altitudes are $r_1\,a$, $r_2\,a$, etc., their contents are therefore $F_1\,r_1\,a$, $F_2\,r_2\,a$, etc., and consequently the volume of the whole body $A\ B\ D\ D_1\ B_1\ A_1$ is

$$V = F_1\,r_1\,a + F_2\,r_2\,a + \ldots. = (F_1\,r_1 + F_2\,r_2 + \ldots)\,a.$$

If $y = CS$ is the distance of the centre of gravity S of the generating surface from the axis of rotation, we have

$$(F_1 + F_2 + \ldots)\, y = F_1 r_1 + F_2 r_2 + \ldots,$$

and consequently the volume of the whole body

$$V = (F_1 + F_2 + \ldots)\, y\, a.$$

But $F_1 + F_2 + \ldots$ is the area of the surface F, and $y\,a$ is the arc $S\,S_1 = w$ described by the centre of gravity; hence it follows that $V = F\,w$, which is what was to be proved.

This formula is also applicable to the case of the rotation of a line, since the latter can be considered as a surface of infinitely small width. In this instance we have $F = l\,w$, I.E. the surface of rotation is the product of the generating line (l) and the space (w) described by its centre of gravity.

EXAMPLE 1. If the semi-axes of the elliptical cross section $A\ B\ E\ D$, Fig. 167, of a half ring are $C\,A = a$ and $C\,B = b$, and if the distance $C\,M$ of its centre C from the axis $X\,\overline{X} = r$, the elliptical generating surface will be $F = \pi\,a\,b$, and the space described by its centre of gravity (C) will be $w = \pi\,r$. Hence the volume of this half ring is $V = \pi^2\,a\,b\,r$, and that of the whole ring is $V_1 = 2\,V = 2\,\pi^2\,a\,b\,r$.

If the dimensions are $a = 5$ inches, $b = 3$ inches and $r = 6$ inches, the volume of one-quarter of the ring is

$$\tfrac{1}{2} \cdot \pi^2 \cdot 5 \cdot 3 \cdot 6 = 9{,}8696 \cdot 5 \cdot 9 = 444{,}132 \text{ cubic inches.}$$

FIG. 167.

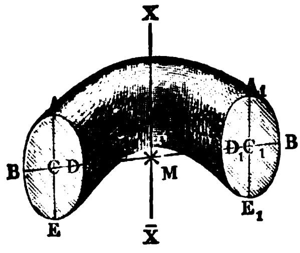
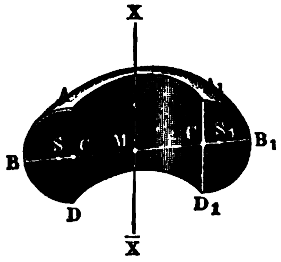


FIG. 168.



EXAMPLE 2. The volume of a ring with the semi-circular cross section $A\ B\ D$, Fig. 168, is, when $C\,A = C\,B = a$ denotes the radius of this cross section and $M\ C = r$ that of the hollow space,

$$V = \frac{\pi\,a^2}{2} \cdot 2\,\pi \left(r + \frac{4\,a}{3\,\pi}\right) = \pi\,a^2\,(\pi\,r + \tfrac{4}{3}\,a).$$

EXAMPLE 3. If the segment of a circle $A\,D\,B$, Fig. 169, revolves about the diameter $E\,F$ parallel to its chord $A\ B$, it describes a sphere $A\,D_1 B$ with a cylindrical hole $A\ B\,B_1\ A_1$ in it. If A is the area of the segment and s the length of its chord $A B = A_1 B_1$, we have (§ 114) for the distance of its centre of gravity S from the centre C

$$C S = y = \frac{s^3}{12 A},$$

and consequently the volume of the sphere with the cylindrical hole is

$$V = 2 \pi y A = 2 \pi \frac{s^3}{12} = \frac{\pi s^3}{6}.$$

For a complete sphere we have the chord or height of the hole equal to the diameter d of the sphere, and consequently its volume

$$V = \frac{\pi d^3}{6},$$

as we know.

EXAMPLE 4. We are required to find the area of the surface and the contents of the cupola $A D B$, Fig. 170, of a cloistered arch, when the half width $M A = M B = a$ and the altitude $M D = h$ are given. From the two given dimensions we obtain the radius $C A = C D$ of the generating circle

FIG. 169.

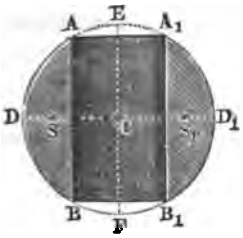
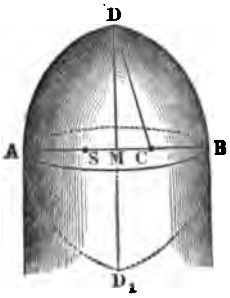


FIG. 170.



$$r = \frac{a^2 + h^2}{2 a}.$$

The central angle $A C D = \alpha$ is given by the formula

$$sin. \alpha = \frac{h}{r}.$$

The centre of gravity S of an arc $D A D_1 = 2 A D$ is determined by the distances

$$C S = r \,.\, \frac{\text{chord } M D}{\text{arc } A D} = \frac{r \, sin. \alpha}{\alpha} \text{ and } C M = r \cos. \alpha;$$

consequently the distance of the centre of gravity S from the axis $M D$ is

$$M S = \frac{r \, sin. \alpha}{\alpha} - r \cos. \alpha = r \left(\frac{sin. \alpha}{\alpha} - \cos. \alpha\right),$$

and the space described by the centre of gravity in describing the surface $A D B$ is

$$w = 2 \pi r \left(\frac{sin. \alpha}{\alpha} - \cos. \alpha\right).$$

The generatrix $D A D_1$ is $2 r a$, consequently its half is $A D = r a$, and the surface of rotation $A D B$ generated by the latter is

$$O = r a \,.\, 2 \pi r \left(\frac{sin.\, a}{a} - cos.\, a\right) = 2 \pi r^2 \,(sin.\, a - a \, cos.\, a).$$

Very often we have $a^\circ = 60^\circ$, or

$$a = \frac{\pi}{3},\ sin.\, a = \tfrac{1}{2}\sqrt{3} \text{ and } cos.\, a = \tfrac{1}{2};$$

hence the required area is

$$O = \pi r^2 \left(\sqrt{3} - \frac{\pi}{3}\right) = 2{,}1515 \,.\, r^2.$$

The distance of the centre of gravity of the segment $D A D_1 = A = r^2 (a - \tfrac{1}{2} sin.\, 2 a)$ from the centre C is

$$= \frac{(2 \,.\, M D)^3}{12 A} = \frac{2}{3} \cdot \frac{r^3 \, sin.^3 \, a}{A},$$

and, therefore, its distance from the axis is

$$M S = C S - C M = \frac{2}{3} \frac{r^3 \, sin.^3 \, a}{A} - r \, cos.\, a,$$

and the space described by this centre of gravity in one revolution around $M D$ is

$$w = \frac{2 \pi r}{A} (\tfrac{2}{3} r^3 \, sin.^3 \, a - A \, cos.\, a) = \frac{2 \pi r^3}{A} [\tfrac{2}{3} sin.^3 \, a - (a - \tfrac{1}{2} sin.\, 2 a) \, cos.\, a].$$

The volume of the body generated by the revolution of the segment $D A D_1$ is found by multiplying this space by A, and the volume of the cupola by dividing the last product by two. The latter volume is

$$V = \pi r^3 [\tfrac{2}{3} sin.^3 \, a - (a - \tfrac{1}{2} sin.\, 2 a) \, cos.\, a]$$

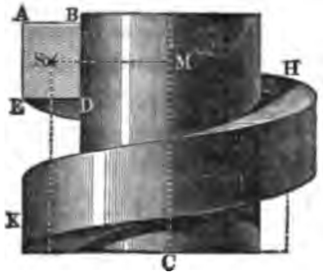
E.G., if $a^\circ = 60^\circ$, we have

$$a = \frac{\pi}{3},\ sin.\, a = \tfrac{1}{2}\sqrt{3},\ sin.\, 2 a = \tfrac{1}{2}\sqrt{3},\ cos.\, a = \tfrac{1}{2}, \text{ and therefore}$$

$$V = \pi r^3 \left(\tfrac{3}{8}\sqrt{3} - \frac{\pi}{6}\right) = 0.3956 \,.\, r^3.$$

§ 127. The properties of Guldinus are also applicable to bodies formed by the motion of the centre of gravity of the generating surface along any curve, as long as the surface remains at right-angles to the curve; for every curve can be regarded as composed of an infinite number of infinitely small arcs of circles. The volume of the body is here also equal to the product of the generating surface and of the space described by its centre of gravity. The properties can also be made use of, when the generating surface in moving forwards is always at right angles to the projection of the path of its centre of gravity upon any plane. In this case the generating surface is to be multiplied not by the space described, but by its projection.

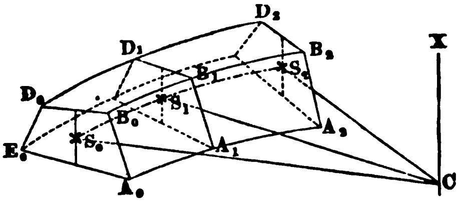
FIG. 171.



Hence, for example, the volume of one turn of the thread $A\,H\,K$, Fig 171, of a screw is determined by the product of its cross section $A\,B\,D\,E$ by the circumference of the circle, whose radius is the distance $M\,S$ of the centre of gravity S of the surface $A\,B\,D\,E$ from the axis $C\,M$ of the screw.

In many cases we can combine the use of the properties of Guldinus with that of S[illegible]pson's rule. E.G., to find the contents of the curved embankment $A_0\,D_0\,B_1\,D_2\,A_2$, Fig. 172, we need only know the central angles $S_0\,C\,S_2 = 2\,S_0\,C\,S_1 = 2\,S_1\,C\,S_2 = \beta$, the cross sections $A_0\,D_0 = F_0$, $A_1\,D_1 = F_1$, $A_2\,D_2 = F_2$ and the distances $C\,S_0 = r_0$, $C\,S_1 = r_1$ and $C\,S_2 = r_2$ of the centres of gravity S_0, S_1 and S_2 of these cross sections from the central axis $C\,X$. The volume V of the body is determined by the formula

FIG. 172.



$$V = \beta\left(\frac{F_0\,r_0 + 4\,F_1\,r_1 + F_2\,r_2}{6}\right) = \frac{\beta^\circ\,\pi}{180^\circ}\left(\frac{F_0\,r_0 + 4\,F_1\,r_1 + F_2\,r_2}{6}\right)$$

$$= 0{,}01745\,\beta^\circ\left(\frac{F_0\,r_0 + 4\,F_1\,r_1 + F_2\,r_2}{6}\right).$$

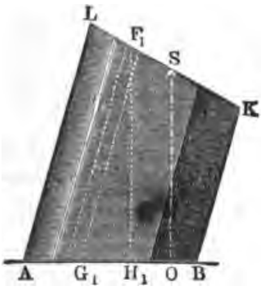
If the radii r_0, r_1 and r_2 are equal to each other, or if they differ but little, we can put $r_0 = r_1 = r_2 = r$ and therefore

$$V = 0{,}01745\,\beta^\circ\,r\left(\frac{F_0 + 4\,F_1 + F_2}{6}\right).$$

§ **128.** The following is another application of the theory of the centre of gravity, which is closely allied to the foregoing.

We can assume that every obliquely truncated prismatic body $A\,B\,K\,L$, Fig. 173, is composed of infinitely thin prisms, such as $\overline{F_1\,G_1}$. If G_1, G_2, etc., are the bases and $h_1\,h_2$, etc., the altitudes of these prismatic elements, we have the contents

FIG. 173.



$$G_1\,h_1,\; G_2\,h_2, \text{ etc.}$$

and consequently the volume of the whole obliquely truncated prism

$$V = G_1\,h_1 + G_2\,h_2 + \ldots.$$

Now an element F_1 of the oblique section $K\,L$ is to the element G_1 of the base $A\,B = G$ as the whole oblique surface F is to the base G; hence we have

$$G_1 = \frac{G}{F}\,F_1,\; G_2 = \frac{G}{F}\,F_2, \text{ etc., and}$$

$$V = \frac{G}{F}\,(F_1\,h_1 + F_2\,h_2 + \ldots).$$

Finally, since $F_1\,h_1 + F_2\,h_2 + \ldots$ is the moment $F\,h$ of the whole oblique section, we can put

$$V = \frac{G}{F} \cdot F\,h = G\,h,$$

I.E., the volume of an obliquely truncated prism is equal to the volume of a complete prism, which stands on the same base and whose altitude is equal to the distance $S\,O$ of the centre of gravity S of the oblique section from the base.

The distance of the centre of gravity of the oblique section of a right triangular prism, which is truncated obliquely, from the base is

$$h = \frac{h_1 + h_2 + h_3}{3},$$

and consequently the volume of this prism is

$$V = G\,h = G\,\frac{(h_1 + h_2 + h_3)}{3}.$$

CHAPTER III.

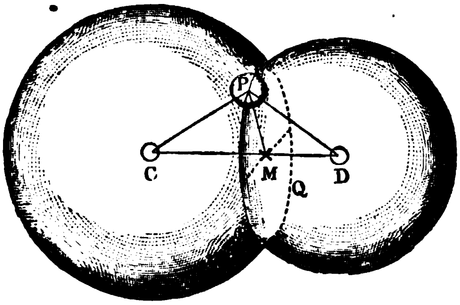
EQUILIBRIUM OF BODIES RIGIDLY FASTENED AND SUPPORTED.

§ **129. Method of Fastening.**—The propositions relative to the equilibrium of rigid systems of forces, demonstrated in the first chapter of this section, are applicable to solid bodies subjected to the action of forces, when we consider *the weight of the body as a force applied at the centre of gravity and acting vertically downwards.*

Bodies, which are held in equilibrium by forces, are capable of moving freely, I.E., they can obey the influence of the forces, or they are in one or more points rigidly fastened, or they are supported by other bodies.

If a point C, Fig. 174, of a solid body is rigidly fastened, any other point P of the body, when put in motion, will describe a path, which lies upon the surface of a sphere, whose centre is the fixed point C and whose radius is the distance $C\,P$ of the other point from C. If, on the contrary, we fasten a body in two points C and D, the paths described by all other points in consequence of any possible motion would be circles; for the path of each point is the intersection $O\,P\,Q$ of two spherical surfaces described from the two fixed points.

FIG. 174.



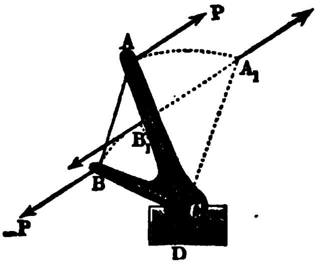
The planes of these circles are parallel to each other and perpendicular to the straight line joining the two fixed points. The points upon the latter line remain immovable; the body, therefore, revolves around this line $C D$, which is called, for this reason, the *axis* of rotation or revolution of the body.

The planes perpendicular to this axis, and in which the different points revolve, are called the *planes* of rotation or revolution of the body. We obtain the radius $M P$ of the circle $O P Q$ by letting fall a perpendicular upon the axis of revolution $C D$. The greater this perpendicular is, the greater is the circle, in which the point revolves.

If three points of a body, not in the same straight line, are firmly fastened, then the body does not move in any direction, since the three spherical surfaces, in which the body must move, cut each other only in a point.

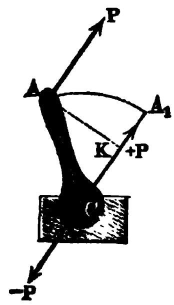
130. Equilibrium of Supported Bodies.—Every force passing through the fixed point of a body, E.G., through the centre of a ball and socket joint, is counteracted by the support of the body, and has, therefore, no influence upon the state of equilibrium of the body. In like manner, if a body is supported in two points or bearings, every force whose direction cuts the axis passing through these fixed points is counteracted by the supports, without producing any other effect on the body. A couple would also be counteracted by the supports of a body, if the plane of the couple contains the axis of revolution passing through these points, or is parallel to the same. Every other couple $(P, -P)$, Fig. 175, produces, on the contrary, a revolution of the body $A\ C\ B$ about the axis of revolution C, if it is not balanced by another couple (see § 95 and § 97). If the couple retains its direction during the rotation, its lever arm and consequently its moment is variable, and both become $= 0$, when the body occupies a certain position. If a body $A\ C\ B$, Fig. 175, is rigidly fastened at C, and if the direction of the force forms the angle $B A P = a$ with the line $A\ B$ passing through the two points of application, a rotation $A\ C A_1 = \beta = 180° - a$ is necessary to annul the moment of the couple $(P, -P)$; the same is also true of a body rigidly fastened in an axis and acted upon by a couple, whose plane is perpendicular to this axis.

FIG. 175.



If a body $A\ B$, Fig. 176, rigidly fastened at C, is acted on by a force P, whose direction does not pass through C, we can, by the addition of two opposite forces P and $-P$, decompose this force into a couple $(P, -P)$ and a force $+P$, applied in C and counteracted by the point of support. The relations are the same, when the axis of a body is rigidly fastened and a force acts upon it in a plane of revolution. Here, however, the force $+P$ is divided between the two points of support. If a is the distance CA of the point of application A of the force from the axis C and a the angle $A\ C\ A_1$, formed by the line CA with the direction of the force, we have the moment of the couple $(P, -P)$, which tends to turn the body, $M = Pa \sin. a$. If the direction of the force P remains unchanged during the rotation, M changes with a and is a maximum for $a = 90°$ and for $a = 0°$ or $180°$ it is $= 0$. The work done by the force P or by the couple $(P, -P)$ during the rotation of the body is

FIG. 176.



$$A = P\ .\ \overline{KA_1} = Pa\,(1 - \cos.\ a).$$

131. Stability of a Suspended Body.—If the force acting upon a body, supported at one point or in a line, consists only of its weight, the conditions of equilibrium require, that the centre of gravity shall be supported, I.E., that the vertical line of gravity shall pass through the point of support.

If the centre of gravity coincides with the point of support, we have a case of *indifferent* equilibrium (Fr. équilibre indifférent, Ger. indifferentes Gleichgewicht); for the body remains in equilibrium, no matter how we may turn it. If, on the contrary, the body is

FIG. 177.

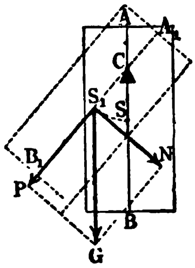
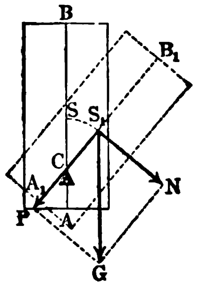


FIG. 178.

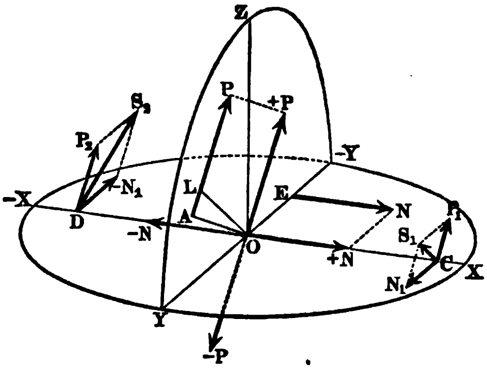


rigidly fastened or supported at a point C, lying above the centre of gravity S, the body is in *stable* equilibrium (Fr. stable, Ger. sicheres or stabiles); for, if we bring the body into another position, one of the components N of the weight S causes the body to return to its original position, and the other component P is counteracted by the fixed point C. If finally the body A B, Fig. 178, is fastened at a point C, which lies below the centre of gravity, the body is in *unstable* equilibrium (Fr. éq. instable, Ger. unsicheres or labiles Gleichgewicht); for if we move the centre of gravity out of the vertical line passing through C, the weight G is resolved into two components, one N of which, instead of tending to bring the body back to its original position, moves it more and more from it, until the centre of gravity comes vertically below the point of support.

The circumstances are the same, when a body is supported in two points or in an axis; it is either in indifferent, stable or unstable equilibrium as the centre of gravity coincides with, or is vertically below or above the point of support. If a body is supported at a point or in a horizontal axis, the moment with which the body seeks to return to its position of stable equilibrium is $M = G\, a\, sin.\, \alpha$, in which formula G denotes the weight, a the distance $C\, S_1$ of the centre of gravity S_1 from the axis C and α the angle of revolution $S\, C\, S_1$. The work done is $A = G\, a\, (1 - cos.\, \alpha)$.

§ 132. Pressure upon the Points of Support of a Body.—When a body C A D, Fig. 179, supported in two points C and

FIG. 179.

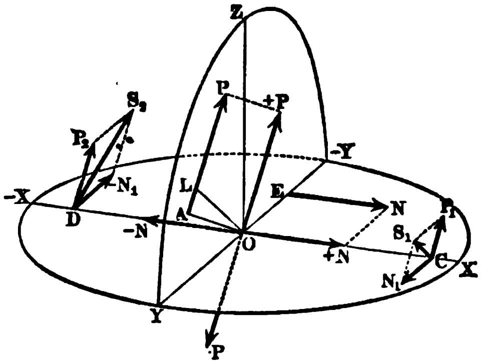


D, is acted upon by a system of forces, in order to determine the conditions of its equilibrium we refer (according to § 97) the whole system to two forces, the direction of one of which is parallel to the axis, while that of the other lies in a plane normal to this line. Let $\overline{EN} = N$, Fig. 180, be the force parallel to the axis $X\overline{X}$ passing through the points of support C and D and $\overline{AP} = P$ the other force, whose direction lies in a plane $Y Z \overline{Y}$ perpendicular to $X\overline{X}$. We can resolve the first force into a force $+ N$, tending to displace the axis in its own direction, and a couple $(N, - N)$, which is transmitted to the points of support in the shape of another couple $(N_1, - N_1)$, the components of which are

$$N_1 = \frac{d}{l} N \text{ and } - N_1 = - \frac{d}{l} N,$$

d denoting the distance $O E$ of the parallel force N from the axis $C D$ and l the distance $C D$ of the two points of support from each other.

FIG. 180.



In like manner we decompose the force P into a force $+ P$ and a couple $(P, - P)$, and the former again into its components P_1 and P_2, the first applied in C and the second in D. Designating the distances $C O$ and $D O$ of the points of application O from the two points of support C and D by l_1 and l_2, we have

$$P_1 = \frac{l_2}{l} P \text{ and } P_2 = \frac{l_1}{l} P,$$

and it is now easy, by employing the parallelogram of forces, to find the resultant S_1 of the forces N_1 and P_1 at C, and also the resultant S_2 of the forces $- N_1$ and P_2 at D.

If we put the angle $Y O (+ P)$ formed by the plane $N O X$ with the direction of the force P or $+ P$, $= a$, we have also the angle $N_1\,C\,P_1 = \alpha$ and $\overline{N_1}\,D\,P_2 = 180^\circ - \alpha$, and consequently the resulting pressures in C and D are

$$S_1 = \sqrt{N_1^2 + P_1^2 + 2\,N_1\,P_1 \cos.\,\alpha}$$

and
$$S_2 = \sqrt{N_1^2 + P_2^2 - 2\,N_1\,P_2 \cos.\,\alpha}.$$

If, finally, a denotes the perpendicular $O\,L$ to the direction of the force, the moment of the couple $(P, -P)$, which tends to turn the body, is $M = P\,a$. If the body is in a state of equilibrium, a must naturally be $= 0$, and therefore P must pass through the axis $C\,D$.

EXAMPLE.—Let the entire system of forces acting on a body rigidly supported in the axis $X\,\overline{X}$ be reduced to the normal force $P = 36$ pounds, and the parallel force $N = 20$ pounds; let the distance of the latter force from the axis be $O\,E = d = 1\frac{1}{2}$ feet, and the distance $C\,D$ between the two points of support be $l = 4$ feet; required the pressure upon the axis or on the fixed points C and D supposing that the direction of the force P forms an angle $\alpha = 65^\circ$ with the plane $X\,Y$, and that its point of application O is at a distance $C\,O = l_1 = 1$ foot from the point C.

The force $N = 20$ produces in the axis in its own direction a thrust $N = 20$ pounds and also the forces

$$N_1 = \frac{d}{l}\,N = \frac{1{,}5}{4}\,.\,20 = 7{,}5 \text{ pounds and } -N_1 = -\,7{,}5 \text{ pounds,}$$

which are counteracted by the supports C and D. The force P gives rise to the forces

$$P_1 = \frac{l_2}{l}\,P = \frac{4-1}{4}\,.\,36 = 27 \text{ pounds and } P_2 = \frac{l_1}{l}\,P = \tfrac{1}{4}\,.\,36 = 9 \text{ pounds.}$$

Combining the latter with the former force, we obtain the resultants

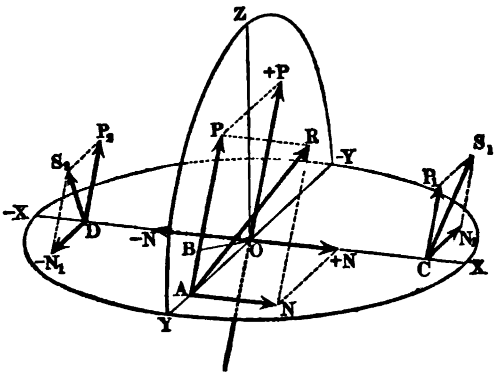
$$S_1 = \sqrt{7{,}5^2 + 27^2 + 2\,.\,7{,}5\,.\,27\,.\cos.\,65^\circ} = \sqrt{56{,}25 + 729 + 171{,}160}$$
$$= \sqrt{956{,}410} = 30{,}926 \text{ pounds, and}$$
$$S_2 = \sqrt{7{,}5^2 + 9^2 - 2\,.\,7{,}5\,.\,9\,.\cos.\,65^\circ} = \sqrt{56{,}25 + 81 - 57{,}054}$$
$$= \sqrt{80{,}196} = 8{,}955 \text{ pounds.}$$

§ **133**. If a body $C\,B\,D$, Fig. 181, firmly supported in two points C and D, is acted upon by a single force R, whose direction forms an angle $P\,A\,R = \beta$ with the plane of rotation $Y\,O\,Z$, we can decompose this force into the components

$$\overline{A\,P} = P = R \cos.\,\beta \text{ and}$$
$$\overline{A\,N} = N = R \sin.\,\beta,$$

the first of which acts in the plane of rotation and the second parallel to the axis, and we can treat these forces in exactly the same manner as the resultants P and N of the system of forces in the last paragraph. Here the force which the axis must counteract in its own direction is $N = R \sin. \beta$, and the components of

FIG. 181.



the couple $(N_1, -N_1)$, which act in C and D in opposite directions and at right angles to $C\,D$, are

$$N_1 = \frac{d}{l} N = \frac{d}{l} R \sin. \beta \text{ and } - N_1 = - \frac{d}{l} R \sin. \beta,$$

l denoting the distance $C\,D$ of the two points of support C and D from each other and d the distance $O\,A$ of the point of application A of the force R from the point O on the axis.

In like manner the force acting in O at right angles to $C\,D$ is $+ P = R \cos. \beta$ and its components in C are

$$P_1 = \frac{l_2}{l} P = \frac{l_2}{l} R \cos. \beta, \text{ and in } D$$

$$P_2 = \frac{l_1}{l} P = \frac{l_1}{l} R \cos. \beta,$$

l_1 and l_2 again denoting the distances $C\,O$ and $D\,O$ of the points C and D from the plane of rotation $Y\,Z\,\overline{Y}$.

Substituting the values of N_1, P_1, and P_2 in the formulas

$$S_1 = \sqrt{N_1^2 + P_1^2 + 2\,N_1\,P_1 \cos. a}$$

$$S_2 = \sqrt{N_1^2 + P_2^2 - 2\,N_1\,P_2 \cos. a}$$

for the normal pressures in C and D, in which we designate by a the angle $Y\,A\,P$ formed by the component P with the plane $A\,C\,D$, we obtain

$$S_1 = \frac{R}{l} \sqrt{(d \sin. \beta)^2 + (l_1 \cos. \beta)^2 + 2 \, d \, l_1 \sin. \beta \cos. \beta \cos. \alpha}$$

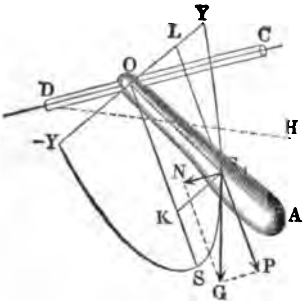
$$S_2 = \frac{R}{l} \sqrt{(d \sin. \beta)^2 + (l_1 \cos. \beta)^2 - 2 \, d \, l_1 \sin. \beta \cos. \beta \cos. \alpha}$$

The moment of the remaining couple $(P, -P)$ is

$$P . \overline{O B} = P a = R \, d \sin. \alpha \cos. \beta.$$

These formulas are applicable to the discussion of the stability of a body $O \, A$, Fig. 182, revolving about an inclined axis $C \, D$. R is here the weight G of the body, d the distance $O \, S = O \, S_1$ of its centre of gravity from the axis of rotation, α the angle $S \, O \, S_1 = O \, S_1 \, L$, which the centre of gravity has described in turning from its position of equilibrium S in the plane $Y \, S \, Y$ perpendicular to $C D$, and β the angle $G \, S_1 \, P$ formed by the plane of revolution with the vertical line, or that formed by the axis of revolution $C \, D$ with the horizontal line $D \, H$.

FIG. 182.



The work done, when the body is brought back by its weight to its position of equilibrium and S_1 to S, is

$$A = G . \overline{K S} \cos. \beta = G \, d \cos. \beta \, (1 - \cos. \alpha).$$

§ **134. Equilibrium of Forces around an Axis.**—The resultant P is produced by all the component forces, whose directions lie in one or more planes normal to the axis. But in this case (according to § 89) the statical moment $P \, a$ is equal to the sum $P_1 \, a_1 + P_2 \, a_2 + \ldots$ of the statical moments of the components, and, when the forces are in equilibrium, the arm a is $= 0$; for this force then passes through the axis itself, and consequently this sum

$$P_1 \, a_1 + P_2 \, a_2 + \ldots = 0;$$

I.E., a body rigidly supported in an axis is in equilibrium, and therefore remains without turning, when the sum of the statical moments of all the forces in relation to this axis is $= 0$, or when the sum of the moments of the forces acting in one direction of rotation is equal to the sum of the moments of those acting in the other.

By the aid of the last formula any element of a balanced system of forces, such as a force or an arm, can be found, and any force of rotation reduced from one arm to another.

If we wish to produce a state of equilibrium in a body movable about its axis, and whose moment of rotation is $P\,a$, we have only to apply a force of rotation Q or a couple, the moment of which $Q\,b = P\,a$, the difference in the two cases being that by the addition of the couple $(Q, -Q)$ the pressure on the axis is not changed, while by that of a force Q a force $+\ Q$ is added to the pressure on the axis. If the force Q or its lever arm b is given, we can calculate either

$$b = \frac{P\,a}{Q} \text{ or } Q = \frac{P\,a}{b}.$$

In the latter case we call Q the force P reduced from the arm a to the arm b, and we can thus reduce the given force of rotation P to any arbitrary arm, or we can replace or balance it by another force acting with any arbitrary arm.

We can also, by means of the formula

$$Q = \frac{P_1\,a_1 + P_2\,a_2 + \ldots}{b},$$

reduce a whole system of forces to one and the same arm.

EXAMPLE.—The forces $P_1 = 50$ pounds and $P_2 = -\ 35$ pounds act on a body movable about an axis with the arms $a_1 = 1\frac{1}{4}$ feet and $a_2 = 2\frac{1}{2}$ feet; required the force P_3 which must act with an arm $a_3 = 4$ feet, in order to produce equilibrium or to prevent motion about the axis. We have

$$50 \,.\, 1{,}25 - 35 \,.\, 2{,}5 + 4\,P_3 = 0, \text{ and}$$

$$P_3 = \frac{87{,}5 - 62{,}5}{4} = 6{,}25 \text{ pounds.}$$

§ 135. The Lever.—A body movable about a fixed axis and acted on by forces is called a *lever* (Fr. levier, Ger. Hebel). If we imagine it imponderable, we have a *mathematical* lever; but if not, it is a *material* lever.

We generally assume the forces of a lever to act in a plane at right angles to the axis and substitute for the axis a fixed point called the *fulcrum* (Fr. point d'appui, Ger. Ruhe, Dreh, or Stützpunkt). The perpendiculars let fall from this point upon the direction of the forces are called (§ 89) the *arms of the lever.* If the directions of the forces of a lever are parallel, the arms of the lever form a single right line, and the lever is then called a straight lever (Fr. levier droit, Ger. geradliniger or gerader Hebel). The straight lever acted on by two forces only is one or two armed, according as the points of application of the forces lie upon the same or upon opposite sides of the fulcrum. We distinguish also levers of the first, second and third sort, calling the two-armed lever a lever of the first sort, the one-armed lever a lever of the second sort or of the third sort, according as the force (load), which acts vertically downwards, or that (power), which acts vertically upwards, is nearest the fulcrum.

§ 136. The theory of the equilibrium of the lever has been completely demonstrated in what precedes, and we have only to make special applications of it.

For the two-armed lever $A\ C\ B$, Fig. 183, when the arm $C\ A$ of the force P is denoted by a and that $C\ B$ of the other force Q, which is generally called the load, by b, we have, according to the general theory $P\ a = Q\ b$, I.E. *the moment of the force is equal to the moment of the load*, or also $P : Q = b : a$, I.E. *the force is to the load as the arm of the latter is to the arm of the former*. The pressure on the fulcrum is $R = P + Q$.

FIG. 183.

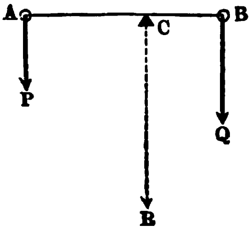
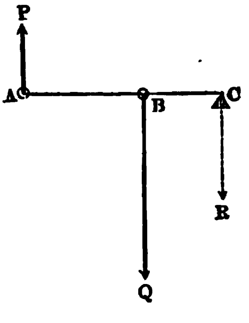


FIG. 184.



For the one-armed lever $A\ B\ C$, Fig. 184 and $B\ A\ C$, Fig. 185, the relations between force (P) and load (Q) are the same, but the direction of the power is opposite to that of the load, and therefore the pressure on the fulcrum is equal to the difference of the two; in the first case we have

$$R = Q - P, \text{ and in the second } R = P - Q.$$

If in the bent lever $A\ C\ B$ the arms are $C\ N = a$ and $C\ O = b$, Fig. 186, we have again $P : Q = b : a$, but in this case the

FIG. 185.

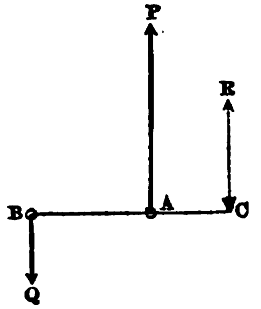
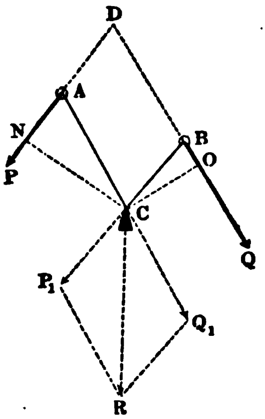


FIG. 186.



pressure R on the fulcrum is the diagonal R of the parallelogram $C\ P_1\ R\ Q_1$, constructed with the force P, the load Q and with the angle $P_1\ C\ Q_1 = P\ D\ Q = a$ formed by their directions with each other.

If G is the weight of the lever and $C\ E = e$, Fig. 187, the distance of the fulcrum C from the vertical line $S\ G$ passing through the centre of gravity S of the lever, we must put $P\ a \pm G\ e = Q\ b$, and we must employ the plus sign of G, when the centre of gravity lies on the same side as the force P, and the minus sign, when upon that of the load Q.

The theory of the lever is often applicable to tools and machinery. The knee lever $A\ B\ C\ D$, Fig. 188, which is sometimes cited as a peculiar sort of lever, is simply a bent lever. The arm, which is movable around an axis C, is acted upon by a force at its

FIG. 187.

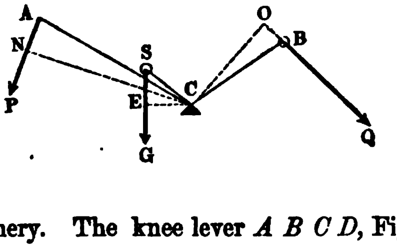
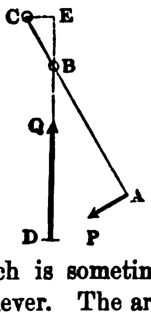


FIG. 188.



end A, and acts by means of a rod $B\ D$, (which forms with the arm an acute angle $A\ B\ D = C\ B\ E = a$) upon the load, which is applied at D. If a denotes the length of the arm $C\ A$ and b the length of the arm $C\ B$, we have the lever arm of Q

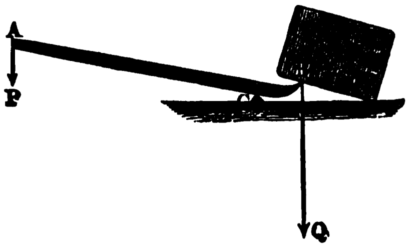
$$\overline{C\ E} = b\ sin.\ a, \text{ whence}$$
$$P\ a = Q\ b\ sin.\ a, \text{ or}$$
$$P = \frac{b}{a}\ Q\ sin.\ a, \text{ and inversely}$$
$$Q = \frac{a}{b} \frac{P}{sin.\ a}.$$

This lever is employed for pressing together materials. The pressure increases directly with P and $\frac{a}{b}$, and inversely as $sin.\ a$. By diminishing the angle a this force Q can be arbitrarily increased.

EXAMPLE—1) If the end A of a crowbar $A\ C\ B$, Fig. 189, be pressed down with a force P of 60 pounds, and if the arm $C\ A$ of the power is 12 times as great as the arm $C\ B$ of the load, then the latter, or rather the force Q developed in B, is 12 times as great as P, and we have

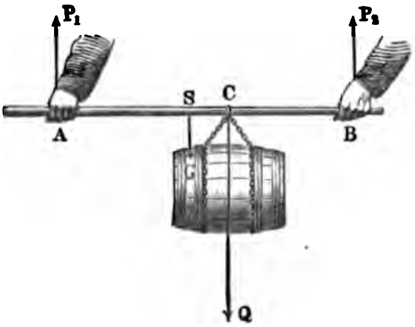
$$Q = 12 \,.\, 60 = 720 \text{ pounds.}$$

FIG. 189.



2) If a load Q, Fig. 190, hanging from a bar, be carried by two workmen, one of whom takes hold at A and the other at B, we can determine how much weight each has to sustain. Let the load be $Q = 120$ pounds, the weight of the rod be $G = 12$ pounds, the distance $A\ B$ of the two workmen from each other be $= 6$ feet, the distance of the load from one of them B be $B\ C =$ 2½ feet and the distance of the centre of gravity of the bar S from the same point be $B\ S =$ 3½ feet. If we regard B as the fulcrum, the force P_1 at A must balance the load Q and G, and therefore we have

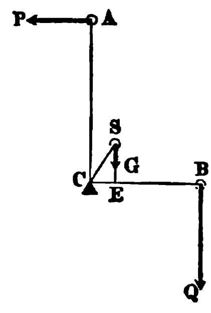
FIG. 190.



$$P_1 . \overline{BA} = Q . \overline{BC} + G . \overline{BS}, \text{ i.e.,}$$
$$6\,P_1 = 2{,}5 . 120 + 3{,}5 . 12 = 300 + 42 = 342,$$
and therefore
$$P_1 = \frac{342}{6} = 57 \text{ pounds.}$$

If, on the contrary, A be regarded as the fulcrum, we can put
$$P_2 . \overline{AB} = Q . \overline{AC} + G . \overline{AS}, \text{ or in numbers}$$
$$6\,P_2 = 3{,}5 . 120 + 2{,}5 . 12 = 420 + 30 = 450,$$
and the force exerted of the second workman is
$$P_2 = \frac{450}{6} = 75 \text{ pounds.}$$
The sum of the forces, which act upwards, is therefore correctly
$$P_1 + P_2 = 57 + 75 = 132 \text{ pounds,}$$
or as great as the sum of those acting downwards
$$Q + G = 120 + 12 = 132 \text{ pounds.}$$

FIG. 191.



3) The load upon a bent lever $A\ C\ B$, Fig. 191, weighing 150 pounds, acts vertically downwards and is $Q = 650$ pounds, and its arm $C\ B = 4$ feet, and, on the contrary, the arm of the force P, $C\ A = 6$ feet and that of the weight $C\ E = 1$ foot: required the force P necessary to produce equilibrium and the pressure R on the bearings. We have
$$\overline{CA} . P = \overline{CB} . Q + \overline{CE} . G, \text{ i.e.,}$$
$$6\,P = 4 . 650 + 1 . 150 = 2750,$$
and consequently
$$P = \frac{2750}{6} = 458\tfrac{1}{3} \text{ pounds.}$$
The pressure on the bearings is composed of the vertical force $Q + G = 650 + 150 = 800$ pounds, and of the horizontal force $P = 458\tfrac{1}{3}$ pounds, and consequently we have
$$R = \sqrt{(Q + G)^2 + P^2}$$
$$= \sqrt{(800)^2 + (458\tfrac{1}{3})^2}$$
$$= \sqrt{850070} = 922 \text{ pounds.}$$

§ **137.** More than two forces P and Q may act on a lever; it also is not necessary that these forces act upon the lever in one and the same plane of rotation. If Q_1, Q_2, Q_3 are the loads on a lever $A\ C\ B_3$, Fig. 192, and b_1, b_2, b_3 their lever arms $C\ B_1$, $C\ B_2$, $C\ B_3$, while the power acts with the lever arm $C\ A = a$, we have
$$P\,a = Q_1\,b_1 + Q_2\,b_2 + Q_3\,b_3;$$
and if the lever is straight, the pressure on the fulcrum is
$$R = P + Q_1 + Q_2 + Q_3.$$
If the several forces of a lever act in different planes of rotation

upon the lever $A\ C\ D\ B_1\ B_2$, Fig. 193, the formula for the moment $P\,a = Q_1\,b_1 + Q_2\,b_2 + \ldots$ does not therefore change, but a different distribution of the total pressure $R = P + Q_1 + Q_2 + Q_3$ upon the axis takes place between the two points of support or bearings C and D. If we denote by l the length of the axis $C\,D$ of the lever or the distance of the fulcrums from each other and by $l_0, l_1, l_2, \ldots$ the distances $C\,O$, $C\,O_1$, $C\,O_2$ of the planes of revolution from the fulcrum C, the pressures R_2 and R_1 on the bearings at D and C are determined by the following formulas

FIG. 192.

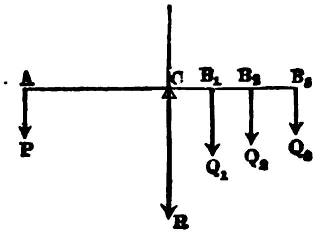
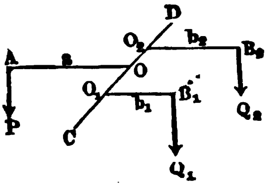


FIG. 193.



$$R_2 = \frac{P\,l_0 + Q_1\,l_1 + Q_2\,l_2 + \ldots}{l}, \text{ and}$$

$$R_1 = R - R_2 = \frac{P\,(l - l_0) + Q_1\,(l - l_1) + Q_2\,(l - l_2)}{l}$$

If the forces acting upon a bent lever are not parallel, the expression $P\,a = Q_1\,b_1 + Q_2\,b_2 + \ldots$ remains unchanged, but the pressures in the axis reduced to the fulcrum, E.G., $\frac{P\,l_0}{l}$, $\frac{Q\,l_1}{l}$, $\frac{Q_2\,l_2}{l}$, act in different directions and cannot, therefore, be combined by simple addition, but, on the contrary, we must combine them in the same manner as several forces applied to a point and acting in the same plane (see §§ 79 and 80).

EXAMPLE.—The lever represented in Fig. 193 supports the loads $Q_1 = 300$ pounds and $Q_2 = 480$, acting at the distances $C\,O_1 = l_1 = 12$ inches and $C\,O_2 = l_2 = 24$ inches from the bearing C with the arms $O_1\,B_1 = b_1 = 16$ inches and $O_2\,B_2 = b_2 = 10$ inches; required the force P, which, acting with the arm $O\,A = a = 60$ inches, is necessary to produce equilibrium, and the pressure on the bearings at C and D, under the assumption, that the force acts at a distance $C\,O = l_0 = 18$ inches from the journal C, and that the length of the entire axis is $C\,D = l = 32$ inches.

The force required is

$$P = \frac{Q_1\,b_1 + Q_2\,b_2}{a} = \frac{300\,.\,16 + 480\,.\,10}{60} = \frac{30\,.\,16 + 480}{6} = 80 + 80 = 160$$

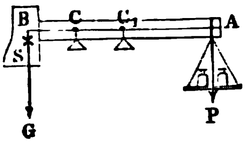
pounds, and the pressures on the bearings are

$$R_2 = \frac{160 \,.\, 18 + 300 \,.\, 12 + 480 \,.\, 24}{32} = 562{,}5 \text{ pounds and}$$

$$R_1 = R - R_2 = 300 + 480 + 160 - 562{,}5 = 377{,}5 \text{ pounds.}$$

REMARK.—The action of gravity on the lever can be employed with advantage to determine the centre of gravity S and the weight G of a body $A\ B$, Fig. 194. We support the body first at a point C and then at a point C_1 at a distance $C\ C_1 = d$ from the former, and each time we bring the body into equilibrium by a force acting at the distances $C\ A = a$ and $C_1\ A = a_1 = a - d$. If the force necessary in the first case be $= P$ and in the second case $= P_1$, and if the weight of the body be G and the distance of its centre of gravity S from A be $A\ B = x$, we have

FIG. 194.



$$P\ a = G\ (x - a) \text{ and } P_1\ a_1 = G\ (x - a_1), \text{ whence}$$

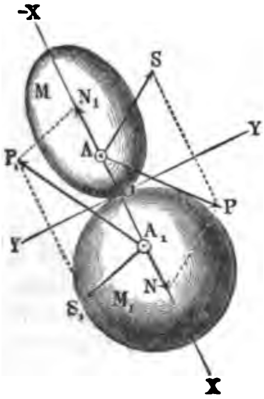
$$x = \frac{(P - P_1)\ a\ a_1}{P\ a - P_1\ a_1} \text{ and}$$

$$G = \frac{P\ a - P_1\ a_1}{a_1 - a_1}.$$

§ 138. Pressure of Bodies upon one another.

—The law deduced from experiment and announced in § 65: "Action and reaction are equal to each other," is the basis of the whole mechanics of machines, and we must here explain at greater length its meaning. If two bodies M and M_1, Fig. 195, act upon each other with the forces P and P_1, the directions of which do not coincide with that of the common normal $X\ \overline{X}$ to the two surfaces of contact, a decomposition of the forces always occurs; only that force N or N_1, whose direction is that of the normal, is transmitted from one body to the other, the other component force S or S_1, on the contrary, remains in the body and must be counteracted by some other force or obstacle, when the bodies are to be held in equilibrium. But according to the principle announced, the two normal components N and N_1 must be exactly equal. If the direction of the force P forms an angle $N\ A\ P = \alpha$ with the normal $A\ X$ and an angle $S\ A\ P = \beta$ with the direction of the other component S, we have (see § 78)

FIG. 195.



$$N = \frac{P \sin. \beta}{\sin. (\alpha + \beta)},\ S = \frac{P \sin. \alpha}{\sin. (\alpha + \beta)}.$$

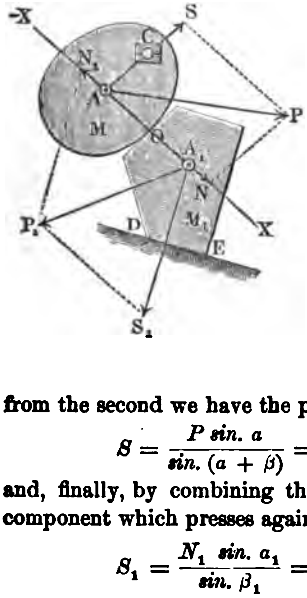
Designating in like manner $N_1 A_1 P_1$ by α_1 and $S_1 A_1 P_1$ by β_1, we have also

$$N_1 = \frac{P_1 \sin. \beta_1}{\sin. (\alpha_1 + \beta_1)} \text{ and } S_1 = \frac{P_1 \sin. \alpha_1}{\sin. (\alpha_1 + \beta_1)},$$

and, finally, since $N = N_1$

$$\frac{P \sin. \beta}{\sin. (\alpha + \beta)} = \frac{P_1 \sin. \beta_1}{\sin. (\alpha_1 + \beta_1)}.$$

FIG. 196.



EXAMPLE.—How are the forces decomposed, when a body M_1, Fig. 196, held fast by an impediment $D E$, is pressed upon by another body M, movable about its axis C, with a force $P = 250$ pounds? The angles formed by the directions are the following:

$$P A N = \alpha = 35^\circ$$
$$P A S = \beta = 48^\circ$$
$$P_1 A_1 N_1 = \alpha_1 = 65^\circ$$
$$P_1 A_1 S_1 = \beta_1 = 50^\circ.$$

The normal pressure between the two bodies is determined by the first formula and is

$$N = N_1 = \frac{P \sin. \beta}{\sin. (\alpha + \beta)}$$
$$= \frac{250 \sin. 48^\circ}{\sin. 83^\circ} = 187{,}18 \text{ pounds};$$

from the second we have the pressure on the axis or bearing C

$$S = \frac{P \sin. \alpha}{\sin. (\alpha + \beta)} = \frac{250 \sin. 35^\circ}{\sin. 83^\circ} = 144{,}47 \text{ pounds};$$

and, finally, by combining the third and fourth formula we obtain the component which presses against the impediment $D E$

$$S_1 = \frac{N_1 \sin. \alpha_1}{\sin. \beta_1} = \frac{187{,}18 \sin. 65^\circ}{\sin. 50^\circ} = 221{,}46 \text{ pounds.}$$

§ 139. In consequence of the equality of action and reaction, the equilibrium of a supported body is not changed, when, instead of the support, we substitute a force, which counteracts the pressure or tension transmitted to the support, and which is, therefore, equal in magnitude and opposite in direction to it. After having introduced this force, any body supported or partially retained may be considered as entirely free, and consequently its state of equilibrium can be treated in the same manner as that of a free body or of a rigid system of forces.

If, E.G., a body M, Fig. 197, is movable around its axis C, the force N is transmitted to a second body M_1, the force S is counteracted by the axis C and we can assume, that the body is entirely free and that besides P two other forces $-N$ and $-S$ act upon it. If the body M_1 presses upon M with the force N_1 and against the fixed plane $D\ E$ with the force S_1, the equilibrium would not be disturbed, if instead of these impediments we should substitute two opposite forces $-N_1$ and $-S_1$ and combine the same with the forces (E.G. with P_1), which act upon the body. In a state of equilibrium the resultant of the forces in the one as well as that in the other body must be null, and therefore the resultant of $-N$ and $-S$ must be counteracted by P and the resultant of $-N_1$ and $-S_1$ by P_1.

FIG. 197.

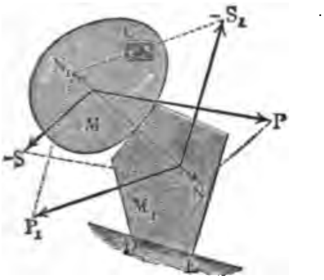
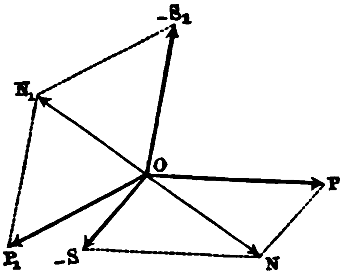


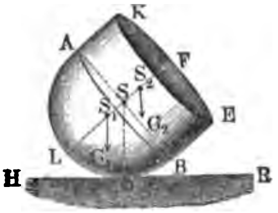
FIG. 198.



Since the forces N and N_1, with which the two bodies act upon each other, are in equilibrium, the forces P, $-S$, P_1 and $-S_1$ must be in equilibrium, when the combination of the two bodies (M, M_1) is in equilibrium. The forces N, N_1 are called the interior and the forces P, $-S$, P_1 and $-S_1$ the exterior or extraneous forces of the combination of bodies or of the system of forces, and we can therefore assert that not *only the interior forces are in equilibrium, but that the exterior forces are so also*, when, as is represented in Fig. 198, we suppose the forces applied in any point O.

§ **140. Stability.**—When a body supported upon a horizontal plane is acted on by no other force than that of gravity, it has no tendency to move forwards; for its weight, acting vertically downwards, is completely counteracted by this plane, but a rotation of the body may be produced. If the body *A D B F*, Fig. 199, rests with the point *D* on the horizontal plane *H R*, it will remain at rest as long as its centre of gravity *S* is supported, i.e., as long as it lies in the vertical line (vertical line of gravity), passing through the point of support *D*. But if a body is supported in two points upon the horizontal surface of another body, the conditions of equilibrium require, that the vertical line of gravity shall pass through the line joining the two points of support. If, finally, a body rests upon three or more points on a horizontal plane, equilibrium exists. when the vertical line of gravity passes through the triangle or polygon formed by joining these points by straight lines.

Fig. 199.



We must also distinguish for supported bodies, stable and unstable equilibrium. The weight *G* of a body *A B*, Fig. 200, draws the centre of gravity *S* of the same downwards; if there is no obstacle to the action of this force, it produces a rotation of the body, which continues until the centre of gravity has assumed its lowest position and the body has assumed a state of equilibrium. We can assert that the equilibrium is stable, when the centre of gravity occupies its lowest position (Fig. 201), that it is unstable, when it occupies its highest position (Fig. 202), and that finally the equilibrium is *indifferent*, when the centre of gravity remains at the same height, no matter what may be the position of the body (Fig. 203).

Fig. 200.

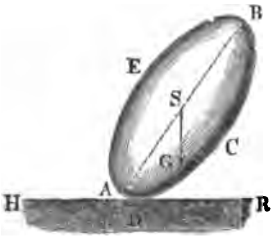


Fig. 201.

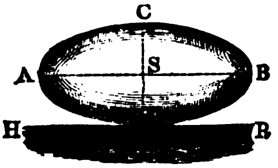


Fig. 202.

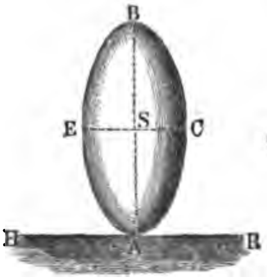
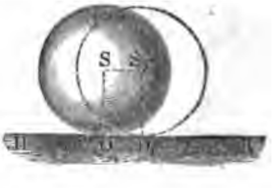
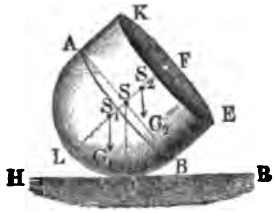


Fig. 203.



EXAMPLES.—1) The homogeneous body $A\ D\ B\ F$, Fig. 204, composed of a hemisphere and a cylinder, rests upon a horizontal plane $H\ R$. Required the height $S\ F = h$ of the cylindrical portion in order that this body shall be in equilibrium. Any radius of a sphere is perpendicular to the tangent plane corresponding to it, but the horizontal plane is such a plane, and consequently the radius $S\ D$ must be perpendicular to it and contain the centre of gravity. The axis $F\ S\ L$ passing through the centre of the sphere is also a line of gravity; the centre S, as intersection of the two lines of gravity, is therefore the centre ot gravity of the body. If we put the radius of the sphere and of the cylinder $S\ A = S\ B = S\ L = r$, and the altitude of the cylinder $S\ F = B\ E = h$, we have for the volume of the hemisphere $V_1 = \frac{2}{3}\ \pi\ r^3$, and for the volume of the cylinder $V_2 = \pi\ r^2\ h$, for the distance of the centre of gravity of the sphere S_1, $S\ S_1 = \frac{3}{8}\ r$ and for that of the centre of gravity of the cylinder S_2, $S\ S_2 = \frac{1}{2}\ h$. In order that the centre of gravity of the whole body fall in S we must make the moment of the hemisphere $\frac{2}{3}\ \pi\ r^3 . \frac{3}{8}\ r$ equal to the moment of the cylinder $\pi\ r^2\ h . \frac{1}{2}\ h$, whence we have

Fig. 204.



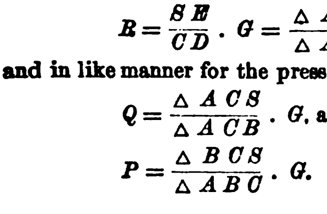
$$h^2 = \frac{1}{2} r^2 \text{ or } h = r\ \sqrt{\tfrac{1}{2}} = 0{,}7071\ r.$$

If the body is not homogeneous, but on the contrary the hemispherical portion has the specific gravity ε_1 and the cylindrical portion the specific gravity ε_2, then the moments of these portions are $\frac{2}{3}\ \pi\ r^3 . \varepsilon_1\ \frac{3}{8}\ r$ and $\pi\ r^2\ h\ \varepsilon_2 . \frac{1}{2}\ h$, and consequently by equating them we have

$$2\ \varepsilon_2\ h^2 = \varepsilon_1\ r^2, \text{ or } h = r \sqrt{\frac{\varepsilon_1}{2\ \varepsilon_2}} = 0{,}7071 \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} . r.$$

2) The pressure, which each of three legs A, B, C, Fig. 205, of an arbitrarily loaded table has to bear, can be determined in the following manner. Let S be the centre of gravity of the loaded table, and $S\ E$, $C\ D$ perpendiculars upon $A\ B$. Designating the weight of the entire table by G and the pressure in C by R, we can treat $A\ B$ as an axis and put the moment of $R =$ the moment of G, I.E., $R . C\ D = G . S\ E$, from which we obtain

Fig. 205.



$$R = \frac{S\ E}{C\ D} . G = \frac{\triangle\ A\ B\ S}{\triangle\ A\ B\ C} . G;$$

and in like manner for the pressure in B, we have

$$Q = \frac{\triangle\ A\ C\ S}{\triangle\ A\ C\ B} . G, \text{ and for that in } A$$

$$P = \frac{\triangle\ B\ C\ S}{\triangle\ A\ B\ C} . G.$$

§ **141**. Let us now investigate more fully the case of a body resting with one base upon a horizontal plane. Such a body possesses stability or is in stable equilibrium, when its centre of gravity is supported, I.E. when the vertical line passing through its centre of gravity passes also through its base, since in this case the rotation, which the weight of the body tends to produce, is prevented by the resistance of the body. If the vertical line passes through the periphery of the base, the body is in unstable equilibrium; and if it passes outside of the base, the body is not in equilibrium, but will rotate around one of the sides of the periphery of its base and be overturned. The triangular prism *A B C*, Fig. 206, is consequently in stable equilibrium, since the vertical line *S G* passes through a point *N* of its base *B C*. The parallelopipedon *A B C D*, Fig. 207, is in unstable equilibrium, because the vertical line *S G* passes through one of the edges *D* of the base *C D*. Finally, the cylinder *A B C D*, Fig. 208, is without stability; for *S G* does not pass through its base *C D*.

FIG. 206.

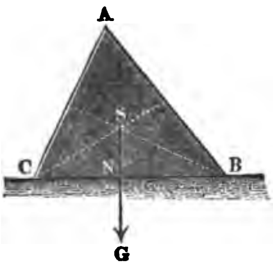
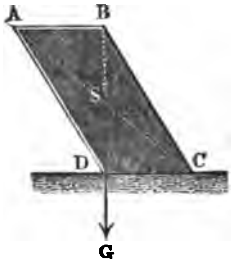
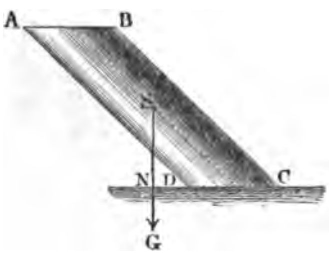


FIG. 207.



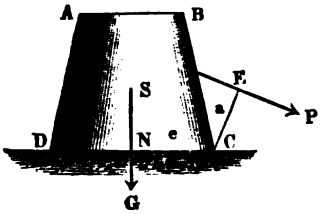
Stability (Fr. stabilité, Ger. Stabilität or Standfähigkeit) is the capacity of a body to maintain by its weight alone its position and to resist any cause of rotation. If we wish to select a measure for the stability of a body, it is necessary to distinguish the case of simply moving the body from that of actually overturning it. Let us first consider the former case alone.

FIG. 208.



§ **142**. **Formulas for Stability.**—A force *P* whose direction is not vertical tends not only to overturn, but also to push forward the body *A B C D*, Fig. 209. Let us suppose that there is an obstacle to its pushing or pulling the body forwards, and let us consider only the rotation around an edge C. If from this edge we let fall a perpendicular $C E = a$ upon the direction of the force and another perpendicular $C N = e$ upon the vertical line of gravity $S G$ of the body, we have then a bent lever $E C N$, to which the formula $P a = G e$ or $P = \frac{e}{a} G$ is applicable. If, therefore, the exterior force P is slightly greater than $\frac{G e}{a}$, the body begins to turn around C and thus loses its stability. Its stability is therefore dependent upon the product ($G e$) of the weight of the body and the smallest distance of a side of the periphery of the base from the vertical line passing through the centre of gravity, and $G e$ can therefore be considered as *a measure of stability,* and we will henceforth call it simply the *stability.* Hence we see that the *stability* increases equally with the weight G and with the distance e, and consequently we can conclude that under the same circumstances a wall, etc., whose weight is two or three tons, does not possess any more stability than one, whose weight is one ton and in which the distance or arm of the lever e is two or three fold.

FIG. 209.



§ 143. 1) The weight of a parallelopipedon $A B C D$, Fig. 210, whose length is l, whose breadth is $A B = C D = b$ and whose height is $A D = B C = h$, is $G = V \gamma = b h l \gamma$, and its stability

$$St = G . \overline{D N} = G . \tfrac{1}{2} \overline{C D} = \frac{G b}{2} = \tfrac{1}{2} b^2 h l \gamma,$$

γ denoting the heaviness of the material of the parallelopipedon.

FIG. 210.

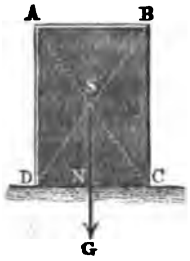
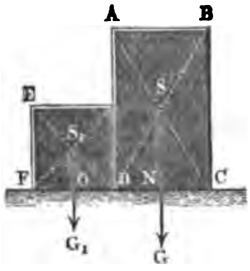


FIG. 211.



2) The stabilities of a body $B D E$, Fig. 211, composed of two

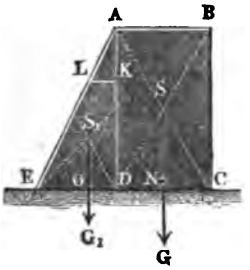
parallelopipedons, in reference to the two edges of the base C and F, are different from each other. If the heights are $B\,C$ and $E\,F = h$ and h_1 and the widths $C\,D$ and $D\,F = b$ and b_1, we have the weights G and G_1 of the two portions $= b\,h\,l\,\gamma$ and $b_1\,h_1\,l\,\gamma$; the arms in reference to C are $C\,N = \frac{1}{2}\,b$ and $C\,O = b + \frac{1}{2}\,b_1$, and those in reference to F are $b_1 + \frac{1}{2}\,b$ and $\frac{1}{2}\,b_1$, and the stability is, first, for a rotation around C

$$St = \tfrac{1}{2}\,G\,b + G_1\,(b + \tfrac{1}{2}\,b_1), = (\tfrac{1}{2}\,b^2\,h + b\,b_1\,h_1 + \tfrac{1}{2}\,b_1^2\,h_1)\,l\,\gamma,$$

and, secondly, for a rotation about F

$$St_1 = G\,(b_1 + \tfrac{1}{2}\,b) + \tfrac{1}{2}\,G_1\,b_1 = (\tfrac{1}{2}\,b_1^2\,h_1 + b\,b_1\,h + \tfrac{1}{2}\,b^2\,h)\,l\,\gamma.$$

Fig. 212.



The latter stability is $St_1 - St = (h - h_1)\,b\,b_1\,l\,\gamma$ greater than the former. If we wish to increase the stability of a wall $A\,C$ by offsets $D\,E$, we must put them upon the side of the wall, towards which the force of rotation (wind, water, pressure of earth, etc.) acts. The stability of a wall $A\,B\,C\,E$, Fig. 212, which is battered on one side, is determined as follows. Let the length of the wall be l, the width on top $A\,B = b$, the height $B\,C = h$ and the batter $= n$, I.E. when the height $A\,K = 1$ foot the batter $K\,L = n$, or for a height h feet, $= n\,h$. The weight of the parallelopipedon $A\,C$ is $G = b\,h\,l\,\gamma$, that of the triangular prism $A\,D\,E = G_1 = \frac{1}{2}\,n\,h\,.\,h\,l\,\gamma$; the arms for a rotation about E are $E\,N = E\,D + \frac{1}{2}\,b = n\,h + \frac{1}{2}\,b$ and $E\,O = \frac{2}{3}\,E\,D = \frac{2}{3}\,n\,h$. Hence the stability is

$$St = G\,(n\,h + \tfrac{1}{2}\,b) + \tfrac{2}{3}\,G_1\,n\,h = (\tfrac{1}{2}\,b^2 + n\,h\,b + \tfrac{1}{3}\,n^2\,h^2)\,h\,l\,\gamma.$$

A parallelopipedical wall of the same volume is $b + \frac{1}{2}\,n\,h$ wide, and its stability is

$$St_1 = \tfrac{1}{2}\,(b + \tfrac{1}{2}\,n\,h)^2\,h\,l\,\gamma = (\tfrac{1}{2}\,b^2 + \tfrac{1}{2}\,n\,h\,b + \tfrac{1}{8}\,n^2\,h^2)\,h\,l\,\gamma;$$

the stability is therefore $St - St_1 = (b + \frac{5}{12}\,n\,h)\,.\,\frac{1}{2}\,n\,h^2\,l\,\gamma$ smaller than that of a battering wall.

The stability of a wall with a batter on the other side is

$$St_2 = (b^2 + n\,h\,b + \tfrac{1}{3}\,n^2\,h^2)\,.\,\tfrac{1}{2}\,h\,l\,\gamma,$$

and consequently smaller than St by an amount

$$St - St_2 = (b + \tfrac{1}{3}\,n\,h)\,.\,\tfrac{1}{2}\,n\,h^2\,l\,\gamma,$$

but greater by an amount $St_2 - St_1 = \frac{1}{24}\,n^2\,h^3\,l\,\gamma$ than the stability of a parallelopipedical wall of the same volume.

EXAMPLE.—What is the stability per running foot of a stone wall 10 feet high, 1¼ feet wide on top and with a batter of ⅛ of a foot on its back? The density of this wall can be put (§ 61) = 2,4, consequently its heaviness is $\gamma = 62{,}4 \, . \, 2{,}4 = 149{,}76$ pounds; but we have $l = 1$, $h = 10$, $b = 1{,}25$ and $n = \frac{1}{5} = 0{,}2$, and consequently the required stability is

$$St = [\tfrac{1}{2} \, . \, (1{,}25)^2 + 0{,}2 \, . \, 1{,}25 \, . \, 10 + \tfrac{1}{3} (0{,}2)^2 \, . \, 10^2] \, 10 \, . \, 1 \, . \, 149{,}76$$
$$= (0{,}78125 + 2{,}5 + 1{,}3333) \, 1497{,}6 = 4{,}6146 \, . \, 1497{,}6 = 6911 \text{ foot-pounds.}$$

If the same quantity of materials is used, under the same circumstances the stability of a parallelopipedical wall would be

$$St_1 = [\tfrac{1}{2} \, . \, (1{,}25)^2 + \tfrac{1}{2} \, . \, 0{,}2 \, . \, 1{,}25 \, . \, 10 + \tfrac{1}{8} (0{,}2)^2 \, . \, 10^2] \, . \, 149{,}76 \, . \, 10$$
$$= (0{,}78125 + 1{,}25 + 0{,}5) \, 1497{,}6 = 2{,}531 \, . \, 1497{,}6 = 3790 \text{ foot-pounds.}$$

The stability of the same wall with a batter on its front would be

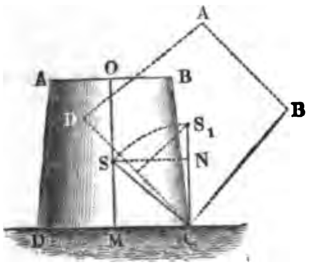
$$St_2 = [\tfrac{1}{2} (1{,}25)^2 + \tfrac{1}{2} \, . \, 0{,}2 \, . \, 1{,}25 \, . \, 10 + \tfrac{1}{6} (0.2)^2 \, . \, 10^2] \, 149{,}76 \, . \, 10$$
$$= (0{,}78125 + 1{,}25 + 0{,}666) \, 1497{,}6 = 2{,}6979 \, . \, 1497{,}6 = 4040 \text{ foot-pounds.}$$

Remark.—We see from the above that we economize material by battering the wall, by furnishing it with counterforts or offsets, by building it on plinths, etc. This subject will be treated more in detail in the second volume, where the pressure of earth, arches, bridges, etc., will be considered.

§ **144. Dynamical Stability.**—We must distinguish from the measure of stability given in the last paragraph another measure of the stability of a body, in which we bring into consideration the mechanical effect necessary to overturn the body. The work done is equal to the product of the force and the space; the force in a heavy body is its weight, and the space is the vertical projection of the space described by the centre of gravity, and, consequently, in the latter sense the product $G \, s$ can be employed as the measure of the stability of a body, when s is the vertical height, which the centre of gravity of the body must rise, in order to bring the body from its state of stable into one of unstable equilibrium.

Let C be the axis of rotation and S the centre of gravity of a body $A \, B \, C \, D$, Fig. 213, whose dynamical stability is to be determined. If we cause the body to rotate, so that its centre of gravity S comes to S_1, i.e. vertically above C, the body is in unstable equilibrium; for if it is caused to revolve a little more, it will tumble over.

FIG. 213.



If we draw the horizontal line $S \, N$, it will cut off the height $N S_1 = s$, which the centre of gravity has ascended, by the aid of which we obtain the dynamical stability $G \, s$. If now we have $C S = C S_1 = r$, $C M = N S = e$ and the altitude $C N = M S = a$, we obtain

$$S_1 N = s = r - a = \sqrt{a^2 + e^2} - a,$$

and the stability in the second sense is

$$St = G\,(\sqrt{a^2 + e^2} - a).$$

The factor $s = \sqrt{a^2 + e^2} - a$ gives, for $a = 0$, $s = e$, for $a = e$, $s = e\,(\sqrt{2} - 1) = 0.414\,e$, for $a = n\,e$, $s = (\sqrt{n^2 + 1} - n)\,e$, approximatively $= (n + \frac{1}{2n} - n)\,e = \frac{e}{2n}$, thus for $a = 10\,e$, $s = \frac{e}{20}$ and for $a = \infty$, $s = \frac{e}{\infty} = 0$; this stability, therefore, becomes greater and greater as the centre of gravity becomes lower and lower, and it approaches more and more to zero as the centre of gravity is elevated more and more above the base. Sleds, wagons, ships etc. should therefore be loaded in such a manner, that the centre of gravity shall lie not only as low as possible, but also as near as possible above the centre of the base.

If the body is a prism with a symmetrical trapezoidal section, such as is represented in Fig. 213, and if the dimensions are the following: length $= l$, height $M\,O = h$, lower breadth $C\,D = b_1$, upper breadth $A\,B = b_2$, we have

$$M\,S = a = \frac{b_1 + 2\,b_2}{b_1 + b_2} \cdot \frac{h}{3} \text{ (§ 110) and}$$

$$C\,M = e = \tfrac{1}{2}\,b_1, \text{ whence}$$

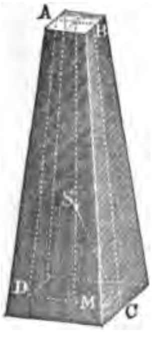
$$C\,S = r = \sqrt{\left(\frac{b_1}{2}\right)^2 + \left(\frac{b_1 + 2\,b_2}{b_1 + b_2} \cdot \frac{h}{3}\right)^2},$$

and the dynamical stability or the mechanical effect necessary to overturn this body is

$$St = G\left[\sqrt{\left(\frac{b_1}{2}\right)^2 + \left(\frac{b_1 + 2\,b_2}{b_1 + b_2} \cdot \frac{h}{3}\right)^2} - \frac{b_1 + 2\,b_2}{b_1 + b_2} \cdot \frac{h}{3}\right].$$

Example.—What is the stability of, or what is the mechanical effect necessary to overturn, the granite obelisk $A\,B\,C\,D$, Fig. 214, when its height is $h = 30$ feet, its upper length and breadth $l_1 = 1\frac{1}{2}$ and $b_1 = 1$ foot and its lower length and breadth $l_2 = 4$ feet and $b_2 = 3\frac{1}{2}$ feet? The volume of this body is

Fig. 214.



$$V = (2\,b_1\,l_1 + 2\,b_2\,l_2 + b_1\,l_2 + b_2\,l_1)\,\frac{h}{6}$$
$$= (2 \cdot \tfrac{3}{2} \cdot 1 + 2 \cdot 4 \cdot \tfrac{7}{2} + 1 \cdot 4 + \tfrac{3}{2} \cdot \tfrac{7}{2})\,\tfrac{30}{6}$$
$$= 40{,}25 \cdot 5 = 201{,}25 \text{ cubic feet.}$$

If a cubic foot of granite weighs $\gamma = 3 \cdot 62{,}4 = 187.2$ pounds, we have for the total weight of the body

$$G = 201{,}25 \cdot 187{,}2 = 37674.$$

The height of its centre of gravity above the base is

$$a = \frac{b_2\, l_2 + 3\, b_1\, l_1 + b_2\, l_1 + b_1\, l_2}{2\, b_2\, l_2 + 2\, b_1\, l_1 + b_2\, l_1 + b_1\, l_2} \cdot \frac{h}{2}$$
$$= \frac{4 \,.\, \frac{7}{2} + 3 \,.\, \frac{3}{2} \,.\, 1 + 1 \,.\, 4 + \frac{3}{2} \,.\, \frac{7}{2}}{40{,}25} \,.\, \frac{30}{2} = \frac{27{,}75 \,.\, 15}{40{,}25} = 10{,}342 \text{ feet.}$$

Supposing a rotation around the longer edge of the base, we have the horizontal distance of the centre of gravity from this edge, $e = \frac{1}{2}\, b_2 = \frac{1}{2} \,.\, \frac{7}{2} = \frac{7}{4}$ feet, and therefore the distance of the centre of gravity from the axis is

$$C\,S = r = \sqrt{a^2 + e^2} = \sqrt{(1{,}75)^2 + (10{,}342)^2} = \sqrt{110{,}002} = 10{,}489;$$

hence the height that centre of gravity must be lifted is

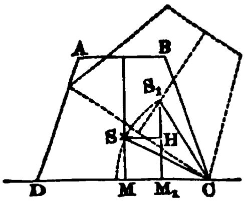
$$s = r - a = 10{,}489 - 10{,}342 = 0{,}147 \text{ feet,}$$

and the work to be done or the stability

$$S\,t = G\,s = 87674 \,\,.\,\, 0{,}147 = 5538 \text{ foot-pounds.}$$

§ 145. **Work Done in Moving a Heavy Body.**—In order to find the mechanical effect, which is necessary to change the position of a heavy body by causing a rotation, we must pursue the same course as in calculating its dynamical stability. If we cause a heavy body $A\ C$, Fig. 215, to rotate about a horizontal axis to such an extent, that the inclination $M\,C\,S = a$ of the line of gravity $C\,S = r$ becomes $M\,C\,S_1 = a_1$, the centre of gravity S will describe the vertical space $H\,S_1 = M_1\,S_1 - M\,S = s_1 = r\,(sin.\ a_1 - sin.\ a)$, and therefore if we designate by G the weight of the body, the mechanical effect required is

FIG. 215.



$$A_1 = G\,s_1 = G\,r\,(sin\ a_1 - sin.\ a).$$

If the axis of rotation is not horizontal, but inclined at an angle β to the horizon, we have

$$s_1 = r\ cos.\ \beta\ (sin.\ a_1 - sin.\ a) \text{ and}$$
$$A_1 = G\,s_1 = G\,r\ cos.\ \beta\ (sin.\ a_1 - sin.\ a). \text{ (Compare § 133.)}$$

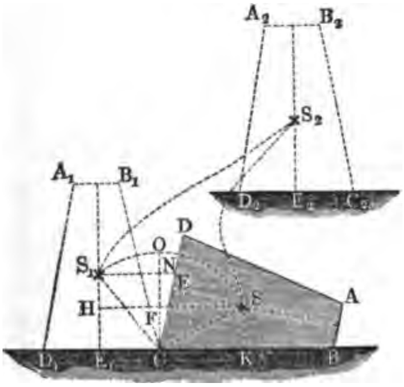
If in addition the body is moved in such a manner as not to change its position in relation to the direction of gravity, and if its centre of gravity and all its parts describe one and the same space, the vertical projection of which is $= s_2$, then the moving of the body will require, in addition to the above mechanical effect, an amount of work $A_2 = G\,s_2$, and consequently the total work done will be

$$A = A_1 + A_2 = G\,[r\ cos.\ \beta\ (sin.\ a_1 - sin.\ a) + s_2.]$$

The space described by the body in a horizontal direction does not enter into the question, when we suppose the motion to be slow, in which case the work of inertia can be put equal to zero.

If a body $A\ C$, Fig. 216, lying upon a horizontal plane $B\ C$ is to be placed upright upon another plane $C_2\ D_2$, we have $\beta = 0°$, or $\cos. \beta = 1$; and if a and e denote the horizontal and vertical co-ordinates of the centre of gravity of the body, when it is in an upright position, the radius $C\ S_1 = r = \sqrt{a^2 + e^2}$, and the height $E_1\ S_1 = a = r \sin.\ a_1$. If a is the angle of inclination $B\ C\ S$ formed by the side $B\ C$ of the body with the line of gravity $C\ S$, we have the original height of the centre of gravity above the surface on which the body rests

FIG. 216.



$$K\,S = C\,S \sin.\ B\,C\,S = r \sin.\ a = \sqrt{a^2 + e^2} \sin.\ a,$$

and consequently the height, which the centre of gravity is raised, while the body is being placed upright is

$$H\,S_1 = s_1 = E_1\,S_1 - E_1\,H = a - \sqrt{a^2 + e^2} \sin.\ a.$$

If now s_2 is the vertical distance of the plane $C_2\ D_2$ above the first plane $B\ C$, we have for the entire work done in placing the body upon $C_2\ D_2$

$$A = G\,(a - \sqrt{a^2 + e^2} \sin.\ a + s_2).$$

This determination of the work necessary to move the body is perfectly correct only, when the centre of gravity is raised by a continuous movement from S to S_2. If, on the contrary, the body is first placed upright and then raised, the mechanical effect necessary is

$$A = G\,(\overline{F\,O} + s_2) = G\,(\overline{C\,O} - K\,S + s_2) = G\,[\,\sqrt{a^2 + e^2}\,(1 - \sin.\ a) + s_2];$$

for the work $G\,.\,\overline{O\,N}$ which the body performs, when the centre of gravity sinks from O to S_1, is lost.

§ 146. Stability of a Body on an Inclined Plane.—A body $A\ C$, Fig. 217, resting upon an inclined plane (Fr. plan incliné, Ger. schiefe Ebene), can assume two motions; it can slide down the inclined plane, or it can overturn by a revolution around one of the edges of its base. If the body is left to itself the weight G is decomposed into a force N at right angles to and a force P parallel to the base; the first is counteracted entirely by the inclined plane, the latter, however, moves the body down the plane. If we put the angle of inclination of the plane to the horizon $= a$, we have also the angle $G\ S\ N = a$, and consequently the normal pressure

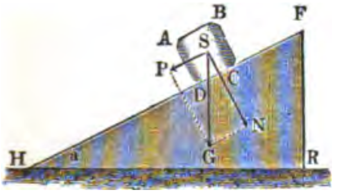
$$N = G \cos. a \text{ and}$$

the sliding force

$$P = G \sin. a.$$

If the vertical line of gravity $S\ G$ passes through the base $C\ D$, as is shown in Fig. 217, the sliding motion alone can take place; but if the line of gravity, as in Fig. 218, passes without the base, the body will be overturned and is without stability.

FIG. 217.



The stability of a body $A\ C$ upon an inclined plane $F\ H$, Fig. 219, is different from that of a body upon a horizontal plane $H\ R$. If $D\ M = e$ and $M\ s = a$ are the rectangular co-ordinates of the centre of gravity S, we have for the arm of the stability

$$D\ E = D\ O - M\ N = e \cos. a - a \sin. a,$$

while, on the contrary, it is $= e$, when the body stands upon a horizontal plane. Since $e > e \cos. a - a \sin. a$, the stability in reference to the lower edge D is always smaller upon the inclined plane, and become null, when $e \cos. a = a \sin. a$, I.E. when $\text{tang. } a = \frac{e}{a}$. If, then, a body, whose stability is $G\ e$ when standing upon a horizontal plane, is placed upon an inclined plane, whose angle of inclination corresponds to the expression $\text{tang. } a = \frac{e}{a}$, it loses its sta-

FIG. 218.

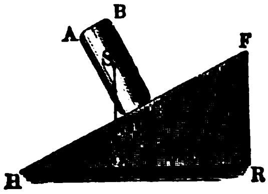
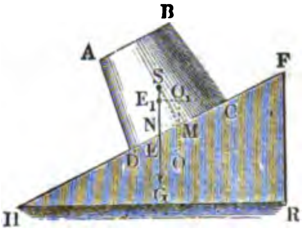


FIG. 219.



bility. On the other hand, a body can acquire stability upon an inclined plane, although wanting it when placed upon a horizontal one. For a rotation about the upper edge C the arm is $C E_1 = C O_1 + M N = e_1 \cos. a + a \sin. a$, while for the same position on a horizontal plane it is $C M = e_1$. If, however, e_1 is negative, the body possesses no stability as long as it rests upon a horizontal plane; but if placed upon an inclined plane, the angle of inclination a of which is such that we have $tang.\ a > \frac{e_1}{a}$, the body acquires a position of stable equilibrium. If, in addition to the force of gravity, another force P acts upon the body $A\ B\ C\ D$, Fig. 209, it retains its stability, if the direction of the resultant N of the weight G of the body and of the force P passes through the base $C\ D$ of the body.

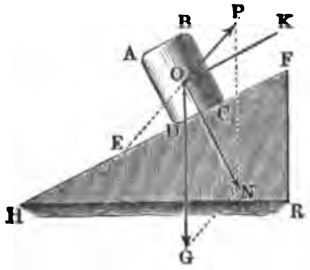
EXAMPLE.—In the obelisk in the example of paragraph 144, $e = \frac{7}{4}$ and $a = 10{,}342$ feet; consequently it will lose its stability, when placed upon an inclined plane, for whose angle of inclination we have

$$tang.\ a = \frac{7}{4\,.\,10{,}342} = \frac{7000}{41368} = 0{,}16922,$$

and whose angle of inclination is therefore

$$a = 9^\circ\ 36'.$$

FIG. 220.



§ 147. Theory of the Inclined Plane.—Since the inclined plane counteracts only the pressure perpendicular to it, the force P, necessary to retain the body, which is prevented from turning over, on the inclined plane, is determined by the consideration, that the resultant N, Fig. 220, of P and G must be perpendicular to the inclined plane. According to the theory of the parallelogram of forces, we have

$$\frac{P}{G} = \frac{\sin.\ P\ N\ O}{\sin.\ P\ O\ N};$$

but the angle $P\ N\ O =$ angle $G\ O\ N = F\ H\ R = a$, and the angle $P\ O\ N = P\ O\ K + K\ O\ N = \beta + 90^\circ$, when we denote the angle $P\ E\ F = P\ O\ K$ formed by the direction of the force with the inclined plane by β; hence we have

$$\frac{P}{G} = \frac{\sin.\ a}{\sin.\ (\beta + 90)}, \text{ I.E. } \frac{P}{G} = \frac{\sin.\ a}{\cos.\ \beta},$$

and the force, which holds the body on the inclined plane, is

$$P = \frac{G \sin. \alpha}{\cos. \beta}.$$

For the normal pressure we have

$$\frac{N}{G} = \frac{\sin. O\,G\,N}{\sin. O\,N\,G}$$

or, since the angle $O\,G\,N = 90° - (\alpha + \beta)$ and $O\,N\,G = P\,O\,N = 90 + \beta$,

$$\frac{N}{G} = \frac{\sin. [90^0 - (\alpha + \beta)]}{\sin. (90^0 + \beta)} = \frac{\cos. (\alpha + \beta)}{\cos. \beta},$$

and the normal pressure against the inclined plane is

$$N = \frac{G \cos. (\alpha + \beta)}{\cos. \beta}.$$

If $\alpha + \beta$ is $> 90°$ or $\beta > 90° - \alpha$, N becomes negative, and then, as is represented in Fig. 221, the inclined plane $H\,F$ must be placed above the body O, to which the force P is applied. If the force P is parallel to the inclined plane, β becomes $= 0$ and $\cos. \beta = 1$, and we have

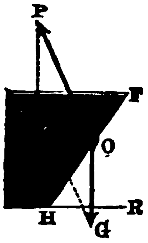
$$P = G \sin. \alpha \text{ and } N = G \cos. \alpha.$$

If the force P acts vertically $\alpha + \beta$ is $= 90°$, and we have

$$\cos. \beta = \sin. \alpha,\ \cos. (\alpha + \beta) = 0,$$

$P = G$ and $N = 0$. In this case the inclined plane has no influence upon the body.

Fig. 221.



Finally, if the force is horizontal, β becomes $= -\alpha$ and $\cos. \beta = \cos. \alpha$, and we have

$$P = \frac{G \sin. \alpha}{\cos. \alpha} = G \operatorname{tang.} \alpha \text{ and } N = \frac{G \cos. 0}{\cos. \alpha} = \frac{G}{\cos. \alpha}.$$

Example.—In order to retain a body weighing 500 pounds upon a plane inclined to the horizon at an angle of 50°, a force is employed, whose direction forms an angle of 75° with the horizon: required the intensity of the force and the pressure of the body upon the inclined plane. The force is

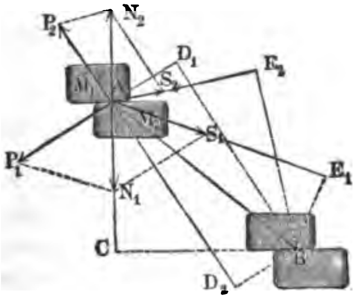
$$P = \frac{500 \sin. 50°}{\cos. (75° - 50°)} = \frac{500 \sin. 50°}{\cos. 25°} = 422{,}6 \text{ pounds}.$$

and the pressure upon the plane is

$$N = \frac{500 \cos. 75°}{\cos. 25°} = 142{,}8 \text{ pounds}.$$

§ 148. The Principle of Virtual Velocities.—If we combine the principle of the equality of action and reaction, explained in § 138, with the principle of virtual velocities (§ 83 and § 98), we obtain the following law. If two bodies M_1 and M_2 hold each other in equilibrium, *then, for a finite rectilinear or for an infinitely small curvilinear motion of the point A of pressure or contact, not only the sum of the mechanical effects of the forces of each separate body, but also the sum of the mechanical effects of the exterior forces acting upon the two bodies (taken together) is equal to zero.* If P_1 and S_1 are the forces in one body and P_2 and S_2 those in the other, when the point of contact is moved from A to B, the spaces described by these forces are $A\,D_1, A\,E_1, A\,D_2$ and $A\,E_2$, and according to the law announced above we have

FIG. 222.



$$P_1 . \overline{A\,D_1} + S_1 . \overline{A\,E_1} + P_2 \overline{A\,D_2} + S_2 . \overline{A\,E_2} = 0,$$

or without reference to the direction

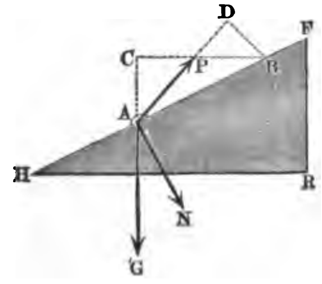
$$P_1 \overline{A\,D_1} + S_1 . \overline{A\,E_1} = P_2 . \overline{A\,D_2} + S_2 . \overline{A\,E_2}.$$

The correctness of this law can be demonstrated as follows. Since the normal forces N_1 and N_2 are equal, their mechanical effects $N_1 . \overline{A\,C}$ and $N_2 . \overline{A\,C}$ must also be equal to each other, the only difference being, that one of the forces is positive and the other negative. But according to what we have already seen, the mechanical effect of the resultant $N_1 . \overline{A\,C}$ is equal to the sum of those $P_1 \overline{A\,D_1} + S_1 . \overline{A\,E_1}$ of its components, and in like manner $N_2 \overline{A\,C} = P_2 . \overline{A\,D_2} + S_2 . \overline{A\,E_2}$; consequently we have

$$P_1 . \overline{A\,D_1} + S_1 . \overline{A\,E_1} = P_2 . \overline{A\,D_2} + S_2 . \overline{A\,E_2}.$$

This more general application of the principle of virtual velocities is of great importance in researches in statics, the determination of formulas for equilibrium being much simplified by it. If, E.G., we move a body A upon an inclined plane, $F\,H$, Fig. 223, a distance $A\,B$, the space described by its weight G is $= A\,C = A\,B\,sin.\,A\,B\,C =$ $A\,B\,sin.\,F\,H\,R = A\,B\,sin\,\alpha$, and, on the contrary, the space de-

FIG. 223.



scribed by the force P is $= A\,D = A\,B\, cos.\, B\,A\,D = A\,B\, cos.\, \beta$, and finally that described by the normal force N is $= 0$; but the work done by N is equal to the work done by G plus the work done by P, and we can therefore put

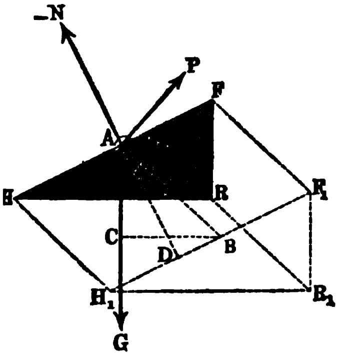
$$N\,.\,0 = -\,G\,.\,\overline{A\,C} + P\,.\,\overline{A\,D},$$

consequently the force, which holds the body upon an inclined plane, is

$$P = \frac{A\ \ C}{A\ \ D}\,.\,G = \frac{G\ sin.\ \alpha}{cos.\ \beta},$$

a result, which agrees perfectly with that obtained in the foregoing paragraph.

Fig. 224.



On the contrary, to find the normal force N, we must move the inclined plane $H\,F$, Fig. 224, an arbitrary distance $A\ B$ at right angles to the direction of the force P, determine the space described by the exterior forces and then put the mechanical effect of the weight G and of the force P equal to the mechanical effect of the pressure N upon the inclined plane.

The space described by N is

$$A\,D = A\,B\, cos.\, B\,A\,D = A\,B\, cos.\, \beta;$$

that described by G is

$$A\ \ C = A\ \ B\ cos.\ B\ A\ \ C = A\ \ B\ cos.\ (\alpha + \beta),$$

and that described by the force P is $= 0$, hence the mechanical effect is

$$N\,.\,\overline{A\,D} = G\,.\,\overline{A\,C} + P\,.\,0,$$

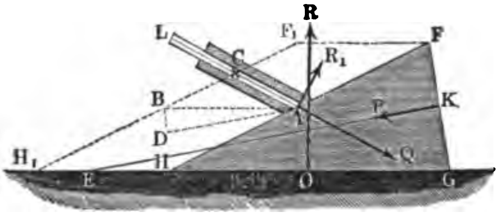
and $$N = \frac{G\,.\,\overline{A\,C}}{A\,D} = G\,.\,\frac{cos.\ (\alpha + \beta)}{cos.\ \beta},$$

as we found in the foregoing paragraph.

§ 149. **Theory of the Wedge.**—We can now deduce very simply the theory of the wedge. The wedge (Fr. coin, Ger. Keil) is a movable inclined plane formed by a three-sided prism $F\ H\ G$,

Fig. 225. The force $\overline{KP} = P$ acts generally at right angles to the back $F\,G$ of the wedge and balances another force or weight

FIG. 225.



$A\,Q = Q$, which presses against a side $F\,H$ of the wedge. If the angle, which measures the sharpness of the wedge, is $F\,H\,G = \alpha$ and the angle formed by the direction $K\,P$ or $A\,D$ of the force with the side $G\,H$ is $G\,E\,K = B\,A\,D = \delta$, and, finally, if the angle $L\,A\,H$ formed by the direction of the load Q with the side $F\,H$ is $= \beta$, the spaces described, when the wedge is moved from the position $F\,H\,G$ to the position $F_1\,H_1\,G_1$, are found in the following manner. The space described by the wedge is

$$A\,B = F\,F_1 = H\,H_1,$$

that described by the force is

$$A\,D = A\,B \cos. B\,A\,D = A\,B \cos. \delta,$$

and that described by the rod $A\,L$ or by the load Q is

$$A\,C = \frac{A\,B \sin. A\,B\,C}{\sin. A\,C\,B} = \frac{A\,B \sin. \alpha}{\sin. H\,A\,C} = \frac{A\,B \sin. \alpha}{\sin. \beta}.$$

On the contrary, the space described by the reaction R of the base $E\,G$ as well as that described by the reaction corresponding to the pressure against the guides of the rod is $= 0$.

Now putting the sum of the mechanical effects of the exterior forces P, Q, R and $R_1 = 0$, we have

$$P \,.\, \overline{A\,D} - Q \,.\, \overline{A\,C} + R \,.\, 0 + R_1 \,.\, 0 = 0,$$

from which we obtain the equation of condition

$$P = \frac{Q \,.\, \overline{A\,C}}{\overline{A\,D}} = \frac{Q \,.\, \overline{A\,B} \sin. \alpha}{\overline{A\,B} \cos. \delta \sin. \beta} = \frac{Q \sin. \alpha}{\sin. \beta \cos. \delta}.$$

If the direction $K\,E$ of the force passes through the edge H of the wedge and bisects the angle $F\,H\,G$, we have $\delta = \frac{\alpha}{2}$, and therefore

$$P = \frac{Q \sin. \alpha}{\sin. \beta \cos. \frac{\alpha}{2}} = \frac{2\, Q \sin. \frac{\alpha}{2}}{\sin. \beta}$$

If the direction of the force is parallel to the base or side $G\,H$, we have $\delta = 0$, and consequently

$$P = \frac{Q \sin. \alpha}{\sin. \beta},$$

and if the direction of the load is also perpendicular to the side $F\,H$, we have $\beta = 90°$, and consequently

$$P = Q \sin. \alpha.$$

EXAMPLE.—The sharpness $F\,H\,G = \alpha$ of a wedge is 25°, the direction of the force is parallel to the base, and therefore δ is $= 0$, and the load acts at right angles to the side $F\,H$, I.E., β is $= 90°$: required the relations of the force and load to each other; in this case we have

$$P = Q \sin. \alpha \text{ or } \frac{P}{Q} = \sin. 25° = 0{,}4226.$$

If the load is $Q = 130$ pounds, the force is

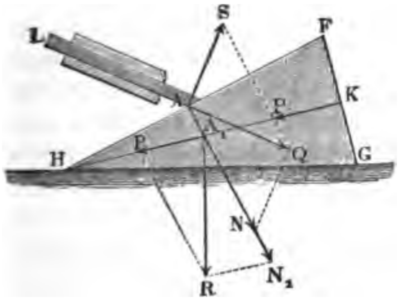
$$P = 130 \,.\, 0{,}4226 = 54{,}938 \text{ pounds.}$$

In order to move the load or rod a foot, the wedge must describe the space

$$A\,B = \frac{A\,C}{\sin. \alpha} = \frac{1}{0{,}4226} = 2{,}3662 \text{ feet.}$$

REMARK 1. The relation between the force P and the load Q of the wedge $F\,G\,H$, Fig. 226, can be determined by the application of the parallelogram of forces in the following manner. The load upon the rod $\overline{A\,Q} = Q$ is decomposed into a component $\overline{A\,N} = N$ perpendicular to the side $F\,H$ and into a component $\overline{A\,S} = S$ perpendicular to the axis of the rod. While S is counteracted by the guides of the rod, $\overline{A\,N} = N$ is transmitted to the wedge and combines there as $\overline{A_1\,N_1}$ with the force $\overline{K\,P} = \overline{A_1\,P} = P$ of the wedge to form a resultant $\overline{A_1 R} = R$, whose direction must be perpendicular to the base $G\,H$ of the wedge, in which case it will be transmitted completely to the support of the wedge. The parallelogram of forces $A_1\,P\,R\,N_1$ gives

FIG. 226.



$$\frac{P}{N_1} = \frac{sin.\, R\, A_1\, N_1}{sin.\, A_1\, R\, N_1} = \frac{sin.\, F\, H\, G}{sin.\, P\, A_1\, R} = \frac{sin.\, \alpha}{cos.\, \delta},$$

and from the parallelogram of forces $A\, N\, Q\, S$ we have

$$\frac{N}{Q} = \frac{sin.\, N\, Q\, A}{sin.\, A\, N\, Q} = \frac{sin.\, Q\, A\, S}{sin.\, L\, A\, H} = \frac{1}{sin.\, \beta};$$

but since N_1 is $= N$, we obtain by multiplying these proportions together,

$$\frac{P}{N} \cdot \frac{N}{Q} = \frac{P}{Q} = \frac{sin.\, \alpha}{sin.\, \beta\, cos.\, \delta}, \text{ or}$$

$$P = \frac{Q\, sin.\, \alpha}{sin.\, \beta\ cos.\, \delta},$$

as was found in the large text of this paragraph.

REMARK 2. The theory of the lever, inclined plane and wedge will be discussed at length in the fifth chapter, when the influence of friction will also be taken into consideration.

CHAPTER IV.

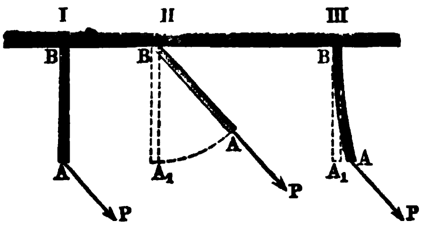
EQUILIBRIUM IN FUNICULAR MACHINES.

§ 150. Funicular Machines.—We have previously considered the solid bodies to be perfectly rigid or stiff bodies (Fr. corps rigides; Ger. starre or steife Körper); I.E., as bodies, whose volume and form are unchanged by the action of exterior forces upon them. Very often in the practical application of mechanics the supposition, that bodies are perfectly rigid, is not permissible, and it becomes necessary, therefore, to consider these bodies in two other states. These states are those of perfect flexibility and of perfect elasticity, and consequently we distinguish flexible bodies (Fr. corps flexible; Ger. biegsame Körper) and elastic bodies (Fr. corps élastiques; Ger. elastische Körper). Flexible bodies counteract without change of form forces in one direction only and follow perfectly those acting in other directions; elastic bodies, on the contrary, yield to a certain extent to every force acting upon them.

A rigid body $A\ B$, Fig. 227, I, counteracts completely the force

P, a flexible body *A B*, Fig. 227, II, follows the direction of the force *P*, which acts upon it, in such a manner, that its axis assumes

FIG. 227.



the direction of the force, and an elastic body *A B*, Fig. 227, III, resists the force *P* to a certain extent only, so that its axis undergoes a certain deflection. Cords, ropes, straps and in a certain sense chains are representatives of flexible bodies, although they do not possess perfect flexibility. These bodies will be the subject of the present chapter; elastic bodies, or rather the elasticity of rigid bodies, will be treated of in the fourth section.

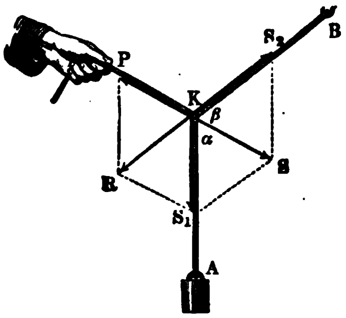
We understand by a *funicular machine* (Fr. machine funiculaire; Ger. Seilmaschine) a cord or a combination of cords (the word cord being employed in a general sense), which is stretched by forces, and we will occupy ourselves in this chapter with the theory of the equilibrium of this machine. The point of the funiculaire machine to which a force is applied, and where, consequently, the cord forms an angle or undergoes a change of direction is called a *knot* (Fr. noeud; Ger. Knoten). The same is either *fixed* (Fr. fixe; Ger. fest) or *movable* (Fr. coulant; Ger. beweglich). Tension (Fr. tension; Ger. Spannung) is the force propagated in the direction of its axis by a stretched cord. The tensions at the ends of a straight cord or piece of cord are equal and opposite (§ 86). A straight cord cannot propagate any other force but the tension acting in the direction of its axis; for if it did, it would bend and would no longer be straight.

§ 151. Equilibrium in a Knot.—Equilibrium exists in a funicular machine, when each of its knots is in equilibrium. Consequently we must begin with the study of the conditions of equilibrium in a single knot.

Equilibrium exists in a knot *K* formed by a piece of cord

$A\ K\ B$, Fig. 228, when the resultant $\overline{K\ S} = S$ of the two tensions of the cord $\overline{K\ S_1} = S_1$ and $\overline{K\ S_2} = S_2$ is equal and opposite to the force P applied at the knot; for the tensions of the cord S_1 and S_2 produce the same effect in the knot K as two forces equal to them and acting in the same direction as they do, and the three forces are in equilibrium, when one of them is equal and opposite to the resultant of the other two (§ 87). In like manner the resultant R of the force P and of one of the tensions S_1 is equal and opposite to the second tension S_2, etc. We can profit by this equality to determine two conditions, E.G., the tension and direction of one of the ropes. If, E.G., the force P, the tension S_1 and the angle formed by them

FIG. 228.



$$A\ K\ P = 180° - A\ K\ S = 180° - \alpha$$

are given, we have for the other tension

$$S_2 = \sqrt{P^2 + S_1^2 - 2\ P\ S_1\ cos.\ \alpha}$$

and for its direction or for the angle $B\ K\ S = \beta$ formed by it with $K\ S$

$$sin.\ \beta = \frac{S_1\ sin.\ \alpha}{S_2}.$$

EXAMPLE.—If the rope $A\ K\ B$, Fig. 228, is fastened at its end B and stretched at its end A by a weight $G = 135$ pounds and at its centre K by a force $P = 109$ pounds, whose direction is upwards at an angle of 25° to the horizon, what will be the direction of the tension in the piece of cord $K\ B$?

The intensity of the required tension is

$$S_2 = \sqrt{109^2 + 135^2 - 2\ .\ 109\ .\ 135\ cos.\ (90° - 25°)}$$
$$= \sqrt{11881 + 18225 - 29430\ .\ cos.\ 65°} = \sqrt{17668,3} = 132,92 \text{ pounds.}$$

For the angle β we have

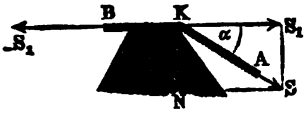
$$sin.\ \beta = \frac{S_1\ sin.\ \alpha}{S_2} = \frac{135\ .\ sin.\ 65°}{132,92},\ log\ sin.\ \beta = 0,96401 - 1,$$

whence $\beta = 67°\ 0'$, and the inclination of the piece of cord to the horizon is

$$\beta° - 25° = 67°\ 0' - 25°\ 0' = 42°\ 0'.$$

§ 152. If a cord $A\ K\ B$, Fig. 229, forms a fixed knot K in consequence of one portion of the cord $B\ K$ lying upon a firm support M, while the other portion of the cord is stretched by a force $\overline{K\ S} = S$, whose direction forms a certain angle $S\ K\ S_1 = \alpha$ with the direction of the first portion of the cord, we have the tension in the portion $K\ B$ of the cord

FIG. 229.



$$\overline{K\ S_1} = S_1 = S\ cos.\ \alpha,$$

while the second component $\overline{K\ N} = N = S\ sin.\ \alpha$ is counteracted by the support M. We have also

$$S_1 = S\ \sqrt{1 - (sin.\ \alpha)^2},$$

and therefore, when the angle of divergence is small,

$$S_1 = \left(1 - \frac{1}{2}\,(sin.\ \alpha)^2\right) S = \left(1 - \frac{\alpha^2}{2}\right) S,$$

or inversely

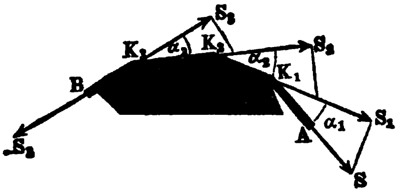
$$S = \frac{S_1}{1 - \frac{\alpha^2}{2}} = \left(1 + \frac{\alpha^2}{2}\right) S_1.$$

If a cord is laid upon a prismatical body, and its directions thus changed successively an amount measured by the angles α_1, α_2, α_3, the foregoing decomposition of the force is repeated, so that in the knot K_1 the tension S is changed into $S_1 = S\ cos.\ \alpha_1$, and in the knot K_2 the tension S_1 into

$$S_2 = S_1\ cos.\ \alpha_2 = S\ cos.\ \alpha_1\ cos.\ \alpha_2,$$

and in the knot K_3 the tension S_2 into

FIG. 230.



$$S_3 = S_2\ cos.\ \alpha_3 = S\ cos.\ \alpha_1\ cos.\ \alpha_2\ cos.\ \alpha_3.$$

If the angles α_1, α_2, α_3 are equal to each other and $= \alpha$, we have

$$S_3 = S\,(cos.\ \alpha)^3$$

$$S_n = S\,(cos.\ \alpha)^n.$$

If the prism M becomes a cylinder, α is infinitely small and n infinitely great, and consequently

$$S_n = \left(1 - \frac{\alpha^2}{2}\right)^n S = \left(1 - \frac{n\,\alpha^2}{2}\right) S,$$

or if we denote the total angle of divergence $n\ \alpha$ by β, we have

$$S_n = \left(1 - \frac{\alpha\beta}{2}\right) S, \text{ i.e.}$$

$S_n = S$, because α and consequently $\frac{\alpha\beta}{2}$ is infinitely small compared with 1.

If, therefore, a cord is laid upon a smooth body so as to cover a portion of the periphery of its cross section, its tension is not changed thereby; and when a state of equilibrium exists the tension at both ends of the cord are equal to each other.

§ 153. If the knot K is movable, if, E.G., the force P is applied by means of a ring to the cord $A\ K\ B$, Fig. 231, which is passed through it, the resultant S of the tensions S_1 and S_2 of the cord is equal and opposite to the force P applied to the ring; besides the tensions of the cord are equal to each other. This equality is a consequence of § 152, but it can also be proved in the following manner. If we pull the rope a certain distance through the ring, one of the tensions S_1 describes the space s, the other tension S_2 the space $-\ s$, and the force P the space 0. If, therefore, we assume perfect flexibility, the work done is

$$P \cdot 0 = S_1 \cdot s - S_2 \cdot s, \text{ i.e. } S_1 s = S_2 s \text{ or } S_1 = S_2.$$

The equality of the angles $A\ K\ S$ and $B\ K\ S$, formed by the direction of the resultant S with the directions of the rope, is also a consequence of this equality of the tensions. Putting this angle $= \alpha$ the resolution of the rhomb $K\ S_1\ S\ S_2$ gives

$$S = P = 2\ S_1 \cos. \alpha, \text{ and inversely}$$

$$S_1 = S_2 = \frac{P}{2 \cos. \alpha}.$$

FIG. 231.

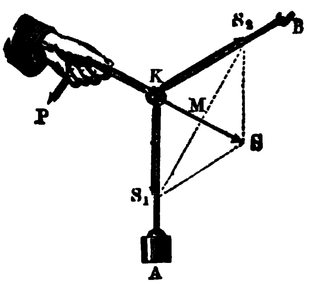
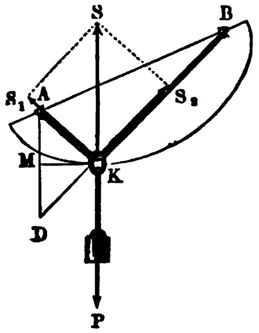


FIG. 232.

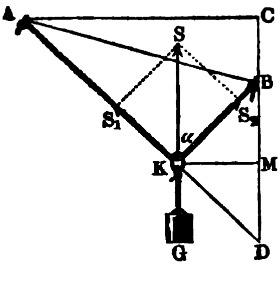


If A and B, Fig. 232, are fixed points of a cord $A\ K\ B$ of a given length ($2\,a$) with a movable knot K, we can find the positions of this knot by constructing an ellipse, whose foci are at A and B and whose major axis is equal to the length of the rope $2\,a$, and by drawing a tangent to this curve perpendicular to the given direction of the force. The point of tangency thus found is the position of the knot; for the normal $K\,S$ to the ellipse forms equal angles with the radii vectores $K\,A$ and $K\,B$, exactly as the resultant S does with the tensions S_1 and S_2 of the cord.

If we draw $A\,D$ parallel to the direction of the given force, make $B\,D$ equal to the given length of the cord, divide $A\,D$ in two equal parts at M and erect the perpendicular $M\,K$, we obtain the position K of the knot without constructing an ellipse; for the angle $A\,K\,M =$ angle $D\,K\,M$ and $A\,K = D\,K$, and consequently the angle $A\ K\ S =$ angle $B\ K\ S$ and $A\ K + K\,B = D\,K + K\,B = D\,B$.

EXAMPLE.—Between the points A and B, Fig. 238, a cord 9 feet long is stretched by a weight $G = 170$ pounds, hung upon it by means of a ring. The horizontal distance of the two points from each other is $A\ C = 6\frac{1}{2}$ feet and the vertical distance of the same $C\,B = 2$ feet: required the position of the knot as well as the tensions and directions of the two portions of the cord. From the length $A\ D =$ 9 feet as hypothenuse and the horizontal distance $A\ C = 6\frac{1}{2}$ feet, we obtain the vertical line

FIG. 238.



$$C\,D = \sqrt{9^2 - 6{,}5^2} = \sqrt{81 - 42{,}25}$$
$$= \sqrt{38{,}75} = 6{,}225 \text{ feet,}$$

and from this the base of the isosceles triangle $B\,D\,K$

$$B\,D = C\,D - C\,B = 6{,}225 - 2 = 4{,}225 \text{ feet.}$$

On account of the similarity of the triangles $D\,K\,M$ and $D\,A\,C$, we have

$$D\,K = B\,K = \frac{D\,M}{D\,C} \cdot D\ A = \frac{4{,}225 \cdot 9}{2 \cdot 6{,}225} = 3{,}054 \text{ feet,}$$

whence

$$A\,K = 9 - 3{,}054 = 5{,}946 \text{ feet.}$$

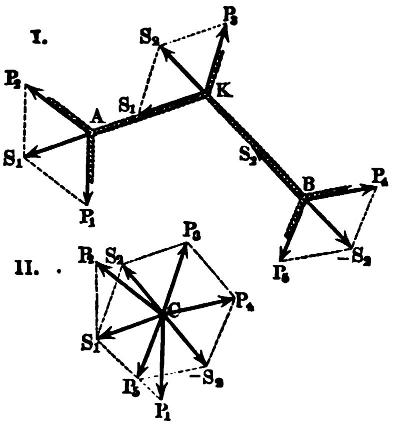
Hence for the angle α formed by the two portions of the cord with the vertical line we have

$$\cos.\ \alpha = \frac{B\,M}{B\,K} = \frac{2{,}1125}{3{,}054} = 0{,}6917, \text{ whence } \alpha = 46^\circ\ 14',$$

and finally the tension in the cord is

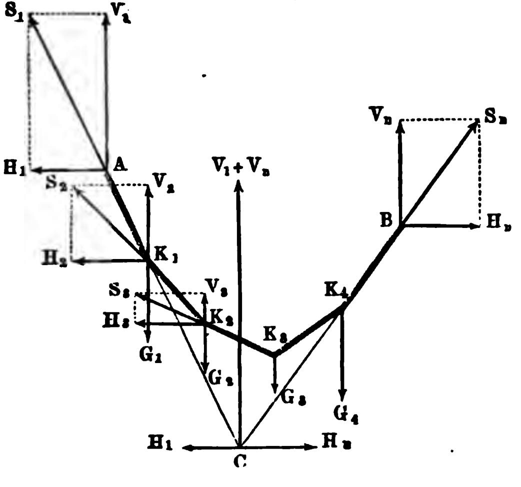
$$S_1 = S_2 = \frac{G}{2\cos.\ \alpha} = \frac{170}{2 \cdot 0{,}6917} = 122{,}9 \text{ pounds.}$$

FIG. 234.



§ 154. Equilibrium of a Funicular Polygon.—The conditions of equilibrium of a funicular polygon, I.E. of a stretched cord acted upon in different points by forces, are the same as those of the equilibrium of forces applied at the same point. Let $A\,K\,B$, Fig. 234, I, be a cord stretched by the forces P_1, P_2, P_3, P_4, P_5; P_1 and P_2 being applied in A, P_3 in K and P_4 and P_5 in B. Let us denote the tension of the portion of the cord $A\ K$ by S_1 and that of the portion $B\ K$ by S_2, then we have S_1 as the resultant of the two forces P_1 and P_2 applied in A. Transferring the point of application of this tension from A to K, we have S_2 as resultant of S_1 and P_3 or of P_1, P_2 and P_3. Transferring the point of application of the force S_2 from K to B, we have S_2 as the resultant of P_4 and P_5; now, since S_2 is the resultant of P_1, P_2, and P_3, this system of forces is in equilibrium; we can therefore assert, that *if certain forces P_1, P_2, P_3, etc., of a funicular polygon are in equilibrium, they will also hold each other in equilibrium, when they are applied without change of direction or intensity to a single point,* E.G. *to C (II).* If the rope $A\,K_1\,K_2 \ldots B$, Fig. 235, is stretched in the knots K_1, K_2, etc., by the weights G_1, G_2, etc., and if its extremities are held fast by the vertical forces V_1 and V_n and by the hori-

FIG. 235.



zontal forces H_1 and H_n, the sum of the vertical forces is

$$V_1 + V_n - (G_1 + G_2 + G_3 + \dots),$$

and the sum of the horizontal forces is $H_1 - H_n$. The conditions of equilibrium require both these sums to be $= 0$, and therefore we have

$$1)\ V_1 + V_n = G_1 + G_2 + G_3 + \dots \text{ and}$$

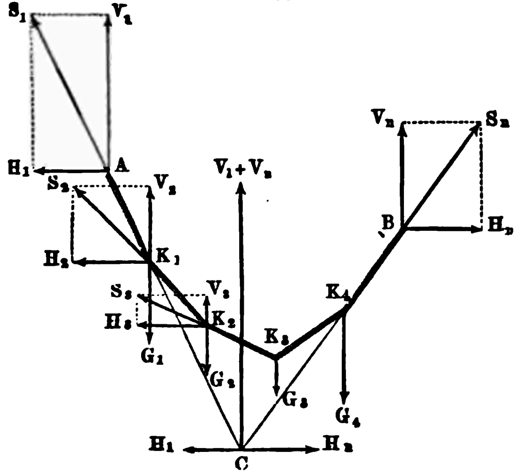
$$2)\ H_1 = H_n, \text{ I.E.}$$

the sum of the vertical forces or tensions at the extremities of the ropes of a funicular polygon stretched by weights is equal to the sum of weights hung upon it, and the horizontal tension at one extremity is equal and opposite to that at the other.

If we prolong the directions of the tensions S_1 and S_n at the extremities A and B, until they cut each other in C, and if we transfer the point of application of these tensions to this point, we obtain a single force $P = V_1 + V_n$; for the horizontal forces H_1 and H_n balance each other. Since this force balances the sum $G_1 + G_2 + G_3 + \dots$ of the weights attached to it, the point of application or centre of gravity of these weights must be in the direction of this force, I.E. in the vertical line passing through C.

§ 155. From the tension S_1 of the first portion $A\ K_1$ of the rope and from the angle of inclination $S_1\ A\ H_1 = \alpha_1$ we obtain the vertical tension $V_1 = S_1\ sin.\ \alpha_1$ and the horizontal tension $H_1 = S_1\ cos.\ \alpha_1$. If we transfer the point of application of these forces from A to K_1, we have, in addition to them, the weight G_1, which acts vertically downwards, and the vertical tension in the following portion $K_1\ K_2$ of the rope is $V_2 = V_1 - G_1 = S_1\ sin.\ \alpha_1 - G_1$, while the horizontal tension $H_2 = H_1 = H$ remains unchanged. The two latter forces, when combined, give the axial tension of the second portion of the rope

FIG. 285.



$$S_2 = \sqrt{V_2^2 + H^2},$$

and its inclination α_2 is determined by the formula

$$tang.\, \alpha_2 = \frac{V_2}{H} = \frac{S_1\, sin.\, \alpha_1 - G_1}{S_1\, cos.\, \alpha_1}, \text{ i.e.}$$

$$tang.\, \alpha_2 = tang.\, \alpha_1 - \frac{G_1}{H}.$$

Transferring the point of application of V_2 and H_2 from K_1 to K_2, we have, by the addition of the weight G_2, a new vertical force

$$V_3 = V_2 - G_2 = V_1 - (G_1 + G_2) = S_1\, sin.\, \alpha_1 - (G_1 + G_2),$$

which is that of the third portion of the rope, while the horizontal force $H_3 = H$ remains unchanged. The total tension in this third portion of the cord is

$$S_3 = \sqrt{V_3^2 + H^2},$$

and its angle of inclination α_3 is determined by the formula

$$tang.\, \alpha_3 = \frac{V_3}{H} = \frac{S_1\, sin.\, \alpha_1 - (G_1 + G_2)}{S_1\, cos.\, \alpha_1}, \text{ i.e.}$$

$$tang.\, \alpha_3 = tang.\, \alpha_1 - \frac{G_1 + G_2}{H}.$$

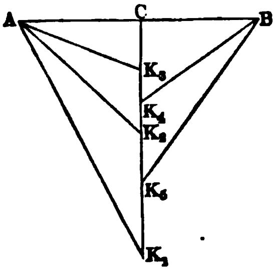
For the angle of inclination of the fourth portion of the cord we have

$$tang.\, \alpha_4 = tang.\, \alpha_1 - \frac{G_1 + G_2 + G_3}{H}, \text{ etc.}$$

If $\frac{G_1 + G_2 + G_3}{H}$ becomes $>$ *tang.* α_1 or $G_1 + G_2 + G_3 > V_1$, then *tang.* α_4 and consequently α_4 becomes negative, and the corresponding side $K_3 K_4$ of the polygon is no longer directed downward, but upward. The conditions are the same for any point, for which $G_1 + G_2 + G_3 + \ldots$ is $> V_1$.

The tensions S_1, S_2, S_3, etc., as well as the angles of inclination α_1, α_2, α_3, etc., of the different portions of the rope can easily be represented geometrically. If we make the horizontal line $CA = CB$, Fig. 236, $=$ the horizontal tension H and the vertical line $CK_1 =$ the vertical tension V_1 at the point of suspension A, the hypothenuse $A K_1$ will give the total tension S_1 of the first portion of the rope, and the angle $C A K_1$ the inclination of the same to the horizon. If, now, we lay off upon $C K_1$ the weights G_1, G_2, G_3, etc., as the divisions $K_1 K_2$, $K_2 K_3$, etc., and draw the transverse lines $A K_2$, $A K_3$,

FIG. 236.



the latter will indicate the tensions of the different succeeding portions of the cord, and the angles $C\,A\,K_2$, $C\,A\,K_3$ etc., the angles of inclination a_2, a_3, etc., of these portions.

§ **156.** From the investigations in the foregoing paragraph we can deduce the following law for the equilibrium of a cord stretched by weights:

1) *The horizontal tension is in all parts of the cord one and the same, viz.:*

$$H = S_1 \cos. a_1 = S_n \cos. a_n.$$

2) *The vertical tension in any portion is equal to the vertical tension of the cord at the end above it minus the sum of the weights suspended above it, or*

$$V_m = V_1 - (G_1 + G_2 + \dots G_{m-1}).$$

This law can be expressed more generally thus: The vertical tension in any point is equal to the tension in any other lower or higher point plus or minus the sum of the weights suspended between them.

If we know besides the weights the angle a_1 and the horizontal tension H, we obtain the vertical tension at the extremity A by means of the formula

$$V_1 = H \,.\, tang.\, a_1,$$

and that at the extremity B is

$$V_n = (G_1 + G_2 + \dots + G_n) - V_1.$$

If, on the contrary, the two angles of inclination a_1 and a_n at the two points of suspension A and B are known, the horizontal and vertical tensions are determined in the following manner; we have

$$\frac{V_n}{V_1} = \frac{tang.\, a_n}{tang.\, a_1},$$

and therefore $$V_n = \frac{V_1\, tang.\, a_n}{tang.\, a_1}.$$

But since $V_1 + V_n = G_1 + G_2 + \dots$ i.e.,

$$\left(\frac{tang.\, a_1 + tang.\, a_n}{tang.\, a_1}\right) V_1 = G_1 + G_2 \dots,$$

we have

$$V_1 = \frac{(G_1 + G_2 + \dots)\, tang.\, a_1}{tang.\, a_1 + tang.\, a_n} = (G_1 + G_2 + \dots)\frac{\sin.\, a_1 \cos.\, a_n}{\sin.\, (a_1 + a_n)}$$

and

$$V_n = \frac{(G_1 + G_2 + \dots)\, tang.\, a_n}{tang.\, a_1 + tang.\, a_n} = (G_1 + G_2 + \dots)\frac{\sin.\, a_n \cos.\, a_1}{\sin.\, (a_1 + a_n)},$$

and consequently

$$H = V_1 \, cotg. \, a_1 = V_n \, cotg. \, a_n = (G_1 + G_2 + \dots) \frac{cos. \, a_1 \, cos. \, a_n}{sin. \, (a_1 + a_n)}.$$

If the two ends of the cord have the same inclination, we have $V_1 = V_n = \frac{G_1 + G_2 + \dots + G_n}{2}$; then one end A carries as much as the other end B.

These formulas are applicable to any pair of points or knots of the funicular polygon, when we substitute instead of $G_1 + G_2 + \dots$ the sum of the weights, etc., suspended to the cord between the two points. The vertical tensions of a cord, on which a weight G_m is hung and the angles of inclination of which are a_m and a_{m+1}, are

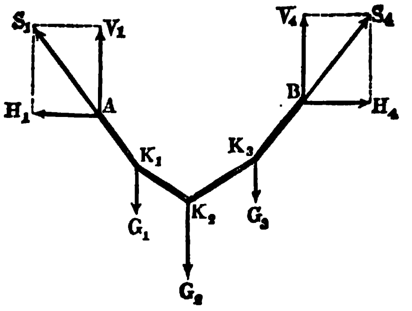
$$V_m = G_m \frac{sin. \, a_m \, cos. \, a_{m+1}}{sin. \, (a_m + a_{m+1})} = \frac{G_m}{1 + cotg. \, a_m \, tang. \, a_{m+1}} \text{ and}$$

$$V_{m+1} = G_m \frac{sin. \, a_{m+1} \, cos. \, a_m}{sin. \, (a_m + a_{m+1})} = \frac{G_m}{1 + tang. \, a_m \, cotg. \, a_{m+1}}.$$

These laws are applicable to any funicular polygon stretched by parallel forces, when we substitute instead of the vertical the direction of the forces.

EXAMPLE.—The funicular polygon $A \, K_1 \, K_2 \, K_3 \, B$, Fig. 237, is stretched by three weights $G_1 = 20$, $G_2 = 30$ and $G_3 = 16$ pounds as well as by the horizontal force $H_1 = 25$ pounds; required the axial tensions, supposing the extremities A and B to have the same angle of inclination. The vertical tensions at the ends are equal and are

FIG. 237.



$$V_1 = V_4 = \frac{G_1 + G_2 + G_3}{2} = \frac{20 + 30 + 16}{2} = 33 \text{ pounds.}$$

The vertical tension of the second portion of the cord is $V_2 = V_1 - G_1 = 33 - 20 = 13$ pounds; that of the third is,

$$V_3 = V_4 - G_3 \text{ (or } G_1 + G_2 - V_1) = 33 - 16 = 17 \text{ pounds.}$$

The angles of inclination a_1 and a_4 of these extremities are determined by the formulas

$$tang. \, a_1 = tang. \, a_4 = \frac{V_1}{H} = \frac{33}{25} = 1{,}32;$$

those of the second and third portions by the formulas

$$tang. \, a_2 = tang. \, a_1 - \frac{G_1}{H} = 1{,}32 - \frac{20}{25} = 0{,}52 \text{ and}$$

$$tang. \, a_3 = tang. \, a_4 - \frac{G_3}{H} = 1{,}32 - \frac{16}{25} = 0{,}68;$$

— [illegible] all Original (in German).

whence we have

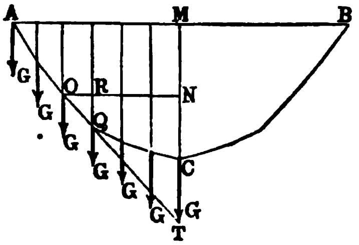
$$\alpha_1 = \alpha_4 = 52^\circ\ 51',\ \alpha_2 = 27^\circ\ 28',\ \alpha_3 = 34^\circ\ 13'.$$

Finally the axial tensions are

$$S_1 = S_4 = \sqrt{V_1^2 + H^2} = \sqrt{33^2 + 25^2} = \sqrt{1714} = 41{,}40 \text{ pounds},$$
$$S_2 = \sqrt{V_2^2 + H^2} = \sqrt{13^2 + 25^2} = \sqrt{794} = 18{,}18 \text{ pounds and}$$
$$S_3 = \sqrt{V_3^2 + H^2} = \sqrt{17^2 + 25^2} = 30{,}23 \text{ pounds}.$$

§ 157. **The Parabola as Catenary.**—Let us suppose, that the cord *A C B*, Fig. 238, is stretched by the weights G_1, G_2, G_3, etc., hung at equal horizontal distances from each other. Let us denote the horizontal distance *A M* between the point of suspension *A* and the lowest point *C* by *b* and the vertical distance *C M* by *a*; let us also put the similarly placed co-ordinates of a point *O* of the funicular polygon $O N = y$ and $C N = x$. If the vertical tension in *A* is $= V$, that in *O* is consequently $= \frac{y}{b} \cdot V$, and therefore we have for the angle of inclination to the horizon $N O T = R O Q = \phi$ of the portion of the cord *O Q*

FIG. 238.



$$tang.\ \phi = \frac{y}{b} \cdot \frac{V}{H},$$

in which *H* designates the horizontal tension.

From this we obtain $Q R = \overline{O R} \cdot tang.\ \phi = \overline{O R} \cdot \frac{y}{b} \cdot \frac{V}{H}$, which is the difference of height of two neighboring corners of the funicular polygon. If we put *y* successively $= \overline{O R}$, $2\ \overline{O R}$, $3\ \overline{O R}$, etc., the latter formula gives the difference of height of the first, second, third, etc., corners, counting from the lowest point upwards; if now we add all these values, whose number we can suppose to be $= m$, we obtain the height *C N* of the point *O* above the lowest point *C*. Here we have

$$x = C N = \frac{V}{H} \cdot \frac{\overline{O R}}{b}(\overline{O R} + 2\,\overline{O R} + 3\,\overline{O R} + \ldots + m \,.\, \overline{O R})$$
$$= \frac{V}{H} \cdot \frac{\overline{O R}^2}{b}(1 + 2 + 3 + \ldots + m) = \frac{V}{H} \cdot \frac{m\,(m + 1)}{1 \,.\, 2} \cdot \frac{\overline{O R}^2}{b}$$

in accordance with the rule for summing an arithmetical series.

Finally, putting $O R = \frac{y}{m}$, we obtain

$$x = \frac{V}{H} \cdot \frac{m\,(m+1)}{2\,m^2} \cdot \frac{y^2}{b},$$

or substituting for the value of the tangent of the angle of inclination a of the end A of the rope $tang.\ a = \frac{V}{H}$

$$x = \frac{m\,(m+1)\,y^2\,tang.\ a}{2\,m^2\,b}.$$

If the number of the weights is very great, we can put $m + 1 = m$, and consequently

$$x = \frac{V}{H} \cdot \frac{y^2}{2\,b} = \frac{y^2}{2\,b}\,tang.\ a.$$

For $x = a$, $y = b$, and consequently we have

$$a = \frac{V}{H} \cdot \frac{b}{2} = \frac{b\,tang.\ a}{2}$$

or more simply $\frac{x}{a} = \frac{y^2}{b^2}$,

which is the equation of a parabola.

If, therefore, an imponderable string is stretched by an infinite number of equal weights applied at equal horizontal distances from each other, the funicular polygon becomes a parabola.

For the angle of inclination ϕ we have

$$tang.\ \phi = \frac{y}{b} \cdot \frac{2\,a}{b} = 2\,y \cdot \frac{a}{b^2} = 2\,y \cdot \frac{x}{y^2} = \frac{2\,x}{y} \text{ and}$$

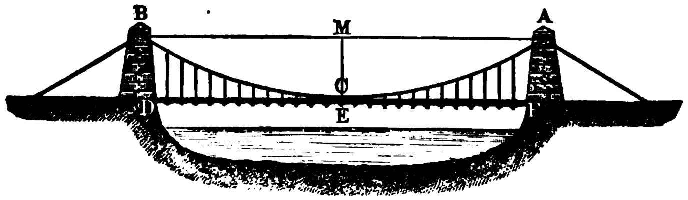
$$tang.\ a = \frac{2\,a}{b}.$$

The subtangent for the point O is

$$\overline{N\,T} = \overline{O\,N}\,tang.\ \phi = y\,\frac{2\,x}{y} = 2\,x = 2\,\overline{C\,N}.$$

If the chains and rods of a chain bridge $A\ B\ D\ F$, Fig. 239, were

FIG. 239.



without weight or very light in proportion to that of the loaded bridge $D\ E\ F$, the latter weights alone would have to be considered, and the chain $A\ C\ B$ would form a parabola.

EXAMPLE.—The entire load of the chain bridge in Fig. 239 is $G = 2\,V = 320000$ pounds, the span is $A\,B = 2\,b = 150$ feet, the height of the arc $C\,M = a = 15$ feet; required the tension and other conditions of the chain. The inclination of the chain to the horizon is determined by the formula

$$tang.\ \alpha = \frac{2\,a}{b} = \frac{30}{75} = \frac{2}{5} = 0{,}4,\ \text{whence}\ \alpha = 21^\circ\,48'.$$

The vertical tension in each point of suspension is

$$V = \tfrac{1}{2}\ \text{weight} = 160000\ \text{pounds},$$

the horizontal tension is

$$H = V\,cotg.\ \alpha = 160000\,.\,\frac{1}{0{,}4} = 400000\ \text{pounds},$$

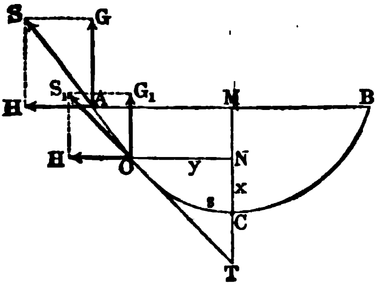
and the total tension at one end is

$$S = \sqrt{V^2 + H^2} = V\sqrt{1 + cotg.^2\,\alpha} = 160000\,.\,\sqrt{1 + \left(\frac{1}{0{,}4}\right)^2}$$

$$= 160000\,\sqrt{\frac{29}{4}} = 80000\,\sqrt{29} = 430813\ \text{pounds.}$$

§ **158. The Catenary.**—If a perfectly flexible and inextensible cord, or a chain composed of short links, is stretched by its own weight, the axis of the same will form a curved line, which has received the name of the *catenary curve* (Fr. chainette, Gr. Kettenlinie). The strings, ropes, ribbons, chains, etc., which we meet with in practice, are imperfectly elastic and extensible, and consequently form curves, which only approach the catenary, but which can generally be treated as such. From what precedes we know, that the horizontal tension in the catenary is equal at all points, while, on the contrary, the vertical tension in one point is equal to the vertical tension in the point of attachment above it minus the weight of the portion of the chain between this point and the point of suspension. Since the vertical tension at the vertex, where the catenary is horizontal, is = 0, or since the vertical tension at the point of suspension is equal to the weight of the chain from the point of attachment to the vertex, the vertical tension in any point is equal to the weight of the portion of the chain or cord below it.

FIG. 240.



If equal portions of the chain are equally heavy, the curve produced is the *common catenary*, which is the only one we will discuss here. If a portion of the chain or cord one foot long weighs γ, and if the arc corresponding to the co-ordinates $C M = a$ and $M A = b$, Fig. 240, is $A\ O\ C = l$, we have for the weight of the portion $A\ O\ C$ of the chain $G = l\,\gamma$.

If, on the contrary, the length of the arc corresponding to the co-ordinates $C N = x$ and $N O = y$ is $= s$, we have for the weight of this arc $V = s\,\gamma$. Putting, finally, the length of a similar piece of chain, whose weight is equal to the horizontal tension $H, = c$, we have $H = c\,\gamma$, and we have for the angles of inclination a and ϕ in the points A and O

$$tang.\, a = tang.\, S A H = \frac{G}{H} = \frac{l\,\gamma}{c\,\gamma} = \frac{l}{c} \text{ and}$$

$$tang.\, \phi = tang.\, N O T = \frac{V}{H} = \frac{s\,\gamma}{c\,\gamma} = \frac{s}{c}$$

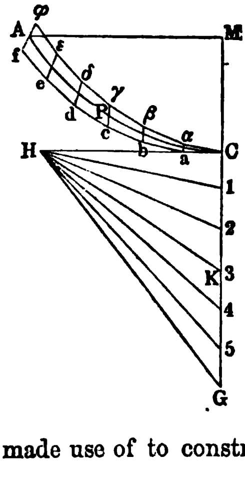
§ **159.** If we make the horizontal line $C H$, Fig. 241, equal to the length c of the portion of the chain measuring the horizontal tension and $C G$ equal to the length l of arc of the chain on one side, in accordance with § 155, the hypothenuse $G H$ gives the intensity and direction of the tension of the cord at the point of suspension A; for

$$tang.\, C H G = \frac{C G}{C H} = \frac{l}{c} \text{ and}$$

$$\overline{G H} = \sqrt{\overline{C G}^2 + \overline{C H}^2} = \sqrt{l^2 + c^2}, \text{ or}$$

$$S = \sqrt{G^2 + H^2} = \sqrt{l^2 + c^2}\,.\,\gamma = \overline{G H}\,.\,\gamma.$$

FIG. 241.



If we divide $C\ G$ into equal parts and draw from H to the points of division 1, 2, 3, etc., straight lines, the latter give the intensity and direction of the tensions obtained by dividing the length of the arc of the chain $A\ C$ into as many equal parts. For example, the line $H K$ gives the magnitude and direction of the tension or tangent at the point of division (P) of the arc $A\ P\ C$, since at this point the vertical tension $= \overline{C K}\,.\,\gamma$, while the horizontal tension is constant and $= c\,.\,\gamma$, and therefore for this point we have

$$tang.\, \phi = \frac{\overline{C K}\,.\,\gamma}{c\,\gamma} = \frac{C K}{C H}$$

as is really shown by the figure.

This peculiarity of the catenary can be made use of to construct mechanically, approximatively correctly, this curve. After having divided the given length of the catenary to be constructed in very many equal parts and laid off the line $C H = c$, which measures the horizontal tension, we draw the transverse lines $H 1$, $H 2$, $H 3$, etc., and lay off on $C H$ a division $\overline{C 1}$ of the arc of the curve as $C a$, pass through the point of division (a) thus obtained a parallel to the transverse line $\overline{H 1}$ and cut off again from it a part $a b = \overline{C 1}$. In like manner we draw through the point (b) thus obtained a parallel to the transverse line $\overline{H 2}$ and cut off from it $b c = \overline{C 1}$ equal to a division of the arc. We now draw through the new point (c) a parallel to $\overline{H 3}$ and make $c d$ equal to a division of the arc and continue in this way, until we have obtained the polygon $C a b c d e f$. We now construct another polygon $C \alpha \beta \gamma \delta \varepsilon \phi$ by drawing $C \alpha$ parallel to $\overline{H 1}$, $\alpha \beta$ to $\overline{H 2}$, $\beta \gamma$ to $\overline{H 3}$, etc., and by making $C \alpha = \alpha \beta = \beta \gamma$, etc., $= \overline{C 1} = \overline{12} = \overline{23}$, etc. If, finally, we pass through the centre of the lines $a \alpha$, $b \beta$, $c \gamma \ldots f \phi$ a curve, we obtain approximatively the catenary required.

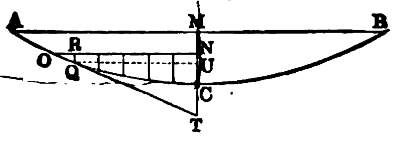
For practical purposes we can often obtain accurately enough a catenary corresponding to given conditions, E.G. to a given width and height of the arc or to a given width and length of arc, etc., by hanging a chain with small links against a vertical wall.

§ 160. Approximate Equation of the Catenary.

—In many cases, and particularly in its application to architecture and machinery, the horizontal tension of the catenary is very great compared to its vertical one, and therefore the height of the arc is small, compared with its width. Under this assumption, an equation for this curve can be found in the following manner:

Let s denote the length, x the abscissa $C N$ and y the ordinate $N O$ of a very low arc $C O$, Fig. 242. We can, according to the remark upon page 298, put approximatively

FIG. 242.



$$s = \left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right] y,$$

and therefore the vertical tension in a point O of a low arc of a catenary is

$$V = \left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right] y \gamma,$$

and the tangent of the tangential angle $T\,O\,N = \phi$ is

$$tang.\, \phi = \frac{s}{c} = \left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]\frac{y}{c}.$$

If we divide the ordinate y into m equal parts, we find the portion $R\,Q = N\,U$ of the abscissa X corresponding to such a division $O\,R$ by putting

$$\overline{R\,Q} = \overline{O\,R}\; tang.\, \phi = \overline{O\,R}\,.\frac{y}{c}\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right].$$

Since x is very small compared to y, we have approximatively $\overline{R\,Q} = \overline{O\,R}\,.\frac{y}{c}$. Substituting now $O\,R = \frac{y}{m}$ and successively for y the values $\frac{y}{m}, \frac{2\,y}{m}, \frac{3\,y}{m}$, etc., we obtain one after the other the different portions of x, the sum of which is

$$x = \frac{y^2}{c\,m^2}(1 + 2 + 3 + \ldots + m) = \frac{y^2}{c\,m^2}.\frac{m\,(m+1)}{2}\;(\S\,157) = \frac{y^2}{2\,c};$$

the latter equation is that of the parabola.

If we proceed more accurately and substitute in the formula

$$\overline{Q\,R} = \overline{O\,R}.\frac{y}{c}\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right],$$

instead of x, the value $\frac{y^2}{2\,c}$ just found, we obtain

$$\overline{Q\,R} = \overline{O\,R}.\frac{y}{c}\left(1 + \frac{1}{6}.\frac{y^2}{c^2}\right) = \frac{O\,R}{c}\left(y + \frac{1}{6}.\frac{y^3}{c^2}\right).$$

Putting y again successively equal to $\frac{y}{m}, \frac{2\,y}{m}, \frac{3\,y}{m}$, etc., and instead of $O\,R$, $\frac{y}{m}$, we obtain successively the different portions of x, and consequently their sum

$$x = \frac{y}{c\,m}\left[\frac{y}{m}\,(1+2+3+\ldots+m) + \frac{1}{6\,c^2}.\left(\frac{y}{m}\right)^3(1^3+2^3+3^3+\ldots+m^3)\right].$$

When the number of members is very great, the sum of the cardinal numbers $1 + 2 + 3 \ldots + m$ is $= \frac{m^2}{2}$ and the sum of their cubes is $= \frac{m^4}{4}$ (see "Ingenieur," page 88). Hence we have

$$x = \frac{y}{c}\left(\frac{y}{2} + \frac{1}{6\,c^2}.\frac{y^3}{4}\right), \text{ I.E.}$$

$$1)\; x = \frac{y^2}{2\,c} + \frac{y^4}{24\,c^3} = \frac{y^2}{2\,c}\left[1 + \frac{1}{12}.\left(\frac{y}{c}\right)^2\right],$$

the equation of very powerfully stretched catenary.

By inversion we obtain

$$y^2 = 2\,c\,x - \frac{y^4}{12\,c^2} = 2\,c\,x - \frac{4\,c^2\,x^2}{12\,c^2} = 2\,c\,x - \frac{x^2}{3}, \text{ whence}$$

$$2)\ y = \sqrt{2\,c\,x - \frac{x^2}{3}}, \text{ or approximatively,}$$

$$y = \sqrt{2\,c\,x}\left(1 - \frac{x}{12\,c}\right).$$

The measure of the horizontal tension is given by the formula

$$c = \frac{y^2}{2\,x} + \frac{y^4}{2\,x\,.\,12\,c^2} = \frac{y^2}{2\,x} + \frac{y^4}{24\,x}\cdot\frac{4\,x^2}{y^4}, \text{ I.E.}$$

$$3)\ c = \frac{y^2}{2\,x} + \frac{x}{6}.$$

The tangential angle is determined by the formula

$$tang.\,\phi = \frac{y}{c}\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right] = \frac{y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]}{\frac{y^2}{2\,x}\left[1 + \frac{1}{3}\left(\frac{x}{y}\right)^2\right]}$$

$$= \frac{2\,x}{y}\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]\left[1 - \frac{1}{3}\left(\frac{x}{y}\right)^2\right], \text{ I.E.}$$

$$4)\ tang.\,\phi = \frac{2\,x}{y}\left[1 + \frac{1}{3}\left(\frac{x}{y}\right)^2\right].$$

The formula for the rectification of the curve is

$$5)\ s = y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right] = y\left[1 + \frac{1}{6}\left(\frac{y}{c}\right)^2\right].$$

EXAMPLE—1) The length of the catenary for a width of arc $2\,b = 16$ feet and for a height of arc $a = 2\tfrac{1}{2}$ feet is

$$2\,l = 2\,b\left[1 + \frac{2}{3}\left(\frac{a}{b}\right)^2\right] = 16\,.\left[1 + \frac{2}{3}\left(\frac{2,5}{8}\right)^2\right]$$

$$= 16 + 16\,.\,0,065 = 17,04 \text{ feet};$$

and the length of the portion of the chain, which measures the horizontal tension, is

$$c = \frac{b^2}{2\,a} + \frac{a}{6} = \frac{64}{5} + \frac{5}{12} = 12,8 + 0,417 = 13,217 \text{ feet};$$

the tangent of the angle of inclination at the point of suspension is

$$tang.\,\alpha = \frac{2\,a}{b}\left[1 + \frac{1}{3}\left(\frac{a}{b}\right)^2\right] = \frac{5}{8}\cdot\left[1 + \frac{1}{3}\left(\frac{5}{16}\right)^2\right] = \frac{5\,.\,1,03255}{8} = 0,6453...,$$

whence the angle itself is $\alpha = 32^\circ\ 50'$.

2) If a chain is 10 feet long and the width of span is $9\tfrac{1}{2}$ feet, the height of arc is

$$a = \sqrt{\frac{3}{2}(l - b)\,b} = \sqrt{\frac{3}{2}\frac{(10 - 9\frac{1}{2})}{2}\cdot\frac{9\frac{1}{2}}{2}} = \sqrt{\frac{3}{2}\cdot\frac{19}{16}} = \sqrt{\frac{57}{32}}$$

$$= \sqrt{1,7812} = 1,335 \text{ feet,}$$

and the measure of the horizontal tension is

$$c = \frac{b^2}{2a} + \frac{a}{6} = \frac{4.75^2}{2\,.\,1,335} + \frac{1,335}{6} = 8,678 \text{ feet.}$$

3) If a string 30 feet long and weighing 8 pounds is stretched as nearly horizontal as possible by a force of 20 pounds, the vertical tension is

$$V = \tfrac{1}{2}\, G = 4 \text{ pounds, and the horizontal force}$$

$$H = \sqrt{S^2 - V^2} = \sqrt{20^2 - 4^2} = \sqrt{384} = 19,596 \text{ pounds,}$$

the tangent of the angle of inclination at the point of suspension is

$$tang.\,\phi = \frac{V}{H} = \frac{4}{19,596} = 0,20412,$$

and the angle itself is 11° 32′; the measure of the horizontal tension is

$$c = \frac{H}{\gamma} = H : \frac{8}{30} = \frac{30}{8} H = 73,485 \text{ feet,}$$

the width of the span is

$$2b = 2l\left[1 - \frac{1}{6} \cdot \left(\frac{l}{c}\right)^2\right] = 30\,.\left[1 - \frac{1}{6} \cdot \left(\frac{15}{73,48}\right)^2\right] = 30 - 0,208 = 29,792 \text{ ft.,}$$

and the height of the arc

$$a = \sqrt{\frac{3}{2}\, b\,(l - b)} = \sqrt{\frac{3}{2}\,\frac{29,792\,.\,0,208}{2\,.\,2}} = \sqrt{29,792\,.\,0,078} = 1,524 \text{ feet.}$$

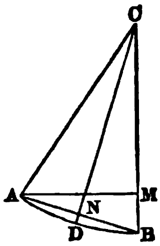
Remark 1.—We find from the radius $CA = CB = CD = r$ and the ordinate $A\,M = y$ of an arc of a circle $A\,B$, Fig. 243, the ordinate $A\,N = B\,N = y_1$ of half the arc $A\,D = B\,D$, by putting

$$\overline{A\,B}^2 = \overline{A\,M}^2 + \overline{B\,M}^2 = \overline{A\,M}^2 + (C\,B - C\,M)^2$$

$$= \overline{A\,M}^2 + (CB - \sqrt{\overline{C\,A}^2 - \overline{A\,M}^2})^2 = 2\,CA^2 - 2\,CA\sqrt{\overline{C\,A}^2 - \overline{A\,M}^2},$$

$$\text{i.e. } 4\,y_1{}^2 = 2\,r^2 - 2\,r\,\sqrt{r^2 - y^2}.$$

FIG. 243.



Hence we have

$$y_1 = \sqrt{\frac{r^2 - r\sqrt{r^2 - y^2}}{2}},$$ or approximatively, if y is small compared with r,

$$y_1 = \sqrt{\tfrac{1}{2}\left[r^2 - r\left(r - \frac{y^2}{2\,r} - \frac{y^4}{8\,r^3}\right)\right]}$$

$$= \sqrt{\frac{y^2}{4}\left(1 + \frac{y^2}{4\,r^2}\right)} = \frac{y}{2}\left(1 + \frac{y^2}{8\,r^2}\right).$$

By repeated application of this formula we find the ordinate of a quarter of the arc

$$y_2 = \frac{y_1}{2}\left(1 + \frac{y_1{}^2}{8\,r^2}\right) = \frac{y}{4}\left(1 + \frac{y^2}{8\,r^2}\right)\left(1 + \tfrac{1}{4} \cdot \frac{y^2}{8\,r^2}\right)$$

and that of an eighth of the arc

$$y_3 = \frac{y_2}{2}\left(1 + \frac{y_2{}^2}{8\,r^2}\right) = \frac{y}{8}\left(1 + \frac{y^2}{8\,r^2}\right)\left(1 + \tfrac{1}{4} \cdot \frac{y^2}{8\,r^2}\right)\left(1 + (\tfrac{1}{4})^2\,\frac{y^2}{8\,r^2}\right)$$

$$= \frac{y}{8}\left(1 + [1 + \tfrac{1}{4} + (\tfrac{1}{4})^2]\,\frac{y^2}{8\,r^2}\right).$$

Since the ordinates of very small arcs can be put equal to the arcs themselves, we obtain for the arc $A\,B$ approximatively

$$s = 8 \, . \, y_3 = y\left(1 + \left[1 + \tfrac{1}{4} + (\tfrac{1}{4})^2\right] \frac{y^2}{8\,r^2}\right), \text{ or more accurately}$$
$$= y\left(1 + \left[1 + \tfrac{1}{4} + (\tfrac{1}{4})^2 + (\tfrac{1}{4})^3 + \ldots\right] \frac{y^2}{8\,r^2}\right).$$

But $1 + \frac{1}{4} + (\frac{1}{4})^2 + (\frac{1}{4})^3 + \ldots$ is $= \dfrac{1}{1 - \frac{1}{4}} = \frac{4}{3}$ (see Ingenieur, page 82), and therefore

$$s = \left(1 + \frac{y^2}{6\,r^2}\right) y;$$

or substituting instead of r the abscissa $B\,M = x$ by putting $2\,r\,x = y^2$, we obtain

$$s = \left[1 + \tfrac{2}{3}\left(\frac{x}{y}\right)^2\right] y.$$

This formula is not only applicable to the arc of a circle, but also to all low arcs of curves.

REMARK 2. If we compare the equation

$$y = \sqrt{2\,c\,x - \frac{x^2}{3}},$$

found above, with the equation of the ellipse

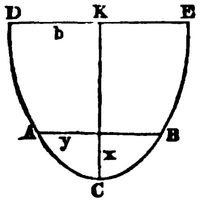
$$y = \frac{b}{a}\sqrt{2\,a\,x - x^2}$$

(see Ingenieur, page 169), we find

$$\frac{b^2}{a} = c \text{ and } \frac{b^2}{a^2} = \tfrac{1}{3}, \text{ and consequently}$$

$$a = 3\,c \text{ and } b = a\,\sqrt{\tfrac{1}{3}} = c\,\sqrt{3}.$$

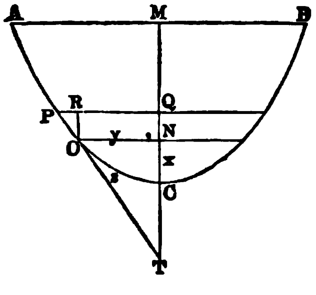
FIG. 244.



The curve formed by a powerfully stretched string can therefore be considered as the arc $A\,C\,B$, Fig. 244, of an ellipse, the major axis of which is $K\,C = a = 3\,c$ and the minor axis is $K\,D = K\,E = b = c\,\sqrt{3} = a\,\sqrt{\tfrac{1}{3}} = 0{,}577\,a$.

(§ **161.**) **Equation of the Catenary.**—The complete equation of the catenary can be found in the following manner by the aid of the calculus. According to § 158, we have for the angle of suspension $T\;O\;N = \phi$, Fig. 245, formed by the tangent $O\;T$ to a point O of the catenary $A\;C\,B$ with the horizontal co-ordinate $O\;N$, when the arc $C\;O$ is denoted by s and the horizontal tension by $H = c\,\gamma$,

FIG. 245.



$$tang.\,\phi = \frac{s}{c}$$

But ϕ is also equal to the angle $O\;P\;R$ formed by the element of the arc $O\,P = d\,s$ with the element $P\,R = d\,y$ of the ordinate $O\,N = y$, and hence

$$tang.\ O\,P\,R = \frac{O\,R}{P\,R} = \frac{d\,x}{d\,y},$$

in which $O\,R$ is considered as an element $d\,x$ of the abscissa $C\,N = x$. From the above it follows, that

$$\frac{d\,x}{d\,y} = \frac{s}{c}, \text{ or } \frac{d\,y^2}{d\,x^2} = \frac{c^2}{s^2}.$$

But $d\,s^2$ is $= d\,x^2 + d\,y^2$, or $d\,y^2 = d\,s^2 - d\,x^2$, whence

$$\frac{d\,s^2 - d\,x^2}{d\,x^2} = \frac{c^2}{s^2}.$$

Clearing the equation of fractions and transposing, we obtain

$$d\,x^2\,(s^2 + c^2) = s^2\,d\,s^2, \text{ or } d\,x = \frac{s\,d\,s}{\sqrt{s^2 + c^2}}.$$

Putting $s^2 + c^2 = u$, we have

$$2\,s\,d\,s = d\,u \text{ and } d\,x = \frac{\frac{1}{2}\,d\,u}{u^{\frac{1}{2}}} = \tfrac{1}{2}\,u^{-\frac{1}{2}}\,d\,u.$$

By integration we obtain (according to Article 18 of the Introduction to the Calculus)

$$x = \tfrac{1}{2}\int u^{-\frac{1}{2}}\,d\,u = \tfrac{1}{2}\,.\,\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + Const. = \sqrt{u} + Const.$$
$$= \sqrt{s^2 + c^2} + Const.$$

Finally, since for $x = 0$, s is also $= 0$, we have $0 = \sqrt{c^2} + Const.$, I.E. $Const. = -\,c$ and

1) $x = \sqrt{s^2 + c^2} - c$, or inversely

$$s = \sqrt{(x + c)^2 - c^2} = \sqrt{2\,c\,x + x^2}, \text{ and}$$
$$c = \frac{s^2 - x^2}{2\,x}$$

EXAMPLE.—If a chain $A\ C\ B$, 10 feet long and weighing 30 pounds, is suspended in such a manner that the height of the arc is $C\,M = 4$ feet, we have

$$\gamma = \tfrac{30}{10} = 3 \text{ pounds},$$
$$c = \frac{s^2 - x^2}{2\,x} = \frac{5^2 - 4^2}{8} = \tfrac{9}{8},$$

and consequently the horizontal tension

$$H = c\,\gamma = 3\,.\,\tfrac{9}{8} = 3\tfrac{3}{8} \text{ pounds}.$$

(**§ 162.**) As in the last paragraph by eliminating $d\,y$ we obtained an equation between the arc s and the abscissa x, in like manner by eliminating $d\,x$ we can deduce an equation between the arc s and the ordinate y. For this purpose we substitute in the equation

$$\frac{d\,y^2}{d\,x^2} = \frac{c^2}{s^2}, d\,x^2 = d\,s^2 - d\,y^2$$

and obtain the equation

$$\frac{s^2}{c^2} = \frac{d\,s^2 - d\,y^2}{d\,y^2}, \text{ or } d\,y^2\,(s^2 + c^2) = c^2\,d\,s^2, \text{ whence}$$

$$d\,y = \frac{c\,d\,s}{\sqrt{s^2 + c^2}}.$$

Dividing the numerator and denominator by c and putting $\frac{s}{c} = v$, we obtain

$$d\,y = \frac{c\,d\left(\frac{s}{c}\right)}{\sqrt{1 + \left(\frac{s}{c}\right)^2}} = \frac{c\,d\,v}{\sqrt{1 + v^2}},$$

and the formula XIII, in Article 26 of the Introduction to the Calculus, gives us the corresponding integral

$$y = c \int \frac{d\,v}{\sqrt{1 + v^2}} = c\,l\,(v + \sqrt{1 + v^2}), \text{ i.e.}$$

$$2)\ y = c\,l\left(\frac{s + \sqrt{s^2 + c^2}}{c}\right).$$

Substituting in this formula $s = \sqrt{2\,c\,x + x^2}$, we obtain the proper equation for the co-ordinates of the common catenary

$$3)\ y = c\,l\left(\frac{c + x + \sqrt{2\,c\,x + x^2}}{c}\right), \text{ or}$$

$$4)\ y = c\,l\left(\frac{s + x}{s - x}\right) = \frac{s^2 - x^2}{2\,x}\,l\left(\frac{s + x}{s - x}\right).$$

Finally, by inverting 2 and 3, we obtain

$$5)\ s = \left(e^{\frac{y}{c}} - e^{-\frac{y}{c}}\right) \cdot \frac{c}{2} \text{ and}$$

$$6)\ x = \left[\tfrac{1}{2}\left(e^{\frac{y}{c}} + e^{-\frac{y}{c}}\right) - 1\right] c,$$

e denoting the base 2,71828 ... of the Naperian system of logarithms (see Article 19 of the Introduction to the Calculus).

Example.—The two corresponding co-ordinates of a point of the catenary are $x = 2$ and $y = 3$; required the horizontal tension c of this curve.

Approximatively, according to No. 3 of paragraph 160, we have

$$c = \frac{y^2}{2x} + \frac{x}{6} = \frac{9}{4} + \frac{2}{6} = 2{,}58.$$

But according to No. 3 of this paragraph (162), we have exactly

$$y = c\, l\left(\frac{c + x + \sqrt{2\, c\, x + x^2}}{c}\right), \text{ I.E.}$$

$$3 = c\, l\left(\frac{c + 2 + \sqrt{4\, c \cdot + 4}}{c}\right).$$

Substituting for *c*, 2,58, we find the error

$$f = 3 - 2{,}58\, l\left(\frac{4{,}58 + 2\sqrt{3{,}58}}{2{,}58}\right) = 3 - 2{,}58\, l\left(\frac{8{,}3642}{2{,}58}\right)$$
$$= 3 - 3{,}035 = -\, 0{,}035.$$

If, however, we assume $c = 2{,}53$, we find the error

$$f_1 = 3 - 2{,}53\, l\left(\frac{4{,}53 + 2\sqrt{3{,}53}}{2{,}53}\right) = 3 - 2{,}58\, l\left(\frac{8{,}2876}{2{,}53}\right)$$
$$= 3 - 3{,}002 = -\, 0{,}002.$$

In order to find the true value of *c*, we put according to a well known rule (see Ingenieur, page 76)

$$\frac{c - 2{,}58}{c - 2{,}53} = \frac{f}{f_1} = \frac{0{,}035}{0{,}002} = 17{,}5\,;$$

whence it follows that $16{,}5\,.\,c = 17{,}5\,.\,2{,}53 - 2{,}58 = 41{,}69$ and

$$c = \frac{41{,}69}{16{,}5} = 2{,}527 \text{ feet.}$$

Remark.—We can express very simply *s*, *x* and *y* for the common catenary in terms of the angle of suspension ϕ; for from what precedes we have

$$s = c\, tang.\, \phi = \frac{c\, sin.\, \phi}{cos.\, \phi}$$

$$x = c\,(\sqrt{1 + tang.^2\, \phi} - 1) = \frac{c\,(1 - cos.\, \phi)}{cos.\, \phi} \text{ and}$$

$$y = c\, l\,(tang.\, \phi + \sqrt{1 + tang.^2\, \phi}) = c\, l\left(\frac{1 + sin.\, \phi}{cos.\, \phi}\right).$$

By means of these formulas we can easily calculate the lengths of the arcs and of the co-ordinates for different angles of suspension, and a useful table, such as is given in the Ingenieur, page 353, may be thus prepared. For this purpose we need adopt as base but a single catenary, and in this case the best one is that, in which the measure of the horizontal tension is $= 1^{\cdot}$; to obtain *s*, *x* and *y* for another catenary corresponding to the horizontal tension *c*, we have but to multiply the values of *s*, *x* and *y* given in the table by *c*. If *tang.* ϕ were not $= \frac{s}{c}$, but to $\frac{y}{c}$, we would have the common parabola, for which

$$s = \frac{c}{2}\left[\frac{sin.\, \phi}{cos.^2\, \phi} + l\, tang.\left(\frac{\frac{1}{2}\pi + \phi}{2}\right)\right],$$

$$x = \frac{c}{2}\, tang.^2\, \phi = \frac{c}{2}\left(\frac{sin.\, \phi}{cos.\, \phi}\right)^2 \text{ and}$$

$$y = c\, tang.\, \phi = \frac{c\, sin.\, \phi}{cos.\, \phi}.$$

§ 163. Equilibrium of the Pulley.—Ropes, belts, etc., are the ordinary means employed to transmit forces to the pulley and the wheel and axle. We will here discuss only the most general part of the theory of these two apparatuses, so far as it can be done without taking into consideration the friction and the rigidity of cordage.

A pulley (Fr. poulie; Ger. Rolle) is a circular disc or sheave $A\ B\ C$, Figs. 246 and 247, movable about an axis and around whose circumference a string is laid, the extremities of which are pulled by the forces P and Q. The block (Fr. chape; Ger. Gehäuse or Lager) of a fixed pulley (Fr. p. fixe; Ger. feste R.), in which the axles or journals rest, is immovable. That of a movable pulley (Fr. p. mobile; Ger. lose R.) on the contrary is free to move.

FIG. 246.

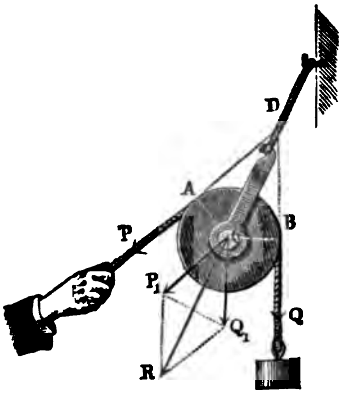
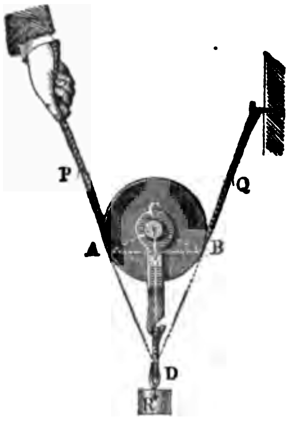


FIG. 247.



When a pulley is in equilibrium, the forces P and Q at the extremities of the cord are equal to each other; for every pulley is a lever with equal arms, which we obtain by letting fall from the axis C the perpendiculars $C\ A$ and $C\ B$ upon the directions $D\ P$ and $D\ Q$ of the forces or cords. It is also evident, that during any rotation about C the forces P and Q describe equal spaces $r\ \beta$, when r denotes the radius $C A = C B$ and β° the angle of rotation, and from this we can conclude, that P and Q are equal. The forces P and Q give rise to a resultant $\overline{C R} = R$, which is counteracted by the journal or axle and is dependent upon the angle $A\ D\ B = a$ formed by the directions of the cords, it is given by construction as the diagonal of the rhomb $C P_1\ R\ Q_1$ constructed with P and a; its value is

$$R = 2\ P\ cos.\ \frac{a}{2}.$$

§ 164. The weight to be raised or the resistance Q to be overcome in a fixed pulley, Fig. 246, acts exactly in the same manner as the force P, and the force is therefore equal to the resistance, and the use of this pulley produces no other effect than a change of direction.

On the contrary, in a movable pulley, Fig. 247, the weight R acts on the hook-shaped end of the bearings of the axle, while one end of the rope is made fast to some immovable object; here the force is

$$P = \frac{R}{2 \cos. \frac{a}{2}}.$$

Designating the chord $A\ M\ B$ corresponding to the arc covered by the string by a and the radius $C\,A = C\,B$, as before, by r, we have

$$a = 2\,\overline{A\,M} = 2\,\overline{C\,A} \cos.\, C\,A\,M = 2\,\overline{C\,A} \cos.\, A\,D\,M = 2\,r \cos. \frac{a}{2}$$

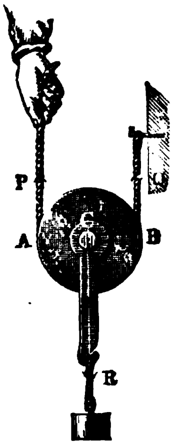
and therefore

$$\frac{r}{a} = \frac{1}{2 \cos. \frac{a}{2}} \text{ and } \frac{P}{R} = \frac{r}{a}.$$

Hence, in a movable pulley, the force is to the load as the radius of the pulley is to the chord of the arc covered by the string.

If $a = 2\,r$, I.E. if the string covers a semicircle, Fig. 248, the force is a minimum and is $P = \frac{1}{2} R$; if $a = r$ or if 60° of the pulley is covered by the string, we have $P = R$. The smaller a becomes, the greater is P; I.E., when the arc covered by the cord is infinitely small, the force P is infinitely great. The relation is inverted, when we consider the spaces described; if s is the space described by P, while R describes the space h, we have $Ps = Rh$, whence

FIG. 248.



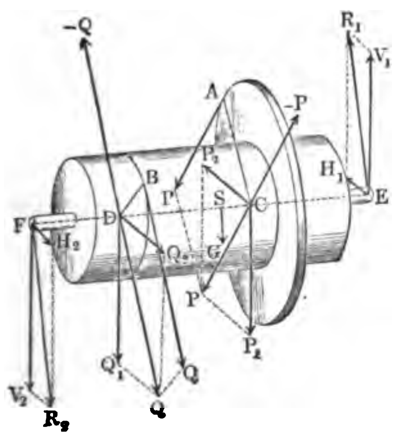
$$\frac{s}{h} = \frac{a}{r}.$$

The movable pulley is a means of changing the force, and is used to gain power; by means of it we can, E.G., raise a given load with a smaller force; but in the same ratio as the force is increased the space described is diminished.

REMARK.—The combinations of pulleys, such as block and tackle, etc., as well as the influence of friction and of the rigidity of cordage upon the state of equilibrium of pulleys, will be treated in the third volume.

§ 165. Wheel and Axle.—The wheel and axle (Fr. roue sur l'arbre, Ger. Radwelle) is a rigid combination $A\ B\ F\ E$, Fig. 249, of two pulleys or wheels movable about a common axis. The smaller of these wheels is called the axle (Fr. arbre, Ger. Welle), and the larger the wheel (Fr. roue, Ger. Rad). The rounded ends E and F, upon which the apparatus rests, are called the journals (Fr. tourillons, Ger. Zapfen). The axis of revolution of a wheel and axle is either horizontal, vertical or inclined. We will now discuss only the wheel and axle, movable around a horizontal axis. We will also suppose, that the forces P and Q or the force P and the weight Q act at the ends of perfectly flexible ropes, which are wound around the circumferences of the wheel and of the axle. The questions to be answered are, what is the relation between the force P and the weight Q, and what is the pressure upon the bearings at E and F?

FIG. 249.



If at the point C, where the plane of rotation of the force P cuts the axis $E\ F$, we imagine two equal opposite forces $C P = P$ and $C \overline{P} = -P$ to be acting in a direction parallel to that of the force of rotation P, we obtain by the combination of these three forces a force $C\ P = P$, which acts upon the axis, and a couple $(P, -P)$, whose moment is $= P\ .\ \overline{C A} = P\ a$, when a designates the arm of the force $A\ P = P$ or the radius $C\ A$ of the wheel. Now if we imagine the two forces $D\ Q = Q$ and $D\ \overline{Q} = -Q$ to be applied at the point D, where the plane of revolution of the weight Q cuts the axis $E\ F$, we obtain also a force $D\ Q = Q$ acting upon the axis and a couple $(Q, -Q)$, whose moment is $= Q\ .\ \overline{D\ B} = Q\,b$, when b designates the arm of the weight Q applied in B or the radius $D\,B$ of the axle. Since the axial forces $C\,P = P$ and $D\,Q = Q$ are counteracted by the bearings, and consequently can have no influence upon the revolution of the machine, it is necessary, in order to have a state of equilibrium, that the two couples, which act in parallel planes, shall have equal moments (compare § 94), or that

$$P\,a = Q\,b, \text{ or}$$
$$\frac{P}{Q} = \frac{b}{a}.$$

In every wheel and axle which is in equilibrium, whatever may be its length, the moment P a of the power is, as in the lever, equal to the moment Q b of the load, or the ratio of the power to the load is equal to that of the arm of the load to the arm of the power.

If more than two forces act upon the wheel and axle, the sum of moments of the forces tending to turn it in one direction is naturally equal to the sum of those tending to turn it the other.

§ **166.** The axial forces $C\,P = P$ and $D\,Q = Q$ can be decomposed into the vertical forces $C\,P_1 = P_1$ and $D\,Q_1 = Q_1$ and into the horizontal forces $C\,P_2 = P_2$ and $D\,Q_2 = Q_2$; the first two forces combined with the weight of the machine G, which acts at the centre of gravity S of the machine, give the total vertical pressure on the bearings, which is

$$V_1 + V_2 = P_1 + Q_1 + G,$$

while the horizontal forces P_2 and Q_2 produce the lateral pressures H_1 and H_2 on the bearings. If α is the angle of inclination $P\,C\,P_2$ of the direction of the force P to the horizon and β that $Q\,D\,Q_2$ of the load, we have

$$P_1 = P\,sin.\,\alpha \text{ and } P_2 = P\,cos.\,\alpha, \text{ as well as}$$
$$Q_1 = Q\,sin.\,\beta \text{ and } Q_2 = Q\,cos.\,\beta.$$

If now l is the total length of the axis $\overline{E\,F}$, d the distance $\overline{C\,E}$, e the distance $D\,E$ and c the distance $\overline{S\,E}$ of the points of the axis C, D and S from one extremity E of the axis, we have, according to the theory of the lever:

1) When we consider E as fulcrum of the lever $E\,F$, which is acted on by the forces P_1, Q_1 and G,

$$V_2 \,.\, \overline{EF} = P_1 \,.\, \overline{E\,C} + Q_1 \,.\, \overline{ED} + G \,.\, \overline{E\,S}, \text{ i.e.}$$
$$V_2\,l = P_1\,d + Q_1\,e + G\,s,$$

whence we obtain the vertical pressure

$$V_2 = \frac{P_1 d + Q_1 e + G s}{l}, \text{ and}$$

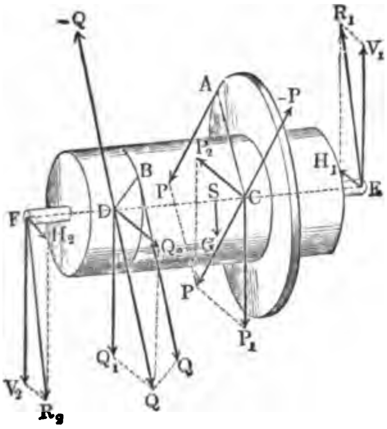
2) considering F as the fulcrum of the supposed lever,

$$V_1 . \overline{FE} = P_1 . \overline{FC} + Q_1 . \overline{FD} + G . \overline{FS}, \text{ I.E.}$$
$$V_1 l = P_1 (l - d) + Q_1 (l - e) + G (l - s),$$

whence we deduce the vertical pressure

$$V_1 = \frac{P_1 (l - d) + Q_1 (l - e) + G (l - s)}{l}.$$

FIG. 249.



The horizontal pressures H_1 and H_2 are found, as follows, from the horizontal forces P_2 and Q_2.

1) Considering E as the fulcrum of the lever $E F$ acted on by the forces P_2 and Q_2, we obtain

$$H_2 . \overline{EF} = P_2 . \overline{EC} - Q_2 . \overline{ED}, \text{ I.E.}$$
$$H_2 l = P_2 d - Q_2 e,$$

whence we obtain the horizontal pressure

$$H_2 = \frac{P_2 d - Q_2 e}{l}.$$

2) Considering F as the fulcrum, we have

$$H_1 . \overline{FE} = P_2 . \overline{FC} - Q_2 . \overline{FD}, \text{ I.E.}$$
$$H_1 l = P_2 (l - d) - Q_2 (l - e),$$

from which we deduce the horizontal pressure

$$H_1 = \frac{P_2 (l - d) - Q_2 (l - e)}{l}.$$

By the application of the parallelogram of forces, we obtain the total pressures R_1 and R_2 upon the bearings E and F, and they are

$$R_1 = \sqrt{V_1^2 + H_1^2} \text{ and } R_2 = \sqrt{V_2^2 + H_2^2}.$$

Finally, if δ_1 and δ_2 are the angles $R_1 E H_1$ and $R_2 F H_2$ formed by these pressures with the horizon, we have

$$tang.\, \delta_1 = \frac{V_1}{H_1} \text{ and } tang.\, \delta_2 = \frac{V_2}{H_2}.$$

EXAMPLE.—The weight Q, suspended to a wheel and axle, acts vertically and weighs 365 pounds; the radius of the wheel is $a = 1\frac{3}{4}$ feet; the radius of the axle is $b = \frac{3}{4}$ foot; the weight of the wheel and axle together is 200 pounds; the distance of its centre of gravity from the journal E is $1\frac{1}{2}$ feet; the centre of the wheel is at a distance $d = \frac{3}{4}$ from this journal E, and the vertical plane, in which the weight acts, is $e = 2$ feet distant from the same point, while the whole length of the axis is $E F = l = 4$ feet; now if the force necessary to produce equilibrium acts downwards at an angle of inclination to the horizon of $\alpha = 50°$, how great must it be and what are the pressures upon the bearings? Here we have $Q = 365$, $\beta = 90°$, and consequently $Q_1 = Q \sin. \beta = Q$ and $Q_2 = Q \cos. \beta = 0$, P is unknown, and α is $= 50°$, whence $P_1 = P \sin. \alpha = 0{,}7660 \,.\, P$ and P_2 is $= P \cos. \alpha = 0{,}6428 \,.\, P$, but a is $= 1\frac{3}{4} = \frac{7}{4}$ and $b = \frac{3}{4}$, whence

$$P = \frac{b}{a}\, Q = \tfrac{3}{7} \,.\, 365 = 156{,}4 \text{ pounds, } P_1 = 119{,}8 \text{ and } P_2 = 100{,}5 \text{ pounds.}$$

Since $l = 4$, $d = \frac{3}{4}$, $e = 2$ and $s = \frac{3}{2}$, we have $l - d = \frac{13}{4}$, $l - e = 2$ and $l - s = \frac{5}{2}$.

1) On the bearing F the vertical pressure is

$$V_2 = \frac{119{,}8 \,.\, \frac{3}{4} + 365 \,.\, 2 + 200 \,.\, \frac{3}{2}}{4} = 280{,}0 \text{ pounds,}$$

and the horizontal pressure is

$$H_2 = \frac{100{,}5 \,.\, \frac{3}{4} - 0 \,.\, 2}{4} = 18{,}8 \text{ pounds,}$$

and consequently the resulting pressure is

$$R_2 = \sqrt{V_2^2 + H_2^2} = \sqrt{280^2 + 18{,}8^2} = 280{,}6 \text{ pounds,}$$

and its inclination to the horizon is determined by the formula

$tang.\, \delta_2 = \frac{280{,}0}{18{,}8}$, $log\, tang.\, \delta_2 = 1{,}17300$, from which we obtain $\delta_2 = 86°\, 9'\, 5''$.

2) For the bearing at E

$$V_1 = \frac{119.8 \,.\, \frac{13}{4} + 365 \,.\, 2 + 200 \,.\, \frac{5}{2}}{4} = 404{,}8 \text{ pounds and}$$

$$H_1 = \frac{100{,}5 \,.\, \frac{13}{4} - 0}{4} = 81{,}7 \text{ pounds,}$$

and consequently the resulting pressure is

$$R_1 = \sqrt{V_1^2 + H_1^2} = \sqrt{404{,}8^2 + 81{,}7^2} = 413{,}0 \text{ pounds,}$$

and for its inclination δ_1 to the horizon we have

$$tang.\ \delta_1 = \frac{404,8}{81,7},\ log\ tang.\ \delta_1 = 0,69502 \text{ or } \delta_1 = 78^\circ\ 35'.$$

We see that these results are correct, for we have

$$V_1 + V_2 = 280 + 404,8 = 684,8 = P_1 + Q_1 + G, \text{ and}$$
$$H_1 + H_2 = 81,7 + 18,8 = 100,5 = P_2 + Q_2.$$

CHAPTER V.

THE RESISTANCE OF FRICTION AND THE RIGIDITY OF CORDAGE.

§ 167. Resistance of Friction.—Heretofore we have supposed (§ 138) that two bodies could act upon one another only by forces perpendicular to their common plane of contact. If these bodies were perfectly rigid and their surfaces of contact mathematical planes, I.E. unbroken by the smallest hills or hollows, this law would also be confirmed by experiment; but since every material body possesses a certain degree of elasticity or even of softness, and since the surface of all bodies, even the most highly polished, contains small hills and valleys and in consequence of the porosity of matter does not form a perfectly continuous plane, when two bodies press upon each other their points of contact penetrate, producing an adhesion of the parts, which can only be overcome by a particular force, whose direction is that of the plane of contact. This adhesion of bodies in contact, produced by their mutual penetration and grasping of each other, is what is called *friction* (Fr. frottement, Ger. Reibung). Friction presents itself in the motion of a body as a passive force or resistance, since it can only hinder or prevent motion, but can never produce or aid it. In investigations in mechanics it can be considered as a force acting in opposition to every motion, whose direction lies in the plane of contact of the two bodies. Whatever the direction may be in which we move a body resting upon a horizontal or inclined plane, the friction will always act in the opposite direction to that of the motion, E.G., when we slide the body down an inclined plane, it will appear as motion up the same. If a system of forces is in equilibrium, the smallest additional force produces motion as long as the friction does not come into play; but when friction is called into existence a greater addition of force, the amount of which depends upon the friction, is necessary to disturb the equilibrium.

§ **168.** In overcoming the friction, the parts which come in contact are compressed, the projecting parts bent over, or perhaps torn away, broken off, etc. The friction is therefore dependent not only upon the roughness or smoothness of the surfaces, but also upon the nature of the material of which the bodies are composed.

The harder metals generally cause less friction than the softer ones. We cannot establish *á priori* any general rules for the dependence of friction upon the natural properties of bodies; it is in fact necessary to make experiments upon friction with different materials, in order to be able to determine the friction existing between bodies under other circumstances. The unguents (Fr. les enduits; Ger. die Schmieren) have a great influence upon the friction and upon the wearing away of bodies in contact. The pores of the bodies are filled and the other roughnesses diminished by the fluid or half fluid unguents, such as oil, tallow, fat, soaps, etc., and the mutual penetration of the bodies much diminished; for this reason they diminish very considerably the friction.

But we must not confound friction with adhesion, I.E., with that union of two bodies which takes place when the bodies come in contact in very many points without the existence of any pressure between them. The adhesion increases with the surface of contact and is independent of the pressure, while for friction the reverse is true. When the pressures are small, the adhesion appears to be very great compared with the friction, but if the pressures are great, it becomes but a very small portion of the friction and can generally be neglected. Unguents generally increase the adhesions, since they produce a greater number of points of contact.

§ **169. Kinds of Friction.**—We distinguish two kinds of friction, viz., *sliding* and *rolling* friction. The sliding friction (Fr. frottement de glissement; Ger. gleitende Reibung) is that resistance of friction produced, when a body slides, I.E., moves so that all its points describe parallel lines. Rolling friction (Fr. f. de roulement; Ger. rollende or wälzende Reibung) on the contrary, is that resistance developed, when a body rolls, I.E., when every point of the body at the same time progresses and revolves and when the point of contact describes the same space upon the moving body as upon the immovable one. A body M, Fig. 250, supported on the plane $H\ R$, slides, for example, upon the plane and must overcome sliding friction, when all points such as A, B, C etc., describe the parallel trajectories $A\ A_1$, $B\ B_1$, $C\ C_1$, etc., and therefore the same points of the moving body come in contact with different ones of the support. The body *M*, Fig. 251, rolls upon the plane *H R* and must therefore overcome rolling friction, when the points *A*, *B*, etc., of its surface move in such a manner, that the space $A\ E\ B_1$ = the space $A\ D\ B = A_1\ D_1\ B_1$ and also that space $A\ E$ is = the space $A\ D$ and the space $B_1\ E = B_1\ D_1$, etc.

FIG. 250.

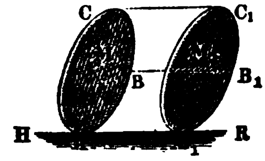
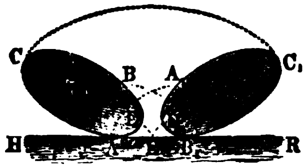


FIG. 251.



A particular kind of friction is the friction of axles or journals which is produced, when a cylindrical axle, journal or gudgeon revolves in its bearing. We distinguish two kinds of axles, horizontal and vertical. The horizontal axle, journal or gudgeon (Fr. tourillon; Ger. liegende Zapfen) moves in such a manner that different points of the gudgeon, etc., come successively in contact with the same point of the support. The vertical axle or pivot (Fr. pivot; Ger. stehende Zapfen) presses with its circular base upon the step, on which the different points of it revolve in concentric circles.

Particular kinds of friction are produced, when a body oscillates upon an edge, as, E.G., a balance, or when a vibrating body is supported upon a point, as, E.G., the needle of a compass.

Friction can also be divided into immediate (Fr. immédiat; Ger. unmittelbare) and mediate (Fr. médiat; Ger. mittelbare). In the first case the bodies are in immediate contact; in the latter, on the contrary, they are separated by unguents, as, E.G., a thin layer of oil.

We distinguish also the friction of repose or quiescence (Fr. f. de répos; Ger. R. der Ruhe), which must be overcome when a body at rest is put in motion, from the friction of motion (Fr. f. de mouvement; Ger. R. der Bewegung), which resists the continuance of a motion.

§ **170. Laws of Frictions.**—1. The friction is proportional to the normal pressure between the rubbing bodies. If we press a body twice as much against its support as before, the friction becomes double. A triple pressure gives a triple friction, etc. If this law varies slightly for small pressures, we must ascribe the variations to the proportionally greater influence of the adhesion.

2. The friction is independent of the rubbing surfaces or surfaces of contact. The greater the rubbing surfaces the greater is, it is true, the number of the rubbing parts, but the pressure upon each part is so much the smaller, and consequently the resistance of friction upon it is less. The sum of the frictions of all the parts is therefore the same for a large and for a small surface, when the pressure and other circumstances are the same. If the surfaces of the sides of a parallelopipedical brick are of the same nature, the force necessary to move the brick on a horizontal plane is the same whether it lies on the smallest, medium, or greatest surface. When the surfaces are very great and the pressures very small, this rule appears to be subject to exceptions on account of the effect of the adhesion.

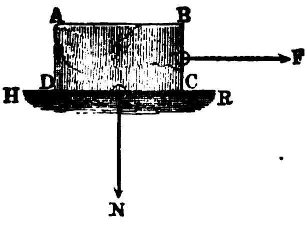
3. The friction of quiescence is generally greater than that of motion, but the latter is independent of the velocity; it is the same for high and low velocities.

4. The friction of greased surfaces (mediate friction) is generally smaller than that of ungreased surfaces (immediate friction) and depends less upon the rubbing bodies themselves than upon the unguent.

5. The friction on axles is less than the ordinary friction of sliding. The rolling friction between smooth surfaces is in most cases so small, that we need scarcely take it into account in comparison with the friction of sliding.

REMARK.—The foregoing rules are strictly true only, when the pressure upon the unit of surface of the bearings is a medium one, and when the velocity of the circumference of the journal does not exceed certain limits. This medium pressure is from 250 to 500 pounds per square inch, and the mean velocity of the circumference should be 2 to 10 inches. When the pressure is much smaller, the adhesion forms a very sensible portion of the resistance which then becomes dependent upon the magnitude of the rubbing surfaces, and where the pressure and velocity are very great a large quantity of heat is developed, which volatilizes the unguents, thus causing the journals to cut very quickly. When, as in the case of turbines, railroad cars, etc., we cannot avoid these great velocities, we must counteract this heating of the axle by increasing the rubbing surfaces, I.E., by increasing the length and thickness of the axles.

FIG. 252.



§ 171. Co-efficient of Friction.

—From the first law of the foregoing paragraph we can deduce the following. If in the first place a body

$A\ C$, Fig. 252, presses with a force N against its support, and if to move it along, I.E., to overcome its friction, we require the force F, and if in the second place, when pressing with the force N_1 a force F_1 is necessary to transfer it from a state of rest into one of motion, we will have, according to the foregoing paragraph,

$$\frac{F}{F_1} = \frac{N}{N_1}, \text{ whence } F = \frac{F_1}{N_1} \cdot N.$$

If by experiment we have found for a certain pressure N_1 the corresponding friction F_1, we see from the above, that if the rubbing bodies and other circumstances are the same, *the friction F corresponding to another pressure N can be found by multiplying this pressure by the ratio $\left(\frac{F_1}{N_1}\right)$ between the values F_1 and N_1 corresponding to the first observation.*

This ratio of the friction to the pressure or the friction for a pressure $= 1$, E.G. pound, is called the coefficient of friction (Fr. coëfficient du frottement; Ger. Reibüngscoefficient) and will in future be designated by ϕ. Hence we can put in general

$$F = \phi\, N.$$

The coefficient of friction is different for different materials and for different conditions of the same material and must therefore be determined by experiments undertaken for that purpose. If the body $A\ C$ is pulled along a distance s upon its support, the work to be performed is $F s$. The mechanical effect $\phi\, N\, s$ absorbed by the friction is equal to the product of the coefficient of friction, the normal pressure and the space described. If the support is also movable, we must understand by $s = s_1 - s_2$ the relative space described by the body, and $F s = \phi\, N s$ is the work done by the friction between the two bodies. The body that moves the most quickly must perform, while describing the space s_1, the mechanical effect $\phi\, N s_1$ and the body which moves slower gains in consequence of the friction while describing the space s_2 the mechanical effect $\phi\, N s_2$; the loss of mechanical effect caused by the friction between the two bodies is

$$\phi\, N s_1 - \phi\, N s_2 = \phi\, N (s_1 - s_2) = \phi\, N s.$$

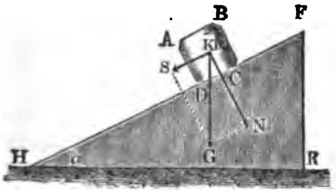
EXAMPLES—1. If for a pressure of 260 pounds the friction is 91 pounds, the corresponding coefficient of friction is $\phi = \frac{91}{260} = \frac{7}{20} = 0{,}35$.

2. In order to pull forward a sled weighing 500 pounds on a horizontal and very smooth snow-covered road, when the coefficient of friction is $\phi = 0{,}04$, a force $F = 0{,}04 \,.\, 500 = 20$ pounds is necessary.

3. If the coefficient of friction of a sled loaded with 500 pounds and

pulled over a paved road is 0,45, the mechanical effect required to move the sled 480 feet is $\phi N s = 0{,}45 \,.\, 500 \,.\, 480 = 108000$ foot-pounds.

FIG. 253.



§ 172. The Angle of Friction or of Repose and the Cone of Friction.—If a body $A\,C$, Fig. 253, lies upon an inclined plane $F\,H$, whose angle of inclination is $F\,H\,R = \alpha$, we can decompose its weight into the normal pressure $N = G\ cos.\ \alpha$, and into the force $S = G\ sin.\ \alpha$ parallel to the plane. The first force causes the friction $F = \phi\ G\ cos.\ \alpha$, which resists every motion upon the plane; consequently the force necessary to push the body up the plane is

$$P = F + S = \phi\ G\ cos.\ \alpha + G\ sin.\ \alpha$$
$$= (sin.\ \alpha + \phi\ cos.\ \alpha)\ G,$$

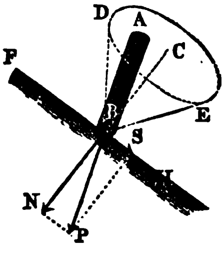
and the force necessary to push it down the same is

$$P_1 = F - S = (\phi\ cos.\ \alpha - sin.\ \alpha)\ G.$$

The latter force becomes $= 0$, I.E. the body holds itself upon the inclined plane by its friction when $sin.\ \alpha = \phi\ cos.\ \alpha$, I.E. when $tang.\ \alpha = \phi$. As long as the inclined plane has an angle of inclination, whose tangent is less than ϕ, so long will the body remain at rest upon the inclined plane; but if the tangent of the angle of inclination is a little greater than ϕ, the body will slide down the inclined plane. We call this angle, I.E. the one whose tangent is equal to the coefficient of friction, the angle of friction or of repose or of resistance (Fr. angle du frottement, Ger. Reibungs—or Ruhewinkel). Hence we obtain the coefficient of friction (for the friction of quiescence) by observing the angle of friction ρ and putting $\phi = tang.\, \rho$.

In consequence of the friction, the surface $F\,H$, Fig. 254, of a body counteracts not only the normal pressure N of another body $A\,B$, but also any oblique pressure P when the angle $N\,B\,P = \alpha$ formed by its direction with the normal to the surface does not exceed the angle of friction; since the force P gives rise to the normal pressure $\overline{B\,N} = P\ cos.\ \alpha$, and to the lateral or tangential pressure $\overline{B\,S} = S = P\ sin.\ \alpha$ and since the normal pressure $P\ cos.\ \alpha$ produces the friction $\phi\ P\ cos.\ \alpha$, which opposes

FIG. 254.



every movement in the plane *F H*, *S* can produce no motion as long as we have

$$\phi\, P\, cos.\, a > P\, sin.\, a \text{ or } \phi\, cos.\, a > sin.\, a, \text{ I.E.}$$

$$tang.\, a < \phi \text{ or } a < \rho.$$

If we cause the angle of friction $C\,B\,D = \rho$ to revolve about the normal *C B*, it describes a cone, which we call the cone of friction or of resistance (Fr. cone de fr., Ger. Reibungskegel). The cone of friction embraces the directions of all the forces, which are completely counteracted by the inclined plane.

EXAMPLE.—In order to draw a full bucket weighing 200 pounds up a wooden plane inclined to the horizon at an angle of 50°, the coefficient of friction being $\varphi = 0{,}48$, we would require a force

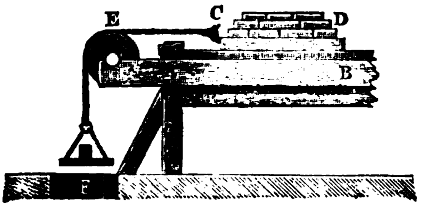
$$P = (\phi\, cos.\, a + sin.\, a)\, G = (0{,}48\, cos.\, 50^\circ + sin.\, 50^\circ)\,.\,200$$
$$= (0{,}308 + 0{,}766)\,.\,200 = 215 \text{ pounds.}$$

In order to let it down or to prevent its sliding down, we would have need of a force

$$P_1 = (\phi\, cos.\, a - sin.\, a)\, G = -\,(sin.\, 50^\circ - 0{,}48\, cos.\, 50^\circ)\,.\,200$$
$$= -\,(0{,}766 - 0{,}308)\,.\,200 = -\,91{,}5 \text{ pounds.}$$

§ 173. **Experiments on Friction.**—Experiments on friction have been made by many persons; those, which were most extended and upon the largest scale, are the experiments of Coulomb and Morin. Both these experimenters employed, for the determination of the coefficient of friction of sliding, a sled movable upon a horizontal surface and dragged along by a rope passing over a fixed pulley, to the end of which a weight was attached, as is shown in Fig. 255, in which *A B* is the surface, *C D* the sled, *E* the pulley, and *F* the weight. In order to obtain the coefficients of frictions for different substances, not only the runners of the sled, but also the surface upon which it slid, were covered with the smoothest possible plates of the material to be experimented on, such as wood, iron, etc. The coefficients of friction of rest were given by the weight necessary to bring the sled from a state of rest into motion, and the coefficients of friction of motion were determined by aid of the time required by the sled to describe a certain space *s*. If *G* is the weight of the sled and *P* the weight necessary to move the same, we have the friction

FIG. 255.



$= \phi\, G$, the moving force $= P - \phi\, G$ and the mass $M = \frac{P + G}{g}$, whence, according to § 68, the acceleration of the uniformly accelerated motion engendered is

$$p = \frac{P - \phi\, G}{P + G}\, g,$$

and inversely the coefficient of friction is

$$\phi = \frac{P}{G} - \frac{P + G}{G} \cdot \frac{p}{g}.$$

But we have also (§ 11) $s = \frac{1}{2} p\, t^2$, whence $p = \frac{2s}{t^2}$ and

$$\phi = \frac{P}{G} - \frac{P + G}{G} \cdot \frac{2\, s}{g\, t^2}.$$

If we allow the sled to slide down an inclined plane, the moving force is $= G\, (sin.\, a - \phi\, cos.\, a)$, and the accelerated mass is $= \frac{G}{g}$; consequently the acceleration is

$$p = \frac{2\, s}{t^2} = \frac{G\, (sin.\, a - \phi\, cos.\, a)}{\frac{G}{g}} = g\, (sin.\, a - \phi\, cos.\, a)$$

or $\frac{2\, s}{g\, t^2} = sin.\, a - \phi\, cos.\, a$, and consequently the coefficient of sliding friction is $\phi = tang.\, a - \frac{2\, s}{g\, t^2\, cos.\, a}$.

If h denotes the altitude, l the length and a the base of the inclined plane, we have also $\phi = \frac{h}{a} - \frac{2\, s\, l}{g\, a\, t^2}$.

In order to determine the coefficient of friction for the friction of axles or journals, they employed a fixed pulley $A\ C\ B$, Fig. 256, around which a rope was wound, to which the weights P and Q were suspended; from the sum of the weights $P + Q$ we have the pressure R upon the axle, and from their difference $P - Q$ the force at the periphery of the pulley, which is held in equilibrium by the friction $F = \phi\, (P + Q)$ on the surface of the axle. If now $C\, A = a =$ the radius of the axle and $C\, D = r =$ the radius of the journal, we have, since the statical moments are equal,

$$(P - Q)\, a = F\, r = \phi\, (P + Q)\, r,$$

and consequently the coefficient of friction of rest

$$\phi = \frac{P - Q}{P + Q} \cdot \frac{a}{r}$$

and, on the contrary, when the weight P falls and Q rises in the time t a distance s, the coefficient of friction of motion is

$$\phi = \left(\frac{P - Q}{P + Q} - \frac{2\,s}{g\,t^2}\right)\frac{a}{r}.$$

The engineer Hirn employed in his (the latest) experiments upon friction of journals the apparatus represented in Fig. 257,

FIG. 256.

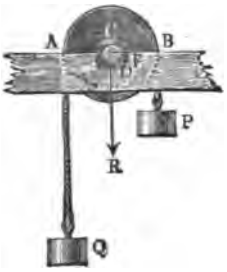
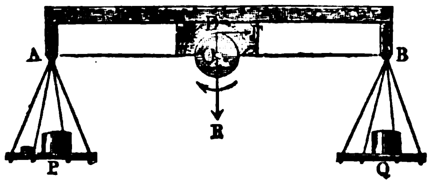


FIG. 257.



which he called a friction balance (Fr. balance de frottement, Ger. Reibungswage). Here C is an axle, which is kept in constant rotation, as, E.G., by a water-wheel, D is the bearing, and $A\ D\ B$ is a lever of equal arms, which produces the pressure between the journal and its bearing by means of the weights P and Q. The pressure on the axle $R = P + Q$ produces the friction

$$F = \phi\,R = \phi\,(P + Q)$$

between the journal and its bearing. With this force the revolving shaft seeks to turn the bearing and the lever $A\ D\ B$, which is attached to it, in the direction of the arrow; and therefore, in order to keep the whole in equilibrium, we must make the weight P on one side A so much greater than the weight Q on the other, that $P - Q$ will balance the friction. But the friction F acts with the arm $C\,D = r =$ the radius of the bearing and the difference of the weights $P - Q$ with the arm $C\,A = a$, which is equal to the horizontal distance between the axis C of the shaft and the vertical line through the point of suspension A, and therefore we have

$$F\,r = \phi\,R\,r = \phi\,(P + Q)\,r = (P - Q)\,a,$$

and the coefficient of friction required

$$\phi = \frac{P - Q}{P + Q} \cdot \frac{a}{r}.$$

REMARK.—Before Coulomb, Amontons, Camus, Bülffinger, Muschenbrock, Ferguson, Vince and others had studied the subject of friction and made experiments upon it. The results of all these researches have, however, little practical value; for the experiments were made upon too small a scale. The same objection applies to those of Ximenes, which were made about the same time as those of Coulomb. The results of Ximenes are to be found in the work "Teoria e Pratica delle resistenze de' solidi ne' loro attriti, Pisa, 1782." Coulomb's experiments are described in detail in the work: "Théorie des machines simples, etc., par Coulomb. Nouv. édit., 1821." An abstract from it is to be found in the prize essay of Metternich, "Vom Widerstande der Reibung, Frankfurt und Mainz, 1789." The later experiments on friction were made by Rennie and Morin. Rennie employed in his experiments in some cases a sled, which slid upon a horizontal surface, and in others an inclined plane, down which he caused the bodies to slide, and from the angle of inclination determined the amount of the friction. Rennie's experiments were made with most of the substances, which we meet with in practice, such as ice, cloth, leather, wood, stone and the metals; they also give important data in relation to the manner in which bodies wear, but the apparatus and the manner of conducting these experiments do not allow us to hope for as great accuracy as Morin seems to have attained in his experiments. A German translation of Rennie's Experiments is to be found in the 17th volume (1832) of the Wiener Jahrbücher des K. K. Polytechnischen Institutes, and also in the 34th volume (1829) of Dingler's Polytechnisches Journal. The most extensive experiments and those, which probably give the most accurate results, are those made by Morin, although it cannot be denied that they leave certain points doubtful and uncertain, and that here and there there are points, upon which more information could be desired. This is not the place to describe the method and apparatus employed in these experiments; we can only refer to Morin's writings: "Nouvelles Expériences sur le frottement," etc. A capital discussion of the subject "friction," and a rather full description of almost all the experiments upon it, Morin's included, is given by Brix in the transactions of the Society for the Advancement of Industry in Prussia, 16th and 17th Jahrgang—Berlin, 1837 and 1838. Later experiments on mediate friction, with particular reference to the different unguents, made by M. C. Ad. Hirn, are described in the "Bulletin de la société industrielle de Mulhouse, Nos. 128 and 129, 1855," under the title of "Etudes sur les principaux phénomènes que présentent les frottements médiats, etc.;" an abstract of it is to be found in the "Polytechnisches Centralblatt, 1855. Lieferung, 10." The latest researches upon friction by Bochet are described under the title, "Nouv. Recherches expérimentales sur le frottement de glissement, par M. Bochet," in the Annales des Mines, Cinq. Série, Tome XIX., Paris, 1861. Prof. Rühlmann gives some information in regard to the experiments with Waltjen's friction balance in the "Polytechnisches Centralblatt, 1861. Heft 10."

§ **174. Friction Tables.**—The following tables contain a condensed summary of the coefficients of friction of the substances, most generally employed in practice.

TABLE I.

COEFFICIENTS OF FRICTION OF REST.

Name of the rubbing bodies.		Condition of the surfaces and nature of the unguents. Dry.	Moistened with water.	With olive oil.	Hog's lard.	Tallow.	Dry soap.	Polished and greasy.	Greasy and moistened.
Wood upon wood	Minimum value.	0,30	0,65	—	—	0,14	0,22	0,30	
	Mean "	0,50	0,68	—	0,21	0,19	0,36	0,35	
	Maximum "	0,70	0,71	—	—	0,25	0,44	0,40	
Metal upon metal	Minimum value.	0,15	—	0,11					
	Mean "	0,18	—	0,12	0,10	0,11	—	0,15	
	Maximum "	0,24	—	0,16					
Wood on metal		0,60	0,65	0,10	0,12	0,12	—	0,10	
Hemp in ropes, plaits, etc., on wood	Mini'm value.	0,50							
	Mean "	0,63	0,87						
	Max'm "	0,80							
Thick sole leather as packing on wood or cast iron . . .	On edge . . .	0,43	0,62	0,12					
	Flat	0,62	0,80	0,13	—	—	—	—	0,27
Black leather straps over drums	Made of wood.	0,47							
	" metal	0,54	—	—	—	—	—	0,28	0,38
Stone or brick upon stone or brick, well polished	Mini'm value.	0,67							
	Max'm "	0,75							
Stone upon wrought iron	Min. val.	0,42							
	Max. "	0,49							
Pearwood upon stone		0,64							

TABLE II.

COEFFICIENTS OF FRICTION OF MOTION.

Name of the rubbing bodies.		Condition of the surfaces and nature of the unguents.								
		Dry.	With water.	Olive oil.	Hog's lard.	Tallow.	Hog's fat and plumbago.	Pure wagon grease.	Dry soap.	Greasy.
Wood upon wood	Min. value.	0,20	—	—	0,06	0,06	—	—	0,14	0,08
	Mean "	0,36	0,25	—	0,07	0,07	—	—	0,15	0,12
	Max. "	0,48	—	—	0,07	0,08	—	—	0,16	0,15
Metal upon metal	Min. value.	0,15	—	0,06	0,07	0,07	0,06	0,12	—	0,11
	Mean "	0,18	0,31	0,07	0,09	0,09	0,08	0,15	0,20	0,13
	Max. "	0,24	—	0,08	0,11	0.11	0,09	0,17	—	0,17
Wood upon metal	Min. value.	0,20	—	0,05	0,07	0,06	—	—	—	0,10
	Mean "	0,42	0,24	0,06	0,07	0,08	0,08	0,10	0,20	0,14
	Max. "	0,62	—	0,08	0,08	0,10	—	—	—	0,16
Hemp in ropes, etc	On wood.	0,45	0,33							
	On iron .	—	—	0,15	—	0,19				
Sole leather flat upon wood or metal	Raw . . .	0,54	0,36	0,16	—	0,20				
	Pounded.	0,30	—							
	Greasy . .	—	0,25							
The same on edge for piston packing.	Dry . . .	0,34	0,31	0,14	—	0,14				
	Greasy . .	—	0,24							

REMARK.—More complete tables of the coefficients of friction are to be found in the "Ingenieur," page 403, etc. The coefficients of friction of loose granular masses will be given in the second volume, when the theory of the pressure of earth is treated.

§ **175. The Latest Experiments on Friction.**—From the experiments of Bochet upon sliding friction, we find, that the results obtained by the older experimenters Coulomb and Morin must undergo some important modifications. The former experiments were made with railroad wagons weighing from 6 to 10 tons, which were caused to slide on a horizontal railroad either upon their wheels, which were made fast, or upon a kind of shoe (patin). The shoes were fastened to the frame of the wagon before, between and behind the wheels, and in the different series of experiments they were covered with soles of different materials, such as wood, leather, iron, etc., on which a pressure of 2, 4, 6, 10 and 15 kilograms per square centimetre could be produced. The wagon, thus transformed into a sled, was moved by a locomotive attached in front by means of a spring dynameter, which gave the pull or force, which balanced the sliding friction. In order to prevent, as much as possible, the resistance of the air, the wagon, which preceded the sled, had a greater cross-section than the latter.

The correctness of the formula $F = \phi N$, according to which the friction F is proportional to the pressure, is proved anew by these experiments; but it was found, that the co-efficient of friction was dependent not only upon the nature and state of the rubbing surfaces, but also upon other circumstances, viz.: the velocity of the sliding body and the specific pressure, I.E., the pressure per unit of surface. Bochet puts

$$\phi = \frac{\kappa - \gamma}{1 + a v} + \gamma,$$

in which v denotes the velocity of sliding, κ the value of ϕ for infinitely slow and γ the value ϕ for a very rapid motion. According to this formula the coefficient decreases gradually from κ to γ as the velocity increases. The mean value of the coefficient a is $= 0{,}3$, when v is expressed in meters, and on the contrary $= 0{,}091$, when v is given in feet. Hence we can assume the co-efficient of friction to be constant only, when the velocities vary from 0 to at most 1 foot and when the other circumstances remain the same. The co-efficients κ and γ are different for different materials and depend upon the degree of smoothness of the rubbing surfaces, upon the unguents, upon the specific pressure etc.

The co-efficient of friction κ attains its maximum value for wood, particularly soft wood, leather and gutta-percha sliding upon dry and ungreased iron rails. Here we have $\kappa = 0{,}40$ to $0{,}70$. The mean value for soft wood is $\kappa = 0{,}60$ and for hard wood $\kappa = 0{,}55$.

The value κ is also very different for the friction of iron upon iron. If the surfaces are not polished we have $\kappa = 0{,}25$ to $0{,}60$; and, on the contrary, for polished surfaces we have $\kappa = 0{,}12$ to

0,40. The friction of iron upon iron is not diminished by sprinkling it with water, but the friction of wood, leather and gutta-percha is considerably diminished by wetting the rail. When the surfaces are oiled, κ sinks to from 0,05 to 0,20.

The co-efficient γ is always smaller than κ. When the velocities are great, the surfaces smooth, the unguent properly applied and the specific pressure a medium one, γ has nearly the same value for all substances.

The friction of rest is greater only in those cases where wood or leather slide upon wet or greased rails, and then it is twice as great.

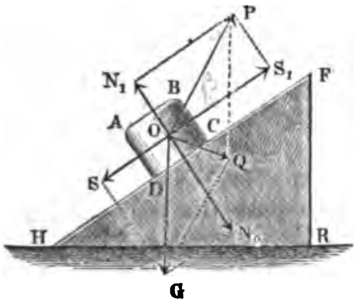
According to these experiments, we have

1. for dry soft wood, when the pressure is at least 10 kilograms per square centimeter or 142 pounds per square inch,
$$\phi = \frac{0{,}30}{1 + 0{,}3\, v} + 0{,}30;$$
2. for dry hard wood under the same pressure
$$\phi = \frac{0{,}30}{1 + 0{,}3\, v} + 0{,}25\cdot$$
3. for half polished iron, dry or wet, under a pressure of more than 300 kilograms per square centimeter or 4267 pounds per square inch,
$$\phi = \frac{0{,}15}{1 + 0{,}3\, v} + 0{,}15;$$
4. for the same either dry, under a pressure of at least 100 kilograms per square centimeter or polished and greased under specific pressure of at least 20 kilograms, and also for resinous wood with water as unguent under the same pressure,
$$\phi = \frac{0{,}175}{1 + 0{,}3\, v} + 0{,}075;$$
5. for wood properly polished and rubbed with fatty water or fat under a pressure of at least 20 kilograms per square centimeter (284 pounds per square inch),
$$\phi = \frac{0{,}10}{1 + 0{,}3\, v} + 0{,}06.$$

If v is given in feet, we must substitute in the denominator $0{,}091\, v$ instead of $0{,}3\, v$.

REMARK.—It is very desirable that these experiments, made on so large a scale and giving results which differ so much from those already known, should be repeated.

FIG. 258.



§ 176. Inclined Plane.—One of the most important applications of the theory of sliding friction is to the determination of the conditions of equilibrium of a body $A\ C$ upon an inclined plane $F\ H$, Fig. 258. If, as in § 146, $F\ H\ R = \alpha$ is the angle of inclination of the inclined plane and $P\ O\ S_1 = \beta$ the angle formed by the direction of the force P with the inclined plane, we have the normal force due to the weight G

$$N_0 = G\ cos.\ \alpha,$$

the force which tends to move the body down the plane $= S = G\ sin.\ \alpha$, the force N_1, with which the force P seeks to raise the body from the plane, $= P\ sin.\ \beta$ and the force S_1 with which it draws the body up the plane $= P\ cos.\ \beta$. The resulting normal force is

$$N = N_0 - N_1 = G\ cos.\ \alpha - P\ sin.\ \beta,$$

and consequently the friction is

$$F = \phi\ (G\ cos.\ \alpha - P\ sin.\ \beta).$$

If we wish to find the force necessary to draw the body up the plane, the friction must be overcome, and therefore we have

$$S_1 = S + F, \text{ I.E. } P\ cos.\ \beta = G\ sin.\ \alpha + \phi\ (G\ cos.\ \alpha - P\ sin.\ \beta).$$

But if the force necessary to prevent the body from sliding down the plane is required, as the friction assists the force, we will have

$$S_1 + F = S, \text{ I.E. } P\ cos.\ \beta + \phi\ (G\ cos.\ \alpha - P\ sin.\ \beta) = G\ sin.\ \alpha.$$

From these equations we obtain in the first case

$$P = \frac{sin.\ \alpha\ +\ \phi\ cos.\ \alpha}{cos.\ \beta\ +\ \phi\ sin.\ \beta} \cdot G, \text{ and in the second case,}$$

$$P = \frac{sin.\ \alpha\ -\ \phi\ cos.\ \alpha}{cos.\ \beta\ -\ \phi\ sin.\ \beta} \cdot G.$$

If we introduce the angle of friction or of repose ρ by putting

$$\phi = tang.\ \rho = \frac{sin.\ \rho}{cos.\ \rho}, \text{ we obtain}$$

$$P = \frac{sin.\ \alpha\ cos.\ \rho\ \pm\ cos.\ \alpha\ sin.\ \rho}{cos.\ \beta\ cos.\ \rho\ \pm\ sin.\ \beta\ sin.\ \rho} \cdot G,$$

or according to a well-known trigonometrical formula

$$P = \frac{sin. (\alpha \pm \rho)}{cos. (\beta \mp \rho)} \cdot G;$$

the upper signs are for the case, when motion is to be produced, and the lower ones, when motion is to be prevented.

As long as we have

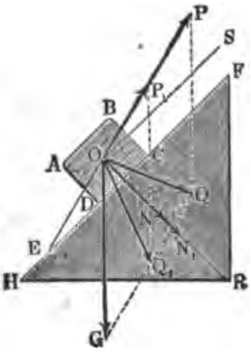
$$P > \frac{sin. (\alpha - \rho)}{cos. (\beta + \rho)} G \text{ and } < \frac{sin. (\alpha + \rho)}{cos. (\beta - \rho)} G,$$

the body will move neither up nor down.

If α is $< \rho$, the force necessary to push the body down the plane is

$$P = \frac{sin. (\rho - \alpha)}{cos. (\rho + \beta)} G.$$

FIG. 259.



The latter formula can be found by the simple application of the parallelogram of forces $O\ P\ Q\ G$, Fig. 259. Since a body counteracts any force from another body, when the angle of divergence of the direction of the force from that of the normal to the surface is equal to the angle of friction ρ (§ 172), a state of equilibrium will exist in the foregoing case, when the resultant $\overline{O\ Q} = Q$ of the forces P and G forms an angle $N\ O\ Q = \rho$ with the normal. If, in the general formula

$$\frac{P}{G} = \frac{sin.\ G\ O\ Q}{sin.\ P\ O\ Q},$$

we substitute $G\ O\ Q = G\ O\ N + N\ O\ Q = \alpha + \rho$ and $P\ O\ Q = P\ O\ S + S\ O\ Q = \beta + 90^\circ - \rho$, we obtain

$$\frac{P}{G} = \frac{sin. (\alpha + \rho)}{sin. (\beta - \rho + 90^\circ)} = \frac{sin. (\alpha + \rho)}{cos. (\beta - \rho)}.$$

If the force P_1 is to prevent the body from sliding down the inclined plane, the resultant Q_1 falls on the lower side of the normal $O\ N$, and the angle of friction ρ enters in the calculation with a negative sign, and consequently we have

$$\frac{P}{G} = \frac{sin. (\alpha - \rho)}{cos. (\beta + \rho)},$$

If the body lies upon a horizontal plane, α is $= 0$, and the force necessary to move it forward becomes

$$P = \frac{\phi\, G}{cos.\beta + \phi\, sin.\beta} = \frac{G\ sin.\ \rho}{cos.\ (\beta - \rho)}.$$

If the force acts parallel to the inclined plane, I.E., in the direction of its slope, we have $\beta = 0$, and therefore

$$P = (sin.\ \alpha \pm \phi\ cos.\ \alpha)\ G = \frac{sin.\ (\alpha \pm \rho)}{cos.\ \rho}\ .\ G.\ \text{(Compare § 172.)}$$

If, finally, the force acts horizontally, we have $\beta = -\alpha$, $cos.\ \beta = cos.\ \alpha$ and $sin.\ \beta = -\ sin.\ \alpha$, and consequently

$$P = \frac{sin.\ \alpha \pm \phi\ cos.\ \alpha}{cos.\ \alpha \mp \phi\ sin.\ \alpha}\ .\ G = \frac{tang.\ \alpha \pm \phi}{1 \mp \phi\ tang.\ \alpha}\ .\ G,\ \text{I.E.}$$

$P = tang.\ (\alpha \pm \rho)\ G$, which is also given by the direct resolution of the parallelogram $O\ P\ Q\ G$.

Further, the force necessary to push the body up the plane becomes a minimum, when the denominator $cos.\ (\beta - \rho)$ becomes a maximum, that is, when it is $= 1$, or when $\beta - \rho$ is $= 0$, I.E. when $\beta = \rho$. When the angle formed by the direction of the force with that of the inclined plane is equal to the angle of friction, this force is a minimum and is $P = sin.\ (\alpha + \rho)\ .\ G$.

EXAMPLE.—What is the pressure along the axis of a wooden prop $A\ E$, Fig. 260, which prevents the mass of rock $A\ B\ C\ D$, weighing $G = 5000$ pounds, from sliding down an inclined plane (the floor of a mine), when the inclination of the prop to the horizon is 35°, that of the inclined plane $C\ D$, 50° and when the coefficient of friction ϕ is $= 0{,}75$? Here we have

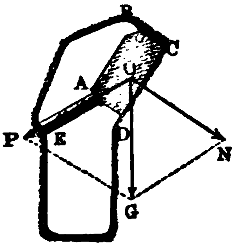
$$G = 5000,\ \alpha = 50°,\ \beta = 35° - 50° = -\ 15° \text{ and } \phi = 0{,}75,$$

and the formula gives

$$P = \frac{sin.\ \alpha - \phi\ cos.\ \alpha}{cos.\ \beta - \phi\ sin.\ \beta}\ .\ G = \frac{sin.\ 50° - 0.75\ cos.\ 50°}{cos.\ 15° + 0{,}75\ sin.\ 15°}\ .\ 5000$$

$$= \frac{0{,}766 - 0{,}482}{0{,}966 + 0{,}194}\ .\ 5000 = \frac{1420}{1{,}160} = 1224 \text{ pounds.}$$

FIG. 260.



If the prop was horizontal, we would have $\beta = -\ 50°$ and $tang.\ \rho = 0{,}75$, or $\rho = 36°\ 52'$, from which we obtain

$P = G\ tang.\ (\alpha - \rho) = 5000\ tang.\ (50° - 36°\ 52')$
$= 5000\ tang.\ 13°\ 8' = 5000\ .\ 0{,}2333 = 1166$ pounds.

In order to push the same mass of rock by means of a horizontal force up the floor, when the other circumstances are the same, a force

$P = G\ tang.\ (\alpha + \rho) = 5000\ tang.\ 86°\ 52'$
$= 5000\ .\ 18{,}2676 = 91338$ pounds would be necessary.

177. The normal pressure, with which a body $A\ C$ presses upon the inclined plane $F\ H$, Fig. 261, while being pushed up it, is

$$N = Q\ cos.\ \rho = \frac{G\ sin.\ O\ P\ Q}{sin.\ P\ O\ Q}\ cos.\ \rho = \frac{G\ sin.\ (90° - \alpha - \beta)}{sin.\ (\beta + 90° - \rho)}\ cos.\ \rho$$

$$= \frac{G\ cos.\ (\alpha + \beta)\ cos.\ \rho}{cos.\ (\beta - \rho)}$$

and, on the contrary, when we prevent its sliding down, we have

$$N_1 = Q_1\ cos.\ Q_1\ O\ N_1 = Q_1\ cos.\ \rho = \frac{G\ cos.\ (\alpha + \beta)\ cos.\ \rho}{cos.\ (\beta + \rho)}.$$

If the direction of the force is parallel to the direction of the plane, we have $\beta = 0$ and $N = G\ cos.\ \alpha$, and when its direction is horizontal, we have $\beta = -\alpha$ and

$$N = \frac{G\ cos.\ \rho}{cos.\ (\alpha \pm \rho)}.$$

FIG. 261.

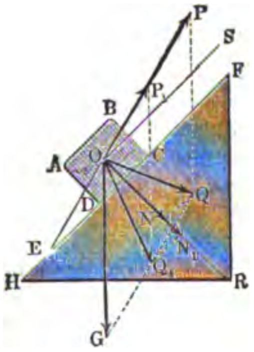
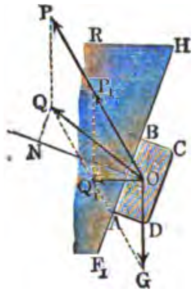


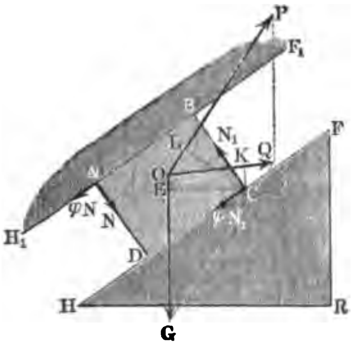
FIG. 262.



The normal pressure becomes null, when $cos.\ (\alpha + \beta) = 0$ or $\alpha + \beta = 90°$, and becomes negative, when $\alpha + \beta$ is $> 90°$ or β is $> 90° - \alpha$. In the latter case the inclined plane is not under but over the body, as is represented in Fig. 262. Here again the two extreme cases of equilibrium exist when the resultant Q or Q_1, which is transmitted to the inclined plane $F\ H$, diverges from the normal either above or below it at an angle, which is that of the friction $N\ O\ Q = N\ O\ Q_1 = \rho$.

In the foregoing development of the formulas for the equilibrium of a body upon an inclined plane it is supposed, that the resultant Q can be completely transmitted from the body $A\ C$ to the support $F\ H\ R$, which forms the inclined plane; this is only possible (according to § 146), when the direction of this force passes through the supporting surface $C\,D$ of the body $A\,C$. Otherwise the body $A\,C$, Fig. 263, has a tendency to revolve or overturn about the outer edge C, and this tendency increases with the distance $C\,K = e$ of this edge from the direction $O\,Q$ of the resultant Q.

Fig. 263.



If a denotes the distance $C\,L$ of the direction $O\,P$ of the force and b the distance $C\,E$ of the vertical line of gravity $O\,G$ of the body from the outer edge C, then the moment, with which the body seeks to turn from left to right about C, is $Q\,e = P\,a - G\,b$.

If $P\,a$ were $= G\,b$ or $\frac{P}{G} = \frac{b}{a}$, the resultant Q would pass through the edge C and would be counteracted by the inclined plane; if $P\,a$ were $< G\,b$, the body would have a tendency to turn from right to left, which turning would be prevented by its impenetrability.

If, on the contrary, $P\,a$ is $> G\,b$ the body must receive a second support or be guided by a second inclined plane $F_1\,H_1$. If this second inclined plane counteracts in A the force N and the friction $\phi\,N$ caused by it, the inclined plane $F_1\,H_1$ will react upon the body in A with the opposite forces $-N$ and $-\phi\,N$, which prevent the turning of the body about C, and the sum of the moments of these forces must be equal to the moment of rotation of the force Q, i.e. $N\,l + \phi\,N\,d = Q\,e = P\,a - G\,b$, or

$$1)\ N\,(l + \phi\,d) = P\,a - G\,b,$$

l and d designating the distances $C\,D$ and $C\,B$ of the edge A from C in the directions parallel and at right angles to the inclined plane.

If, further, N_1 is the pressure of the body upon the inclined plane $F\,H$ at C and $\phi\,N_1$ the friction caused by it, we can put

$$2)\ P \cos. \beta = G \sin. \alpha + \phi\,(N + N_1) \text{ and}$$

$$3)\ P \sin. \beta = G \cos. \alpha + N - N_1.$$

Eliminating N_1 from the last two equations we obtain the equation of condition.

$$P\,(\cos.\beta + \phi\sin.\beta) = G\,(\sin.\alpha + \phi\cos.\alpha) + 2\,\phi\,N,$$

and substituting the value $N = \frac{P\,a - G\,b}{l + \phi\,d}$ from equation (1) we have the equation

$$P\,(\cos.\beta + \phi\sin.\beta) = G\,(\sin.\alpha + \phi\cos.\alpha) + \frac{2\,\phi\,(Pa - G\,b)}{l + \phi\,d}$$

or
$$P\left(\frac{l + \phi\,d}{2}\,(\cos.\beta + \phi\sin.\beta) - \phi\,a\right)$$
$$= G\left(\frac{l + \phi\,d}{2}\,(\sin.\alpha + \phi\cos.\alpha) - \phi\,b\right),$$

from which we obtain finally

$$P = \frac{(l + \phi\,d)\,(\sin.\alpha + \phi\cos.\alpha) - 2\,\phi\,b}{(l + \phi\,d)\,(\cos.\beta + \phi\sin.\beta) - 2\,\phi\,a}\,G$$
$$= \frac{(l + \phi\,d)\sin.(\alpha + \rho) - 2\,\phi\,b\cos.\rho}{(l + \phi\,d)\cos.(\beta - \rho) - 2\,\phi\,a\cos.\rho}\,.\,G.$$

If N is $= 0$, we have $P\,a = G\,b$ and

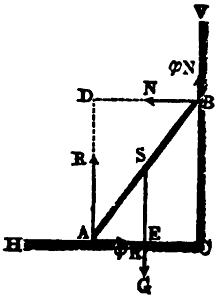
$$\frac{\sin.(\alpha + \rho)}{\cos.(\beta - \rho)} = \frac{b}{a}, \text{ whence}$$
$$P = \frac{\sin.(\alpha + \rho)}{\cos.(\beta - \rho)}\,G,$$

as we found before.

§ 178. The Theory of the Equilibrium of Supported Bodies referred to the Equilibrium of Free Bodies.—In investigating the conditions of equilibrium of a body, taking into consideration the friction, we will accomplish more surely our object, if we imagine the body entirely free and suppose, that every body, with which it comes in contact, acts upon it with two forces, viz.: with one force N, which proceeds from it and is normal to the surface of contact, and with another force $\phi\,N$, which opposes the supposed motion of the point of contact on this surface and which is caused by the friction between the two bodies. In this way we obtain a rigid system of forces, whose state of equilibrium can easily be determined according to the rules given in § 90, as is shown in the following special case.

A prismatical bar $A\,B$, Fig. 264, is so placed, that its lower end rests upon a horizontal floor $C\,H$ and that its upper end leans against the vertical wall $C\,V$; at what inclination $B\,A\,C = \alpha$ does it lose its equilibrium? We can here express the reactions of the floor upon the body by a vertical force R and by the friction $\phi\,R$, which acts horizontally, and, on the contrary, the reaction of the wall by a horizontal force N and by a friction $\phi\, N$ acting upwards. Hence, if G is the weight of the rod acting at its centre of gravity S, we have here a system of vertical forces G, R, $\phi\, N$ and a system of horizontal ones N and $\phi\, R$.

FIG. 264.



When these forces are in equilibrium, we have

1) $G = R + \phi\, N$,

2) $\phi\, R = N$ and

3) $G\,.\,\overline{A\,E} = N\,.\,\overline{A\,D} + \phi\, N\,.\,\overline{A\,C}$.

But the arm $A\,E$ is $= A\;S\; cos.\; a = \frac{1}{2}\, A\; B\; cos.\; a$, the arm $A\; D = A\; B\; sin.\; a$ and the arm $A\; C = A\; B\; cos.\; a$, hence the third equation becomes simply

$$\tfrac{1}{2}\; G\; cos.\; a = N\,(sin.\; a + \phi\; cos.\; a).$$

Combining the first two equations, we obtain

$$G = R + \phi^2\, R = (1 + \phi^2)\, R, \text{ whence}$$

$$R = \frac{G}{1 + \phi^2} \text{ and } N = \frac{G\,\phi}{1 + \phi^2}.$$

Substituting this value of N in the equation (3), we have

$$\tfrac{1}{2}\; G\; cos.\; a = \frac{\phi\; G}{1 + \phi^2}\,(sin.\; a + \phi\; cos.\; a), \text{ or}$$

$$\frac{1 + \phi^2}{2\,\phi} = tang.\; a + \phi,$$

and the tangent of the required angle of inclination is

$$tang.\; a = \frac{1 + \phi^2 - 2\,\phi^2}{2\,\phi} = \frac{1 - \phi^2}{2\,\phi} = \frac{1 - tang.^2\,\rho}{2\; tang.\; \rho}$$

$$= \frac{cos.^2\,\rho - sin.^2\,\rho}{2\; sin.\; \rho\; cos.\; \rho} = \frac{cos.\; 2\,\rho}{sin.\; 2\,\rho} = cotg.\; 2\,\rho$$

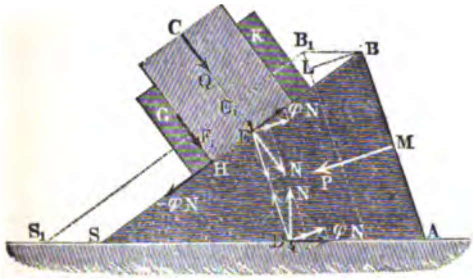
$$= tang.\; (90^\circ - 2\,\rho)\text{; therefore}$$

$$\angle\; B\,A\,C = a = 90^\circ - 2\,\rho \text{ and } \angle\; A\,B\,C = \beta = 2\,\rho.$$

§ 179. **Theory of the Wedge.**—Friction has also a great influence upon the conditions of equilibrium of the wedge (see § 149). Let us suppose, that its cross section forms an isosceles triangle $A\; B\; S$, Fig. 265, the acute angle of which $A\; S\; B = a$, that the force acts in the centre M of the back of the wedge $A\; B$

and at right angles to it and that the body *C H K* presses with a certain force *N* against the surface of the wedge *B S*, while the wedge reposes with its surface *A S* upon a horizontal plane. The body *C H K* is also inclosed in two guides *G* and *K*, which compel it, when the wedge is pushed forward upon the horizontal plane, to rise with the load *Q* in the direction *E C* perpendicular to the surface *B S* of the wedge.

FIG. 265



Since the direction of the force *P* forms equal angles with the two surfaces *A S* and *B S* of the wedge, the normal pressures *N*, *N*, and consequently the frictions $\phi\ N$, $\phi\ N$ caused by them, are equal to each other, and the forces *P*, *N*, *N*, $\phi\ N$ and $\phi\ N$ must hold each other in equilibrium. If we decompose each of the last four forces into two components, one parallel and the other perpendicular to the direction of the force *P*, the sum of the forces having the same direction as *P* must, of course, be in equilibrium with *P*. But the directions of the forces *N*, *N* form, with the direction *M S* of the force *P*, an angle $90 - \frac{a}{2}$, and those of the forces $\phi\ N$, $\phi\ N$ an angle $\frac{a}{2}$, and therefore the components of *N*, *N* in the direction *M S* are $N\ sin.\ \frac{a}{2}$ and $N\ sin.\ \frac{a}{2}$, and those of $\phi\ N$ and $\phi\ N$ are $\phi\ N\ cos.\ \frac{a}{2}$, and $\phi\ N\ cos.\ \frac{a}{2}$, and consequently we can put

$$P = 2\ N\ sin. \frac{a}{2} + 2\ \phi\ N\ cos.\ \frac{a}{2} = 2\ N\left(sin.\ \frac{a}{2} + \phi\ cos. \frac{a}{2}\right).$$

In consequence of the friction $\phi\ N$ between the surface *B S* of the wedge and the base of the body *C H K*, this body is pressed with an opposite force $-\ \phi\ N$ against the guide *G H*, which causes a friction $F_1 = \phi_1 . \phi\ N = \phi\ \phi_1\ N$, which resists the upward movement of the body *C H K*; hence we have

$$N - F_1 = Q \text{ or } N\,(1 - \phi\,\phi_1) = Q \text{ and}$$

$$N = \frac{Q}{1 - \phi\ \phi_1}.$$

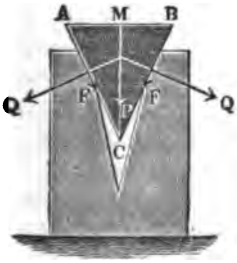
Substituting this value for N in the above equation, we obtain the force necessary to raise the weight Q

$$P = \frac{2\,Q}{1 - \phi\,\phi_1}\left(sin.\,\frac{a}{2} + \phi\,cos.\,\frac{a}{2}\right), \text{ approximatively}$$
$$= 2\,Q\,(1 + \phi\,\phi_1)\left(sin.\,\frac{a}{2} + \phi\,cos.\,\frac{a}{2}\right)$$
$$= 2\,Q\left(sin.\,\frac{a}{2} + \phi\,cos.\,\frac{a}{2} + \phi\,\phi_1\,sin.\,\frac{a}{2}\right),$$

or putting the coefficient of friction ϕ along the guides equal to that along the surfaces $A\ S$ and $B\ S$ of the wedge, we obtain

$$P = \frac{2\,Q}{1 - \phi^2}\left(sin.\,\frac{a}{2} + \phi\,cos.\,\frac{a}{2}\right), \text{ approximatively}$$
$$= 2\,Q\left((1 + \phi^2)\,sin.\,\frac{a}{2} + \phi\,cos.\,\frac{a}{2}\right).$$

Fig. 266.



When a wedge $A\ B\ C$, Fig. 266, is used for splitting or compressing bodies, the force upon the back $A\ B$ corresponding to the normal pressure Q against the sides $A\ C$ and $B\ C$ is

$$P = 2\,Q\left(sin.\,\frac{a}{2} + \phi\,cos.\,\frac{a}{2}\right).$$

Example.—Let the load on the wedge represented in Fig. 265 be $Q = 650$, the sharpness of the wedge $a = 25°$ and the coefficient of friction $\phi = \phi_1 = 0{,}36$; required the mechanical effect necessary to move the load Q ½ foot along its guides.

The force is

$$P = \frac{2\,.\,650}{1 - (0{,}36)^2}\,(sin.\ 12\tfrac{1}{2}° + 0{,}36\ cos.\ 12\tfrac{1}{2}°)$$
$$= \frac{1300}{1 - 0{,}1296}\,(0{,}2164 + 0{,}36\,.\,0{,}9763)$$
$$= \frac{1300}{0{,}8704}\,(0{,}2164 + 0{,}3515) = \frac{737{,}27}{0{,}8704} = 848{,}2 \text{ pounds.}$$

The space described by the load is $E\,E_1 = s_1 = \frac{1}{4}$ foot, and that described by the force is

$$B\,L = s = B\,B_1\,cos.\,\frac{a}{2} = \frac{E\,E_1}{sin.\ a}\,cos.\,\frac{a}{2} = \frac{s_1}{2\,sin.\,\frac{a}{2}} = \frac{0{,}25}{sin.\ 12\tfrac{1}{2}°}$$
$$= \frac{0{,}25}{0{,}2164} = 1{,}155 \text{ feet,}$$

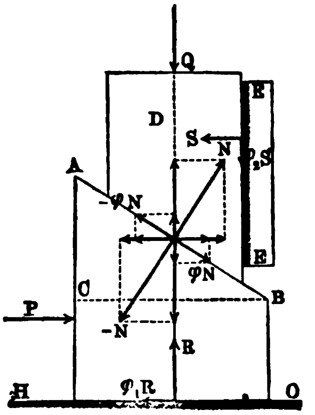
and consequently the mechanical effect necessary is

$$P\,s = 848{,}2\,.\,1{,}155 = 979{,}6 \text{ foot-pounds.}$$

If we neglected the friction, the work done would be $P\,s = Q\,s_1 = \frac{1}{2}\,.\,650 = 325$; consequently the friction nearly triples the mechanical effect necessary to raise Q.

§ 180. In the same way we can find the force P required, when a wedge $A\ B\ C$, Fig. 267, raises a load Q vertically upwards, while moving forward itself upon a horizontal plane $H\ O$. Let the normal pressure between the wedge $A\ B\ C$ and the block D, which is pressed vertically downwards by the load Q, be $= N$, the normal pressure of the wedge upon the support $H\ O$ be $= R$ and the normal pressure of the block against the guide $E\ E$ be $= S$. Then P must balance the forces R, $\phi_1\ R$, $-\ N$ and $-\ \phi\ N$, and Q the forces S, $\phi_2\ S$, N and $\phi\ N$.

FIG. 267.



If a is the angle of inclination $A\ B\ C$ of the surface $A\ B$ of the wedge, we can decompose N into the vertical force $N\ cos.\ a$ and the horizontal force $N\ sin.\ a$, and $\phi\ N$ into the vertical force $\phi\ N\ sin.\ a$ and the horizontal force $\phi\ N cos.\ a$, and therefore we can put

1) $P = \phi_1\ R + N sin.\ a + \phi\ N cos.\ a$,
2) $R = N\ cos.\ a - \phi\ N\ sin.\ a$,
3) $Q = N cos.\ a - \phi\ N sin.\ a - \phi_2\ S$ and
4) $S = N\ sin.\ a + \phi\ N\ cos.\ a$.

From the first two equations we obtain

$$P = [(1 - \phi\ \phi_1)\ sin.\ a + (\phi + \phi_1)\ cos.\ a]\ N,$$

and from the last two

$$Q = [(1 - \phi\ \phi_2)\ cos.\ a - (\phi + \phi_2)\ sin.\ a]\ N;$$

and dividing the first by the second, we have

$$\frac{P}{Q} = \frac{(1 - \phi\ \phi_1)\ sin.\ a + (\phi + \phi_1)\ cos.\ a}{(1 - \phi\ \phi_2)\ cos.\ a - (\phi + \phi_2)\ sin.\ a}.$$

If $\phi = \phi_1 = \phi_2$, we have, since $\phi = tang.\ \rho$ and

$$\frac{2\ \phi}{1 - \phi^2} = tang.\ 2\ \rho,$$

$$\frac{P}{Q} = \frac{sin.\ a + cos.\ a\ tang.\ 2\ \rho}{cos.\ a - sin.\ a\ tang.\ 2\ \rho} = \frac{tang.\ a + tang.\ 2\ \rho}{1 - tang.\ a\ tang.\ 2\ \rho}$$
$$= tang.\ (a + 2\ \rho).$$

If we disregard the friction upon the points of support, we can put ϕ_1 and $\phi_2 = 0$, and consequently

$$\frac{P}{Q} = \frac{sin.\ a + \phi\ cos.\ a}{cos.\ a - \phi\ sin.\ a} = \frac{tang.\ a + \phi}{1 - \phi\ tang.\ a} = tang.\ (a + \rho).\ \text{(Comp. § 176.)}$$

When the load Q acts at right angles to the surface of the wedge, the equations (3) and (4) must be replaced by the following

$$Q = N - \phi_2 S \text{ and}$$
$$S = \phi N,$$

whence $Q = (1 - \phi \phi_2) N$, or inversely,

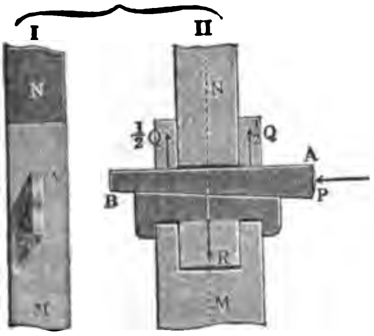
$$N = \frac{Q}{1 - \phi \phi_2} \text{ and}$$
$$\frac{P}{Q} = \frac{(1 - \phi \phi_1) \, sin. \, a + (\phi + \phi_1) \, cos. \, a}{1 - \phi \phi_2}.$$

When ϕ is $= \phi_1 = \phi_2$, it becomes

$$\frac{P}{Q} = sin. \, a + cos. \, a \, . \, tang. \, 2 \, \rho.$$

The formula $P = Q \, tang. \, (a + 2 \rho)$ is applicable to the determination of the conditions of equilibrium, when two bodies M and N are fastened together by means of a key A B, Fig. 268, I. and II. The force P applied to the back of the wedge causes the tension, with which the two bodies are drawn against one another,

FIG. 268.



$$Q = P \, cotg. \, (a + 2 \rho).$$

On the contrary, the force, with which we must press upon the bottom B of the key in order to loosen it, I.E. to drive it back in the direction B A, is, since a is negative here,

$$P_1 = Q \, tang. \, (2 \rho - a),$$

or substituting the former value of Q, we have

$$P_1 = P \frac{tang. \, (2 \rho - a)}{tang. \, (2 \rho + a)}.$$

In order to prevent the wedge from jumping back of itself, a must $< 2 \rho$.

§ 181. Coefficients of Friction of Axles.—For axles the friction of motion alone is important, and for this reason only the results of experiments upon it are given.

TABLE III.

COEFFICIENTS OF FRICTION OF AXLES, ACCORDING TO MORIN.

Name of the rubbing bodies.	Condition of the surfaces and nature of the unguents.							
	Dry or slightly greasy.	Greasy and moistened with water.	Anointed and moistened with water.	Oil, Tallow, or Lard.		Very soft, purified wagon grease.	Hog's fat, with plumbago.	Greasy.
				In the common way.	Well maintained.			
Bell metal upon bell metal.	—	—	—	0,097	—	—	—	—
" " cast iron...	—	—	—	—	0,049	—	—	—
Wro't iron " bell metal.	0,251	0,189	—	0,075	0,054	0,090	0,111	—
" " cast iron...	—	—	—	0,075	0,054	—	—	—
Cast iron " " ..	—	0,137	0,079	0,075	0,054	—	—	0,137
" " bell metal..	0,194	0,161	—	0,075	0,054	0,065	—	0,166
Wro't iron " lig. vitæ...	0,188	—	—	0,125	—	—	—	—
Cast iron " " ..	0,185	—	—	0,100	0,092	—	0,109	0,140
Lign'm vitæ " cast iron ..	—	—	—	0,116	—	—	—	0,153
" " lig. vitæ...	—	—	—	—	0,070	—	—	—

From this table the following practically important conclusions can be drawn: for axles, journals or gudgeons of wrought or cast iron running in bearings of cast iron or bell-metal (brass), greased with oil, tallow or lard, the coefficient of friction

is, when the lubrication is well sustained, = 0,054,
and with ordinary lubrication, = 0,070 to 0,080.

The values found by Coulomb differ in some respects from the above.

REMARK.—By his experiments upon mediate friction, by means of the friction balance, Hirn obtained several results, which differ somewhat from those already known. The axle employed by him, consisting of a hollow cast-iron drum 0,23 metres in diameter, and 0,22 metres long, was lubricated upon the outer surface by dipping it in oil and kept cool by causing water to pass through its interior. The bronze bearing (8 of copper and 1 of tin) was pressed upon it by means of a lever $1\frac{1}{4}$ metre long and weighing 50 kilogr. while the axle made 50 to 100 revolutions per minute. It is easy to see, that in the experiments made with this apparatus the fluidity and adhesion of the oil employed as unguent must have played an important part, since not only the velocity of revolution, but also the rubbing surface was very great compared to the pressure. The velocity at the circumference of the drum, since its circumference was 72 centimetres and since it revolved $\frac{5}{6}$ to $\frac{10}{6}$ times in a second, was 60 to 120 centimetres, or 24 to 48 inches, while in machines it is generally but from 2 to 6 inches. The horizontal section of the axle was 22 . 23 = 506 square centimetres, and consequently the pressure on each square centimetre of this section was only $\frac{50}{506} = 0{,}1$ kilogram, I.E. 6,45 . 0,220 = 1,42 pounds upon a square inch, while this pressure in ordinary machines is generally more than one hundred pounds. Hirn's experiments were consequently made under conditions different from those generally met with in very large and powerful machinery, and under which the other experiments, such as, E.G., those of Morin, were tried, and therefore the variation in the results obtained is perfectly explicable. The principal results of Hirn's experiments are the following.

The mediate friction is dependent not only upon the pressure and the nature and character of the rubbing surfaces and of unguent, but also upon the velocity and upon the temperature of the rubbing surfaces and of the surroundings, as well as upon the magnitude of these surfaces. The friction is directly proportional to the velocity, when the temperature is constant; and if the temperature is disregarded, it increases with the square root of the velocity. From other experiments Hirn concludes, that the mediate friction is also proportional to the square root of the rubbing surfaces as well as to the square root of the pressure. In regard to the particular influence of the temperature, the following formula was given by these experiments:

$$F = \frac{F_0}{1{,}0492^t},$$

in which t denotes the temperature of the rubbing surface, F_0 the friction at 0°, and F that at t degrees of temperature.

One of the principal results of these experiments was the determination of the mechanical equivalent of heat. This subject will be treated more at length, when we discuss the theory of heat.

§ 182. Work Done by the Friction of Axles.—If we know the pressure R between the axle and its bearing, and if the radius r of the axle, Fig. 269, is given, we can easily calculate the work done by the friction on the axle during each revolution. The friction is $F = \phi\, R$, the space described is the circumference $2\,\pi\, r$ of the axle, and consequently the mechanical effect lost by the friction is $A = \phi\, R \,.\, 2\,\pi\, r = 2\,\pi\,\phi\, R\, r$. If the axle makes u revolutions per minute, the mechanical effect expended in each second is

$$L = 2\,\pi\,\phi\, R\, r \,.\, \frac{u}{60} = \frac{\pi\, u\, \phi\, R\, r}{30} = 0{,}105 \,.\, u\, \phi\, R\, r.$$

The work done by the friction increases, therefore, with the pressure on the axle, with the radius of the axle and with the number of revolutions. We have therefore the following practical rule, not to increase unnecessarily the pressure on the axles in rotating machines, to make them as small as possible without endangering their solidity and durability and not to allow them to make too many revolutions in a minute, at least, when the other circumstances do not require it.

FIG. 269.

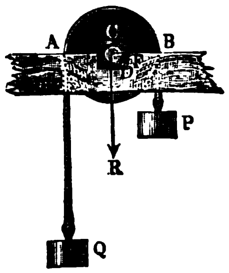
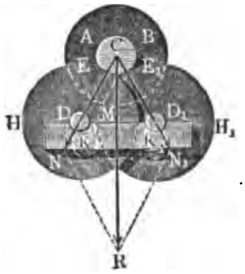


FIG. 270.



By the use of friction-wheels instead of plumber-blocks, the work done by the friction is diminished. In Fig. 270 $A\ B$ is a shaft, whose journal $C\ E\ E_1$ rests upon the circumferences $E\ H$ and $E_1\ H_1$ of the wheels (friction-wheels), which revolve around D and D_1 and lie close behind one another. The given pressure R upon the shaft gives rise to the pressure

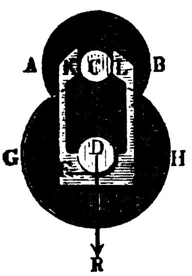
$$N = N_1 = \frac{R}{2 \cos. \frac{a}{2}}.$$

Here a denotes the angle $D\ C\ D_1$ included between the lines joining the centres, which are also lines of pressure. In consequence of the rolling friction between the axle C and the circumference of the wheels, the latter revolve with this axle, and the frictions $\phi\ N$ and $\phi\ N_1$ are produced on the bearings D and D_1, the sum of which is $F = \phi\ (N + N_1) = \frac{\phi\ R}{\cos. \frac{a}{2}}$. If the radius $D\ E = D_1\ E_1$ be denoted by a_1 and the radius of the axle by r_1, we obtain the force, which must be exerted at the circumference of the wheels or at that of the axle C to overcome the friction, and it is

$$F_1 = \frac{r_1}{a_1} F = \frac{r_1}{a_1} \cdot \frac{\phi R}{\cos. \frac{a}{2}},$$

while, on the contrary, it is $= \phi R$, when the axle lies directly on a step.

FIG. 271.



If we neglect the weight of the friction-wheels, the work done when these wheels are employed is $\psi = \frac{r_1}{a_1 \cos. \frac{a}{2}}$ times as great as when the shaft revolves in a plumber-block.

If we oppose a single friction-wheel $G H$, Fig. 271, to the pressure R of the axles and if we counteract the lateral forces, which in other respects can be neglected, by the fixed cheeks K and L, a becomes $= 0$, $\cos. \frac{a}{2} = 1$ and the above ratio $\psi = \frac{r_1}{a_1}$.

EXAMPLE.—A water-wheel weighs 80000 pounds, the radius of its circumference a is 16 feet and that of its gudgeon is $r = 5$ inches; how much force is required at the circumference of the wheel to overcome the friction or to maintain the wheel in uniform motion, when running empty, and how great is the corresponding expenditure of mechanical effect, when it makes 5 revolutions per minute? We can here assume a coefficient of friction $\phi = 0{,}075$, and consequently the friction is $\phi R = 0{,}075 \,.\, 30000 = 2250$ pounds. Since the radius of the wheel is $\frac{16 \,.\, 12}{5} = \frac{192}{5} = 38{,}4$ times as great as that of the gudgeon or the arm of the friction, the friction reduced to the circumference of the wheel is

$$= \frac{\phi R}{38{,}4} = \frac{2250}{38{,}4} = 58{,}59 \text{ pounds.}$$

The circumference of the gudgeon is $\frac{2 \,.\, 5 \,.\, \pi}{12} = 2{,}618$ feet; and consequently the space described by the friction in a second is

$$\frac{2{,}618 \,.\, 5}{60} = 0{,}2182 \text{ feet,}$$

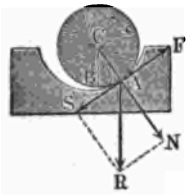
and the work done by the friction during one second is

$$L = 0{,}2182 \,.\, \phi R = 0{,}2182 \,.\, 2250 = 491 \text{ foot-pounds.}$$

If the gudgeon of this wheel is placed on friction wheels, whose radii are but 5 times as great as the radius of the gudgeon, that is, if $\frac{r_1}{a_1} = \frac{1}{5}$, the force necessary at the circumference of the wheel to overcome the friction would be only $\frac{1}{5} . 58{,}59 = 11{,}72$ pounds and the mechanical effect expended but $\frac{491}{5} = 98{,}2$ pounds. But in this case the support would be much less safe.

§ 183. Friction on a Partially Worn Bearing.—The friction of an axle $A\ C\ B$, Fig. 272, upon a bearing, which is partially worn, so that the shaft is supported in a single point A, is smaller than that of a new axle, which touches all points of its bearing.

FIG. 272.



If no rotation takes place, the axle presses upon the point B, through which the direction of the resulting pressure R passes; if the shaft begins to rotate in the direction $A\ B$, the axle rises in consequence of the friction on its bearing, until the force S tending to move it down balances the friction F. The resultant R is decomposed into a normal force N and a tangential one S, N is transmitted to the plumber block and produces the friction $F = \phi\ N$, which acts tangentially, S, however, puts itself in equilibrium with F, and we have, therefore, $S = \phi\ N$. According to the theorem of Pythagoras, we have $R^2 = N^2 + S^2$, whence

$$R^2 = (1 + \phi^2)\ N^2,$$

or inversely the normal pressure

$$N = \frac{R}{\sqrt{1 + \phi^2}} \text{ and the friction } F = \frac{\phi\ R}{\sqrt{1 + \phi^2}},$$

or introducing the angle of friction ρ or $\phi = tang.\ \rho$

$$F = \frac{R\ tang.\ \rho}{\sqrt{1 + tang.^2 \rho}} = R\ tang.\ \rho\ cos.\ \rho = R\ sin.\ \rho.$$

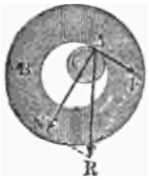
Consequently, when the shaft begins to turn, the point of pressure B moves in its bearing in the opposite direction through an angle $A\ C\ B =$ the angle of friction ρ.

The moment $F . \overline{CA} = F r$ of the friction on the axle is naturally equal to the moment $R\ r\ sin.\ \rho$ of the pressure R upon the bed, both being referred to the axis of revolution C. If there were no motion, we would have

$$F = \phi\ R = R\ tang.\ \rho = \frac{R\ sin.\ \rho}{cos.\ \rho};$$

the friction after the motion begins is $cos.\ \rho$ times as great as before. Generally $\phi = tang.\ \rho$ is scarcely $\frac{1}{10}$ and $cos.\ \rho > 0{,}995$, so that the difference is scarcely $\frac{5}{1000} = \frac{1}{200}$; we can, therefore, in ordinary cases neglect the effect of the motion.

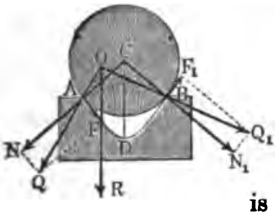
FIG. 273.



If the wheel $A\ B$ revolves with a nave, Fig. 273, about a fixed axle $A\ C$, the friction is the same as if the axle moved in an ordinary plumber-block, but when the nave is worn the arm of the friction is not the radius of the shaft, but that of the opening in the nave.

§ 184. Friction on a Triangular Bearing.—If we lay the axle in a prismatical bearing, we have more pressure on the bearing, and consequently more friction than, when the bearing is circular.

FIG. 274.



If the bearing $A\ D\ B$, Fig. 274, is triangular, the axle is supported at two points A and B and at both of them friction must be overcome. The resulting pressure R is decomposed into two components Q and Q_1, each of which is again decomposed into a normal stress N or N_1 and into a tangential one, which is equal to the friction $F = \phi\ N$ and $F_1 = \phi\ N_1$. According to the foregoing paragraph, we can put these frictions $= Q\ sin.\ \rho$ and $Q_1\ sin.\ \rho$, consequently the total friction is $F + F_1 = (Q + Q_1)\ sin.\ \rho$.

The forces Q and Q_1 are found, by the resolution of a parallelogram of forces formed of Q and Q_1, with the aid of the resultant R, of the angle of friction ρ and of the angle $A\ C\ B = 2\ a$, corresponding to the arc $A\ B$ included between the two points of contact; now we have

$$Q\ O\ R = A\ C D - C\ A\ O = a - \rho \text{ and}$$
$$Q_1\ O\ R = B\ C\ D + C\ B\ O = a + \rho \text{ and therefore}$$
$$Q\ O\ Q_1 = a - \rho + a + \rho = 2\ a.$$

By employing the formula of § 78, we obtain

$$Q_1 = \frac{sin.\ (a - \rho)}{sin.\ 2\ a} \ . \ R \text{ and } Q = \frac{sin.\ (a + \rho)}{sin.\ 2\ a} \ . \ R;$$

whence the required friction is

$$F + F_1 = (Q + Q_1)\ sin.\rho = (sin.\ [a - \rho] + sin.\ [a + \rho])\ \frac{R\ sin.\rho}{sin.\ 2\ a}.$$

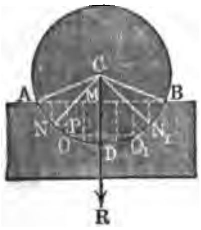
But from trigonometry we know, that $sin.\ (a - \rho) + sin.\ (a + \rho) = 2\ sin.\ a\ cos.\ \rho$, and that $sin.\ 2\ a = 2\ sin.\ a\ cos.\ a$, and we can therefore put

$$F + F_1 = \frac{2\ sin.\ a\ R\ sin.\ \rho\ cos.\ \rho}{2\ sin.\ a\ cos.\ a} = \frac{R\ sin. 2\rho}{2\ cos.\ a},$$

which, owing to the smallness of ρ, we can make $= \frac{R\, sin.\, \rho}{cos.\, a}$. When a triangular bearing is used, the friction becomes $\frac{1}{cos.\, a}$ times greater than when a circular one is employed. If, E.G., $A\ D\ B$ is 60°, $A\ C\ B$ is $180° - 60° = 120°$ and $A\ C\ D = a = 60°$, we have $\frac{1}{cos.\, 60^0}$ times = twice as much friction as for a circular bearing.

§ **185. Friction of a New Bearing.**—By the aid of the latter formula we can find the friction on a new circular bearing, when the axle is supported at all points. Let $A\ D\ B$, Fig. 275, be such a bearing. Let us divide the arc $A\ D\ B$ along which the bearing and axle are in contact into very many parts, such as $A\ N$, $N\ O$, etc., whose projections upon the chord $A\ B$ are equal, and let us suppose that each one of these parts transmits from the axle to the bearing equal portions $\frac{R}{n}$ of the whole pressure R. Here n denotes the number of these parts. According to the foregoing paragraph, the friction of two parts $N\ O$ and $N_1\ O_1$ opposite to each other is

FIG. 275.



$$= \frac{R}{n} \cdot \frac{sin.\, 2\, \rho}{cos.\ N\, C\, D}.$$

But $cos.\ N\, C\, D$ is also $= cos.\ O\, N\, P = \frac{N\, P}{N\, O}$, $N\, P$ representing the projection of the part $N\ O$ on $A\ B$, and therefore

$$N\, P = \frac{\text{chord } A\, B}{n}.$$

consequently the friction corresponding to these two parts $N\ O$ and $N_1\ O_1$ is

$$= \frac{R\, sin.\, 2\, \rho}{n} \cdot \frac{n . \overline{N\, O}}{\text{chord}} = \frac{R\, sin.\, 2\, \rho}{\text{chord}} \cdot \overline{N\, O}.$$

In order to find the friction for the entire arc $A\ D\ B$, we have only to substitute instead of $N\ O$ the arc $A\ D = \frac{1}{2} A\ D\ B$; for the sum of all the frictions is equal to $\frac{R\, sin.\, 2\, \rho}{\text{chord}}$. the sum of all the parts of the arc; consequently the friction on a new bearing is

$$F = R\, sin.\, 2\, \rho . \frac{\text{arc } A\ D}{\text{chord } A\ B},$$

or putting the angle at the centre $A\ C\ B$ corresponding to the arc contained in the bearing $= 2\, a^0$ and the chord $A\ B = 2\ A\ C\ sin.\ a$, we have

$$F = \frac{R\ sin.\ 2\rho}{2} \cdot \frac{a}{sin.\ a} \text{ or approximatively,}$$

assuming $2\,\rho = 2\ sin.\ \rho$,

$$F = R\ sin.\ \rho \cdot \frac{a}{sin.\ a}.$$

Hence the initial friction increases with the depth, that the axle is sunk in its bearing, E.G., if the bearing includes the semi-circumference of the axle, we have $a = \frac{1}{2}\,\pi$ and $sin.\ a = 1$, and therefore $F = \frac{\pi}{2} \cdot R\ sin.\ \rho$ is $\frac{\pi}{2} = 1{,}57$ times as great as it is when a bearing has been worn. If the axle does not lie deep in its bearing, or if a is small, we can put $sin.\ a = a - \frac{a^3}{6} = a\left(1 - \frac{a^2}{6}\right)$, whence it follows that $F = \left(1 + \frac{a^2}{6}\right) R\ sin.\ \rho$ or $= R\ sin.\ \rho$, when a is very small.

(§ **186.**) **Poncelet's Theorem.**—The pressure R on the bearings is generally given as the resultant of two forces P and Q, which act at right angles to each other, and it is consequently $= \sqrt{P^2 + Q^2}$. So far as we need it for the determination of the friction

$$F = \phi\,R = \phi\sqrt{P^2 + Q^2},$$

we can content ourselves with an approximate value of $\sqrt{P^2 + Q^2}$, partly because an exact value of the coefficient ϕ can never be given, as it depends upon so many accidental circumstances, partly, also, because the product $\phi\ R$ is generally but a small fraction of the other forces, which act on the machine, E.G., the lever, pulley, wheel and axle, etc., which is supported by the bearings. The formula for calculating the approximate value of $\sqrt{P^2 + Q^2}$ is known as Poncelet's theorem, and its truth can be demonstrated in the following manner. We have

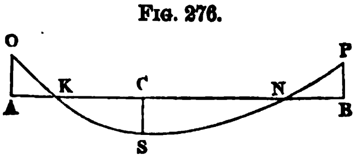
$$\sqrt{P^2 + Q^2} = P\sqrt{1 + \left(\frac{Q}{P}\right)^2} = P\sqrt{1 + x^2},$$

in which $x = \frac{Q}{P}$, and if Q is the smaller force, x is a simple fraction. Now let us put $\sqrt{1 + x^2} = \mu + \nu\,x$, and let us determine the coefficients μ and ν corresponding to certain conditions. The relative error is

$$y = \frac{\sqrt{1 + x^2} - \mu - \nu\,x}{\sqrt{1 + x^2}} = 1 - \frac{\mu + \nu\,x}{\sqrt{1 + x^2}}.$$

This equation corresponds to the curve $O\,S\,P$, Fig. 276, whose ordinate, when the abscissa $x = 0$, is $A\,O = y = 1 - \mu$, and, when the abscissa $A\,B = 1$, is $y = 1 - \frac{\mu + \nu}{\sqrt{2}}$. The curve also cuts the axis of abscissas in two points K and N and at S lies at its greatest distance $C\,S$ from this axis. If we put $y = 0$ or

FIG. 276.



$$\sqrt{1 + x^2} = \mu + \nu\,x,$$

and solve the equation in relation to x, we obtain

$$x = \frac{\mu\,\nu \mp \sqrt{\mu^2 + \nu^2 - 1}}{1 - \nu^2},$$

the values of which are the abscissas $A\,K$ and $A\,N$ of the points K and N, where the curve cuts the axis, and also those values for which the error is $= 0$. In order to find the abscissa $A\,C$ of the maximum negative error $C\,S$, we must put the differential ratio

$$\frac{d\,y}{d\,x} = \frac{(\mu + \nu\,x)\,(1 + x^2)^{-\frac{1}{2}}\,x - \nu\,(1 + x^2)^{\frac{1}{2}}}{1 + x^2} = 0$$

(see Article 13 of the Introduction to the Calculus).

This condition is fulfilled by putting

$$(\mu + \nu\,x)\,(1 + x^2)^{-\frac{1}{2}}x = \nu\,(1 + x^2)^{\frac{1}{2}} \text{ or}$$

$$(\mu + \nu\,x)\,x = \nu\,(1 + x^2), \text{ I.E. } x = \frac{\nu}{\mu}.$$

According to this formula, the abscissa $A\,C = \frac{\nu}{\mu}$ gives the greatest negative ordinate.

$$C\,S = 1 - \frac{\mu + \nu\,.\,\frac{\nu}{\mu}}{\sqrt{1 + \frac{\nu^2}{\mu^2}}} = -\left(\frac{\mu^2 + \nu^2}{\sqrt{\mu^2 + \nu^2}} - 1\right) = -\,(\sqrt{\mu^2 + \nu^2} - 1).$$

In order to have neither a great positive nor a great negative error, let us put the three ordinates $A\,O = 1 - \mu$, $B\,P = 1 - \frac{\mu + \nu}{\sqrt{2}}$ and $C\,S = \sqrt{\mu^2 + \nu^2} - 1$, equal to each other, and determine from them the coefficients μ and ν. We have

$$\mu = \frac{\mu + \nu}{\sqrt{2}}, \text{ I.E., } \nu = (\sqrt{2} - 1)\,\mu = 0{,}414\,\mu \text{ and}$$

$$2 - \mu = \sqrt{\mu^2 + \nu^2}, \text{ I.E., } 2 = \mu\,(1 + \sqrt{1 + 0{,}414^2})$$

and consequently

$$\mu = \frac{2}{1 + \sqrt{1{,}1714}} = 0{,}96 \text{ and } \nu = 0{,}414 \,.\, 0{,}96 = 0{,}40.$$

We can, therefore, put $\sqrt{1 + x^2} = 0{,}96 + 0{,}40 \,.\, x$, and in like manner the resultant

$$R = 0{,}96\, P + 0{,}40\, Q,$$

and we know that in this case the greatest error we can make is $\pm\, y = 1 - \mu = 1 - 0{,}96 = 0{,}04 =$ four per cent. of the true value.

This formula supposes, that we know, which of the two forces is the greater; if this is unknown to us, we assume

$$\sqrt{1 + x^2} = \mu\,(1 + x)$$

and obtain in that way

$$y = 1 - \frac{\mu\,(1 + x)}{\sqrt{1 + x^2}}.$$

In this case not only $x = 0$, but also $x = \infty$ gives an error $1 - \mu$. If we put $x = \frac{\nu}{\mu} = 1$, we have the greatest negative error

$$= -\left(\frac{2\,\mu}{\sqrt{2}} - 1\right) = -\,(\mu\,\sqrt{2} - 1).$$

Putting these errors equal to each other, we obtain

$$1 - \mu = \mu\,\sqrt{2} - 1, \text{ or } \mu = \frac{2}{1 + \sqrt{2}} = \frac{2}{2{,}414} = \frac{1}{1{,}207} = 0{,}828.$$

In case we do not know, which of the forces is the greatest, we can write

$$R = 0{,}83\,(P + Q),$$

then the greatest error we can make is $\pm\, y = 1 - 0{,}83 = 17$ per cent. $= \frac{1}{6}$ of the true value.

If, finally, we know that x is not over 0,2, we do best to neglect x altogether and to put $\sqrt{P^2 + Q^2} = P$; if, however, x is over 0,2, it is better to make

$$\sqrt{P^2 + Q^2} = 0{,}888\, P + 0{,}490 \,.\, Q.$$

In both cases the maximum error is about 2 per cent.*

§ **187. The Lever.**—The theory of friction just given is applicable to the material lever, to the wheel and axle and to other machines. Let us now take up the subject of the lever, discussing at once the most general case, that of the bent lever $A\ C\ B$,

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* Polytechnische Mittheilungen, Vol. I.

FIG. 277.

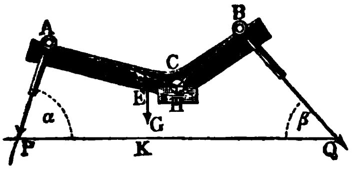


Fig. 277. Let us denote, as formerly (§ 136), the arm of the lever CA of the power P by a, the lever arm CB of the load Q by b and the radius of axle by r, and let us put the weight of the lever $= G$, the arm CE of the same $= s$ and the angles APK and BQK formed by the directions of the forces with the horizon $= \alpha$ and β. The power P produces the vertical pressure $P \sin. \alpha$ and the load Q the vertical pressure $Q \sin. \beta$, and the total vertical pressure is $V = G + P \sin. \alpha + Q \sin. \beta$. The force P produces also the horizontal pressure $P \cos. \alpha$ and the load an opposite pressure $Q \cos. \beta$, and the resulting horizontal pressure is $H = P \cos. \alpha - Q \cos. \beta$, and the total pressure on the axle is

$$R = \mu V + \nu H = \mu (G + P \sin. \alpha + Q \sin. \beta) + \nu (P \cos. \alpha - Q \cos. \beta)$$

in which, however, the second part $\nu (P \cos. \alpha - Q \cos. \beta)$ is never to be taken as negative, and, therefore, when $Q \cos. \beta$ is $> P \cos. \alpha$ the sign must be changed, or rather $P \cos. \alpha$ must be subtracted from $Q \cos. \beta$. In order to find the value of the force corresponding to a state of unstable equilibrium so that for the smallest addition of force motion will take place, we put the statical moment of the power equal to the statical moment of the load plus or minus the moment of the weight of the machine (§ 136) and plus the moment of the friction; thus we have

$$P a = Q b \pm G s + \phi R r$$
$$= Q b \pm G s + \phi (\mu V + \nu H) r, \text{ whence}$$
$$P = \frac{Q b \pm G s + \phi [\mu (G + Q \sin. \beta) \mp \nu Q \cos. \beta] r}{a - \mu \phi r \sin. \alpha \mp \nu \phi r \cos. \alpha}.$$

If P and Q act vertically, we have simply $R = P + Q + G$ and therefore $P a = Q b \pm G s + \phi (P + Q + G) r$. If the lever is one armed, P and Q act in opposite directions to each other and R is $= P - Q + G$ and therefore the friction is less. But R must always enter into the calculation with a positive sign, for the friction ϕR only resists motion and never produces it. We see from this, that a single armed lever is mechanically more perfect than a double armed one.

EXAMPLE.—If the arms of the bent lever represented in Fig. 277 are $a = 6$ feet, $b = 4$ feet, $s = \frac{1}{2}$ foot and $r = 1\frac{1}{2}$ inches, if the angles of inclination are $\alpha = 70°$, $\beta = 50°$, and if the load is $Q = 5600$ pounds and the weight of the lever G is $= 900$ pounds, the force necessary to produce unstable equilibrium is determined as follows. The friction being disregarded, we have $P a + G s = Q b$ and therefore

$$P = \frac{Q b - G s}{a} = \frac{5600 \,.\, 4 - 900 \,.\, \frac{1}{2}}{6} = 3658 \text{ pounds.}$$

If we put $\mu = 0{,}96$ and $\nu = 0{,}40$, we obtain

$\mu\,(G + Q \sin. \beta) = 0{,}96\,(900 + 5600 \sin. 50^{\circ}) = 4982$ pounds,

$\nu\, Q \cos. \beta = 0{,}40 \,.\, 5600 \cos. 50^{\circ} = 1440$ pounds,

$\mu \sin. \alpha = 0{,}96 \,.\, \sin. 70^{\circ} = 0{,}902$ and

$\nu \cos. \alpha = 0{,}40 \,.\, \cos. 70^{\circ} = 0{,}137.$

It is easy to see, that $P \cos. \alpha$ is here smaller than $Q \cos. \beta$; for since P is approximatively 3658 pounds, we have $P \cos. \alpha = 1251$ pounds, while, on the contrary, $Q \cos. \beta$ is $= 3600$ pounds; therefore we must employ in this case for $\nu\, Q \cos. \beta$ and for $\nu\, \phi\, r \cos. \alpha$ the lower sign and put

$$P = \frac{5600 \,.\, 4 - 900 \,.\, \frac{1}{2} + \phi\, r\,(4982 + 1440)}{6 - \phi\, r\,(0{,}902 - 0{,}137)}.$$

Assuming the coefficient of friction $\phi = 0{,}075$, we obtain

$\phi\, r = 0{,}075 \,.\, \frac{1}{8} = 0{,}009375$ and $6422\, \phi\, r = 60$

and the force required

$$P = \frac{22400 - 450 + 60}{6 - 0{,}0717} = \frac{22010}{5{,}9928} = 3673 \text{ pounds.}$$

Here the vertical pressure, when we substitute the force $P = 3658$ pounds determined without reference to the friction, is

$V = 3658 \sin. 70^{\circ} + 5600 \sin. 50^{\circ} + 900 = 3437 + 4290 + 900$
$= 8627$ pounds.

and, on the contrary, the horizontal pressure is

$H = 5600 \cos. 50 - 3658 \cos. 70 = 3600 - 1251 = 2349$ pounds.

Here H is $> 0{,}2\, V$, and therefore we have more correctly

$R = 0{,}888 \,.\, H + 0{,}490\, V = 0{,}888 \,.\, 8627 + 0{,}490 \,.\, 2349 = 8811$, and consequently the moment of the friction is

$= \phi\, r\, R = 0{,}009375 \,.\, 8811 = 82{,}6$ foot-pounds;

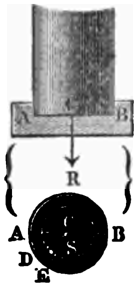
and finally the force

$$P = \frac{22400 - 450 + 82{,}6}{6} = 3672 \text{ pounds,}$$

which value differs very little, it is true, from the one obtained above.

§ 188. Friction of a Pivot.—If in a wheel and axle there is a pressure in the direction of the axis, which is always the case, when the axle is vertical, in consequence of the weight of the machine, friction is produced upon the base of one of the journals. Since there is pressure at all points of the base between the pivot and the step (or footstep), this friction approaches nearer to the ordinary friction of sliding, than to what we have previously considered as axle friction, and we must therefore employ in this case the coefficients of friction given in Table II. (page 320). In order

FIG. 278.



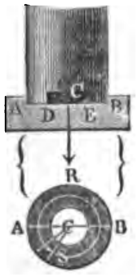
to find the work done by this friction, we must know the mean space described by the base *A B*, Fig. 278, of such a pivot. We assume that the pressure *R* is equally distributed over the whole surface, that is, we suppose that the friction upon equal portions of the base is equally great. If we divide the base by means of the radii *C D*, *C E*, etc., in equal sections or triangles, such as *D C E*, these correspond not only to equal frictions, but also to equal moments, and we need therefore only find the moment of the friction of one of these triangles. The frictions on such a triangle can be considered as parallel forces, since they all act tangentially, I.E., at right angles to the radius *C D*; and since the centre of gravity of a body or of a surface is nothing else than the point of application of the resultant of the parallel forces, which are equally distributed over the body or surface, we can consider the centre of gravity *S* of this sector or triangle *D C E* as the point of application of the resultant of all the frictions upon it. If the pressure on this sector is $= \frac{R}{n}$ and radius $CD = CE = r$, it follows (according to § 113), that the statical moment of the friction of this sector is

$$= \overline{CS} \cdot \frac{\phi R}{n} = \frac{2}{3} r \cdot \frac{\phi R}{n},$$

and finally that the statical moment of entire friction of the pivot is

$$M = n \cdot \frac{2}{3} r \frac{\phi R}{n} = \frac{2}{3} \phi R r.$$

FIG. 279.



Sometimes the rubbing surface is a ring *A B E D*, Fig. 279, If the radii of the same are $CA = r_1$ and $CD = r_2$, we have here to determine the centre of gravity *S* of a portion of a ring. Hence, according to § 114, the arm is

$$CS = \frac{2}{3} \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2},$$

and therefore the moment of the friction is

$$M = \frac{2}{3} \phi R \left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right).$$

If we introduce the mean radius $\frac{r_1 + r_2}{2} = r$ and the breadth of the ring $r_1 - r_2 = b$, we obtain also for the moment of the friction

$$M = \phi R \left(r + \frac{b^2}{12 r}\right).$$

The mechanical effect of the friction is, in the first case,

$A = 2\pi \,.\, \frac{2}{3} \phi R r = \frac{4}{3} \pi \phi R r$, and, in the second case,

$$A = \frac{4}{3} \pi \phi R \left(\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\right) = 2 \pi \phi R \left(r + \frac{b^2}{12 r}\right).$$

From the above data it is easy to calculate the friction upon a journal composed of one or more collars, when a vertical shaft is borne by it. It is also easy to see, that, in order to diminish the loss of mechanical effect, the pivots should be made as small as possible, and that, when the other circumstances are the same, the friction is greater on a ring than on a full circle.

Example.—A turbine, weighing 1800 pounds, makes 100 revolutions per minute, and the diameter of the base of the pivot is 1 inch; how much mechanical effect is consumed in a second by the friction of this pivot? Assuming the coefficient of friction $\phi = 0{,}100$, we obtain

$$\phi R = 0{,}100 \,.\, 1800 = 180 \text{ pounds,}$$

the space described in a revolution is

$$= \frac{4}{3} \pi r = \frac{4}{3} \,.\, 3{,}14 \,.\, \frac{1}{24} = 0{,}1745 \text{ feet,}$$

and therefore the work done in one revolution is

$$= 180 \,.\, 0{,}1745 = 31{,}41 \text{ foot-pounds.}$$

But this machine makes in a second $\frac{100}{60} = \frac{5}{3}$ revolutions, and therefore the required loss of mechanical effect is

$$= \frac{314{,}1}{6} = 52{,}3 \text{ foot-pounds.}$$

§ 189. Friction on Conical Pivots.

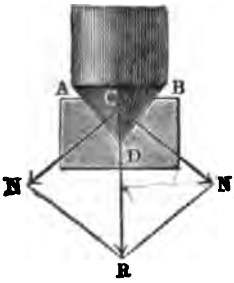
—If the end of the axle $A\,B\,D$, Fig. 280, is conical, the friction is greater than when the pivot is flat, for the axial pressure R is decomposed into the normal forces N, N_1, etc., which produce friction and whose sum is greater than R alone. If half the angle of convergence $A\,D\,C = B\,D\,C = a$, we have

$$2 N = \frac{R}{sin.\, a},$$

and therefore the friction of this conical pivot is

$$F = \phi \frac{R}{sin.\, a}.$$

Fig. 280.



If we denote the radius $C\,A = C\,B$ of the axle at the place of entrance in the step by r_1, we have, in accordance with what precedes, the statical moment,

$$M = \frac{\phi R}{sin.\ a} \cdot \tfrac{2}{3}\, r_1 = \tfrac{2}{3}\, \phi\, \frac{R\, r_1}{sin.\ a};$$

or, since $\frac{r_1}{sin.\ a} = \frac{C\,A}{sin.\ a}$ = the side $D\,A$ of the cone $= a$, we have

$$M = \tfrac{2}{3}\, \phi\, R\, a.$$

If we allow the axle to penetrate a very short distance into the step, the friction is less than for a flat pivot, and for this reason we can employ conical pivots with advantage. If, E.G.,

$$a = \frac{r_1}{sin.\ a} = \frac{r}{2},\ \text{or}\ r_1 = \tfrac{1}{2}\, r\ sin.\ a,$$

the conical pivot, whose radius is r_1, occasions only half as much loss of mechanical effect as the flat pivot, whose radius is r.

If the pivot forms a truncated cone, Fig. 281, friction is produced on the conical surface and on the flat base, and we have for the statical moment of the friction

$$M = \left(r_1^3 + \frac{r^3 - r_1^3}{sin.\ a}\right) \cdot \tfrac{2}{3}\, \frac{\phi\ R}{r^2},$$

when r denotes the radius $C\,A$ at the point, where the pivot enters the step, r_1 the radius of the base and a° half the angle of convergence. In consequence of the great lateral pressure N the step becomes soon so worn that finally only the pressure on the base $E\,F$ remains and the moment of the friction becomes $M = \tfrac{2}{3}\, \phi\, R\, r_1$.

FIG. 281.

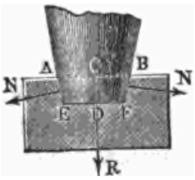


FIG. 282.

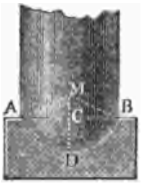
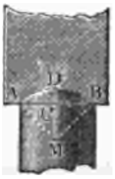


FIG. 283.



Vertical shafts or pivots are very often rounded off as in Figs. 282 and 283. Although by this rounding the friction is not in any way diminished, yet a diminution of the moment of the friction can be produced by diminishing the penetration of the pivot into the step. If we suppose the rounded surface to be spherical, we obtain with the aid of the calculus, for a hemispherical step the moment of friction

$$M = \frac{\phi\,\pi}{2} \cdot R\, r;$$

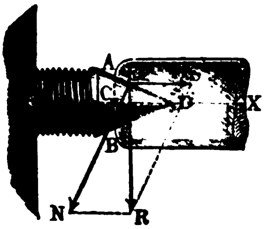
and for a step forming a low segment approximatively

$$M = \tfrac{2}{3}\left[1 + 0{,}3\left(\frac{r_1}{r}\right)^2\right]\phi\, R\, r_1,$$

in which formula r denotes the radius of the sphere $M A = M B$ and r_1 the radius of the step $C A = C B$.

REMARK.—The pressure R upon the centre $A D B$, Fig. 284, of the spindle of a turning-lathe is perpendicular to the direction of the axis $D X$ and is decomposed into a normal pressure N and a lateral pressure S parallel to the axis. Retaining the same notation, that we employed above for conical pivots, we have

FIG. 284.



$$N = \frac{R}{cos.\ a} \text{ and } S = R\ tang\ a.$$

The moment of the friction caused by N is

$$M = \phi\, N\,.\, \tfrac{2}{3}\, r_1 = \tfrac{2}{3}\, \phi\, \frac{R\, r_1}{cos.\ a},$$

or since $r_1 = C A = D A\ sin.\ A D C = a\ sin.\ a$, when a denotes the length $C D$ of the portion of the centre which is buried, we have $M = \tfrac{2}{3}\, \phi\, R\, a\ tang.\ a.$

The lateral force S is entirely or partly counteracted by an opposite force S_1 on the other centre.

EXAMPLE.—If the weight of the shaft and other parts of a whim gin is $R = 6000$ pounds, the radius of its conical pivot is $= r = 1$ inch and the angle of convergence $2\ a$ of the latter is $= 90°$, the statical moment of the friction is

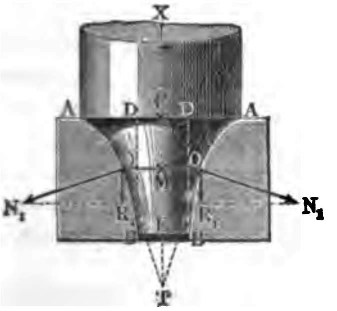
$$M = \tfrac{2}{3}\,.\,\varphi\,.\,\frac{R\,r}{sin.\ a} = \tfrac{2}{3}\,.\,0{,}1\,.\,\frac{6000}{sin.\ 45^\circ}\,.\,\frac{1}{12} = \frac{100}{3\sqrt{\tfrac{1}{2}}} = 47{,}1 \text{ foot-pounds.}$$

If the shaft in hoisting a bucket out of a mine makes $u = 24$ revolutions, the mechanical effect consumed by the friction of the pivot during this time is

$$A = 2\,\pi\,u\,.\,\tfrac{2}{3}\,\phi\,\frac{R\,r}{sin.\ a} = 2\,\pi\,.\,24\,.\,47{,}1 = 7108 \text{ foot-pounds.}$$

§ 190. The so-called Anti-friction Pivots.—Supposing that the axial pressure on a pivot $A\ B\ B\ A$, Fig. 285, is proportional to the surface of the cross-section, we can put the vertical pressure per square inch $R_1 = \frac{R}{G}$, R being the total pressure and G the area of the vertical projection $A\ D\ D\ A$ of the whole rubbing surface $A\ B\ B\ A$. If now a is the angle of inclination $C\ T\ O$ of the element O of the surface to the axis $C\ T$ of the pivot, the normal pressure on each square inch

FIG. 285.



of the bearing, will be $N_1 = \frac{R_1}{sin.\ a}$ and the corresponding friction will be

$$F_1 = \phi\, N_1 = \phi\, \frac{R_1}{sin.\ a} = \frac{\phi\, R}{G\ sin.\ a},$$

and if y denotes the distance or radius of friction $M\,O$, the moment of this friction is

$$F_1\, y = \phi\, \frac{R}{G} \cdot \frac{y}{sin.\ a},$$

or, since $\frac{y}{sin.\ a}$ = tangent $O\ T$,

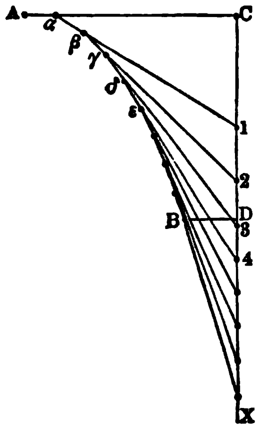
$$F_1\, y = \phi\, \frac{R}{G} \cdot \overline{O\,T}.$$

In order to obtain a regular wearing away of the axle and of its step, the moment $F_1\, y$ must be the same for all positions, and consequently the tangent $O\ T$ must have the same value for all points of the generating curve $A\ O\ B$ of the axle, and therefore the moment of the friction on the whole pivot is when $O\ T = a$

$$M = F_1\, y \,.\, G = \phi\, R\, a.$$

The curve $A\ O\ B$, whose tangent $O\ T$, measured from the point of tangency to the axis $C\ X$, is constant, is a *tractrix or tractory*, and is generated by drawing a heavy point A, Fig. 286, over a horizontal plane by means of a string, whose end moves along a straight line $C\,X$. This string forms the constant tangent lines $A\ C = \alpha\, 1 = \beta\, 2 = \gamma\, 3$, etc. $= a$. In order to construct this curve, we draw $C\,A = a$ perpendicular to the axis $C\ X$ and take in $C\,A$, α near to A, and lay off $\alpha\, 1 = a$, take β in $\alpha\ 1$, near to α and lay off $\beta\, 2 = a$, here again take γ near to β and lay off $\gamma\, 3 = a$, etc., and we then draw a curve tangent to the sides $A\,\alpha$, $\alpha\,\beta$, $\beta\,\gamma$, $\gamma\,\delta \ldots$, etc. This method gives the tractory the more accurately the smaller the sides $A\ \alpha$, $\alpha\ \beta$, $\beta\ \gamma$, $\gamma\ \delta \ldots$, etc., are. Schiele calls this curve the anti-friction curve. (See the Practical Mechanics' Journal, June number, 1849, translated in the Polytechnisches Centralblatt, Jahrgang, 1849.)

FIG. 286.



If, as is represented in Fig. 285, we make the anti-friction curve end at the circumference of the shaft the maximum radius of friction $C A = r$ is at the same time the constant tangent a, and therefore the moment of the friction $M = \phi R r$ is independent of the length of the pivot. When the rubbing surface is flat and of the same radius, the moment of friction is $M_1 = \frac{2}{3} \phi R r$, that is, one third smaller, and it decreases still more in time; for the exterior portions are more worn than the interior ones, and thus the surface of friction becomes less.

The *plugs and chambers of cocks* are sometimes made in the form of the anti-friction curve; for in this case the conditions are the same as in a pivot.

REMARK.—When the pressure R on the pivot is so distributed that the amount of the wearing, measured in the direction of the pressure, is equal in all points of the circumference of the pivot, we have

$$\frac{N_1 y_1}{sin. a_1} = \frac{N_2 y_2}{sin. a_2} = \frac{N_3 y_3}{sin. a_3} \dots$$

and for conical pivots, where

$$a_1 = a_2 = a_3 \dots = a;\ N_1 y_1 = N_2 y_2 = N_3 y_3 \dots$$

If $O_1, O_2, O_3 \dots$ denote the surfaces, upon which the normal pressures $N_1, N_2, N_3 \dots$ act, we have

$$R = N_1 O_1 \, sin. a_1 + N_2 O_2 \, sin. a_2 + N_3 O_3 \, sin. a_3 + \dots$$

or for conical pivots $R = (N_1 O_1 + N_2 O_2 + N_3 O_3 + \dots) \, sin. a$.

The portions of the surface can be considered as rings of the same height $\frac{h}{n}$, whose widths are $\frac{h}{n \, sin. a}$, and whose radii are y_1, y_2, y_3, consequently we have

$$O_1 = 2 \pi y_1 \frac{h}{n \, sin. a},\ O_2 = 2 \pi y_2 \frac{h}{n \, sin. a},\ O_3 = 2 \pi y_3 \frac{h}{n \, sin. a}, \text{ etc.}$$

$$O_2 = \frac{y_2}{y_1} O_1,\ O_3 = \frac{y_3}{y_1} O_1, \text{ etc., and also}$$

$$N_1 O_1 = N_2 O_2 = N_3 O_3 \dots, \text{ and } R = n \,.\, N_1 O_1 \, sin. a.$$

Therefore, under the above assumption, the normal pressure on the equally high rings of the circumference of the pivot are equal.

Inversely we have $N_1 O_1 = \frac{R}{n \, sin. a}$, hence the moment of the friction on the pivot is

$$M = \phi (N_1 O_1 y_1 + N_2 O_2 y_2 + N_3 O_3 y_3 + \dots)$$
$$= \phi N_1 O_1 (y_1 + y_2 + \dots + y_n) = \frac{\phi R}{n \, sin. a} (y_1 + y_2 + \dots + y_n).$$

If we have a truncated conical pivot, whose radii are r_1 and r_2, we must put $y_1 + y_2 + \dots + y_n = \frac{n (r_1 + r_2)}{2}$, from which it follows that $M = \frac{\phi R (r_1 + r_2)}{2 \, sin. a}$

For a complete conical pivot, whose radius is $r_2 = 0$, we have $M =$

$\frac{\phi R r_1}{2 \sin. a}$, while in a foregoing paragraph (§ 189) we found $M = \frac{2}{3} \phi \frac{R r_1}{\sin. a}$.

See the article by Mr. Reye upon the Theory of Friction of Axles in Vol. 6 of the Civilingenieur, as well as the article upon the same subject by Director Grashof in the 5th volume of the Journal of the Association of German Ingenieurs.

§ 191. Friction on Points and Knife-Edges.—In order to diminish as much as possible the friction of the axles of rotating bodies, they are often supported on sharp points, knife-edges, etc. If the bodies employed were perfectly solid and inelastic, no loss of mechanical effect in consequence of the friction would take place by this method, since the space described by the friction is immeasurably small; but since every body possesses a certain degree of elasticity, upon placing it upon the point or knife-edge, a slight penetration takes place and a surface of friction is produced, upon which the friction describes a certain space, which, although small, occasions a loss of mechanical effect. When the rotation or vibration of a body supported in this way has continued some time, such surfaces of friction are arcs developed by the wearing away of the point or knife-edge, and the friction is then to be treated as we have previously done. This mode of support is therefore only employed in instruments such as compasses, balances, etc., where it is important to diminish the friction and where the motion is not constant.

Coulomb made experiments upon the friction of a body supported by a hard steel point and movable around it. According to these experiments, the friction increases somewhat faster than the pressure, and changes with the degree of sharpness of the supporting point. It is a minimum for a surface of garnet, greater for a surface of agate, greater for a surface of rock crystal, still greater for a surface of glass, and the greatest for a steel surface. For very small pressures, as, E.G., in the magnetic needle, the point can be sharpened to an angle of convergence of 10° to 20°. If, however, the pressure is great, we must employ a much larger angle of convergence (30° to 45°). The friction is less, when a body lies with a plane surface upon a point than when the point plays in a conical or spherical hollow. The circumstances are the same for a knife-edge such as that of a balance. Balances, which are to be heavily loaded, have knife-edges with an angle of convergence of 90°. When the balance is light, an angle of 30° is sufficient.

If we assume that the needle *A B*, Fig. 287, has pressed down the point *F C G* an amount *D C E*, the height of which $C M = h$, and the radius of which $D M = r$, and if we suppose the volume

$\frac{1}{3}\,\pi\, r^2\, h$ to be proportional to the pressure R, the measure of the friction can be found in the following manner. If we put $\frac{1}{3}\,\pi\, r^2\, h = \mu\, R$, in which μ is a coefficient given by experiment, and substitute the angle of convergence $D\ C\ E = 2\,a$ or $h = r\ cotg.\ a$, we obtain for the radius of the base

$$r = \sqrt[3]{\frac{3\,\mu\, R\, tang.\, a}{\pi}}, \text{ and}$$

$$\phi\, R\, r = \phi \sqrt[3]{\frac{3\,\mu\, R^4\, tang.\, a}{\pi}} = \phi \sqrt[3]{\frac{3\,\mu}{\pi}} \cdot \sqrt[3]{R^4\, tang.\, a}.$$

From this we see that we can assume, that the friction on a pivot increases with the cube root of the fourth power of the pressure and with the cube root of the tangent of half angle of convergence.

FIG. 287.

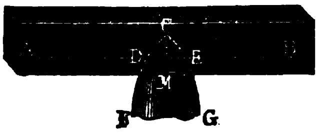
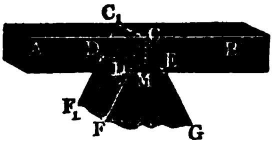


FIG. 288.

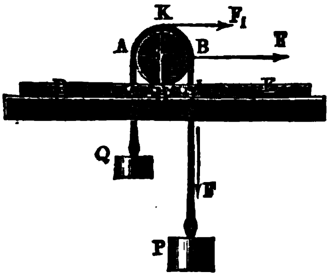


The amount of friction of a beam $A\ B$, Fig. 288, oscillating on a knife-edge $C\ C_1$, can be found in like manner. If a is the half angle of convergence $D\ C\ M$, l the length $C\ C_1$ of the edge and R the pressure, we have

$$\phi\, R\, r = \mu \sqrt{\frac{(R\, tang.\, a)^3}{l}}.$$

§ **192. Friction of Rolling.**—The theory of rolling friction is as yet by no means established upon a firm basis. We know, that the friction increases with the pressure, and that it is greater, when the radius of the roller is small than when it is large; but we cannot yet give the exact algebraical relation of the friction to the pressure and to the radius of the rolling body. Coulomb made a few experiments with rollers of lignum-vitæ and elm from 2 to 10 inches thick, which were rolled upon supports of oak by winding a thin string around the roller and attaching to the ends of it the unequal weights P and Q, Fig. 289. According to the results of these experiments, the rolling friction is directly proportional to the pressure and inversely to the radius of the

FIG. 289.



rollers, so that the force necessary to overcome the rolling friction can be expressed by the formula $F = f \cdot \frac{R}{r}$, R denoting the pressure, r the radius of the roller and f the coefficient of friction to be determined by experiment. If r is given in English inches, we have, according to these experiments, *

For rollers of lignum-vitæ, $f = 0{,}0189$ 0.0184
For rollers of elm, $f = 0{,}0320$. 0.0311

The author found for cast-iron wheels 20 inches in diameter, rolling on cast-iron rails,

$f = 0{,}0183$, and Sectionsrath Rittinger 0.0178
$f = 0{,}0193$. .0187

According to Pambour, we have for iron railroad wheels about 39,4 inches in diameter

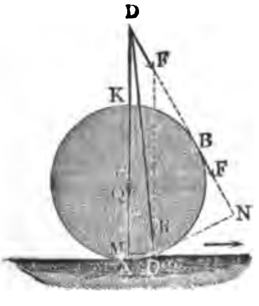
$f = 0{,}0196$ to $0{,}0216$. 0.019 to 0.021

The formula $F = f\frac{R}{r}$ supposes that the force F, which overcomes the friction, acts with a lever-arm $H\,C = H\,L = r$ equal to the radius of the roller, and that it describes the same space as the latter. If, however, it acts on a lever arm $H\,K = 2\,r$, the space described by it is double that described by the roller on the support, and the friction is therefore

$$F_1 = \tfrac{1}{2}\,F = f\frac{R}{2\,r}.$$

The conditions of equilibrium of rolling friction can be found in the following manner. In consequence of the pressure Q of the roller $A\,C\,B$ upon the base $A\,O$, Fig. 290, the latter is compressed; the roller rests, therefore, not upon its lowest point A, but upon the point O which lies a little in front of it. Transferring the points of application A and B of the forces Q and F, of which the latter F is the force necessary to overcome the friction, to their point of intersection D, and constructing with Q and F the parallelogram of forces, we obtain in its diagonal $\overline{D\,R}$ the force R, with which the roller presses upon its support in O, and it is therefore necessary that the moments of the forces of the bent lever $A\ O\ N$ shall be equal to each other. If we put the distance $O\,N$ of the point of support O from the direction of the force $= a$, and the distance $O\,M$ of the same point from the vertical line of grav-

FIG. 290.



* Prussian inches

ity of the body $= f$, we have

$$Fa = Qf,$$

from which we obtain the required equation

$$F = \frac{f}{a} Q.$$

The arm f is a quantity to be determined by experiment and is so small, that we can substitute instead of a the distance of the lowest point A from the direction of the force F, as well as instead of Q the total pressure R.

Hence we have $F = \frac{f}{a} R$, and consequently, when the force acts horizontally and through the centre C, $a = r$ or

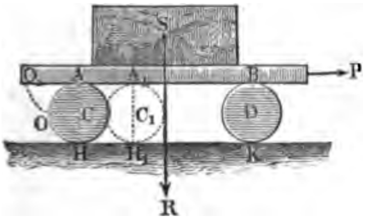
$$F = \frac{f}{r} R,$$

and on the contrary, when this force acts tangentially at the highest point K of the roller, $$F = \frac{f}{2r} R.$$

The so-called coefficient of friction f of rolling friction is therefore no nameless quantity, but a line, and must therefore be expressed in the same unit of measure as a.

If a body $A\ S\ B$ is placed upon two rollers C and D, Fig. 291, and moved forward, the force P required to move the body is very small, as we have only two rolling frictions to overcome, viz., one between $A\ B$ and the rollers and the other between the rollers and the surface $H\ K$. The space described progressively by the rollers is but one-half that described by the load R, so that new rollers must be continually pushed under it in front, for the points of contact A and B between the rollers and the body $A\ B$ move exactly as much backward, in consequence of the rolling, as the axes of the rollers move forward. If the roller $A\ H$ has turned an arc $A\ O$, it has also moved forward the space $A\ A_1$ equal to this arc, O has come in contact with O_1, and the new point of contact O_1 has gone backward behind the former one (A) a distance $A\ O_1 = A\ O$. If we designate the coefficients of friction on $H\ K$ and $A\ B$ by f and f_1, we have for the force necessary to move the body forward

FIG. 291.

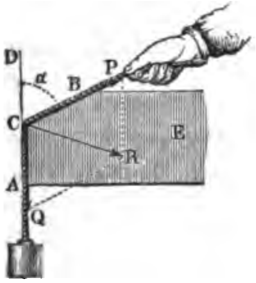


$$P = (f + f_1) \frac{R}{2r}.$$

REMARK.—The extensive experiments of Morin upon the resistance of wagons on roads confirm this law, according to which this resistance increases directly as the pressure and inversely as the thickness of the rollers. Another French engineer, Dupuit, on the contrary, infers from his experiments, that rolling friction increases directly as the pressure and inversely as the square root of the radius of the rollers. The newer experiments of Poirée and Sauvage by means of railroad wagons, also lead to the conclusion, that rolling friction increases inversely as the square root of the radius of the wheel. See Comptes rendues de la société des ingenieurs civils à Paris, 5 et 6 année. Particular theoretical views upon the subject of rolling friction are to be found in Von Gerstner's Mechanics, Vol. I, § 537, and in Brix's treatise on friction, Art. 6. This subject will be treated with more detail in the Third Part, under the head of transportation on roads and railroads.

§ 193. Friction of Cords.—We have now to study the friction of flexible bodies. If a perfectly flexible cord stretched by a force Q is laid over the edge C of a rigid body $A\,B\,E$, Fig. 292, and is thus compelled to deviate from its original direction an angle $D\,C\,B = \alpha°$, a pressure R is produced at this edge, which gives rise to a friction F, in consequence of which a force P, which is either greater or less than Q, is necessary to produce unstable equilibrium. The pressure is (§ 77)

FIG. 292.



$R = \sqrt{P^2 + Q^2 - 2\,P\,Q\,cos.\,\alpha}$, and consequently the friction
$F = \phi\sqrt{P^2 + Q^2 - 2\,P\,Q\,cos.\,\alpha}$.

If now we substitute $P = F + Q$ and P^2 approximatively $= Q^2 + 2\,Q\,F$, we obtain

$$F = \phi\sqrt{Q^2 + 2\,Q\,F + Q^2 - 2\,Q^2\,cos.\,\alpha - 2\,F\,Q\,cos.\,\alpha}$$

$$= \phi\,\sqrt{2\,(1 - cos.\,\alpha)\,(Q^2 + Q\,F)} = 2\,\phi\,sin.\,\frac{\alpha}{2}\sqrt{Q^2 + Q\,F},$$

for which we can write $2\,\phi\,sin.\,\frac{\alpha}{2}\,(Q + \frac{1}{2}\,F)$, when we take into account only the first two members of the square root. Hence we have

$$F = \phi\,F\,sin.\,\frac{\alpha}{2} + 2\,\phi\,Q\,sin.\,\frac{\alpha}{2},$$

and consequently the friction required is

$$F = \frac{2\,\phi\,Q\,sin.\,\frac{a}{2}}{1 - \phi\,sin.\,\frac{a}{2}},$$

for which we can generally write accurately enough

$$F = 2\,\phi\,Q\,sin.\frac{a}{2}\left(1 + \phi\,sin.\,\frac{a}{2}\right),$$ and very often

$$F = 2\,\phi\,Q\,sin.\,\frac{a}{2},$$

when the angle of deviation a is very small. Hence, in order to draw the rope over the edge C, we need a force

$$P = Q + F = \left(1 + \frac{2\,\phi\,sin.\,\frac{a}{2}}{1 - \phi\,sin.\,\frac{a}{2}}\right)Q,$$

and, on the contrary, the force necessary to prevent the weight Q from sinking is

$$P_1 = Q : \left(1 + \frac{2\,\phi\,sin.\,\frac{a}{2}}{1 - \phi\,sin.\,\frac{a}{2}}\right);$$

we can put approximatively

$$P = \left[1 + 2\,\phi\,sin.\,\frac{a}{2}\left(1 + \phi\,sin.\frac{a}{2}\right)\right]Q,$$ or more simply

$$P = \left(1 + 2\,\phi\,sin.\frac{a}{2}\right)Q$$ and

$$P_1 = \frac{Q}{1 + 2\,\phi\,sin.\,\frac{a}{2}\left(1 + \phi\,sin.\,\frac{a}{2}\right)},$$ or

$$P_1 = \frac{Q}{1 + 2\,\phi\,sin.\frac{a}{2}} = \left(1 - 2\,\phi\,sin.\frac{a}{2}\right)Q.$$

If the cord passes over several edges, the forces P and P_1 at the other end of the cord can be calculated by repeated application of these formulas. Let us consider the simple case, where the cord $A\ B\ C$, Fig. 293, is laid upon a body with n edges, and where the deviation at each edge is the same and equal to a. The tension of the first portion of the cord is

$$Q_1 = \left(1 + 2\,\phi\,sin.\frac{a}{2}\right)Q,$$

when that at the end is $= Q$; that of the second is

$$Q_2 = \left(1 + 2\,\phi\, sin. \frac{a}{2}\right) Q_1 = \left(1 + 2\,\phi\, sin. \frac{a}{2}\right)^2 Q,$$

that of the third is

$$Q_3 = \left(1 + 2\,\phi\, sin. \frac{a}{2}\right) Q_2 = \left(1 + 2\,\phi\, sin. \frac{a}{2}\right)^3 Q,$$

and in general the tension at the other end is

$$P = \left(1 + 2\,\phi\, sin. \frac{a}{2}\right)^n Q,$$

when it is required to produce motion in the direction of the force P. Interchanging P and Q, we obtain the force necessary to prevent motion in the direction of the force Q and it is

$$P_1 = \frac{Q}{\left(1 + 2\,\phi\, sin. \frac{a}{2}\right)^n}.$$

FIG. 293.

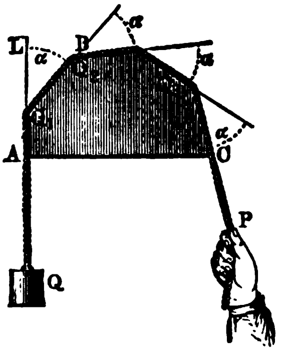
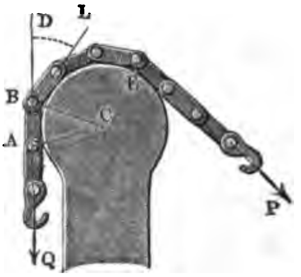


FIG. 294.



The friction in the first case is

$$F = P - Q = \left[\left(1 + 2\,\phi\, sin. \frac{a}{2}\right)^n - 1\right] Q,$$

and in the second

$$F = Q - P_1 = \left[\left(1 + 2\,\phi\, sin. \frac{a}{2}\right)^n - 1\right] P_1$$

$$= \left[1 - \left(1 + 2\,\phi\, sin. \frac{a}{2}\right)^{-n}\right] Q.$$

The same formulas are also applicable to the case of a body composed of links, as, E.G., a chain $A\,B\,E$, Fig. 294, which is passed round a cylindrical body, when n is the number of links lying upon the body. If the length of one joint of the chain is $= l$ and the distance $C\,A$ of the axis A of a link from the centre

C of the arc, which is covered, $= r$, we have for the angle of deviation $D\,B\,L = A\,C\,B = a$, $sin. \frac{a}{2} = \frac{l}{2\,r}$.

EXAMPLE.—How great is the friction on the circumference of a wheel 4 feet high, covered with twenty links of a chain, each five inches long and 1 inch thick, when one of the ends is fastened and the other subjected to a strain of 50 pounds? Here we have

$$P_1 = 50 \text{ pounds}, n = 20, sin. \frac{a}{2} = \frac{5}{48 + 1} = \frac{5}{49};$$

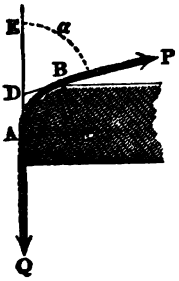
now if we substitute for ϕ the mean value, 0,35, we obtain the friction, with which the chain opposes the revolution of the wheel

$$F = \left[\left(1 + 2\,.\,0{,}35\,.\,\frac{5}{49}\right)^{20} - 1\right].\,50 = \left[\left(1 + \frac{35}{490}\right)^{20} - 1\right].\,50$$

$$= \left[\left(\frac{15}{14}\right)^{20} - 1\right].\,50 = 2{,}974\,.\,50 = 149 \text{ pounds.}$$

§ **194**. If a stretched cord $A\,B$, Fig. 295, lies upon a fixed cylindrically rounded body $A\;C\,B$, the friction can also be found by the rule given in the foregoing paragraph. Here the angle of deviation is $E\,D\,B = a^{\circ} =$ angle at the centre $A\;C\,B$ of the arc $A\;B$ of the cord; if we divide the same in n equal parts and regard the arc $A\;B$ as consisting of n straight lines, we obtain n edges with the deviation $\frac{a^{\circ}}{n}$, and therefore the equation between the power and the load is as in the foregoing paragraph

FIG. 295.



$$P = \left(1 + 2\,\phi\, sin. \frac{a}{2\,n}\right)^n Q.$$

On account of the smallness of the arc $\frac{a}{n}$, $sin. \frac{a}{2\,n}$ can be replaced by $\frac{a}{2\,n}$, and we can put

$$P = \left(1 + \frac{\phi\,a}{n}\right)^n Q.$$

Developing according to the binomial theorem, we obtain

$$P = \left(1 + n\frac{\phi\,a}{n} + \frac{n\,(n-1)}{1\,.\,2}\frac{(\phi\,a)^2}{n^2} + \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\frac{(\phi\,a)^3}{n^3} + \ldots\right)Q,$$

or, since n is very great and we can put $n - 1 = n - 2 = n - 3 \ldots = n$,

$$P = \left(1 + \phi\, a + \frac{1}{1\,.\,2}(\phi\, a)^2 + \frac{1}{1\,.\,2\,.\,3}\,.\,(\phi\, a)^3 + \ldots\right) Q.$$

But $1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \ldots = e^x$, e being the base 2,71828 of the Naperian system of logarithms (see Introduction to the Calculus), and we can therefore write

$$P = e^{\phi\, a}\,.\,Q \text{ or } Q = P\, e^{-\phi\, a}, \text{ and inversely}$$

$$a = \frac{1}{\phi}\, l\, \frac{P}{Q} = \frac{2{,}3026}{\phi}\,(\log P - \log Q).$$

If the arc of the cord is not given in parts of π, but in degrees, then we must substitute $a = \frac{a^\circ}{180^\circ}\,.\,\pi$, and if finally it is expressed by the number u of coils of the rope, we must put $a = 2\,\pi\, u$.

The formula $P = e^{\phi\, a}\,.\,Q$ shows, that the friction of a cord $F = P - Q$ on a fixed cylinder does not depend at all upon the diameter of the same, but upon the number of coils of the cord, and also that it can easily be increased to almost infinity. If we put $\phi = \frac{1}{3}$, we have

for ¼ coils, $P = 1{,}69\ Q$
" ½ " $P = 2{,}85\ Q$
" 1 " $P = 8{,}12\ Q$
" 2 " $P = 65{,}94\ Q$
" 4 " $P = 4348{,}56\ Q.$

(REMARK.)—From the equation $P = \left(1 + 2\,\phi\, \sin.\frac{a}{2}\right) Q$ in § 193, it follows that

$$P - Q = 2\,\phi\, \sin.\frac{a}{2}\, Q,$$

or substituting instead of a the element $d\,a$ of the arc and instead of $P - Q$, the corresponding increase $d\,P$ of the variable tension P of the cord and putting $Q = P$, we obtain

$$dP = 2\,\phi\,\frac{d\,a}{2}\,P, \text{ or } \frac{d\,P}{P} = \phi\, d\,a,$$

whence by integration we obtain

$$l\,P = \phi\, a + Con.$$

In the beginning a is $= 0$ and $P = Q$, and therefore we have

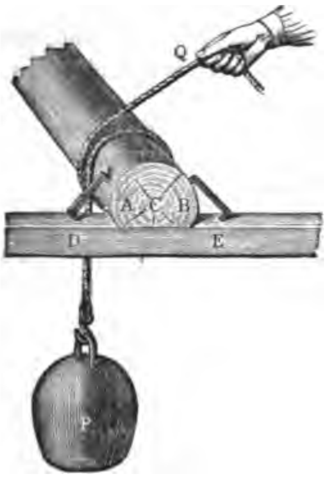
$$l\,Q = 0 + Con. \text{ and } l\,P - l\,Q = l\,\frac{P}{Q} = \phi\, a,$$

or inversely

$$\frac{P}{Q} = e^{\phi\, a}, \text{or } P = e^{\phi\, a}\, Q.$$

EXAMPLE.—In order to let down a shaft a very great but indivisible weight $P = 1200$ pounds, we wind the rope, to which this weight is attached, 1⅜ times around a firmly fastened log $A\,B$, Fig. 296, and we hold the other end of the rope in the hand. What force must be exerted at this end of the rope, when we wish the weight to descend slowly and uniformly? If we put here $\phi = 0{,}3$, we obtain for this force

FIG. 296.



$$Q = P e^{-\phi\alpha} = 1200 \,.\, e^{-0{,}3 \,.\, \frac{11}{8} 2\pi}$$
$$= 1200 \,.\, e^{-\frac{33}{40}\pi}$$
$$l\,Q = l\,1200 - \frac{33}{40}\pi = 7{,}0901 - 2{,}5918$$
$$= 4{,}4983,$$

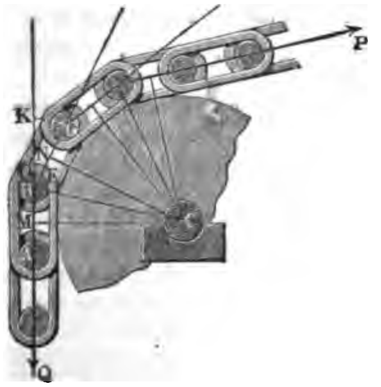
or $\log Q = 1{,}9536$, whence

$$Q = 89{,}9 \text{ pounds.}$$

§ 195. **Rigidity of Chains.**—If ropes or bodies composed of links, etc., are laid on a pulley or a cylinder movable about its axis, the friction of cords and chains considered in the last paragraph ceases, because the circumference of the wheel and the cord have the same velocity, and hence force is only necessary to bend the rope as it lays itself upon the pulley, and sometimes to straighten it as it is unrolled from the pulley.

If it is a chain, which winds itself around a drum, the resistance during the rolling and unrolling consists of the friction of the bolts against the links, since the former are turned through a certain angle in their bearings. If $A\ B$, Fig. 297, is a link of the chain and $B\ G$ the following one, if C is the axis of rotation of the pulley, upon which the chain, stretched by the weight Q, winds, and if finally $C\ M$ and $C\ N$ are perpendiculars let fall upon the major axis of the links $A\ B$ and $B\ G$, then $M\ C\ N = \alpha^{\circ}$ is the angle

FIG. 297.



turned through by the pulley, while a new link lays itself upon it, and $K B G = 180° - A B G$ is the angle described by the link $B G$ with its bolt $B D$ upon the link $A B$ during the same time. If $B D = B E = r_1$ is the radius of the bearing of the bolt, the point D of the pressure or friction describes an arc $D E = r_1 \, a$, while a link lays itself upon the roller, and the work done by the friction at the point D is, $= \phi_1 \, Q \, . \, r_1 \, a$. Supposing the force P_1 necessary to overcome this friction to act in the direction of the greater axis $B G$, we have the space described by it in the same time $s = C N$ multiplied by the arc of the angle $M C N = \overline{C N} \, . \, a$, and therefore the work done $= P_1 \, . \, \overline{C N} \, . \, a$, equating the two mechanical effects, we have $P_1 \, . \, \overline{C N} \, . \, a = \phi_1 \, . \, Q \, r_1 \, a$, and the force required is

$$P_1 = \phi_1 \, Q \frac{r_1}{a},$$

a denoting $C N$ the radius of the drum plus half the thickness of the chain.

If we neglect the friction, the force necessary to turn the pulley would be $P = Q$, but when we take into account the friction caused by the winding of the chain upon the pulley, we have

$$P = Q + P_1 = \left(1 + \phi_1 \frac{r_1}{a}\right) Q.$$

If the chain unwinds from the drum, the resistance is the same; if, therefore, as on a fixed pulley, the rope is wound upon one side and unwound upon the other, the required force is

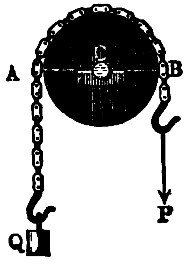
$$P = \left(1 + \phi_1 \frac{r_1}{a}\right)^2 Q, \text{ or approximatively } = \left(1 + 2 \, \phi_1 \frac{r_1}{a}\right) Q.$$

If, finally, the pressure on the axle is $= R$ and the radius of the axle $= r$, the force necessary to overcome all the resistances is

$$P = \left(1 + 2 \, \phi_1 \frac{r_1}{a}\right) Q + \phi \frac{r}{a} R.$$

EXAMPLE.—How great is the force P at the end of a chain passing round a roller $A \; C \; B$, Fig. 298, when the weight acting vertically is $Q = 110$ pounds, the weight of the roller and chain is 50 pounds, the radius a of the roller, measured to the middle of the chain, is $a = 7$ inches, the radius of the axle C is $= \frac{5}{8}$ of an inch and that of the bolts of the chain is $= \frac{3}{8}$ of an inch? If we put $\phi = 0,075$ and $\phi_1 = 0,15$, we obtain, according to the last formula, the force

FIG. 298.



$$P = \left(1 + 2 \, . \, 0,15 \, . \, \frac{3}{8.7}\right) . 110 + 0,075 . \frac{5}{8.7} \, (110 + 50 + P),$$

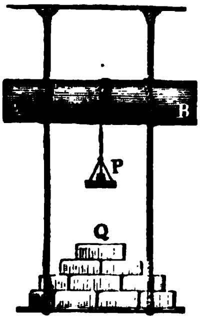
or assuming in the right-hand member P approximatively $= 110$

$$P = 1{,}016 \,.\, 110 + 0{,}0067 \,.\, 270 = 111{,}76 + 1{,}81 = 113{,}6 \text{ pounds.}$$

§ **196. Rigidity of Cordage.**—If a rope is passed over a pulley or winds itself upon a shaft, its rigidity (Fr. roideur, Ger. Steifigkeit) comes into play as a resistance to its motion. The resistance is not only dependent upon the material, of which the rope is made, but also upon the manner, in which it is put together, and upon the thickness of the rope; it can consequently be determined by experiment alone.

The principal experiments for this object are those made by Coulomb and those made more recently by the author himself. While Coulomb employed only small hemp ropes from ¼ to at most 1½ inches in thickness and made them wind upon rollers of 1 to at most 6 inches in diameter, the author employed hemp ropes 2 inches thick and wire ropes from ½ to 1 inch thick and passed them over rollers from 2 to 6½ feet in diameter. Coulomb's experiments were made in two different ways. In one case; like Amonton, he employed the apparatus represented in Fig. 299, where $A\,B$ is a roller around which two ropes are wound, the tension being produced by a weight Q and the rolling down of this roller by a weight P, which pulls upon this roller by means of a thin string. In the other case he laid the ropes around a cylinder rolling upon a horizontal surface and, after having subtracted the rolling friction, calculated the resistance of the rigidity from the difference of the weights, which were suspended to the two ends of the rope and which produced a slow rolling motion.

FIG. 299.



According to the experiments of Coulomb, the resistance of the rigidity increases tolerably regularly with the amount of the tension of the rope; but there is also a constant member K, as might have been expected; for a certain force is necessary to bend an unstretched rope. It was also shown, that this resistance was inversely proportional to the radius of the roller; that for a roller of twice the diameter it is only one-half, for one of three times the diameter, one-third, etc. Finally, the relation between the thickness and rigidity of a rope can only be determined approximatively from these experiments, as we might have supposed; for this rigidity depends upon the nature of the material of the ropes and upon the size of the fibres and strands. When a rope is new, the rigidity is proportional, approximatively, to $d^{1,7}$, and when it is old, to $d^{1,4}$, d denoting the diameter of the rope. The assumption by some authors that it varies with the first power, and that of others that it varies with the square of the thickness of the rope, are therefore only approximatively true.

§ 197. Prony's Formula for the Rigidity of Hemp Ropes.—According to the last paragraph, the rigidity of hemp ropes can be expressed by the following formula:

$$S = \frac{d^n}{a} (K + \nu\, Q),$$

in which d denotes the thickness of the rope, a the radius of the pulley measured to the axis of the rope, Q the tension of the rope, which passes round the pulley, and n, K and ν empirical constants. Prony found from Coulomb's experiments for new ropes

$$S = \frac{d^{1,7}}{a} (2,45 + 0,053\ Q),$$

and for old ones

$$S_1 = \frac{d^{1,4}}{a} (2,45 + 0,053\ Q),$$

in which formulas a and d are expressed in lines and Q and S in pounds. These formulas are, however, based upon Paris measures; for English measures they become, when expressed in inches and pounds,

$$S = \frac{d^{1,7}}{a} (14,39 + 0,289\ Q)$$

$$S_1 = \frac{d^{1,4}}{a} (6,96 + 0,14\ Q).$$

Since even these complicated formulas do not agree as well as could be wished with the results of experiment, we can, as long as we do not take into account the later experiments, write with Eytelwein

$$S = \nu \,.\, \frac{d^2}{a}\, Q = \frac{d^2\, Q}{3604a}.$$

In this formula a must be expressed in English feet and d in English lines, but Q and S may be expressed in any arbitrary system of weights. If we employ the metrical system of measures, we have

$$S = 18,6 \,.\, \frac{d^2\,Q}{a}.$$

The results given by this formula are not sufficiently accurate, except when the tension upon the rope, as is generally the case in practice, is very great.

The rigidity of tarred ropes was found to be about one-sixth greater than that of untarred ones, and wet ropes were found to be about one-twelfth more rigid than dry ones.

EXAMPLE.—If the tension upon a new rope 9 lines thick, which passes round a pulley 5 inches diameter, is 350 pounds, the rigidity, according to Prony, is

$S = \frac{2}{5}\left(\frac{9}{7}\right)^{1,7}(14,39 + 0,289 \,.\, 350) = 0,613 \,.\, 46,216 = 28,33$ pounds,
and according to Eytelwein

$$S = \frac{9^2 \,.\, 350}{3604 \,.\, \frac{5}{24}} = 37,75 \text{ pounds.}$$

If the tension were but $Q = 150$ pounds, we would have, according to Prony,

$$S = 0,613 \,.\, 23,1 = 14,16,$$

and according to Eytelwein

$$S = \frac{81 \,.\, 150}{3604 \,.\, \frac{5}{24}} = 16,2.$$

In this case the formulas give results, which coincide better with each other. We see from the above example, how uncertain these formulas are.

REMARK.—Tables for facilitating the calculation of the resistance due to the rigidity of cordage will be found in the Ingenieur, page 365. According to Morin (see his Leçons de Mécanique Pratique), we have, when n denotes the number of strands in the rope and a the radius of the pulley in centimetres, for untarred ropes

$$d = \sqrt{0,1338\, n} \text{ centimetres and}$$

$$S = \frac{n}{2\,a}(0,0297 + 0,0245\, n + 0,0363\, Q) \text{ kilograms}$$

$$= \frac{d^2}{a}(0,1110 + 0,6848\, d^2 + 0,1357\, Q) \text{ kilograms,}$$

and for tarred ropes

$$d = \sqrt{0,186\, n} \text{ centimetres and}$$

$$S = \frac{n}{2\,a}(0,14575 + 0,0346\, n + 0,0418\, Q) \text{ kilograms}$$

$$= \frac{d^2}{a}(0,3918 + 0,5001\, d^2 + 0,1124\, Q) \text{ kilograms.}$$

If, however, d and a are expressed in inches, and S and Q in pounds, we can put for untarred ropes

$$S = \frac{d^2}{a}\,(0{,}621 + 24{,}70\,d^2 + 0{,}3445\,Q),$$

and for tarred ones

$$S = \frac{d^2}{a}\,(2{,}193 + 18{,}06\,d^2 + 0{,}2889\,Q).$$

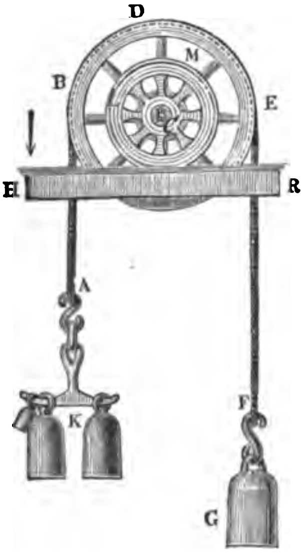
If, E.G., for an untarred rope we have $d = \frac{3}{4}$ inch, $a = \frac{5}{2}$ inches and $Q = 350$ pounds, then

$$S = \frac{9}{16}\cdot\frac{2}{5}\,(0{,}621 + 24{,}70\,.\,\frac{9}{16} + 0{,}3445\,.\,350)$$
$$= \frac{9}{40}\,(0{,}621 + 13{,}893 + 120{,}575) = 30{,}4 \text{ pounds},$$

while in this case (last example) Prony's formula gave $S = 28{,}83$ pounds.

§ 198. Experiments Upon the Rigidity of Thick Ropes.—The author, in his experiments upon the rigidity of cordage, made use of the apparatus represented in Fig. 300. The sheave or roller $B\ D\ E$, over which the rope to be tested is passed, was, together with a pair of iron wheels $C\ L\ M$, fastened upon a shaft or axle C, and these wheels ran upon two horizontal rails $H\ R$. To one end F of the rope a weight G was attached, and to the other end A a cross K, upon which weights were hung until the wheels and pulley began to roll forward slowly. In order to be as independent as possible of errors arising from imperfections in the apparatus, additional weights were afterwards added at F until a rolling motion in the opposite direction was produced. The arithmetical mean of the weights added gave, when the rolling friction was deducted, the rigidity of the rope. The coefficient of rolling friction to be used was determined in the same way, except that a thin string, whose rigidity could be neglected, was employed instead of a rope. The mean value of this coefficient was given in § 192.

FIG. 300.



The resistance due to the rigidity is, according to the author's views, due less to the rigidity proper than to the friction of the different wires or strands upon each other; for in passing over the pulley, they naturally change their relative positions. When a wire rope passes round a fixed pulley, the first part of this resistance is wanting, as the rope, in consequence of its elasticity, gives out, when it straightens itself, as much mechanical effect as was employed in bending it around the pulley. Hence the rigidity of the rope in this case consists solely of the friction of the wires upon one another, a conclusion which is confirmed by the author's experiments; for he found the resistance to be forty per cent. less, when the ropes were freshly oiled or tarred than when they were dry. The conditions are different in the case of hemp ropes, for they do not possess, especially after long use, any elasticity, and the strands and fibres require force not only to bend them, but also to straighten them.

§ 199. New Formulas for the Resistance Due to the Rigidity of Cordage.—Since the rigidity of a rope depends not only upon its thickness, but also upon the amount of bending it is subjected to, and also upon the manner in which it is put together, the author considers, that these conditions can be very well expressed by the formula

$$S = \frac{K + \nu Q}{a};$$

the constants K and ν must be determined specially for each kind of rope. The experiments of the author also showed, that for wire ropes we should put simply K instead of $\frac{K}{a}$, or

$$S = K + \frac{\nu Q}{a}.$$

1. For tarred hemp ropes 1,6 inches thick passing round sheaves from 4 to 6 feet in diameter, he found

$$S = 1{,}5 + 0{,}00565 \frac{Q}{a} \text{ kilograms,}$$

when the radius a is expressed in metres, or

$$S = 3{,}31 + 0{,}222 \frac{Q}{a} \text{ pounds,}$$

when a is expressed in inches.

2. For a new untarred hemp rope ¾ inch thick, upon a pulley 21 inches in diameter, he found

$$S = 0{,}086 + 0{,}00164 \frac{Q}{a} \text{ kilograms} = 0{,}1896 + 0{,}06457 \frac{Q}{a} \text{ pounds.}$$

3. A wire rope 8 lines in diameter, formed of 16 wires, each $1\frac{1}{2}$ lines thick, and weighing 0,68 pound per running foot, was passed around pulleys from 4 to 6 feet in diameter, and gave

$$S = 0{,}49 + 0{,}00238 \frac{Q}{a} \text{ kilograms} = 1{,}08 + 0{,}0937 \frac{Q}{a} \text{ pounds.}$$

4. For a freshly-tarred wire rope, with a hemp centre in each strand and in the rope, which was 7 lines in diameter, was composed of 4.4 = 16 wires, each $1\frac{1}{2}$ lines thick, and weighed 0,67 pound per running foot, he found, with a pulley 21 inches in diameter,

$$S = 0{,}57 + 0{,}000694 \frac{Q}{a} \text{ kilograms} = 1{,}26 + 0{,}0272 \frac{Q}{a} \text{ pounds.}$$

REMARK.—A detailed description of the author's experiments is to be found in the Zeitschrift für Ingenieurwesen (dem Ingenieur), by Bornemann, Brückmann and Röting, Vol. I, Freiberg, 1848. The hemp ropes of 1 were formerly employed in Freiberg for hoisting from the shafts by means of a water-wheel and drum (Ger. Wassergöpel), but of late they have been replaced by the wire ropes of 3 and 4. Both of these kinds of ropes can support with sextuple security a load of 30 cwt. It was shown by the above experiments that, when the load was the same, the resistance due to the rigidity of wire ropes was less than that due to the rigidity of hemp ones. If we assume the tension of the rope to be $Q = 2000$, and the radius of the sheave to be $a = 40$ inches, we have for hemp ropes

$$S = 3{,}81 + 0{,}222 \tfrac{2000}{40} = 14{,}41 \text{ pounds,}$$

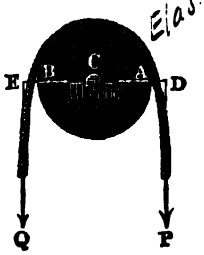
and, on the contrary, for wire ropes

$$S = 1{,}08 + 0{,}0937 \tfrac{2000}{40} = 5{,}8 \text{ pounds.}$$

§ 200. Theory of the Fixed Pulley.

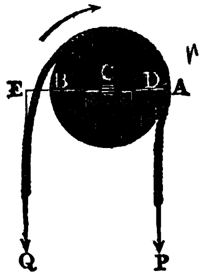
—Let us now apply the principles just enunciated to the theory of the fixed pulley.

FIG. 301.



Elastic

FIG. 302.



non-elastic

Let $A\ C\ B$, Fig. 301 or Fig. 302, be the pulley, and let a be its radius $CA = CB$, r the radius of its axle, G its weight, d the thickness of the rope, Q the weight suspended to one end of the latter, S the resistance due to the rigidity, F the friction upon the axle, reduced to the circumference, and $P = Q + F + S$ the force at the other end of the rope. The rigidity of the rope is shown by the fact that the rope does not immediately assume the curvature of the pulley as it is wound upon the sheave, nor straighten itself immediately, when it is unwound. On the contrary, it approaches the sheave in an arc, the curvature of which constantly increases, and leaves in an arc, the curvature of which constantly diminishes. The difference between the elastic wire ropes and the unelastic hemp ones is that the former leave the sheave somewhat sooner and the latter somewhat later; hence the arm CD of the force in the first case (Fig. 301) is somewhat greater, and in the second case (Fig. 302) somewhat less than the radius $CA = a$ of the sheave. If we neglect the friction upon the axle and put $P = (Q + S)$, we have

$$(Q + S) \,.\, CD = Q \,.\, CE,$$

and consequently the rigidity of the rope is

$$S = \left(\frac{CE - CD}{CD}\right) Q = \left(\frac{CE}{CD} - 1\right) Q,$$

and the ratio of the arms is

$$\frac{CE}{CD} = 1 + \frac{S}{Q},$$

the value of which can easily be calculated by substituting one of the values of S.

We can also determine this force $P = Q + S + F$ without employing the ratio of the arms of the lever by substituting in that formula either with Prony for thin hemp ropes

$$S = \frac{d^n}{a} (K + \nu Q),$$

or with the author for wire or thick hemp ropes.

$$S = K + \frac{\nu Q}{a},$$

and the friction upon the axles reduced to the circumference of the pulley is

$$F = \phi \frac{r}{a} (Q + G + P), \text{ or approximatively,}$$

$$F = \phi \frac{r}{a} (2\,Q + G).$$

Hence, in the first case, we have

$$P = Q + \frac{d^n}{a}(K + \nu Q) + \phi \frac{r}{a}(2\,Q + G)$$

and in the second

$$P = Q + K + \frac{\nu Q}{a} + \phi \frac{r}{a}(2\,Q + G).$$

In the case of the wheel and axle a reduction of the force fi the circumference of the axle to that of the wheel is necessary.

Example.—If a wire rope 3 lines in diameter passes over a pulley 5 feet high, whose axles are 3 inches in diameter, and if the tension upon the rope is 1200 pounds, we have the required force, when the coefficient of friction is $\phi = 0{,}075$ and the weight of the pulley = 1500 pounds

$$P = 1200 + 1{,}08 + 0{,}0987 \,.\, \tfrac{1200}{30} + 0{,}075 \,.\, \tfrac{3}{60}\,(2400 + 1500)$$
$$= 1200 + 1{,}08 + 3{,}748 + 14{,}62 = 1219 \text{ pounds};$$

hence $\frac{19}{12} = 1{,}6$ per cent. of the force is lost in consequence of the rope's passing round the pulley.

If instead of a wire rope we employed a hemp one 1,6 inches thick, we would have

$$P = 1200 + 3{,}81 + 0{,}222 \,.\, \tfrac{1200}{30} + 14{,}62 = 1227$$

and the loss of force would be

$$P - Q = \frac{27}{12} = 2{,}25 \text{ per cent.}$$

FOURTH SECTION.

THE APPLICATION OF STATICS TO THE ELASTICITY AND STRENGTH OF BODIES.

CHAPTER I.

ELASTICITY AND STRENGTH OF EXTENSION, COMPRESSION AND SHEARING.

§ **201. Elasticity.**—The molecules or parts of a solid or rigid body are held together by a certain force, called *cohesion* (Fr. cohésion; Ger. Cohäsion), which must be overcome, when the body changes its form and size, or if it is divided. The first effect, which forces produce upon a body, is a variation in the relative position of its parts, in consequence of which a change of form and volume occurs. If the forces acting upon a body exceed certain limits, a separation of the parts takes place and perhaps a division of the whole body into pieces. The capability of a body to resume its original form, after the force which caused its change of shape has been removed, is called in the most general sense of the word its *elasticity* (Fr. élasticité; Ger. Elasticität). The elasticity of every body has certain limits. If the change of form and volume exceeds a certain amount, the body remains of the same form after such a change, although the forces which have produced the variation have ceased to act. The *limit of elasticity* is very different for different bodies. The bodies, which permit a great change of volume before their limit of elasticity is reached, are called *perfectly elastic;* those, whose limit of elasticity is reached when they have undergone a very slight change of form, are called *inelastic,* although no such bodies really exist. It is an important rule in architecture and in the construction of machinery, not to load the materials employed to such an extent that the change of form produced shall reach, much less exceed, the limit of elasticity.

§ 202. **Elasticity and Strength.**—Different bodies present different phenomena, when they are changed in their form beyond the limit of elasticity. If a body is *brittle* (Fr. cassant; Ger. spröde), it flies in pieces, when its form is changed beyond its limit of elasticity; if, however, it is *ductile* or *malleable* (Fr. ductile; Ger. geschmeidig), as, E.G., many metals, we can cause considerable changes in its form beyond its limit of elasticity, without causing a separation of its parts. Some bodies are *hard* (Fr. dur; Ger. hart), others *soft* (Fr. mou; Ger. weich); while the former oppose great resistance to a separation of their parts, the latter permit it without much difficulty.

We understand by *elasticity*, in the more restricted sense of the word, the resistance with which a body opposes a change of its form, and by *strength* (Fr. résistance, Ger. Festigkeit) the resistance with which a body opposes division. In what follows, both subjects will be treated. According to the manner in which the extraneous forces act upon bodies, we can divide elasticity and strength into

I. *Simple* and

II. *Combined;*

and the former again into

1) *Absolute* or the *elasticity* and *strength of extension,*

2) *Reacting,* or the *elasticity* and *strength of compression,*

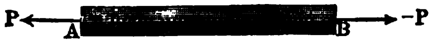
3) *Relative,* or the *elasticity* and *strength of flexure,*

4) The *elasticity* and *strength of sheering* and

5) The *elasticity* and *strength of torsion* or *twisting.*

If two extraneous forces P and $-P$ act by *extension* (Fr. traction, Ger. Zug) in the direction of the axis of a body $A\ B$, Fig. 303, the latter resists the extension and tearing by means of its *absolute elasticity* and *strength* or its *elasticity* and *strength of extension* (Fr. élasticité et résistance de traction, Ger. Zug oder absolute Elasticität

Fig. 303.



und Festigkeit); if, on the contrary, two forces P and $-P$ press the body together in the direction of the axis of the body $A\ B$, Fig. 304, so that the latter is compressed and finally crushed, the *elasticity* and *strength of compression* or the *reacting elasticity* and *strength* (Fr. élasticité et résistance de compression, Ger. Druck or rückwirkende Elasticität und Festigkeit) must be overcome. If, farther, three forces P, Q, R, which balance each other, are applied at three different points A, B, C, in the axis of the body $A\ B$, Fig. 305, and act at right angles to the same, this body would be bent or perhaps broken, and it is the *relative elasticity and strength*, or the *elasticity* and *strength of flexure* (Fr. élasticité et résistance de flexion, Ger. Biegungs oder relative Elasticität und Festigkeit), that must be overcome, in order to bend or break it. If, in the latter case, the points of application A and C lie close together, as is represented in Fig. 305, a distortion is produced in the cross section $D\ D$, between the two points A and C; if the force P is great enough, the body is divided into two parts, and in this case the *elasticity* and *strength of sheering* (Fr. élasticité et résistance par glissement cisaillement ou tranchant, Ger. Elasticität und Festigkeit des Abschierens) is overcome. If two couples $(P, -P)$, $(Q, -Q)$, which balance each other, act upon a body $C\ A$, Fig. 306, in such a manner that their planes are at right angles to the axis of the body, a *twisting* of the body is produced, which may become a *wrenching*, and here the *elasticity* and *strength of torsion* (Fr. élasticité et résistance de torsion, Ger. Drehungs-elasticität und Festigkeit) is to be overcome.

FIG. 304.

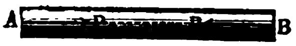


FIG. 305.

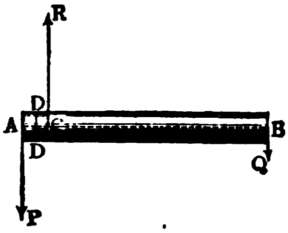
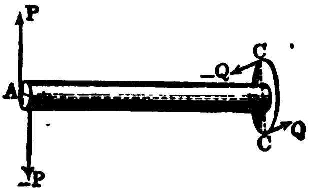


FIG. 306.



If several of the forces here enumerated act at the same time upon a body, the *combined elasticity* and *strength* or a combination of two or more of the simple elasticities and strengths comes into play.

§ 203. **Extension and Compression.**—The most simple case of elasticity and strength is presented by the extension and compression of prismatic bodies, when they are acted upon by forces whose directions coincide with the axis of these bodies. It is of course not necessary that both should be motive forces. The action is the same, when the body is firmly suspended or supported at one end and at the other end subjected to a pull or to a thrust. We can obtain an example of this case either by suspending to a prism *A B C D*, Fig. 307, which hangs vertically, a weight *P*, or by loading with a weight *P* a prism *A B C D*, Fig. 308, which is supported at the bottom. In the first case, the body is extended a certain amount $C C_1 = D D_1 = \lambda$, and in the second case, it undergoes a similar compression; if, therefore, the initial length of the body is $A D = B C = l$, it becomes, in the first case,

FIG. 307.

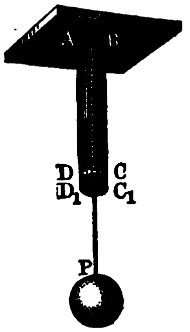
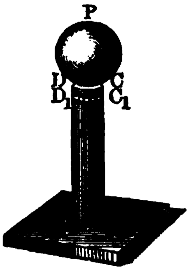


FIG. 308.



$$A D_1 = B C_1 = A D + D D_1 = l + \lambda,$$

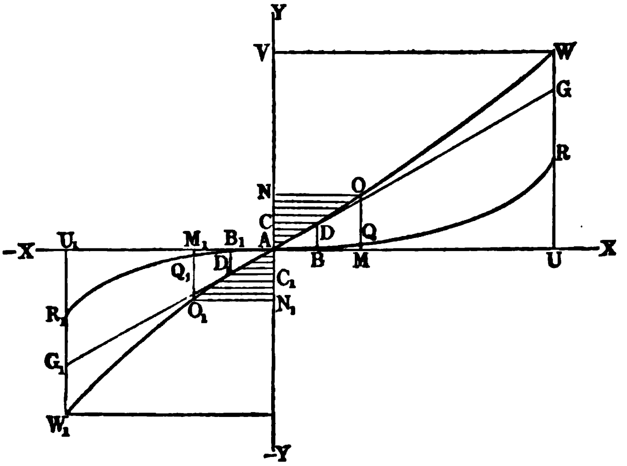
and in the second case,

$$A D_1 = B C_1 = A D - D D_1 = l - \lambda.$$

The extension or compression λ increases with the pull or thrust *P*, and is a function of the same. This function or algebraical relation between *P* and λ cannot be determined *à priori;* it is dependent upon the physical properties of the body, and is different for different materials. If we regard *P* and λ as the co-ordinates of a curve and construct this curve with the corresponding values of *P* and λ determined by experiment, we obtain by this means not only a graphic representation of the law, according to which bodies are extended and compressed by extraneous forces, but also a means of determining the peculiarities of this law.

If we lay off from *A* on the positive side of the axis $X \overline{X}$, Fig. 309, the tensions or tensile forces, which act upon a body, as abscissas *A B*, *A M*, etc., and at their ends the corresponding extensions as ordinates $B\,D$, $M\,O$, etc., parallel to $Y\,\overline{Y}$, we obtain a curve $A\,D\,O\,W$, which represents the law of the extension of

FIG. 309.



this body; and if, on the contrary, we cut off on the negative side of the axis $X\,\overline{X}$ from A the pressures or thrusts as abscissas $A\,B_1$, $A\,M_1$, etc, and at the extremity of the same lay off the corresponding compressions as ordinates $B_1\,D_1$, $M_1\,O_1$, etc, we obtain a curve $A\,D_1\,O_1\,W_1$, by which the law of compression of the body is graphically represented. According to the results of many experiments, these two curves pass without interruption into one another, have consequently at A a common tangent $G\,A\,G_1$, and are therefore properly only branches of the same curved line $W\,O\,D\,A\,D_1\,O_1\,W_1$. Although the curve as a whole differs considerably from a right line, yet in the neigborhood of the origin of co-ordinates A it nearly coincides with the tangent $G\,A\,G_1$, and since for this line the ordinates are proportional to the abscissas, we can also assume that *the small extensions and compressions produced by the pulls or thrusts* $A\,B$, $A\,B_1$, *etc., are proportional to these forces* (Hooks' Law).

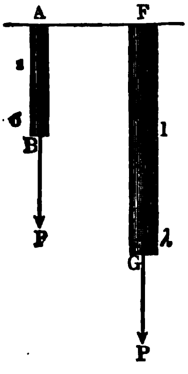
The total extension $M\,O$, produced by the pull $A\,M$, consists of two parts, viz.: the *permanent extension* or *set* $M\,Q$, which remains in the body, when the stress has ceased to act, and the *elastic extension* $Q\,O$, which vanishes with the pull. It is the same for compression. The *total compression* $M_1\,O_1$ is the sum $M_1\,Q_1$ +

$Q_1 O_1$ of the *permanent compression* or *set* $M_1 Q_1$ and of the *elastic one* $Q_1 O_1$. When the forces are small, the permanent change is so very small compared with the total one, that it can be regarded as not existing, and consequently the total extensions and compressions can be treated as the elastic ones. If the force exceeds a certain limit $A B$ ($A B_1$), the so-called *limit of elasticity*, if, E.G., it becomes $A M$ ($A M_1$), the permanent change of length or *set* forms a considerable portion of the total extension $M O$ or of the total compression $M_1 O_1$. If the pull or thrust reaches a certain value $A U$ or $A U_1$, the extensions $U R$, $U W$ and the compressions $U_1 R_1$ and $U_1 W_1$ attain the limit at which the cohesive force of the body is no longer able to balance the pull or thrust, and consequently a tearing asunder or a crushing of the body takes place.

If a body has been subjected to a force, which has not extended or compressed it beyond the limit of elasticity, the body will not assume any further set, when subjected to another pull or thrust, which does not reach the limit of elasticity.

§ 204. Fundamental Laws of Elasticity. Modulus of Elasticity.—The lengthening or extension of a prismatical body, produced by a force P, is proportional, in the first place, to the length l of the body, since we can assume that equally long portions are equally extended, and it is inversely proportional to the cross-section F of the body, since we can suppose the entire stretching force to be equally distributed over the entire cross-section of the body. If, therefore, a body $A B$, Fig. 310, whose length is = unity and whose cross-section = unity, is extended an amount σ by a stress P, the extension produced in another body $F G$ of the same material, whose length is $= l$ and whose cross-section is $= F$, by the same stress is

FIG. 810.

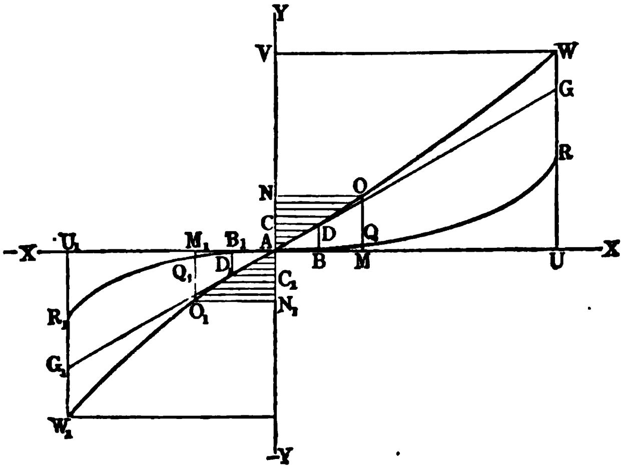


$$\lambda = \frac{\sigma l}{F}.$$

The extension σ is of course dependent upon the pull P alone and is different for different materials; but according to what precedes (§ 203) we can assume that for small pulls, which do not exceed the limits of elasticity, the extension is proportional to the corresponding stress, or that the quotient $\frac{\sigma}{P}$ is a constant quantity.

Now if $A\,B$, Fig. 311, represents the tension P of a prism, whose length is = unity and whose cross-section = unity, within the limits of elasticity and $B\,D$ the corresponding extension σ, and if we denote the angle $G\,A\,U = D\,A\,B$ of the tangent to the curve of extension at A by a, we have also

FIG. 311.



$$tang.\, a = \frac{B\,D}{A\,B} = \frac{\sigma}{P}, \text{ and therefore}$$

$$\sigma = P\, tang.\, a, \text{ whence we obtain}$$

$$1)\ \lambda = \frac{P\, l\, tang.\, a}{F}.$$

The quantity *tang.* a is dependent upon the physical properties of the body and can be determined by experiment only. If we assume $l = 1$, $F = 1$ and $P = 1$, we obtain *tang.* $a = \lambda$, and *this quantity tang. a, to be determined by experiment, is the extension which is produced in a prism, whose length is unity and whose cross-section is unity, by the tensile force unity* (see *Combes : Traité de l'exploitation des mines, tome* I.). If in the formula (1) we assume $F = 1$ and $\lambda = l$, we obtain the expression

$$1 = P\, tang.\, a, \text{ or } \frac{1}{tang.\, a} = cotang.\, a = P;$$

hence $\frac{1}{tang.\, a}$ *is that force, which would stretch a prism, whose cross-section is one square inch* (1), *its own length, were that possible without surpassing the limit of elasticity.*

This hypothetical empirical quantity $\frac{1}{tang.\ \alpha} = cotg.\ \alpha$ is called the *modulus of elasticity* (Fr. coefficient d'élasticité; Ger. Elasticitätsmodul) of the body or material and will hereafter be designated by the letter E.

According to this we have

$$2)\ \lambda = \frac{P\,l}{F\,E},$$

or the relative extension, I.E., its ratio to the entire length of the body

$$3)\ \frac{\lambda}{l} = \frac{P}{F\,E}.$$

Inversely the force corresponding to the extension λ is

$$4)\ P = \frac{\lambda}{l}\,F\,E.$$

The same formulas obtain also for the compression λ, caused by a thrust P, and the modulus of elasticity $E = cotang.\ \alpha$ is the same as for extension as long as the limit of elasticity is not surpassed, although in this case it denotes that force, which would compress a prism of the cross-section unity its whole length, or to an infinitely thin plate, provided that this were possible without exceeding the limits of elasticity.

REMARK 1.—We can also put the modulus of elasticity E equal to the weight of a prism of the same material as the body, upon which E acts, and of the same cross-section unity. If a is the length of this body and γ the heaviness or the weight of one cubic inch of the same material, we have

$$E = a\,\gamma, \text{ and therefore inversely } a = \frac{E}{\gamma}$$

Tredgold (after Young) used this length as the measure of the elasticity (see T. Tredgold on the strength of cast iron and other metals). If E is, E.G., 80000000 pounds for cast steel and $\gamma = 0{,}8$ pounds, we have

$$a = \frac{80000000}{0{,}8} = 100000000 \text{ inches},$$

I.E., a steel rod 100000000 inches long, would extend a steel bar of the same cross-section its whole length, if the law of extension given above were true for all limits.

REMARK 2.—During the extension or compression of a body a change of cross-section takes place, which, according to Wertheim (see Comptes rendues, T. 26), amounts to $\frac{1}{3}$ of the longitudinal extension or compression. If l is the initial length, F the initial cross-section and V the initial volume $F\,l$ of the body, l_1 and F_1 being the length and cross-section during the action of the force P, we have the corresponding volume

$$V_1 = F_1\,l_1 = F\,l + F\,(l_1 - l) - (F - F_1)\,l, \text{ or}$$
$$V_1 - V = F\,(l_1 - l) - (F - F_1)\,l,$$

and the relative change of volume is

$$\frac{V_1 - V}{V} = \frac{l_1 - l}{l} - \frac{F - F_1}{F}$$

But we know that $\frac{F - F_1}{F} = \frac{2}{3}\left(\frac{l_1 - l}{l}\right)$,
whence it follows that

$$\frac{V_1 - V}{V} = \frac{1}{3}\left(\frac{l_1 - l}{l}\right),$$

I.E., *the increase in volume is one-third the increase in length.*

According to the theory of Poisson, $\frac{V_1 - V}{V} = \frac{1}{2}\left(\frac{l_1 - l}{l}\right)$.

EXAMPLE—1) If the modulus of elasticity of brass wire is 14000000 pounds, what force is necessary to stretch a wire 10 feet long and 2 lines thick one line? Here we have

$$l = 10\,.\,12 = 120 \text{ inches}, \lambda = \tfrac{1}{12} \text{ inch and consequently } \frac{\lambda}{l} = \tfrac{1}{1440};$$

but $F = \frac{\pi d^2}{4} = 0{,}7854\,(\tfrac{2}{12})^2 = 0{,}0218$ square inches, hence the force required is

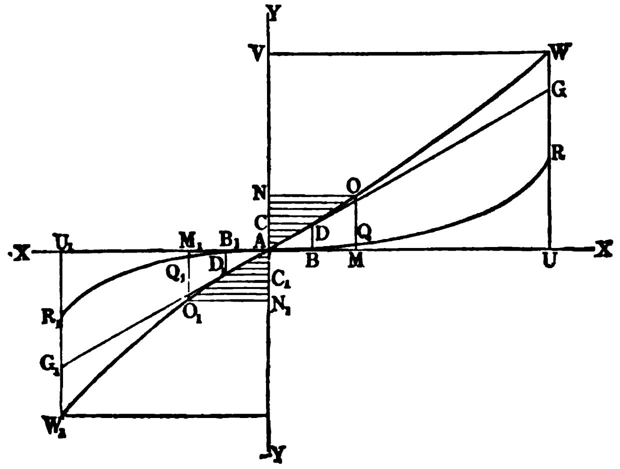
$$P = \tfrac{1}{1440}\,.\,0{,}0218\,.\,14000000 = 212 \text{ pounds.}$$

2) If the modulus of elasticity of iron wire is 31000000 pounds, and an iron surveyor's chain 66 feet long and 0,2 inch thick is submitted to a pull of 150 pounds, the increase in length is

$$\lambda = \frac{150}{0{,}7854\,.\,(0{,}2)^2}\cdot\frac{66\,.\,12}{31000000} = 0{,}122 \text{ inches} = 1{,}464 \text{ lines.}$$

§ 205. Proof Load, Proof Strength, Ultimate Strength.—The force *A B*, Fig. 312, which stretches a prismatical body, whose

FIG. 312.



cross-section is unity, to the limit of elasticity, is called the ***modulus of proof strength of extension***, and will in future be designated by T, while the thrust necessary to compress the same to its limit of elasticity is called the ***modulus of proof strength of compression***, and will hereafter be designated by T_1.

From the moduli of proof strength T and T_1, with the aid of the modulus of elasticity E, the extension σ and the compression σ_1 at the limit of elasticity can easily be found; for we have

$$\frac{\sigma}{1} = \frac{T}{E} \text{ and } \frac{\sigma_1}{1} = \frac{T_1}{E}.$$

If F is the cross section of a prismatical body, whose moduli of proof strength are T and T_1, we have their ***proof strength*** or ***proof load***

$$1) \begin{cases} \text{for a pull,} & P = F\,T \\ \text{and for a thrust,} & P_1 = F\,T_1. \end{cases}$$

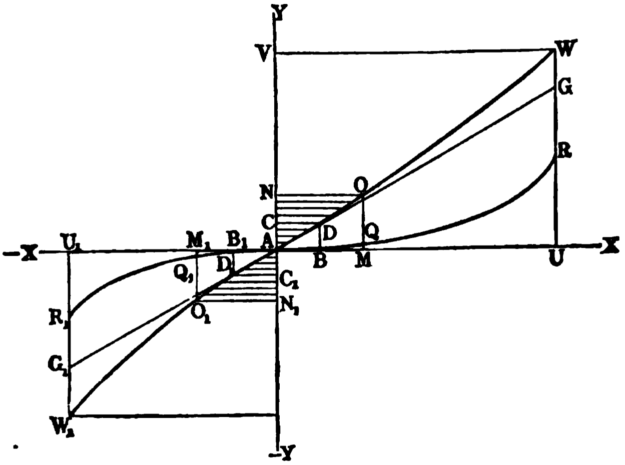
In constructions the bodies should never be loaded beyond their limit of elasticity, and the loads should therefore never surpass the proof strength of the cross-section of the prismatical bodies employed. Cross-sections must therefore be determined by the following formulas:

$$2 \begin{cases} F = \dfrac{P}{T} \text{ and} \\ F_1 = \dfrac{P_1}{T_1}. \end{cases}$$

On account of the accidental overloading and concussions, to which buildings and machines may be subjected, and also on account of the changes, which the bodies undergo in the course of time, owing to the action of air, water, etc., we render these constructions safer by substituting in the foregoing formula, instead of the proof load, only one-half or one-third of the same, I.E. by making the cross-section two or three times as great as those given directly by the formula. In order to have an $m$fold security, we must substitute in the formulas $F = \frac{P}{T}$ or $F_1 = \frac{P_1}{T_1}$, instead of T or T_1, the ***working or safe loads*** $\frac{T}{m}$ or $\frac{T_1}{m}$.

The force $\overline{A\ U}$, Fig. 313, necessary to tear apart a prismatical body, whose cross-section is unity, is called its ***modulus of rupture or of ultimate strength*** of extension, and is denoted by the letter K; and in like manner we call the force $\overline{A\ U_1}$ which crushes a body, whose cross-section is unity, the ***modulus of rupture*** or ***of ultimate*** *strength* of *compression*, and we denote it by K_1. If the cross-section of the prismatical body is F, we have

FIG. 313.



$$3)\quad \begin{cases} P = F\,K \text{ for the force, which will tear the body, and} \\ P_1 = F\,K_1 \text{ for the force, which will crush it.} \end{cases}$$

The cross-section of bodies is often determined from the modulus of rupture by substituting in the formulas

$$4)\quad \begin{cases} F = \frac{P}{K} \text{ and} \\ F_1 = \frac{P_1}{K_1} \end{cases}$$

instead of K the *working load* of rupture, I.E. a small part $\frac{K}{n}$ or $\frac{K_1}{n}$, E.G., a fourth, sixth, tenth, etc., of the numbers determined by experiment. We call n a *factor of safety.* If the proof strength of all substances were the same fraction of the ultimate strength, that is, if the ratios $\frac{A\,B}{A\,U} = \frac{T}{K}$ and $\frac{A\,B_1}{A\,U_1} = \frac{T_1}{K_1}$ were fixed constants, the determination of the cross-section by means of the moduli of proof strength would give the same result as that by means of the working load of rupture; but since this ratio is different for different bodies, the determinations by the aid of the moduli of proof strengths T and T_1, or rather by means of *the working or safe loads* $\frac{T}{m}$ and $\frac{T_1}{m}$, are generally more correct and proper, and the determination by the *working or safe loads* of rupture $\frac{K}{n}$ and $\frac{K_1}{n}$ is only to be employed, when the modulus of proof strength is unknown.

If the cross-section of a body is a circle, whose diameter is d, we have $\frac{\pi d^2}{4} = F$, whence $P = \frac{\pi d^2}{4} T = 0,7854\, d^2\, T$ and

$$d = \sqrt{\frac{4\,F}{\pi}} = 1,128 \sqrt{F} = 1,128 \sqrt{\frac{P}{T}}.$$

EXAMPLE 1.—What weight can a hanging column of fir support, if it is 5 inches wide and 4 inches thick? Assuming the modulus of proof strength to be 3000 pounds, the cross-section being $F = 5 \,.\, 4 = 20$ square inches, we have $P = F\,T = 20 \,.\, 3000 = 60000$ pounds as the proof load of this column. If, however, we assume the modulus of rupture to be $K = 10000$ pounds, and we desire a quadruple security, we have $P = F K = 20 \,.\, \frac{10000}{4} = 50000$ pounds. In order to be secure for a great length of time, we take but a tenth part of K, and obtain thus $P = 20 \,.\, 1000 = 20000$ pounds.

EXAMPLE 2.—A round wrought-iron rod is to be turned so as to bear a weight of 4500 pounds; what should be its diameter? Here T is 18700 pounds, whence $d = 1,128 \sqrt{\frac{4500}{18700}} = 1,128 \sqrt{\frac{45}{187}} = 0,553$ inches. The modulus of rupture of average wrought-iron is $= 58000$ pounds; if, however, we wish five-fold security, we take $K = 11600$ pounds, and we have

$$d = 1,128 \sqrt{\frac{4500}{11600}} = 1,128 \sqrt{\frac{45}{116}} = 0,7025 \text{ inches.}$$

§ 206. Modulus of Resilience and Fragility.—When we stretch a prismatical body by a force, which gradually increases from 0 to $P = A\,M = N\,O$, Fig. 314, and by this means lengthen it from 0 to $\lambda = M\,O = A\,N$, a certain amount of work is done, which is determined by the product of the space or total extension $A\,N$ and the mean value of the pull, which increases gradually from 0 to $P = N\,O$. This product can be expressed by the surface $A\,N\,O$, whose abscissa is the extension $A\,N = \lambda$ and whose ordinate is the pulling stress $N\,O = A\,M = P$. If the extension does not exceed the limit of elasticity, the surface $A\,N\,O$ can be considered as a right-angle triangle, whose base and altitude are λ and P, and the work done, corresponding to it, is

$$L = \tfrac{1}{2}\,\lambda\,P.$$

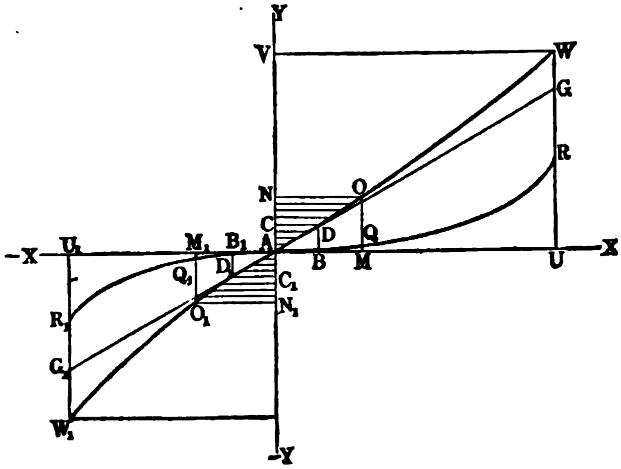
If we substitute in it

$$\lambda = \sigma\, l \text{ and } P = F\,T,$$

we obtain *the work to be done in stretching it to the limit of elasticity* σ

$$L = \tfrac{1}{2}\,\sigma\, l\,.\,F\,T = \tfrac{1}{2}\,\sigma\, T\,.\,F\,l = A\, V,$$

FIG. 314.



in which V denotes the volume $F\,l$ of the body and A a number, given by experiment, which is called *modulus of resilience for extension* and is determined by the expression

$$A = \tfrac{1}{2}\, A\, C\,.\,C\,D = \tfrac{1}{2}\,\sigma\, T = \tfrac{1}{2}\,\frac{T^2}{E} = \tfrac{1}{2}\,\sigma^2 E.$$

In like manner the *work* necessary *to compress it to the limit of elasticity* is

$$L_1 = V\, A_1,$$

in which

$$A_1 = \tfrac{1}{2}\, A\, C_1\,.\,C_1\, D_1 = \tfrac{1}{2}\,\sigma_1\, T_1 = \tfrac{1}{2}\,\frac{T_1^2}{E} = \tfrac{1}{2}\,\sigma_1^2\, E$$

denotes the modulus of *resilience for compression at the limit of elasticity.*

Similar formulas can be employed for the work done in tearing or crushing prismatical bodies; for the first case we have

$$L = V\, B,$$

and for the second,

$$L_1 = V\, B_1,$$

B = the surface $A\ U\, W$ denoting *the modulus of fragility for tearing;* and B_1 = the surface $A\ U_1\ W_1$, *the modulus of fragility for crushing.*

We see from the foregoing that the mechanical effect necessary to stretch or compress a prismatical body to the limit of elasticity, as well as that, which is necessary to produce a tearing or crushing of the same, is not at all dependent upon the different dimensions, but only upon the volume V of the body; that, E.G., for two prisms of the same material the expenditure of mechanical effect in producing rupture is the same, when one is twice as long as the other and the cross-section of the former but one-half that of the latter.

EXAMPLE.—If the modulus of elasticity of wrought iron is $E = 28000000$ pounds and the extension of the same at the limit of elasticity $\sigma = \frac{1}{1500}$, the modulus of proof strength is, since $\sigma = \frac{T}{E}$,

$$T = \sigma E = \frac{28000000}{1500} = 18700, \text{(approximatively)}$$

and consequently the modulus of resilience for extension is

$$A = \tfrac{1}{2} \sigma T = \frac{T^2}{2 E} = \tfrac{1}{2} \sigma^2 E = \frac{18700}{2 \,.\, 1500} = 6{,}28 \text{ pounds.}$$

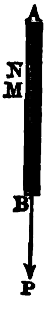
Hence, in order to stretch a prismatical body of wrought iron to the limit of elasticity, the mechanical effect

$$L = A V = 6{,}23\, V \text{ is necessary.}$$

If, E.G., the volume of this body were $V = 20$ cubic inches, the mechanical effect would be $L = 6{,}23 \,.\, 20 = 124{,}6$ inch-pounds $= \frac{124{,}6}{12}$ $= 10{,}38$ foot-pounds.

(§ 207.) **Extension of a Body by its Own Weight.**—If a prismatical body $A B$, Fig. 315, has a considerable length l, it undergoes, in consequence of its weight, a notable extension, which can be determined in the following manner. Let F denote the cross-section of the body, γ its heaviness or the weight of a cubic inch of the matter composing it and x the variable length of a portion of it; the tension in an element $M N$ is produced by the weight of the part of the body $B M$ lying below it, and consequently [according to § 204, (2)] the corresponding extension of the length $M N = \partial x$ of this element is

FIG. 315.



$$d \lambda = \frac{\gamma F x}{F E} d x = \frac{\gamma}{E} x\, d x.$$

By integration we obtain the extension of the entire piece $B M$

$$\lambda = \frac{\gamma}{E} \int x\, d x = \frac{\gamma x^2}{2 E},$$

and consequently that of the entire body $A B$ is

$$\lambda = \frac{\gamma l^2}{2E} = \frac{\gamma F l^2}{2FE} = \frac{\frac{1}{2}G}{FE}\,l,$$

in which $G = \gamma F l$ denotes the weight of the whole body.

If this weight was not equally distributed in the body, but applied at its end B, the extension would be

$$\lambda_1 = \frac{G l}{FE} = 2\,\lambda.$$

The extension $\lambda = \frac{1}{2}\lambda_1$ of a body in consequence of its own weight, is but one half as great as that produced by the same weight at the end of the body.

The same law obtains of course for the compression λ produced in a body by its own weight.

If in either case a pull or thrust P acts upon the body, we have the extension or compression produced

$$\lambda = \frac{Pl}{FE} \pm \frac{1}{2}\frac{Gl}{FE} = \frac{(P \pm \frac{1}{2}G)\,l}{FE},$$

in which the upper sign is to be employed, when the force P acts in the same direction as the weight G, and the lower one, when it acts in the opposite direction. In the latter case, the extension is of course smaller than when P is the only tensile or compressive force.

The total extension or compression is $= 0$, when

$$\tfrac{1}{2}G = P, \text{ or } G = \gamma F l = 2\,P, \text{ or}$$
$$l = \frac{2P}{\gamma F}.$$

The force P, acting at the end of the body, extends it equally in all parts, viz., in the ratio $\frac{\lambda}{l} = \frac{P}{FE}$, while, on the contrary, the weight G stretches or compresses it in the variable ratio $\frac{d\lambda}{dx} = \frac{\gamma x}{E}$. The ratio of the total extension at any point, at the distance x from the point of application of the force P, is

$$\frac{\lambda_1}{l} = \frac{\lambda}{l} \pm \frac{d\lambda}{dx} = \left(\frac{P}{F} \pm \gamma x\right)\frac{1}{E}.$$

If the force P acts in the same direction as G, the maximum ratio of extension or compression is for $x = l$, and it is then

$$\frac{\lambda_1}{l} = \left(\frac{P}{F} + \gamma l\right)\frac{1}{E} = \frac{P+G}{FE}$$

and, on the contrary, the minimum is for $x = 0$, I.E., at the point of application of P, and it is $\frac{\lambda_1}{l} = \frac{P}{FE}$.

If P and G act in opposite directions, we must distinguish the cases, in which $l < \frac{P}{F\gamma}$ and in which $l > \frac{P}{F\gamma}$. In the first case the ratio of extension or compression $\frac{\lambda_1}{l} = \left(\frac{P}{F} - \gamma x\right)\frac{1}{E}$ is a maximum for $x = 0$ and $= \frac{P}{EF}$, and a minimum and $= \left(\frac{P}{F} - \gamma l\right)\frac{1}{E}$ for $x = l$. In the latter case there is a positive maximum $\frac{P}{EF}$ for $x = 0$, and a negative maximum $\left(\gamma l - \frac{P}{F}\right)\frac{1}{E}$ for $x = l$, and, on the contrary, for $x = \frac{P}{F\gamma}$ the function becomes $=$ zero.

In order that the body shall be extended or compressed to the limit of elasticity only, the maximum of the ratio of extension or compression $\left(\frac{P}{F} \pm \gamma x\right)\frac{1}{E}$ should be at most $= \sigma = \frac{T}{E}$, or more simply the maximum of $\left(\frac{P}{F} \pm \gamma x\right) = T$. But, when P and G have the *same direction*, this maximum is

$$= \frac{P}{F} + \gamma l = \frac{P + \gamma F l}{F} = \frac{P + G}{F},$$

and therefore we must put $\frac{P + \gamma F l}{F} = T$, or $P = F(T - \gamma l)$, hence the required cross-section is

$$F = \frac{P}{T - \gamma l}.$$

If, on the contrary, the forces P and G act in opposite directions, we have two maxima, one $= \frac{P}{F}$ and the other $= \left(\gamma l - \frac{P}{F}\right)$, and therefore the corresponding cross-section is equal to the greater of the values

$$F = \frac{P}{T} \text{ and } F = \frac{P}{\gamma l} - T.$$

If in the formulas we substitute K instead of T, we obtain the conditions of tearing and crushing, that is, in the first case,

$P = F(K - \gamma l)$, and in the second either
$P = FK$ or $P = F(\gamma l - K)$.

For $P = 0$ we have either

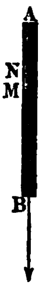
$$\gamma l - T = 0 \text{ and } l = \frac{T}{\gamma} \text{ or}$$

$$\gamma l - K = 0 \text{ and } l = \frac{K}{\gamma};$$

the first formula being applicable to the case, when the body is extended or compressed to the limits of elasticity, and the second to the case, when a tearing or crushing of the body takes place.

REMARK.—The energy stored by a body, which is extended or compressed by its own weight, can be calculated in the following manner. The element $M N$, Fig. 316, whose length is $d x$, is gradually stretched by the weight $\gamma F x$ of the portion of the body $B M$ an amount, which increases gradually from 0 to $d \lambda = \frac{\gamma x d x}{E}$, and the work done in accomplishing it is

FIG. 316.



$$= \tfrac{1}{2} \gamma F x \,.\, \delta \lambda = \tfrac{1}{2} \frac{\gamma^2 F x^2}{E} d x.$$

Integrating this expression, we obtain the expression for the quantity of work done in extending all the elements of the rod from B to M,

$$L = \tfrac{1}{2} \cdot \frac{\gamma^2 F}{E} \int x^2 d x = \tfrac{1}{2} \cdot \frac{\gamma^2 F x^3}{3 E},$$

and that done in extending the entire rod

$$L = \tfrac{1}{2} \cdot \frac{\gamma^2 F l^3}{3 E} = \tfrac{1}{2} \cdot \frac{\gamma^2 F^2 l^2 l}{3 F E} = \tfrac{1}{2} \cdot \frac{G^2 l}{3 F E} = \tfrac{1}{3} G \lambda,$$

in which (according to § 207) $\lambda = \tfrac{1}{2} \frac{G l}{F E}$ denotes the total extension of the rod.

EXAMPLE.—If a lead wire, whose modulus of rupture is $K = 3100$ and the weight of a cubic inch of which is $= 0,412$ pounds, is suspended vertically, it will break by its own weight, when its length is

$$l = \frac{K}{\gamma} = \frac{3100}{0,412} = 7524 \text{ inches} = 627 \text{ feet}.$$

If the modulus of proof strength is $T = 670$, it is stretched to the limit of elasticity, when its length is

$$l_1 = \frac{T}{\gamma} = \frac{670}{0,412} = 1626 \text{ inches} = 135,5 \text{ feet},$$

and if its modulus of elasticity is $E = 1000000$ pounds, we have for the corresponding extension

$$\lambda = \frac{T}{E} l_1 = \frac{670}{1000000} \,.\, 135,5 = 0,090785 \text{ feet} = 1,0894 \text{ inches}.$$

§ 208. Bodies of Uniform Strength.—If the pull or thrust P upon a vertical prismatical body is sensibly augmented by its weight G, we must of course put

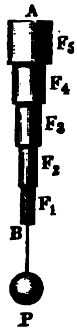
$$P + G = F T \text{ or } P = F T - G = F(T - l \gamma),$$

and determine the cross-section of this body by means of the formula (compare § 207)

$$F = \frac{P}{T - l\gamma}.$$

If this body, as, E.G., $A\ B$, Fig. 317, is composed of prismatical parts, we can save material by giving to each of these parts a cross-section calculated by means of this formula. If the length of these portions of the body are l_1, l_2, l_3, etc., and if the load P is gradually increased by the weights $F_1 l_1 \gamma$, $F_2 l_2 \gamma$, $F_3 l_3 \gamma$, etc., of the portions to P_1, P_2, P_3, etc., the required cross-section of the first portion is

FIG. 317.



$$F_1 = \frac{P}{T - l_1\gamma};$$

that of the second should be

$$F_2 = \frac{P_1}{T - l_2\gamma} = \frac{F_1 T}{T - l_2\gamma};$$

that of the third

$$F_3 = \frac{P_2}{T - l_3\gamma} = \frac{F_2 T}{T - l_3\gamma}, \text{ etc.}$$

If the length of all the parts is the same, or $l_1 = l_2 = l_3$, etc., $= l$, we have more simply

$$F_1 = \frac{P}{T - l\gamma} = \frac{P}{T}\left(\frac{T}{T - l\gamma}\right)$$

$$F_2 = \frac{F_1 T}{T - l\gamma} = \frac{P T}{(T - l\gamma)^2} = \frac{P}{T}\left(\frac{T}{T - l\gamma}\right)^2,$$

$$F_3 = \frac{F_2 T}{T - l\gamma} = \frac{P}{T}\left(\frac{T}{T - l\gamma}\right)^3, \text{ etc.,}$$

or in general for the cross-section of the nth portion

$$F_n = \frac{P}{T}\left(\frac{T}{T - l\gamma}\right)^n.$$

If the cross-section of all the pieces are to be the same, that cross-section should be

$$F = \frac{P}{T - n l\gamma} = \frac{P}{T}\left(\frac{T}{T - n l\gamma}\right).$$

While in this case the volume of the whole body would be

$$V = n F l = \frac{n P l}{T - n l\gamma},$$

in the former case, where every piece has its own proper cross-section, the volume is determined by the geometrical series

$$V_n = (F_1 + F_2 + \ldots + F_n)\, l$$
$$= \frac{P\, l}{T - l\gamma}\left[1 + \frac{T}{T - l\gamma} + \left(\frac{T}{T - l\gamma}\right)^2 + \ldots + \left(\frac{T}{T - l\gamma}\right)^{n-1}\right].$$

But the sum of the geometrical series in the parenthesis is (see Ingenieur, page 82)

$$= \left[\left(\frac{T}{T - l\gamma}\right)^n - 1\right] : \left(\frac{T}{T - l\gamma} - 1\right);$$

whence it follows, that

$$V_n = \frac{P}{\gamma}\left[\left(\frac{T}{T - l\gamma}\right)^n - 1\right] = \frac{(F_n - F_1)\, T}{\gamma},$$

and that the weight of the whole body is

$$G = (F_n - F_1)\, T.$$

If the length l of the parts is very small, and, on the contrary, their number n very great, and if we denote the total length $n\, l$ by a, we have, reasoning as in § 194,

$$(T - l\gamma)^n = \left(T - \frac{a\gamma}{n}\right)^n = T^n\left(1 - \frac{a}{n}\frac{\gamma}{T}\right)^n = T^n e^{-\frac{a\gamma}{T}},$$

in which $e = 2{,}71828$ is the base of the Naperian system of logarithms, and therefore we have

$$F_n = \frac{P}{T}\left(\frac{T}{T - l\gamma}\right)^n = \frac{P}{T e^{-\frac{a\gamma}{T}}} = \frac{P}{T} e^{\frac{a\gamma}{T}} = F_0\, e^{\frac{a\gamma}{T}},$$

in which $F_0 = \frac{P}{T}$ denotes the area of the first cross-section.

We have also approximatively

$$F_n = \frac{P}{T}\left[1 + \frac{a\gamma}{T} + \tfrac{1}{2}\left(\frac{a\gamma}{T}\right)^2\right],$$

and, on the contrary,

$$F = \frac{P}{T}\left[1 + \frac{a\gamma}{T} + \left(\frac{a\gamma}{T}\right)^2\right].$$

The volume of the body, composed of very many small portions, is found in the manner shown above to be

$$V_n = \frac{P}{\gamma}\left[\left(\frac{T}{T - l\gamma}\right)^n - 1\right] = \frac{P}{\gamma}\left(e^{\frac{a\gamma}{T}} - 1\right),$$

approximatively

$$= \frac{P\, a}{T}\left[1 + \tfrac{1}{2}\left(\frac{a\gamma}{T}\right) + \tfrac{1}{6}\left(\frac{a\gamma}{T}\right)^2\right],$$

while on the contrary, the volume of the body with a constant cross-section is approximatively

$$V = \frac{P\,a}{T - a\gamma} = \frac{P\,a}{T}\left[1 + \frac{a\gamma}{T} + \left(\frac{a\gamma}{T}\right)^2\right].$$

The formulas

$$F_x = \frac{P}{T} e^{\frac{a\gamma}{T}} \text{ and } V_x = \frac{P}{\gamma}\left(e^{\frac{a\gamma}{T}} - 1\right)$$

hold good, of course, for every body, such as *A B*, Fig. 318, and *A B*, Fig. 319, in which there is a constant variation of the cross-section. In order to find the cross-section F_x for any position *M* and the volume of the body cut off at the same point, we have only to substitute in this formula for *a* the distance *B M* of the given position from the point of application *B* of the tensile or compressive force. The bodies thus determined have at every point a cross-section corresponding to the load they support, and are therefore called ***bodies of uniform strength*** (Fr. solides d'égale résistance, Ger. Körper von gleichem Widerstande). These bodies have (the other circumstances being the same) the smallest volume, require therefore the least quantity of material and are for this reason generally the cheapest and most advantageous that we can employ. If we compare such a body with a prismatical one, we find from the above approximate formulas, that the economy of volume is

FIG. 318.

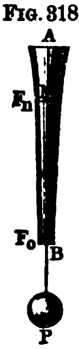
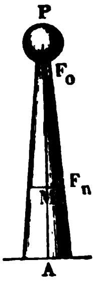


FIG. 319.



$$V - V_n = \frac{P\,a}{T}\left[\frac{1}{2}\frac{a\gamma}{T} + \frac{5}{6}\left(\frac{a\gamma}{T}\right)^2\right] = \frac{P\,a^2\gamma}{2\,T^2}\left(1 + \frac{5}{3}\frac{a\gamma}{T}\right).$$

REMARK.—Since the relative extension and compression of a body of uniform strength is everywhere the same, viz., $\sigma = \frac{T}{E}$, its total extension is $\lambda = \sigma\,a = \frac{T}{E}\,a$, while for a prismatical body it is only

$$\lambda = \frac{(P + \frac{1}{2}\,G)\,a}{F E} = \frac{P + \frac{1}{2}\,G}{P + G} \cdot \frac{T}{E}\,a.$$

EXAMPLE.—What must be the cross-section of a wrought-iron pump rod, whose length is 1000 feet, when, in addition to its own weight, it must support a load $P = 75000$ pounds? If instead of the modulus of proof strength $T = 18600$ we employ for safety a working load $\frac{T}{2} = 9300$ pounds and put the weight of a cubic inch of wrought-iron

$$\gamma = \frac{7{,}70\ .\ 62{,}425}{12\ .\ 12\ .\ 12} = 0{,}2782 \text{ pounds,}$$

the required cross-section is

$$F = \frac{P}{T - a\gamma} = \frac{75000}{9800 - 12000 \,.\, 0{,}2782} = \frac{75000}{5962} = 12{,}58 \text{ square inches,}$$

and the weight of the rod is

$$G = F \,.\, a\,\gamma = 12{,}58 \,.\, 12000 \,.\, 0{,}2782 = 42000 \text{ pounds.}$$

If we could give this rod the form of a body of uniform strength, we would have for the smallest cross-section

$$F_0 = \frac{P}{T} = \frac{75000}{9800} = 8{,}06 \text{ square inches,}$$

and for the greatest

$$F_n = 8{,}06 \,.\, e^{0{,}2782 \,.\, 1{,}29} = 8{,}06\, e^{0{,}3589} = 8{,}06 \,.\, 1{,}432 = 11{,}54 \text{ square inches,}$$

and the weight of the rod would be

$$G_n = V_n \gamma = (F_n - F)\; T = (11{,}54 - 8{,}06)\; 9800 = 32364 \text{ pounds.}$$

If the modulus of elasticity of wrought iron is $E = 28000000$ pounds, the extension of the rod in the latter case would be

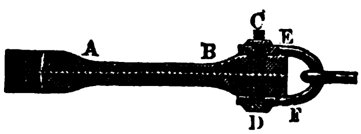
$$\lambda = \frac{T}{E}\, a = \frac{18600 \,.\, 1000}{28000000} = \frac{186}{280} = \frac{93}{140} \text{ feet} = 7{,}97 \text{ inches,}$$

and, on the contrary, in the first case it is

$$\frac{P + \frac{1}{2} G}{P + G}\, \lambda = \frac{75000 + 21000}{75000 + 42000} \,.\, 7{,}97 = \frac{96000}{117000} \,.\, 7{,}97 = 6{,}54 \text{ inches.}$$

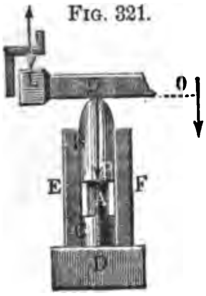
§ 209. Experiments upon Extension and Compression.—In order to study thoroughly the laws of the elasticity of any substance, it is necessary not only to submit prismatical bodies of this substance (which should be made as long as possible) to extension or compression by weights, which are gradually increased in amount until rupture is produced, but also to observe the exact extension or compression produced by each weight. If we place the bodies to be experimented upon in a vertical position, the weights can be hung or laid upon them, and they then give directly the pull or thrust to which the body is subjected. In order to avoid experimenting with too great weights, we generally prefer to let the weights act upon the body by means of a lever with unequal arms; the weights are always hung upon the long arm (a), and the body is acted upon by the shorter arm (b). Multiplying the weight G by the ratio $\frac{a}{b}$ of the arms, we find the corresponding pull or thrust $P = \frac{a}{b}\, G$. The so-called hydraulic press can also be employed with advantage instead of weights to produce very great tensile or compressive forces. In order to observe the amount of the extension or compression, a fine line is drawn upon the bar to be experimented with near each of its ends, or a pair of pointers, with verniers attached, are fastened to it at those points, and in order to determine not only the elastic, but also the permanent extension or set, we measure the distance between these lines or pointers not only before and during the application of the weights, but also after they have been removed, and it is generally preferable to allow several minutes or even hours to elapse between the application or removal of the weights and the measurement; for when the forces are very great the extension and compression do not assume the true value in a moment, but only after a certain time. This distance is measured either with a bar compass or directly by means of a division on the rod itself. The so-called cathometer is also employed for this purpose; it consists essentially of a vertical staff and of a spirit-level, which is capable of sliding up and down the former (see Ingenieur, page 234). In order to observe the compression on long rods, we must enclose them in tube-shaped guides; they must also be well greased from time to time, so that they can slide without resistance in their guides.

FIG. 320.



If we wish to determine the modulus of ultimate strength of a body, we can employ shorter pieces for the experiments. In *experimenting upon rupture by extension* we employ bodies with large heads *A* and *B*, Fig. 320, through which holes are bored exactly in the axis. In the middle of each hole a circular knife-edge is made, so that the body shall be pulled exactly in the line of the axis by means of the bolt *C D* and the clevis *F E*, which is applied to its ends.

FIG. 321.

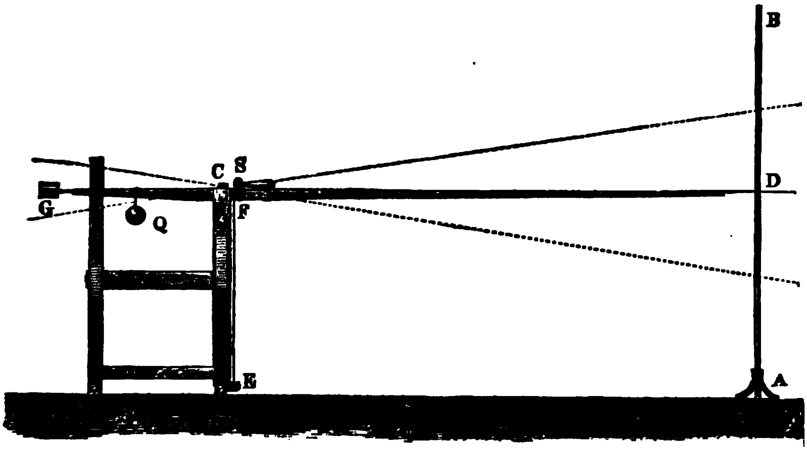


In *experimenting upon rupture by crushing*, the two bases of the body (*A*, Fig. 321) are made parallel, it is then brought between two cylinders *B* and *C*, whose bases are ground flat; while the rounded head of one of the cylinders is acted on by the compressive force, the other is supported by the large bed-plate *D*, and both slide in the interior of cylinder *E F*. The pressure *P* upon the head *H* of the cylinder is produced either by a hydraulic press or by a one-armed lever *L O*, such as is partially represented in the figure.

While the rupture of a body by tearing occurs in the smallest cross-section, and the body is therefore divided in two parts only, the rupture by crushing takes place generally in inclined surfaces, and the body is divided into several pieces. Prismatical bodies are divided, in the first place, into two pyramids, whose bases are those of the body and whose apexes are at its centre, and in the second place, into other pyramidical bodies, whose bases form the sides of the body and whose apexes are also situated at its centre. Bodies, whose structure in different directions is different, of course do not act thus; E.G., a piece of wood would be compressed by a force acting in the direction of the fibres, in such manner, that at its smallest cross-section the fibres would be bent out in a spherical form.

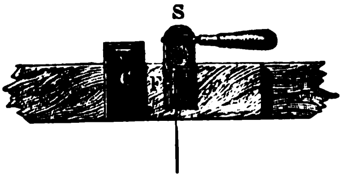
§ **210. Experiments upon Extension.**—We are indebted to Gerstner for the first thorough experiments upon the extension and elasticity of iron wire. He employed in his experiments iron wire from 0,2 to 0,8 lines in diameter and made use of the lever apparatus represented in Fig. 322 with the pointer *C D* 15 feet long, the counter-balance *G* and the sliding weight *Q*. The wire *E F*, which was about 4 feet long, was firmly fastened at one end *E* and the other was wound round a pin *F*, which was turned by the

FIG. 322.



endless screw S, so that the wire could be subjected to any desired strain. The extension of the wire was shown by the pointer D upon a rod $A\ B$ in 54 times its natural size. The knife-edge C of the lever, the pin F, around which the upper end of the wire is wound, and the endless screw S, which turns the pin, are all represented on a larger scale in Fig. 323.

FIG. 323.



Gerstner proves by his experiments, that every extension is the sum of two extensions, one of which (*the elastic extension*) disappears, when the weight is removed, and the other (*the permanent extension*, or *set*) remains, so that the extension λ is not exactly proportional to P within the limits of elasticity, and that it is more proper to replace the formula

$$P = \frac{\lambda}{l} F E \ [\S\ 204\ (4)]$$

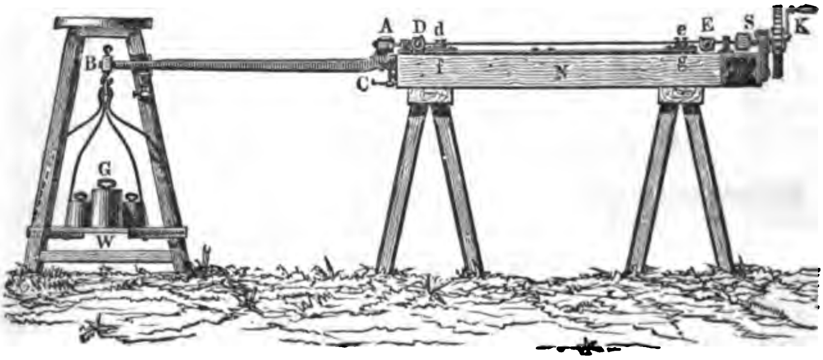
by the following series

$$P = \frac{\lambda}{l}\left[1 + \alpha \frac{\lambda}{l} + \beta \left(\frac{\lambda}{l}\right)^2\right] F E,$$

in which α and β are numbers determined by experiment.

Quite extensive experiments upon the elasticity and strength of wrought iron and iron wire were afterwards made by Lagerhjelm and by Brix. Both experimenters employed in their researches a bent lever $A\ C\ B$, Fig. 324, the longer arm $C\ B$ of which was depressed by the weights G, which were laid upon a scale-pan W, and

FIG. 324.



thus the iron bar or wire $D\,E$, which was fastened to the shorter arm $C\,A$, was stretched to any desired extent. In the apparatus used by Brix, the ratio of the arms of the lever was $\frac{C\,A}{C\,B} = \frac{1}{20}$, and one end D of the wire was attached to the arm $C\,A$ with clamps, hooks and bolts, and the other end was fastened in the same way to a screw S, which was turned by means of a train of wheels by a crank K. The increase in length was given by two verniers, which were screwed fast to the ends of the wire and moved along two scales divided into quarter lines. When the wire had been firmly fastened in the clamps, the scale-pan was gradually loaded with heavy weights, and in each experiment the wire was stretched by turning the crank K until the lever was lifted from its support and the tension of the wire balanced the weight G. The experiments were made with wire $1\frac{1}{3}$ to $1\frac{1}{2}$ lines thick and gave for the average value of the modulus of rupture of unannealed wire $K = 98000$ pounds, and, on the contrary, after annealing, $K = 64500$ pounds. The average modulus of elasticity, on the contrary, for annealed and unannealed wire was found to be $E = 29000000$ pounds; it was also found, that the limit of elasticity was reached, when the strain was $0{,}5\ K$ for unannealed and $0{,}6\ K$ for annealed wire.

When the tensions were greater, the extension became permanent, and the total extension of unannealed wire at the instant of rupture was

$$\frac{\lambda}{l} = 0{,}0034, \text{ and that of annealed wire } \frac{\lambda}{l} = 0{,}0885,$$

or 26 times as much. In the apparatus used by Lagerhjelm the tension on the wire was produced by a hydraulic press, the piston rod of which was attached to the end of the iron bar.

Lagerhjelm employed in his experiments iron rods 36 inches long, $\frac{1}{2}$ inch thick, the cross-sections of which were circular and square. According to his experiments, the average modulus of elasticity for Swedish wrought iron is

$$E = 46000000 \text{ pounds};$$

the modulus of rupture or of ultimate strength is

$$K = \frac{1}{500}\,E = 92000 \text{ pounds};$$

and the modulus of proof strength

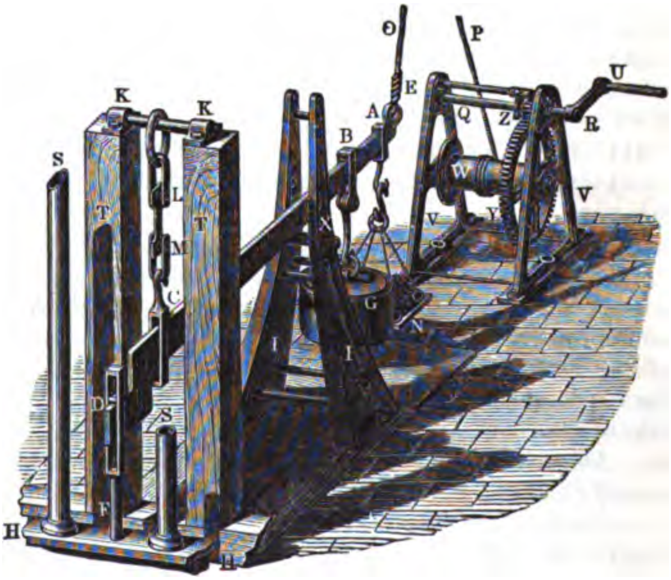
$$T = \sigma\,.\,E = \frac{1}{1600}\,.\,46000000 = 28750 \text{ pounds.}$$

Wertheim, in his experiments upon the elasticity and cohesion of the metals, allowed the wire to hang freely, and fastened to the end of the same a weight-box, which was supported upon the floor by means of feet, which could be raised or lowered by turning a screw. In order to stretch the wire by means of the weights placed in the box, the foot-screws were turned until the box swung freely. A cathometer was employed to determine the extension of the wire.

The experiments were performed at very different temperatures, and with wire made of various metals, such as iron, steel, brass, tin, lead, zinc, silver, etc. The principal results of these experiments will be found in the table given in § 212.

The apparatus, with which Fairbairn performed his experiments, consists essentially of a strong wrought-iron lever or balance-beam *A C D*, Fig. 325, whose fulcrum *D* is firmly retained by a strong bolt *F*, which can be raised or lowered by means of a nut. Two iron pillars give the necessary resistance to the bed-plate *H H*, through which *F* passes. The piece of iron *L M* to be experimented upon is suspended by means of a chain to the support *K K*, which reposes upon the two columns *T T* and is connected by a bolt and clevis to the stirrup *C* of the lever *A C D*. To the longer

FIG. 325.



arm of the latter there is suspended not only a constant weight G, but also a scale-board for the reception of smaller weights; the bolt X serves to support the lever, and the latter is raised by means of a rope $O\ P$, which passes over a pulley and is wound upon the shaft W of a windlass $U\ Y\ Z$. After the weights had been laid on, the arm E of the lever was allowed to sink gradually by turning the crank U, until the piece of iron to be tested was finally subjected to the tension produced by N and G.

REMARK.—Gerstner's experiments upon the elasticity of iron wire, etc., are discussed in Gerstner's Mechanics, Vol. I. For the experiments of Lagerhjelm, see Pfaff's translation of the treatise: Researches for the purpose of determining the density, homogeneity, elasticity, malleability, and strength of bar iron, etc., by Lagerhjelm (Nürnberg, 1829), and the information in regard to the experiments of Brix is to be found in the treatise on the cohesion and elasticity of some of the iron wires employed in the construction of suspension bridges (Berlin, 1837).

The experiments of Wertheim upon the elasticity and cohesion of the metals, etc., as well as of glass and wood, are discussed in "Poggendorf's Annalen der Physik und Chemie," Ergänzungsband II, 1845. In the latter experiments the modulus of elasticity of the bodies named was determined not only by experiments upon extension, but also by experiments upon flexion and vibration. For Fairbairn's experiments on the strength of materials, his "Useful Information for Engineers" can be consulted.

§ 211. Iron and Wood.—The most complete set of experiments upon the elasticity and strength of cast and wrought iron are those more recently made by Hodgkinson. By these we have for the first time acquired a complete knowledge of the laws of extension and compression for these materials, which are of such great importance in their practical applications. Although, according to these experiments, iron produced in different ways has different degrees of elasticity and strength, yet it is possible to express the behavior of this body in regard to extension and compression by means of curves.

The average modulus of elasticity of *cast iron* (Fr. fonte, Ger. Gusseisen) is, according to these experiments, for extension as well as for compression

$E =$ 1000000 kilograms, when the cross-section is one centimeter, and consequently

$E = 14{,}22\ .\ 1000000 =$ 14220000 pounds when the cross-section is one inch.

The extension at the limit of elasticity is

$$\sigma = \frac{\lambda}{l} = \frac{1}{1500}.$$

This extension corresponds to the modulus of proof strength

$$T = \frac{1000000}{1500} = 667 \text{ kilograms, or}$$

$$T = \frac{14220000}{1500} = 9480 \text{ pounds.}$$

The compression at the limit of elasticity, on the contrary, is

$$\sigma_1 = \frac{1}{750},$$

and therefore the modulus of proof strength is

$$T_1 = \frac{1000000}{750} = 1383 \text{ kilograms} = \frac{14220000}{750} = 18960 \text{ pounds.}$$

The modulus of rupture for tearing was found by these experiments to be

$$K = 1300 \text{ kilograms} = 18486 \text{ pounds,}$$

and, on the contrary, that for crushing

$$K_1 = 7200 \text{ kilograms} = 102400 \text{ pounds.}$$

The resistance of cast iron to crushing is, therefore, $5\frac{1}{2}$ times as great as that to tearing.

For *wrought iron* (Fr. fer; Ger. Schmiedeisen) we have for extension as well as compression

$$E = 2000000 \text{ kilograms} = 28440000 \text{ pounds,}$$

and the limit of elasticity is reached, when $\sigma = \frac{\lambda}{l} = \frac{1}{1500}$, whence the modulus of proof strength is

$$T = \frac{2000000}{1500} = 1333 \text{ kilograms} = 18960 \text{ pounds.}$$

Finally the modulus of rupture or of ultimate strength of wrought iron was found to be for tearing

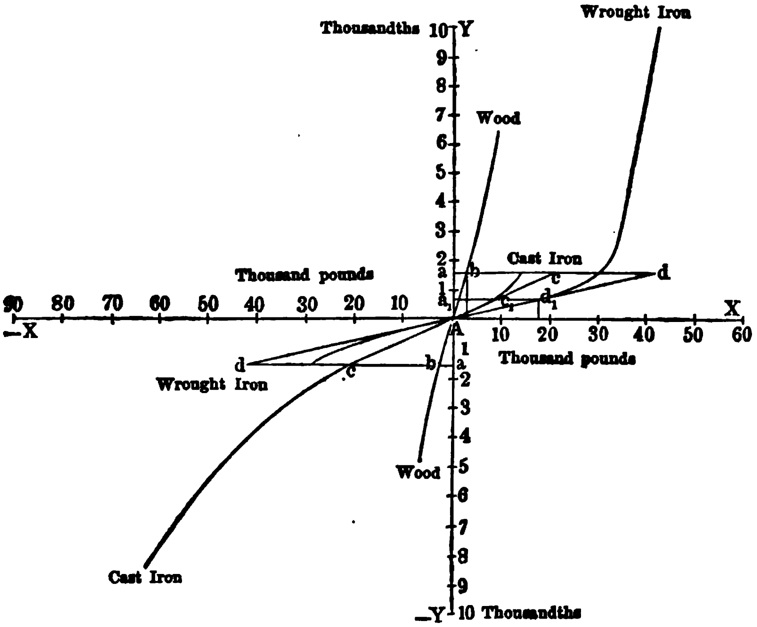
$$K = 4000 \text{ kilograms} = 56880 \text{ pounds,}$$

and for crushing

$$K_1 = 3000 \text{ kilograms} = 42660 \text{ pounds.}$$

The modulus of elasticity of wrought iron is therefore about double that of cast iron, and while the modulus of rupture by tearing of cast iron is but about $\frac{1}{3}$ that of wrought iron, the modulus of rupture by crushing of cast iron is nearly $2\frac{1}{2}$ times as great as that of wrought iron. The relations of the elasticity and strength of cast and wrought iron are graphically represented in Fig. 326. From the origin A on the right-hand side of the axis of abscissas $X\overline{X}$ the tensile forces, given in thousand pounds per square inch, are laid off and on the left-hand side the compressive forces, while the upper half of the axis of ordinates $Y\,\overline{Y}$ represents the corresponding extensions, and the lower half the compressions. It will at once strike the eye, that the curve of cast iron has a great development on the side of compression and that of wrought iron on the side of extension; and we also remark, that the curves form approximatively straight lines near the origin A.

FIG. 826.



As next to iron *wood* (Fr. bois; Ger. Holz) is most generally employed in construction, the relations of the elasticity of fir, beach and oak wood are graphically represented in the figure by a curve. The average modulus of elasticity of these kinds of wood is

$$E = 110000 \text{ kilograms} = 1564200 \text{ pounds}.$$

The limit of elasticity is reached, when $\sigma = \frac{1}{600}$ of the length, and the corresponding modulus of proof strength is

$$T = \frac{110000}{600} = 180 \text{ kilograms} = 2607 \text{ pounds}.$$

Finally, the modulus of rupture for tearing is

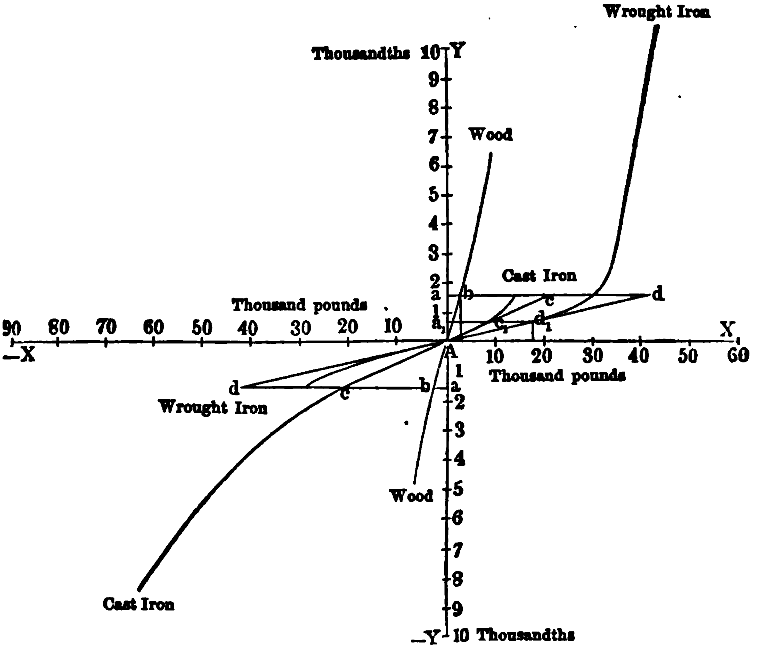
$$K = 650 \text{ kilograms} = 9243 \text{ pounds},$$

and, on the contrary, for crushing

$$K = 450 \text{ kilograms} = 6399 \text{ pounds.}$$

The ratio 156 : 1422 : 2844 approximatively = 1 : 9 : 19 of the moduli of elasticity of wood, cast and wrought iron to each other is expressed in the figure by the subtangents *ab*, *ac* and *ad*.

FIG. 327.



The modulus of resilience $A = \frac{1}{2} \sigma T$ for the limit of elasticity is expressed by the triangles $A\,a\,b$, $A\,a_1\,c_1$ and $A\,a_1\,d_1$, the bases of which are the small ratios of extension $\sigma = A\,a = \frac{1}{600}$ and $\sigma = A\,a_1 = \frac{1}{1500}$ (approximatively).

From the above, we have for wood

$$A = \tfrac{1}{2}\,\sigma\,T = \tfrac{1}{2} \cdot \frac{1}{600} \cdot 180 = 0{,}15 \text{ kilogram centimeters}$$

$$= \tfrac{1}{2} \cdot \frac{1}{600} \cdot 2607 = 2{,}17 \text{ inch-pounds,}$$

for cast iron

$A = \frac{1}{2} \cdot \frac{1}{1500} \cdot 667 = 0{,}222$ kilogram centimeters $= 3{,}16$ inch-pounds, and for wrought iron

$A = \frac{1}{2} \cdot \frac{1333}{1500} = 0{,}444$ kilogram centimeters $= 6{,}32$ inch-pounds.

Properly, a complete series of experiments is necessary to determine the modulus of fragility for tearing or crushing; for this modulus is found by the quadrature (see Art. 29, Introduction to the Calculus) of the complete branches of the curve on either side, and this is especially necessary for the extension of wrought iron and for the compression of cast iron, since the curves corresponding to the changes in these bodies differ considerably from right lines. The extension and compression of wood at the instant of rupture by tearing or crushing is so little known, that we are unable to give with any degree of certainty its moduli of fragility. If we treat the corresponding curve as a right line, we obtain the modulus of resilience for tearing

$B = \frac{1}{2} \frac{K^2}{E} = \frac{650^2}{110000} = 1{,}91$ kilogram centimeters $= 27{,}2$ inch-pounds, and, on the contrary, the modulus of fragility for crushing is

$B = \frac{1}{2} \frac{K_1^2}{E} = \frac{1}{2} \cdot \frac{450^2}{110000} = 0{,}92$ kilogram centimetres $= 13{,}07$ inch-lbs.

When cast iron is ruptured by tearing, assuming the extension to be $\sigma_1 = 0{,}0016$ and the mean value of the force to be 560 kilograms, the modulus of fragility is

$B = 0{,}0016 \,.\, 650 = 1{,}04$ kilogram-centimetres $= 14{,}8$ inch-lbs.

When cast iron is ruptured by crushing, the maximum extension can be assumed to be $\sigma_1 = 0{,}008$ and the mean crushing force to be $= 3600$ kilograms; hence the corresponding modulus of fragility is

$B_1 = 0{,}008 \,.\, 3600 = 29$ kilogram-centimetres $= 411$ inch-lbs.

We can assume as the mean value of σ_1 for the rupture of wrought iron by tearing, 0,008 and for the mean value of the force 3000 kilograms; hence the corresponding modulus of fragility is

$B = 0{,}008 \,.\, 3000 = 24$ kilogram-centimetres $= 341$ inch-lbs.

On the contrary, for the rupture of wrought iron by crushing, we must assume $\sigma = 0{,}0018$ and the mean force to be $= 1300$ kilograms; whence the corresponding modulus of fragility is

$B = 0{,}0018 \,.\, 1300 = 2{,}34$ kilogram-centimetres $= 33{,}3$ inch-lbs.

§ 212. Numbers Determined by Experiment.—In the following tables I and II the mean values of the moduli of elasticity, of proof strength and of ultimate strength of the materials generally employed in constructions are given. The first table is for tensile and the second for compressive forces.

The value of the relative extension $\sigma = \frac{\lambda}{l}$ for the limit of elasticity given in the second column of the tables expresses also the ratio $\frac{T}{E}$ of the values of T and E given in the third and fourth columns. In practice the bodies are only loaded with $\frac{1}{m}\ T$, E.G., $\frac{1}{3}\ T$ to $\frac{1}{2}\ T$, or the cross-section is determined by substituting in the formula

$$F = \frac{P}{K},$$

instead of K, for metals the *modulus of safe load* $\frac{1}{n}\ K = \frac{1}{6}\ K$, for wood and stone $= \frac{1}{10}\ K$, and for masonry but $\frac{1}{20}\ K$. On the contrary, for ropes we can employ $\frac{1}{3}\ K$ to $\frac{1}{5}\ K$. We call n a *factor of safety.*

The lower numbers in the parenthesis $\left\{\ \right\}$ give the values in kilograms, assuming a cross-section of 1 centimetre square; the upper numbers express the values in pounds referred to a cross-section of one square inch.

REMARK.—The moduli given in these tables are for unannealed metals. For *annealed metals* (Fr. metaux cuits, Ger. ausgeglüte Metalle) the modulus of elasticity is generally the same as for unannealed metals, while the modulus of rupture by tearing of annealed metals is generally from 30 to 40 per cent. less than that of unannealed ones. *Tempered* and *annealed steel* (Fr. acier trempé et recuit, Ger. gehärteter und angelassener Stahl) has the same modulus of elasticity as untempered steel, but its modulus of proof strength is 20 to 30 per cent. greater than that of untempered steel. When it is not otherwise stated, the moduli for metals were determined with wire, which had on the outside a harder crust (caused by the drawing) than hammered or cast metal rods. For some materials, E.G. wood, iron, and stone, the moduli of elasticity, of proof strength and of ultimate strength vary so much that in particular cases a value differing 25 per cent. (more or less) from those here given may be found.

TABLE I.

MODULI OF ELASTICITY AND STRENGTH FOR EXTENSION.

Name of the material.	Extension $\sigma = \frac{\lambda}{l}$ at the limit of Elasticity.	Modulus of Elasticity E.	Modulus of proof strength $T = \sigma E$.	Modulus of Resilience $A = \frac{1}{2} \sigma T$.	Modulus of Ultimate Strength K.
Cast iron........	$\frac{1}{1500} = 0{,}000667$	14 220000 1 000000	9480 667	3,16 0,222	18500 1300
Wro't iron in rods.	$\frac{1}{1500} = 0{,}000667$	28 000000 1 970000	18700 1313	6,23 0,44	58200 4090
in wire........	$\frac{1}{1000} = 0{,}001000$	31 000000 2 190000	31000 2190	15,5 1,10	88300 6210
in sheets.......	$\frac{1}{1250} = 0{,}000800$	26 000000 1 830000	20800 1475	8,32 1,18	46800 3290
German steel, tempered and annealed	$\frac{1}{835} = 0{,}001198$	29 000000 2 050000	34730 2460	20,8 1,48	116500 8190
Fine cast steel....	$\frac{1}{450} = 0{,}002222$	41 500000 2 920000	92200 6490	102,4 7,20	145500 10230
Hammered copper	$\frac{1}{4000} = 0{,}000250$	15 640000 1 100000	3910 275	0,49 0,034	33800 2380
Sheet copper.....	$\frac{1}{3650} = 0{,}000274$	15 640000 1 100000	4285 301	0,59 0,041	30400 2140
Copper wire......	$\frac{1}{1000} = 0{,}001000$	1 720000 1 210000	1720 1210	8,60 0,605	60300 4240
Zinc, melted......	$\frac{1}{4150} = 0{,}000241$	13 500000 950000	3250 229	0,392 0,029	7500 526
Brass............	$\frac{1}{1320} = 0{,}000758$	9 100000 640000	6890 485	2,61 0,184	17700 1242
Brass wire........	$\frac{1}{742} = 0{,}001350$	14 000000 987000	18900 1330	12,76 0,90	51960 3654
Bronze, gun metal..	$\frac{1}{1590} = 0{,}000629$	9 800000 690000	6160 434	1,94 0,136	36400 2560
Lead............	$\frac{1}{477} = 0{,}00210$	711000 50000	1490 105	1,56 0,110	1850 130
Lead wire........	$\frac{1}{1500} = 0{,}000667$	1 000000 70000	667 47	0,22 0,016	3100 220

MODULI OF ELASTICITY AND STRENGTH FOR EXTENSION—*Continued.*

Name of the material.	Extension $\sigma = \frac{\lambda}{l}$ at the limit of Elasticity.	Modulus of Elasticity E.	Modulus of proof strength $T = \sigma E$.	Modulus of Resilience $A = \frac{1}{2}\sigma T$.	Modulus of Ultimate Strength K.
Tin	$\frac{1}{900} = 0{,}001111$	5 700000 400000	6300 440	3,50 0,24	5000 350
Silver	$\frac{1}{660} = 0{,}001515$	10 400000 730000	15800 1100	12,00 0,83	41200 2900
Gold	$\frac{1}{600} = 0{,}001667$	11 400000 800000	19000 1300	15,8 1,09	38400 2700
Platina	$\frac{1}{600} = 0{,}001667$	22 800000 1 600000	38000 2700	31,7 2,25	48300 3400
Aluminum	—	9 6000000 675000	— —	— —	28900 2030
Glass	—	10 000000 700000	— —	— —	3530 248
Wood: beach, oak, pine, spruce, fir, in the direction of the fibres	$\frac{1}{600} = 0{,}001667$	1 560000 110000	2600 180	2,17 0,15	9200 650
The same kinds of wood in the direction of the radii to the yearly rings	—	185000 13000	— —	— —	570 40
The same kinds of wood parallel to the yearly rings	—	114000 8000	— —	— —	640 45
Light hemp rope	—	—	—	—	8700 610
Strong hemp rope	—	—	—	—	6830 480
Wire rope	—	—	—	—	47000 3300
Chain cable	—	—	—	—	51900 3650
Leather straps (cow leather)	—	10400 731	— —	— —	4100 290
Sheet iron (riveted with one row of rivets)	—	—	—	—	37000 2600

TABLE II.

THE MODULI OF ELASTICITY AND STRENGTH FOR COMPRESSION.

Name of the material.	Compression $\sigma = \frac{\lambda}{l}$ at the limit of elasticity.	Modulus of elasticity E.	Modulus of proof strength $T = \sigma E$.	Modulus of resilience $A = \frac{1}{2} \sigma T$.	Modulus of ultimate strength K.
Cast iron .	$\frac{1}{750} = 0{,}001333$	14000000 990000	18700 1320	12,44 0,88	104000 7310
Wrought "	$\frac{1}{1500} = 0{,}000667$	28000000 1970000	18700 1320	6,23 0,44	31000 2200
Copper . .	$\frac{1}{4000} = 0{,}000250$	15640000 1100000	3910 275	0,49 0,039	58300 4100
Brass . . .	—	—	—	—	10400 731
Lead. . . .	—	—	—	—	7250 510
Wood in the direction of the fibre. . . .	—	—	—	—	6800 480
Basalt . . .	—	—	—	—	28000 1970
Gneiss and granite . .	—	—	—	—	8300 585
Limestone.	—	—	—	—	5200 365
Sandstone.	—	—	—	—	4150 292
Brick . . .	—	—	—	—	830 59
Mortar. . .	—	—	—	—	526 37

EXAMPLE 1. What should be the cross-section of a wrought-iron rod 1500 feet long, which is subjected to a pull of 60000 pounds?

Neglecting the weight of the rod and allowing a strain of $\frac{T}{2} = 9350$ pounds per square inch, we obtain the required cross-section $F = \frac{60000}{9350} = 6{,}42$ square inches. Taking into account the weight of the rod, the weight of a cubic inch of iron being $\gamma = 0{,}280$ pounds, we have

$$F = \frac{60000}{9350 - 1500 . 12 . 0,280} = \frac{60000}{9350 - 5040} = \frac{6000}{431} = 18,92 \text{ square inches.}$$

The weight of the rod is $G = F l \gamma = 5040 . 13,92 = 70157$ pounds, and the extension of the same by the pull $P = 60000$ pounds and by the weight $G = 70157$ pounds is

$$\lambda = \frac{(P + \frac{1}{2} G) l}{F E} = \frac{95078 . 18000}{13,92 . 28000000} = \frac{142617}{32480} = 4,39 \text{ inches.}$$

Example 2. How thick must the foundation walls of a building 60 feet long and 40 feet wide on the outside, and weighing 85000000 pounds, be made when we employ good cut pieces of gneiss? If we make the thickness of the wall equal to x, we can put the mean length of the wall $= 60 - x$ and the mean breadth $= 40 - x$, and therefore the mean periphery $2 . (60 - x + 40 - x) = 200 - 4 x$, and consequently the base of the whole masonry is $(200 - 4 x) x$ square feet $= 144 (200 - 4 x) x = 576 (50 - x) x$ square inches. The modulus of rupture of gneiss for crushing is 8300 pounds. If, therefore, we assume a coefficient of security of $\frac{1}{20}$ or a factor of safety of 20 for the wall, we can put the allowable pressure upon a square inch $= \frac{8300}{20} = 415$ pounds; hence we have

$$415 . 576 (50 - x) x = 35000000,$$

whence

$$50 x - x^2 = 146,4,$$

and finally the required thickness of the wall

$$x = \frac{146,4 + x^2}{50} = 2,928 + \frac{8,57}{50} = 3,10 \text{ feet.}$$

§ **213. Strength of Shearing.**—The *strength of shearing* (Fr. résistance par glissement ou cisaillement, Ger. Schubfestigkeit or Widerstand des Abdrückens oder Abscheerens), which comes into play when the surface of separation coincides with the direction of the force, can be treated in the same manner as the strength of extension. We have here to consider the action of three parallel forces P, Q, and R, Fig. 328, when the points of application A and C of two of the forces lie so near each other, that bending is not possible, and therefore a separation of the body in two parts takes place between A and C in a surface $D D$ at right angles to the axis of the body. The strength of shearing, like that of tearing and

Fig. 328.

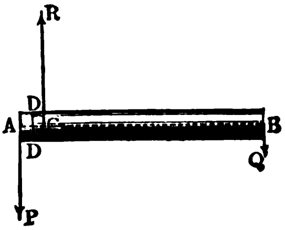
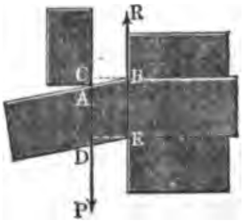


Fig. 329.



crushing, is proportional to the section of the body, or rather to the area F of the surface of separation, and in the case of wrought iron is approximately equal to that for tearing, so that the modulus of rupture K for tearing can also be employed as the modulus of rupture for shearing, and consequently we can put the force necessary to produce rupture by shearing, when the cross-section is F, $P = F K$. In general we have $P = F K_2$, K_2 denoting the ultimate strength of shearing per unit of surface determined by experiment.

The formula $P = \frac{\lambda}{l} F E = \sigma F E$ for tensile and compressive forces within the limit of elasticity can also be employed for the shearing force P, Fig. 329, but here σ denotes the ratio $\iota = \frac{C A}{C B}$ of the displacement $C A$ to the distance $C B$ of the directions $A P$ and $E F$ of the two forces from each other.

The following Table III. contains the modulus of elasticity (C) and that of rupture or ultimate strength (K_2) for all bodies, for which they are known at present, and they correspond to the formulas $P = \iota F C$ and $P_1 = F K_2$ for the elasticity and strength of shearing.

TABLE III.

MODULI OF THE ELASTICITY AND ULTIMATE STRENGTH OF SHEARING.

Names of the Bodies.	Modulus of Elasticity C.	Modulus of Ultimate Strength K_2.
Cast Iron	2840000 200000	32300 2270
Wrought Iron	9000000 630000	50000 3500
Fine Cast Steel	14220000 1000000	92400 6500
Copper	6260000 440000	—
Brass.	5260000 370000	—
Wood of deciduous Trees . .	569000 40000	683 48
Wood of evergreen Trees . .	616000 43300	2290 161

C is generally taken $= \frac{1}{3} E$ and $K_2 = K$.

FIG. 330.

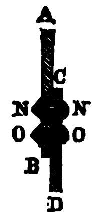
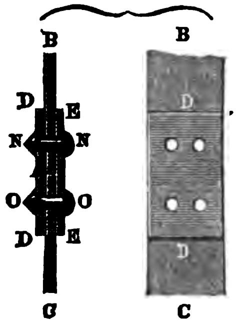


FIG. 331.



The most important application of the formula $P = F K_1$ is to the determination of the thickness d of bolts and rivets, with which plates and other flat bodies are fastened together. There are two modes in which bodies may be fastened together in this way; either the plates $A\,B$ and $C\,D$ to be joined together are laid upon one another, as in Fig. 330, and then fastened together by the bolts or rivets $N\,N$ and $O\,O$, or, as is represented in Fig. 331, the plates are butted together and covered with splicing pieces $D\,D$ and $E\,E$, and they are then fastened together by means of the rivets $N\,N$ and $O\,O$, which pass through both the plate and the splicing pieces. In the first method of joining the plates the tensile stress passes from one plate to the other through the intervention of a couple, which causes both of the plates to undergo in addition to the stretching also a bending, and consequently their safe or working load is diminished. The second method, where no such couple is called into action and where, consequently, no bending takes place, is for this reason to be preferred. Since the plates and splicing pieces, which are thus joined, press upon each other with no inconsiderable force, the strength of the joint is considerably augmented by the friction arising from this pressure. For greater safety we disregard this action in determining the thickness of the rivets. On the other hand, the working load of the plate is diminished by the holes made for the rivets or bolts, and we must therefore take care that it is not exceeded by the working load of the rivets. If d is the thickness of the rivets and ν their number, in the case of the joint in two plates represented in Fig. 331, we have for the working load of the rivets

$$P = \nu \frac{\pi d^2}{4} \frac{K_1}{n}.$$

Now, if b is the width and s the thickness of the pieces to be joined and ν_1 the number of the rivets in one row, the cross-section of the plate submitted to the force P is

$F = (b - \nu_1 d)\, s$, and therefore we have $P = (b - \nu_1 d)\, s \dfrac{K}{n}$, K denoting the modulus of rupture of sheet iron; equating these two values, we obtain

$$\frac{\nu \pi d^2}{4} K_s = (b - \nu_1 d)\, s\, K, \text{ or}$$

$$\nu = \frac{4\,(b - \nu_1 d)\, s}{\pi d^2} \frac{K}{K_s}.$$

When the holes in the plates are punched, the strength of shearing must be overcome, but in this case the surface is not plane, but cylindrical. If s is the thickness of the plate and d the diameter of the hole in it, we have the area of the surface of separation

$$F = \pi d s,$$

and consequently the force necessary to punch the hole is

$$P = F K_s = \pi d s K_s.$$

(Compare in the "Civil Ingenieur," Vol. I, 1854, the article "John Jones' experiments on the force necessary to punch sheet-iron," by C. Borneman).

EXAMPLE—1) An iron rivet 1½ inch thick can resist with safety, if we assume $K_s = \frac{1}{6} . 50000 = 8300$ pounds, a force

$$P = \frac{\pi d^2}{4} K_s = \frac{\pi}{4}\left(\frac{3}{2}\right)^2 . 8300 = \frac{9 . 2075\, \pi}{4} = 14670 \text{ pounds,}$$

and the force necessary to punch the hole through the sheet-iron, which is ½ inch thick, is

$$P_1 = \pi d s . K_s = \pi . \frac{3}{2} . \frac{1}{2} . 50000 = 37500\, \pi = 117810 \text{ pounds.}$$

2) If two pieces of sheet-iron are to be joined together by a row of rivets, and if we denote the thickness of the plate by s and its width for each rivet by b, we have

$$(b - d)\, s = \frac{\pi d^2}{4}, \text{ whence}$$

$$b = d + \frac{\pi d^2}{4 s} = d\left(1 + \frac{\pi d}{4 s}\right);$$

E.G., for $d = \frac{3}{4}$ and $s = \frac{1}{4}$ inch

$$b = \frac{3}{4}\left(1 + \frac{8 \pi}{4}\right) = 5 \text{ inches.}$$

CHAPTER II.

ELASTICITY AND STRENGTH OF FLEXURE OR BENDING.

§ **214. Flexure.**—The most simple case of flexure is that of a body $A\,B\,C$, Fig. 332, acted upon by a force $\overline{A\,P} = P$, whose direction is normal to its axis $A\,B$, while the body at the same time is retained at two points B and C. Let l and l_1 be the distances

$C\,A$ and $C\,B$ of the points of application A and B from the central fulcrum or point of application C, then the force at B is

$$Q = \frac{P\,l}{l_1},$$

and consequently the resultant is

$$R = P + Q = \left(1 + \frac{l}{l_1}\right)P.$$

FIG. 332.

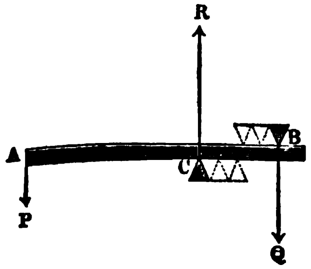
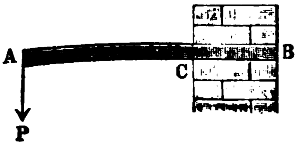


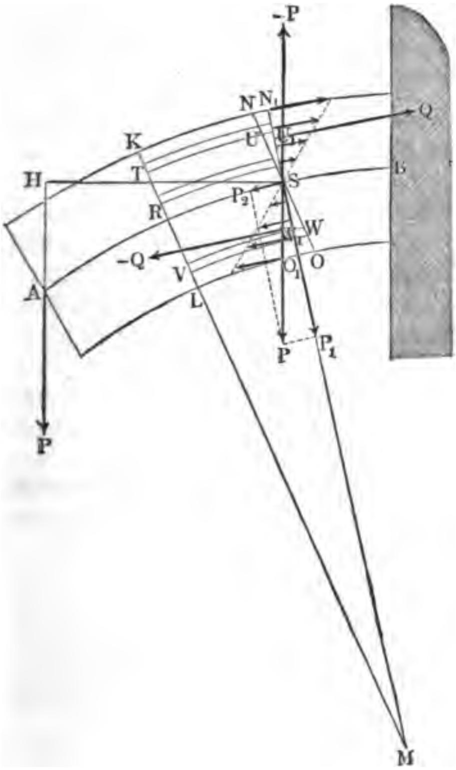
FIG. 333.



If we wish to prevent one portion of the body from bending, we must insert between the two points of support an infinite number of others, or the body must be fastened or solidly walled in along $B\,C$, as is represented in Fig. 333, and we have then to study only the flexure of the free portion $A\;C$ of the body. Let us suppose the body to be a prism, and let us assume, that it is composed of long parallel fibres placed above and alongside of one another and that, when the body is bent, they neither lose their parallelism nor slide upon one another.

By this flexure those fibres, which are on the convex side of the body, are extended, and those on the concave side are compressed, while a certain mean layer undergoes neither extension nor compression. This is called the *neutral surface* of a *deflected beam* (Fr. couche des fibres invariables, Ger. neutrale Axenschicht). The extension and compression of the various fibres above and below this layer are proportional to their distance from it. The extension of the fibres on one side and the compression of those on the other increase gradually, so that the fibres most distant from this surface on the one side undergo the maximum extension, and those on the other the maximum compression. A portion of the body $A\;K\,B$, Fig. 334, bounded before the flexure by the cross-sections $K\,L$ and $N\,O$, assumes, in consequence of the flexure, the form $K\,L\,O_1\;N_1$, by which the cross-section $N\,O$ becomes $N_1\;O_1$, that is, it ceases to be parallel to *K L* and assumes a position perpendicular to the neutral surface *R S*. The length *K N* of the uppermost fibre becomes, in consequence, *K* N_1, and that of *L O* the lowest fibre becomes *L* O_1. The increase in length of the former is therefore *N* N_1, and the decrease of the latter is *O* O_1, while the fibre *R S* in the neutral surface retains its primitive length unaltered. The intermediate fibres, such as *T U*, *V W*, etc., are increased or diminished in length becoming *T* U_1, *V* W_1, etc., and the amount *U* U_1, *W* W_1, etc., of the increase or decrease is determined by the proportions

FIG. 334.



$$\frac{U U_1}{N N_1} = \frac{S U}{S N},$$

$$\frac{W W_1}{O O_1} = \frac{S W}{S O}, \text{etc.}$$

Let us assume the length of the fibre

$$R S = K N = L O = \text{unity } (1),$$

and let us denote the extension or compression of the fibres, which are situated at the distance unity (1) from the neutral surface, by σ, then we have for a fibre, which is situated at a distance *S U* or *S W* = z from this surface, the extension or compression

$$U U_1 \text{ or } W W_1 = \sigma z.$$

If the body is but little bent, so that the limit of elasticity is nowhere surpassed, we can put the strain on the different fibres proportional to their extensions, etc., and we can consequently assume, that these strains are proportional to their distance from the neutral surface, as is represented in the figure by the arrows.

If the cross-section of a fibre is = unity, we have in general the tension upon it = $\sigma z E$; and if the cross-section of the fibre = F, the tensile or compressive strain is expressed by the formula

$$S = \sigma z F E = \sigma E . F z,$$

and its moment in reference to the point S upon the axis is

$$M = z . \sigma z F E = \sigma z^2 F E = \sigma E . F z^2.$$

§ **215. Moment of Flexure.**—The tensile and compressive strains in the cross-section $N_1\ O_1$ balance the bending force P at the end A of the body $A\ B$. We can therefore apply to these forces the well-known laws of equilibrium. If we imagine that there are in action at S two other forces $+ P$ and $- P$, which are not only equal but also parallel in direction to the given force P, we obtain

1) A *couple* $(P, \ - P)$, which produces the flexure or bending around S, and

2) A simple *shearing force* $\overline{S P} = P$, which tends to cut off the portion $A\ S$ of the body in the direction $S\ P$ or $A\ P$. The latter force can be decomposed into two components P_1 and P_2, whose directions lie in the plane of the cross-section $N_1\ O_1$ and in the neutral axis $S\ R$. If a is the angle formed by the cross-section $N_1\ O_1$ with the direction $A\ P$ of the bending force, we have

$$P_1 = P \cos. a \text{ and}$$
$$P_2 = P \sin. a.$$

In ordinary cases in practice the flexure of the body and also a is so small, that we can put $\sin. a = 0$ and $\cos. a = 1$, and consequently we can neglect the component P_2, which tends to tear off the portion $A\ S$ at $N_1\ O_1$, and, on the contrary, we can put the force P_1, which tends to rupture by shearing the piece $A\ S$ in $N_1\ O_1$, equal to the bending stress P.

If F denote the area of the cross-section $N_1\ O_1$ and K_2 the modulus of rupture for shearing, the shearing force is determined by the product $F\ K_2$.

If we are considering a long prismatical body, P is generally so small a portion of $F\ K_2$ that rupture by shearing can scarcely occur, and for this reason it will be considered in particular cases only. (*See the following chapter.*)

Since one couple $(P, \ - P)$ can be balanced only by another couple, it follows, that the tensile strains on one side form with the compressive strains on the other another couple $(Q, \ - Q)$, and that the moments of the two couples must be equal. If F_1, F_2, F_3, etc., are elements or infinitely small portions of the entire surface

F of the cross-section $N\,O = N_1\,O_1$, and if the distance of these portions from the neutral surface or S be denoted by z_1, z_2, z_3, etc., the strains in these elements are

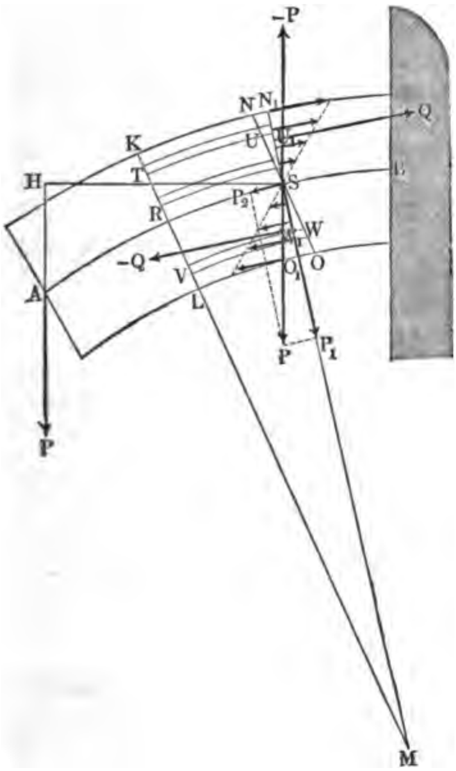
$$\sigma\,E\,.\,F_1\,z_1,\ \sigma\,E\,.\,F_2\,z_2,\ \sigma\,E\,.\,F_3\,z_3,\ \text{etc.},$$

and their moments

$$\sigma\,E\,.\,F_1\,z_1^2,\ \sigma\,E\,.\,F_2\,z_2^2,\ \sigma\,E\,.\,F_3\,z_3^2,\ \text{etc.}$$

Since these forces form a couple $(Q, -Q)$, their sum $\sigma\,E\,(F_1\,z_1 + F_2\,z_2 + F_3\,z_3 + \ldots)$, and consequently $F_1\,z_1 + F_2\,z_2 + F_3\,z_3 + \ldots$ must be $= 0$.

FIG. 334 a.



But this sum can only be $= 0$, when the point S of the axis coincides with the centre of gravity of the surface $F = F_1 + F_2 + F_3 + \ldots$; consequently *the neutral axis of a bent body passes through the centre of gravity S of its cross-section F.* The moment of the couple $(Q, -Q)$,

$$\sigma\,E\,(F_1\,z_1^2 + F_2\,z_2^2 + F_3\,z_3^2 + \ldots),$$

should now be put equal to the moment of the couple $(P, -P)$. If we denote the distance $S\,H$ of the centre of gravity S from the direction $A\,P$ of the bending force by x, we have the moment of the latter couple $= P\,x$, and therefore

$$P\,x = \sigma\,E\,(F_1\,z_1^2 + F_2\,z_2^2 + \ldots).$$

Finally, we have for the *radius of curvature* $M\,R = M\,S$ of the neutral surface the proportion

$$\frac{M\,R}{R\,S} = \frac{S\,U}{U\,U_1},$$

or, substituting $M R = r$, $R S = 1$, $S U = 1$ and $U U_1 = \sigma$,

$$\frac{r}{1} = \frac{1}{\sigma}.$$

Consequently $r \sigma = 1$ or $\sigma = \frac{1}{r}$, whence the moment of force is

$$P x = \frac{E}{r} (F_1 z_1^2 + F_2 z_2^2 + \dots).$$

The radius of curvature at S is therefore

$$r = \frac{E}{P x} (F_1 z_1^2 + F_2 z_2^2 + \dots).$$

The expression $F_1 z_1^2 + F_2 z_2^2 + \dots$ is dependent only upon the form and size of the cross-section, and can therefore be determined by the rules of geometry. We will hereafter denote it by W and we will call the quantity corresponding to it the *measure of the moment of flexure*, and $W E$ the *moment of flexure* itself (Fr. moment de flexion; Ger. Biegungs-moment).*

From the above, we have for the radius of curvature

$$r = \frac{W E}{P x},$$

and we can assert *that the radius of curvature of the neutral axis of a deflected body is directly proportional to the measure W of the moment of flexure and the modulus of elasticity E, and, on the contrary, inversely proportional to the moment $P x$ of the force.*

The curvature itself, being inversely proportional to the radius of curvature, increases with the moment $P x$ of the force, and decreases, when the moment of flexure $W E$ increases.

§ **216. Elastic Curve.**—If we have determined the moments of flexure $W E$ for the cross-sections of the bodies, which generally occur in practice, we can determine by means of these values the curvature and from it the form of the *neutral axis* or of the so-called *elastic curve.* The equation

$$P x r = W E \text{ or } r = \frac{W E}{P x}$$

indicates, that in the case a prismatical body the product of the radius of curvature and the moment of the stress is constant for all parts of the elastic curve $A B$, Fig. 335, and that consequently r becomes greater or less as the arm x of the force is diminished or increased, or as the distance of the point S considered from the end A of the neutral axis is less or greater. At A we have $x = 0$, and consequently the radius of curvature is infinitely great; at the fixed point B, on the contrary, x is a maximum, and the radius of curvature is therefore a minimum; hence the radius of curvature increases by degrees from a certain finite value to infinity, when we proceed from the fixed point B to the end A.

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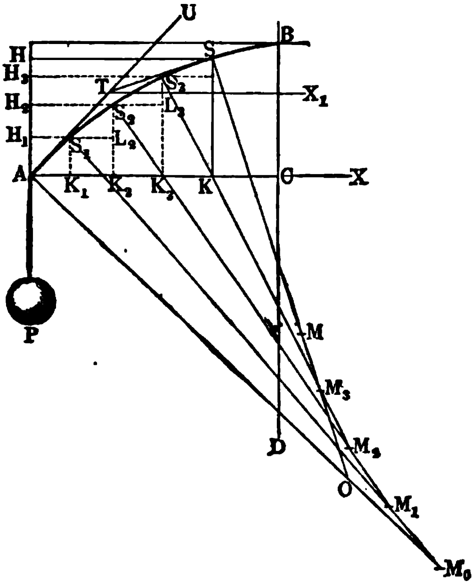
* Moment of flexure is also used for the bending moment $P x$.—Tr.

If we divide a portion $A\ S$ of the elastic curve, the length of which is $= s$, into equal parts, and erect at the end A and at the points of division S_1, S_2, S_3, etc., perpendiculars to the curve, they will intersect each other at the centres M_0, M_1, M_2 of the osculatory circles, and the portions cut off $M_0\ A = M_0\ S_1$, $M_1\ S_1 = M_1\ S_2$, $M_2\ S_2 = M_2\ S_3$, etc., are the required *radii of curvature* r_1, r_2, r_3 of the elastic curve. (See Introduction to the Calculus, Art. 33.) If n is the number of divisions of this line, we have the length of a division $= \frac{s}{n}$; and if we denote the length of the arc (for the radius $= 1$) of the *angles of curvature* $A\ M_0\ S_1 = \delta_1°$, $S_1\ M_1\ S_2 = \delta_2°$, $S_2\ M_2\ S_3 = \delta_3°$, etc., by δ_1, δ_2, δ_3, etc., we can put $\frac{s}{n} = \delta_1 r_1 = \delta_2 r_2 = \delta_3 r_3$, etc., whence we obtain $\delta_1 = \frac{s}{n\,r_1}$, $\delta_2 = \frac{s}{n\,r_2}$, $\delta_3 = \frac{s}{n\,r_3}$, etc.

FIG. 335.



If we suppose the elastic line to be but slightly curved, we can substitute for the divisions of the arc their projections upon the axis of abscissas $A\ X$ perpendicular to the direction of the force, I.E. we can put $A\ K_1 = H_1\ S_1 = K_1\ K_2 = K_2\ K_3$, etc., so that the arms of the force in reference to the points S_1, S_2, S_3, etc., are

$$H_1\ S_1 = \frac{s}{n},$$

$$H_2\ S_2 = H_1\ S_1 + S_1\ L_2 = 2\,\frac{s}{n},$$

$$H_3\ S_3 = H_2\ S_2 + S_2\ L_3 = 3\,\frac{s}{n},\ \text{etc.},$$

and consequently the corresponding moments of the force or the values for $P x$ are $\frac{P s}{n}, \frac{2 P s}{n}, \frac{3 P s}{n}$, etc.

Substituting successively these values for $P x$ in the formula $r = \frac{W E}{P x}$ for the radius of curvature, we obtain the following series of values for the radii of curvature

$$r_1 = n \frac{W E}{P s},\ r_2 = \frac{n}{2} \frac{W E}{P s},\ r_3 = \frac{n}{3} \frac{W E}{P s}, \text{ etc.};$$

hence the corresponding angles which measure the curvature are

$$\delta_1 = \frac{s}{n\, r_1} = \frac{P s^2}{n^2\, W E},\ \delta_2 = \frac{s}{n\, r_2} = 2 \cdot \frac{P s^2}{n^2\, W E},$$

$$\delta_3 = \frac{s}{n\, r_3} = 3 \cdot \frac{P s^2}{n^2\, W E}, \text{ etc.}$$

Summing these angles, we obtain for the angle of curvature $A\ O\ S = \phi^{\circ}$ of the entire arc $A\ S = s = x$

$$\phi = \delta_1 + \delta_2 + \delta_3 + \ldots + \delta_n$$

$$= (1 + 2 + 3 + \ldots + n) \frac{P s^2}{n^2\, W E},$$

or, since we know that $1 + 2 + 3 + \ldots + n = \frac{n^2}{2}$, we have

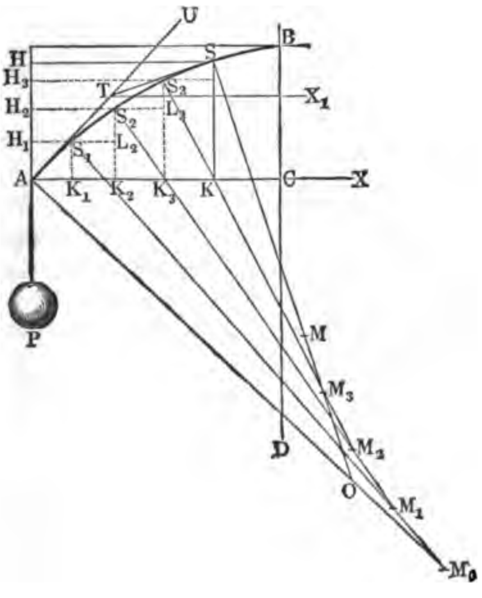
$$\phi = \frac{n^2}{2} \cdot \frac{P s^2}{n^2\, W E} = \frac{P s^2}{2\, W E},$$

for which we can write, according to the above supposition,

$$\phi = \frac{P x^2}{2\, W E}.$$

This arc or angle (since the angle between two lines is equal to that between their normals) is equal to the angle $S\ T\ U$ included between the tangents $A\ T$ and $S\ T$ to the two points A and S or to the angle, which expresses the difference between the inclination of the curve to the axis in A and in S. If we pass from the

FIG. 836.



undetermined point S to the fixed point B, we must substitute instead of s the entire length l of $A\,S\,B$, or approximately the projection $A\,C$ of the same upon the axis of abscissas, and under the supposition that the curve at B is perpendicular to the direction of the stress or parallel to the axis of abscissas, the angle ϕ becomes

$$A\,D\,B = \beta = \frac{P\,l^2}{2\,W\,E},$$

and, on the contrary, the angle of inclination or *tangential angle* $T\,S\,H = S\,T\,X_1$ becomes

$$\alpha = \beta - \phi = \frac{P\,l^2}{2\,W\,E} - \frac{P\,s^2}{2\,W\,E} = \frac{P\,(l^2 - s^2)}{2\,W\,E} = \frac{P\,(l^2 - x^2)}{2\,W\,E}.$$

If the curve at the fixed point B is not perpendicular to the direction of the force, but inclined at a small angle α_1 to the axis, we will have

$$\beta = \alpha_1 + \frac{P\,l^2}{2\,W\,E}, \text{ and therefore}$$

$$\alpha = \alpha_1 + \frac{P\,(l^2 - x^2)}{2\,W\,E}$$

§ **217. Equation of the Elastic Curve.**—By the aid of the latter formula we can now deduce the equation of the elastic curve. The ordinate of the curve $K\,S = y$ is composed of an infinite number (n) of parts, such as $K_1\,S_1$, $L_2\,S_2$, $L_3\,S_3$, etc., which are found by multiplying an element of the arc

$$A\,S_1 = S_1\,S_2 = S_2\,S_3, \text{ etc.} = \frac{s}{n}$$

by the sine of the corresponding tangential angle

$$S_1\,A\,K_1,\ S_2\,S_1\,L_2,\ S_3\,S_2\,L_3, \text{ etc.}$$

Hence we have

$$K\,S = K_1\,S_1 + L_2\,S_2 + L_3\,S_3 + \ldots, \text{ or}$$

$$y = \frac{s}{n}\,(\sin.\ S_1\,A\,K + \sin.\ S_2\,S_1\,L_2 + \sin.\ S_3\,S_2\,L_3 + \ldots).$$

Substituting the abscissa $A\,K = x$ instead of the arc $A\,S = s$, and for the sines the arcs calculated from the formula

$$\alpha = \frac{P\,(l^2 - x^2)}{2\,W\,E},$$

and introducing instead of x successively $\frac{x}{n}$, $\frac{2\,x}{n}$, $\frac{3\,x}{n}$, etc., we obtain

$$y = \frac{x}{n} \cdot \frac{P}{2\,W\,E}\left[l^2 - \left(\frac{x}{n}\right)^2 + l^2 - \left(\frac{2\,x}{n}\right)^2 + l^2 - \left(\frac{3\,x}{n}\right)^2 + \ldots \right.$$
$$\left. + \, l^2 - \left(\frac{n\,x}{n}\right)^2\right].$$

Now we have $l^2 + l^2 + \ldots + l^2 = n\,l^2$ and

$$\left(\frac{x}{n}\right)^2 + \left(\frac{2\,x}{n}\right)^2 + \left(\frac{3\,x}{n}\right)^2 + \ldots + \left(\frac{n\,x}{n}\right)^2$$
$$= (1^2 + 2^2 + 3^2 + \ldots + n^2)\left(\frac{x}{n}\right)^2 = \frac{n^3}{3}\left(\frac{x}{n}\right)^2$$

(see Ingenieur, page 88), whence

$$y = \frac{x}{n}\cdot\frac{P}{2\,W\,E}\left[n\,l^2 - \frac{n^3}{3}\left(\frac{x}{n}\right)^2\right], \text{ or}$$
$$y = \frac{P\,x\,(l^2 - \frac{1}{3}\,x^2)}{2\,W\,E},$$

which is the required *equation of the elastic curve,* when we suppose that the curvature is not very great.

If we substitute in this equation $x = l$, we obtain instead of y the height of the arc or the *deflection*

$$B\,C = a = \frac{P\,l^3}{3\,W\,E}.$$

While the *tangential angle* α increases with the force and with the *square of the length,* the *deflection* increases with the *force* and with *the cube of the length.*

The work done in bending the body is determined, since the force

$$P = \frac{3\,W\,E\,a}{l^3}$$

increases gradually with the space described and its mean value is

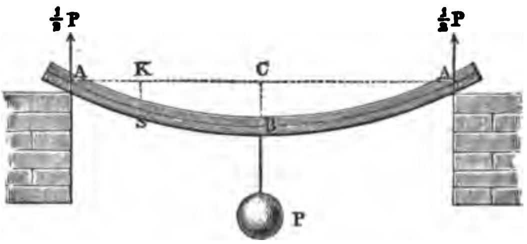
$$\tfrac{1}{2}\,P = \tfrac{3}{2}\,\frac{W\,E\,a}{l^3},$$

by the expresssion

$$L = \tfrac{1}{2}\,P\,a = \tfrac{3}{2}\,\frac{W\,E\,a^2}{l^3} = \tfrac{1}{6}\,\frac{P^2\,l^3}{W\,E}.$$

If a girder $A\,B\,A$, Fig. 337, whose length is l, is supported at both ends and acted on in the centre B by a force P, the ends are bent exactly in the same way as in the case just treated, but in this case we must substitute for the force acting at A, $\frac{1}{2}\,P$

FIG. 337.



and for the length of the arc $A\,B = \frac{1}{2}\,A\,A = \frac{1}{2}\,l$. Consequently the equation for the co-ordinates $A\,K = x$ and $K\,S = y$ is

$$y = \frac{P\,x\,(\frac{1}{4}\,l^2 - \frac{1}{3}\,x^2)}{4\,W\,E} = \frac{P\,x\,(3\,l^2 - 4\,x^2)}{48\,W\,E},$$

so that for $x = \overline{A\,C} = \frac{l}{2}$ the deflection is

$$y = \overline{B\,C} = a_1 = \frac{P\,l^3}{48\,W\,E} = \tfrac{1}{16} \cdot \frac{P\,l^3}{3\,W\,E},$$

I.E., one *sixteenth* of the deflection of a girder (Fig. 333) loaded at one end with an equal weight.

If in the first case the elastic curve $A\,B$, Fig. 336, is inclined at a small angle a_1 to the axis at the fixed point B, we must add to the former expression for y the vertical projection of the portion x of the tangent, I.E., $a_1\,x$, so that we have for the ordinate

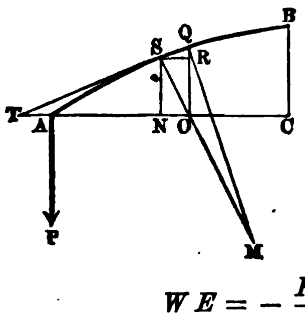
$$y = \left(a_1 + \frac{P\,(l^2 - \frac{1}{3}\,x^2)}{2\,W\,E}\right) x$$

and for the deflection

$$a = \left(a_1 + \frac{P\,l^2}{3\,W\,E}\right) l.$$

(§ 218.) More General Equation of the Elastic Curve.— A more exact equation of the curve $A\,S\,B$, Fig. 338, formed by the neutral axis of a deflected beam, can be deduced in the following manner by the aid of the calculus. If we substitute in the general equation of § 216, $W\,E = P\,x\,r$ the value of radius of curvature (*from Art. 33 of the Introduction to Calculus*),

FIG. 338.



$$r = -\frac{d\,s^3}{d\,x^2\,d\,(tang.\,a)}$$

and in the latter, according to Art. 32,

$$d\,s = \sqrt{1 + (tang.\,a)^2}\,.\,d\,x,$$

we obtain

$$W\,E = -\frac{P\,x\,d\,x\,[1 + (tang.\,a)^2]^{\frac{3}{2}}}{d\,tang.\,a.}$$

When the girder is but moderately deflected, the angle a formed by the tangent with the axis of abscissas is but small, and we can therefore write

$$[1 + (tang.\,a)^2]^{\frac{3}{2}} = 1 + \tfrac{3}{2}\,(tang.\,a)^2,$$

and consequently

$$W E = -\frac{P x \,[1 + \frac{3}{2}\,(tang.\, a)^2]\, d x}{d\,(tang.\, a)},$$

or inversely

$$\frac{P x\, d x}{W E} = -\frac{d\, tang.\, a}{1 + \frac{3}{2}\,(tang.\, a)^2} = -\,[1 - \tfrac{3}{2}\,(tang.\, a)^2]\, d\,(tang.\, a).$$

From the latter we obtain

$$\int \frac{P x\, d x}{W E} = -\int d\,(tang.\, a) + \tfrac{3}{2}\int (tang.\, a)^2\, d\,(tang.\, a),$$

or, according to Art. 18 of the Introduction to the Calculus,

$$\frac{P x^2}{2 W E} = -\, tang.\, a + \tfrac{1}{2}\,(tang.\, a)^3 + Con.$$

But at the vertex B the curve is parallel to the axis of abscissas and $a = 0$; substituting, therefore, the projection $C A = b$ of the elastic line on the axis of abscissas, we obtain

$$\frac{P b^2}{2 W E} = -\, tang.\, 0 + \tfrac{1}{2}\,(tang.\, 0)^3 + Con. = 0 + Con.$$

Subtracting from this the former equation, we have

$$\frac{P\,(b^2 - x^2)}{2 W E} = tang.\, a - \tfrac{1}{2}\,(tang.\, a)^3,$$

or inversely, for the tangential angle $S T N = a$,

$$tang.\, a = \frac{P\,(b^2 - x^2)}{2 W E} + \tfrac{1}{2}\,(tang.\, a)^3$$

$$= \frac{P\,(b^2 - x^2)}{2 W E} + \tfrac{1}{2}\,\frac{P^3\,(b^2 - x^2)^3}{8\, W^3 E^3},$$

i.e., 1) $tang.\, a = \dfrac{P\,(b^2 - x^2)}{2 W E}\left(1 + \dfrac{P^2\,(b^2 - x^2)^2}{8\, W^2 E^2}\right).$

But $tang.\, a = \dfrac{d y}{d x}$, hence we have

$$d y = \left(1 + \frac{P^2\,(b^2 - x^2)^2}{8\, W^2 E^2}\right)\frac{P\,(b^2 - x^2)\, d x}{2 W E}, \text{ and}$$

$$y = \frac{P}{2 W E}\left(\int (b^2 - x^2)\, d x + \frac{P^2}{8\, W^2 E^2}\int (b^2 - x^2)^3\, d x\right)$$

$$= \frac{P}{2 W E}\Big[\int b^2\, d x - \int x^2\, d x$$

$$+ \frac{P^2}{8\, W^2 E^2}\left(\int b^6 d x - \int 3\, b^4 x^2\, d x + \int 3\, b^2 x^4\, d x - \int x^6\, d x\right)\Big]$$

$$= \frac{P}{2 W E}\left[b^2 x - \frac{x^3}{3} + \frac{P^2}{8\, W^2 E^2}\left(b^6 x - b^4 x^3 + \frac{3\, b^2 x^5}{5} - \frac{x^7}{7}\right)\right]$$

$$+ \; Con.$$

Since for $x = 0$, $y = 0$, we have also $Con. = 0$, and

$$2)\ y = \frac{P\,x}{2\,W\,E}\left[b^2 - \frac{x^2}{3} + \frac{P^2}{8\,W^2\,E^2}\left(b^6 - b^4\,x^2 + \tfrac{3}{5}\,b^2\,x^4 - \frac{x^6}{7}\right)\right].$$

At the vertex $x = b$ and y is the deflection $C\,B = a$, and therefore

$$a = \frac{P}{2\,W\,E}\left(\tfrac{2}{3}\,b^3 + \frac{P^2}{8\,W^2\,E^2}\cdot\tfrac{16}{35}\cdot b^7\right),$$

I.E. $$3)\ a = \frac{P\,b^3}{3\,W\,E}\left(1 + \tfrac{8}{35}\,\frac{P^2\,b^4}{W^2\,E^2}\right).$$

From $d\,s = \sqrt{1 + (tang.\,a)^2}\,.\,d\,x = [1 + \tfrac{1}{2}\,(tang.\,a)^2]\,d\,x$ we obtain, by substituting $tang.\,a = \dfrac{P\,(b^2 - x^2)}{2\,W\,E}$,

$$s = \int\left(1 + \tfrac{1}{8}\,.\,\frac{P^2\,(b^2 - x^2)^2}{W^2\,E^2}\right)d\,x$$

$$= \int d\,x + \frac{P^2}{8\,W^2\,E^2}\left[\int\left(b^4\,d\,x - 2\,b^2\,x^2\,d\,x + x^4\,d\,x\right)\right]$$

$$= x + \frac{P^2}{8\,W^2\,E^2}\left(b^4\,x - \frac{2\,b^2\,x^3}{3} + \frac{x^5}{5}\right),$$

I.E., the length of the arc

$$4)\ s = \left[1 + \frac{P^2}{8\,W^2\,E^2}\left(b^4 - \tfrac{2}{3}\,b^2\,x^2 + \frac{x^4}{5}\right)\right]x.$$

If we assume $x = b$, we have the total length of the girder

$$5)\ l = \left(1 + \frac{P^2\,b^4}{15\,W^2\,E^2}\right)b = \left(1 + \tfrac{3}{5}\,.\,\frac{a^2}{b^2}\right)b.$$

Inversely we have

$$6)\ b = \frac{l}{1 + \dfrac{P^2\,b^4}{15\,W^2\,E^2}} = \left(1 - \frac{P^2\,l^4}{15\,W^2\,E^2}\right)l,$$

and therefore

$$a = \frac{P\,l^3}{3\,W\,E}\left(1 - \frac{P^2\,l^4}{15\,W^2\,E^2}\right)^3\left(1 + \tfrac{8}{35}\cdot\frac{P^2\,l^4}{W^2\,E^2}\right), \text{ or}$$

$$= \frac{P\,l^3}{3\,W\,E}\left(1 - \frac{3\,P^2\,l^4}{15\,W^2\,E^2}\right)\left(1 + \tfrac{8}{35}\cdot\frac{P^2\,l^4}{W^2\,E^2}\right),$$

I.E., $$7)\ a = \frac{P\,l^3}{3\,W\,E}\left(1 - \tfrac{4}{35}\cdot\frac{P^2\,l^4}{W^2\,E^2}\right).$$

Neglecting the members containing the higher powers of $\dfrac{P}{W\,E}$, we obtain, as in the last paragraph,

$$tang.\, \alpha = \frac{P\,(l^2 - x^2)}{2\,W\,E} \text{ and } y = \frac{P\,x}{2\,W\,E}\,(l^2 - \tfrac{1}{3}\,x^2), \text{ therefore,}$$

$$\text{for} \quad x = 0,\, tang.\, \alpha = \frac{P\,l^2}{2\,W\,E}, \text{ and for } x = b = l,\, y = a = \frac{P\,l^3}{3\,W\,E}.$$

§ 219. Flexure Produced by two Parallel Forces.—If a girder $A\ A_1\ B$, Fig. 339, I. and II., fixed at one end, is bent by two forces P and P_1, whose points of application A and A_1 are at a distance l from each other, while the point of application A_1 of the force P_1 is at a distance $A_1\,B = l_1$ from the fixed point B, the moment of flexure at a point S of the portion $A\ A_1$ is

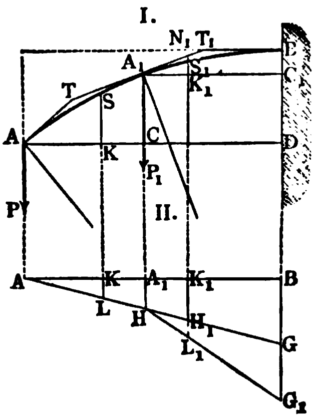
$$M = P\,x,$$

and, on the contrary, that of a point S_1 in the portion $A_1\,B$ is

$$M_1 = P\,(l + x_1) + P_1\,x_1,$$

in which x and x_1 denote the abscissas $A\ K$ and $A_1\ K_1$.

FIG. 339.



In order to obtain a clear idea of the manner in which these moments vary, we can lay off, as in II., their different values for the different points as ordinates, E.G., $M = y = \overline{K\,L}$, $M_1 = y_1 = \overline{K_1\,L_1}$, and join their extremities L, L_1 etc., by a line $A\ L\ H\ L_1\ G_1$, which will limit the values of M and M_1 for the whole length of the beam.

If the girder were subjected to the force P alone, the line bounding all the values of M or $y = P\,x$ would be the straight line $A\ G$, the ordinate of the extremity G of which is $\overline{B\,G} = P\,.\,\overline{A\,B} = P\,(l + l_1)$. By the addition of the force P_1 the portion $H\ G$ of this right line is replaced by the right line $H\ G_1$, whose extremities H and G_1 are determined by the co-ordinates $\overline{A\ A_1} = l$ and $\overline{A_1\,H} = P\ l$, and also $\overline{A\ B} = l + l_1$ and $\overline{B\ G_1} = \overline{B\ G} + \overline{G\ G_1} = P\,(l + l_1) + P_1\,l_1$.

If the force P is *negative*, the moment $M = y = P\,x$ of a point K upon $\overline{A\ A_1} = l$ remains unchanged, while, on the contrary, that of a point K_1 upon $A_1\,B$ becomes $M_1 = y_1 = P\,(l + x_1) - P_1\,x_1$, and the moment of flexure at the fixed point B is $= P\,(l + l_1) -$

$P_1\,l_1$, and it is positive or negative as $P\,(l + l_1)$ is greater or less than $P_1\,l_1$. In both cases the moment of flexure decreases gradually from A_1, remaining in the first case, Fig. 340, positive, and, on the contrary, in the second case, Fig. 341, becoming $= 0$ for a point O at a distance $A_1\,O = x_1 = \frac{P\,l}{P_1 - P}$ from A_1, for greater values it takes the negative sign, and at the fixed point B it is

$$= -\,[P_1\,l_1 - P\,(l + l_1)].$$

FIG. 340.

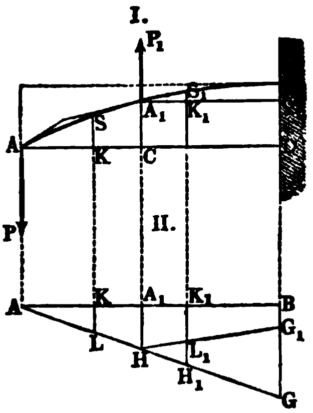
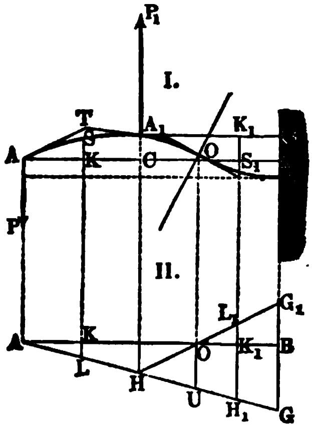


FIG. 341.



In the first case the right line $H\,G_1$, Fig. 340, II., which represents the moment of flexure at a point K_1 between A and B, passes below the base line $A\,B$ and ends at a point G_1, whose ordinate is $\overline{B\,G_1} = P\,(l + l_1) - P_1\,l_1$. In the second case, on the contrary, the right line $H\,G_1$, Fig. 341, II., rises from the point O above $A\,B$, and the ordinates become $K_1\,L_1 = y_1 = -\,[P_1\,x_1 - P\,(l + x_1)]$ and $B\,G_1 = a_1 = -\,[P_1\,l_1 - P\,(l + l_1)]$.

Since the radius of curvature $r = \frac{W\,E}{M}$ of the girder is inversely and consequently the curvature itself is directly proportional to the moment of flexure M, the graphic representations in II. of figures 339, 340 and 341 furnish us also a representation of the variation of the curvature of the girder. In the case represented in Fig. 339, where the forces P and P_1 acting upon the girder have the same direction, the curvature increases gradually in going from A to B, but if P and P_1 have opposite directions, it decreases again gradually as we recede from A_1.

If, as in Fig. 340, $P_1 l_1 < P(l + l_1)$, the beam is bent in one direction only; but if $P_1 l_1 > P(l + l_1)$, there is no flexure at the point A and also at a point O, Fig. 343, where a point of inflection is formed (*see Art.* 14, *Introduction to the Calculus*), and from O towards B the curvature of the girder gradually increases in the opposite direction. If in the second case, Fig. 342, the forces P and P_1 are equal, for a point K_1 between A_1 and B,

FIG. 342.

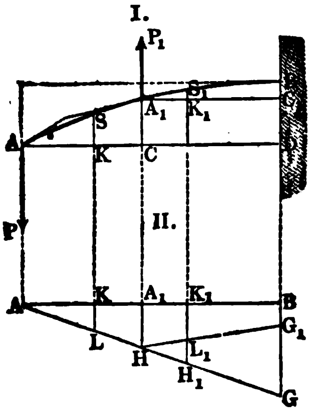
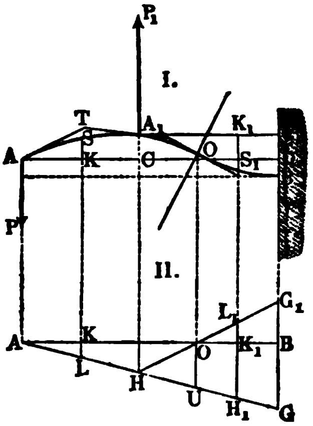


FIG. 343.



$$M = P(l + x_1) - P x_1 = P l$$

is constant, and the curvatures of that portion $A_1 B$ of the girder is the same everywhere, I.E., the elastic curve is a *circle.*

The radius of curvature of the portion $A A_1$ is determined in all three cases by the well-known formula

$$r = \frac{WE}{Px},$$

and that of the portion $A_1 B_1$ in the first case by the formula

$$r_1 = \frac{WE}{P(l + x_1) + P_1 x_1},$$

and, on the contrary, in the second and third cases by the formula

$$r_1 = \frac{WE}{P(l + x_1) - P_1 x_1}.$$

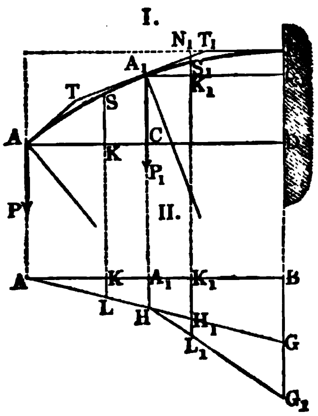
When, in the second case, $P_1 = P$, r_1 becomes $= \frac{WE}{Pl}$ or constant, and in the third case, where $P_1 l_1 > P(l + l_1)$, for the point O, whose abscissa $x_1 = \frac{Pl}{P_1 - P}$, we have $r_1 = \infty$ (infinitely great), and, on the contrary, for the point A_1, $r = \frac{W E}{P l}$, and for the point B,

$$r_1 = - \frac{W E}{P_1 l_1 - P (l + l_1)}.$$

According as $P l$ is greater or less than $P_1 l_1 - P (l + l_1)$ etc., i. e., $P \gtreqless r_1$, in the latter case we have $r \lesseqgtr r_1$ or the curvature at A_1 greater or less than that at B.

§ 220. **The Elastic Curve for Two Forces.**—The equations of the elastic curve, formed by the axis of a girder subjected to the action of two forces P and P_1, can easily be deduced from the formulas found in paragraphs 216 and 217.

FIG. 344.



If α denote the angle of inclination of the elastic line at A_1, we have first for the portion of the curve A A_1, Fig. 344, I, the arc measuring the inclination of the same at S

$$1) \; \alpha = \alpha_1 + \frac{P (l^2 - x^2)}{2 W E},$$

and the ordinate $K S$ corresponding to the abscissa $A K = x$

$$2) \; y = \alpha_1 x + \frac{P x (l^2 - \frac{1}{3} x^2)}{2 W E},$$

(compare § 217).

By putting $x = 0$ in (1), we determine the angle of inclination in A

$$\alpha_0 = \alpha_1 + \frac{P l^2}{2 W E},$$

and, on the contrary, by putting $x = l$ in (2), we obtain the ordinate at A_1

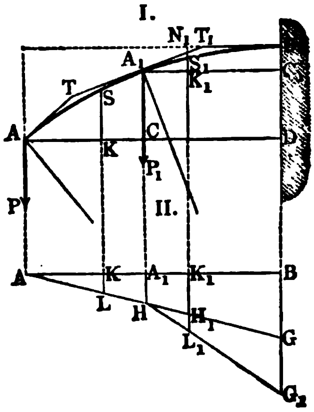
$$A_1 C = a = \alpha_1 l + \frac{P l^3}{3 W E}.$$

For a point in the second portion of the girder $A_1 B$ the moment of flexure $P (l + x_1) + P_1 x_1 = P l + (P + P_1) x_1$ is composed of the two parts $P l$ and $(P + P_1) x_1$, one of which, being constant, bends this portion of the beam in an arc of a circle, whose radius is $r = \frac{W E}{P l}$ and whose angle of inclination at a point S_1 situated a distance $A_1 S_1 = x_1$ from A and $B S_1 = l_1 - x_1$ from B is measured by the arc

$$\beta_1 = \frac{l_1 - x_1}{r} = \frac{P l (l_1 - x_1)}{W E}.$$

The inclination at S of this portion of the girder, due to the flexure produced by the moment $(P + P_1)\, x_1$, is measured by the arc

FIG. 344 a.



$$\beta_2 = \frac{(P + P_1)\,(l_1^2 - x_1^2)}{2\,W E}.$$

and consequently the total inclination at the same point is

$$3)\ \beta = \beta_1 + \beta_2 = \frac{P\,l\,(l_1 - x_1)}{W E} + \frac{(P + P_1)\,(l_1^2 - x_1^2)}{2\,W E}.$$

The deflection of $B\,S_1$, due to the curvature in a circle measured by β_1, is according to the well-known formula for the circle

$$N_1\,S_1 = \frac{B\,S_1^2}{2\,r} = \frac{(l_1 - x_1)^2\,P\,l}{2\,W E};$$

hence that of the entire piece $B\,A_1$ is

$$B\,C_1 = \frac{P\,l\,l_1^2}{2\,W E},$$

and the height of the point S_1 above A_1 is

$$K_1\,S_1 = B\,C_1 - N_1\,S_1 = \frac{P\,l\,[l_1^2 - (l_1 - x_1)^2]}{2\,W E} = \frac{P\,l\,(2\,l_1\,x_1 - x_1^2)}{2\,W E}.$$

According to what precedes (§ 217) the deflection $K_1\,S_1 = \frac{(P + P_1)\,x_1\,(l_1^2 - \frac{1}{3}\,x_1^2)}{2\,W E}$ corresponds to the angle of curvature $\beta_2 = \frac{(P + P_1)\,(l_1^2 - x_1^2)}{2\,W E}$, and the total deflection is therefore

$$4)\ K_1\,S_1 = y_1 = \frac{P\,l\,(2\,l_1\,x_1 - x_1^2) + (P + P_1)\,x_1\,(l_1^2 - \frac{1}{3}\,x_1^2)}{2\,W E}.$$

Substituting in (3) $x_1 = 0$, we obtain the angle of inclination β, which we had assumed as given, and its value is

$$\alpha_1 = \frac{2\,P\,l\,l_1 + (P + P_1)\,l_1^2}{2\,W E}$$

Now if we substitute in (4) $x_1 = l_1$, we obtain by this means the deflection

$$B\,C_1 = a_1 = \frac{3\,P\,l\,l_1^2 + 2\,(P + P_1)\,l_1^3}{6\,W E}.$$

Finally, the total deflection of the whole girder is

$$\overline{BD} = a + a_1 = a_1 l + \frac{P l^3}{3\, W E} + \frac{3\, P l l_1^2 + 2\,(P + P_1)\, l_1^3}{6\, W E}$$

$$= a_1 l + \frac{P l\,(2\, l^2 + 3\, l_1^2) + 2\,(P + P_1)\, l_1^3}{6\, W E}$$

$$= a_1 l + \frac{P\,(2\, l^3 + 3\, l l_1^2 + 2\, l_1^3) + 2\, P_1 l_1^3}{6\, W E}.$$

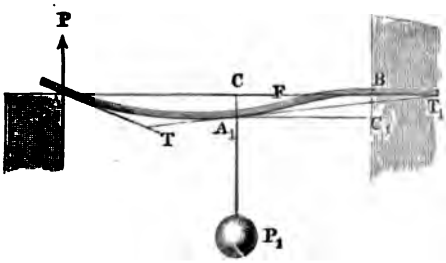
If the beam $A\ B$ is not horizontal at B, but inclined at a certain angle β_0, we must add in (3) β_0 to β, and in (4) to y_1, $\beta_0\, x_1$.

If the force P_1 acts in an opposite direction to P, we must substitute in the fundamental formulas (3) and (4) $P - P_1$ instead of $P + P_1$.

§ 221. Girders Supported at One End.

—The formulas of the foregoing paragraph are applicable to numerous cases in practice. If, for example, a girder $A\ B$, Fig. 345, is at one end imbedded in a wall and at the other merely supported, the question arises, what is the bending force at A, or what force has the support at A to bear, when the beam is loaded with a weight P_1, suspended at an intermediate point A_1?

FIG. 345.



P is here negative, $\beta_0 = 0$ and, since A and B are at the same level, the sum of the deflections $C A_1 = a$ and $C_1\, B = a_1$, is $= 0$,

$$\text{I.E.}\ \left(a_1 + \frac{P l^3}{3\, W E}\right) l + \frac{\frac{1}{2} P l l_1^2 + \frac{1}{3}(P - P_1)\, l_1^3}{W E} = 0,$$

or since $a_1 = \dfrac{P l l_1 + \frac{1}{2}(P - P_1)\, l_1^2}{W E}$, we have

$$P l^2 l_1 + \tfrac{1}{2}(P - P_1)\, l_1^2 l + \tfrac{1}{3} P l^3 + \tfrac{1}{2} P l l_1^2 + \tfrac{1}{3}(P - P_1)\, l_1^3 = 0.$$

From this it follows that

$$P = \frac{(3\, l + 2\, l_1)\, l_1^2}{l^3 + 3\,(l^2 l_1 + l l_1^2) + l_1^3}\,\frac{P_1}{2},$$

E.G., for $l - l_1$, that is, when P_1 is applied in the middle of the girder, we have

$$P = \frac{5}{16} P_1.$$

Hence the moment of flexure at A_1 is

$$P l = \frac{5}{16} P_1 l,$$

and, on the contrary, that at B is

$$P_1 l_1 - 2 P l = \frac{3}{8} P_1 l = \frac{6}{16} P_1 l,$$

or greater than that at A_1.

If $l = l_1$ and the points A and B are not situated upon the same level, if, for example, A lies a distance a_2 higher than B, we must put $a + a_1 = a_2$. But in this case

$$a_1 = \frac{(3 P - P_1) l^2}{2 W E},$$

$$a = a_1 l + \frac{P l^3}{3 W E} = \frac{(11 P - 3 P_1) l^3}{6 W E} \text{ and}$$

$$a_1 = \frac{[3 P + 2 (P - P_1)] l^3}{6 W E} = \frac{(5 P - 2 P_1) l^3}{6 W E};$$

hence we have

$$\frac{(16 P - 5 P_1) l^3}{6 W E} = a_2,$$

and consequently $P = \dfrac{6 W E a_2}{16 l^3} + \dfrac{5}{16} P_1$.

If the moments at A_1 and B should be equal and opposite, we must put $P l = P_1 l - 2 P l$,

or $3 P = P_1$, I.E. $P = \dfrac{P_1}{3}$,

in which case we must make

$$a_2 = \frac{P l^3}{6 W E} = \frac{P_1 l^3}{18 W E}.$$

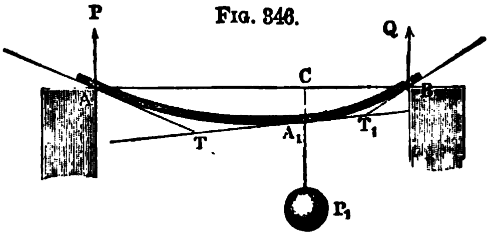
If, therefore, the end of the girder lies $0{,}0555 \dfrac{P_1 l^3}{W E}$ higher than B, the moment of flexure in A and B is $= \pm \dfrac{P_1 l}{3}$, or smaller than when A and B are at the same height.

With the aid of the values found for P we can calculate the radii of curvature, the tangential angles, etc., of the portions $A A_1$ and $A_1 B$ of the curve.

§ 222. Flexure of a Girder supported at both Ends.—

Another case, to which the formulas of the last paragraph are applicable, is that of a girder $A B$, Fig. 346, supported at both ends A and B and acted upon by a force P_1, whose point of application A_1 is at a distance l from one of the points of support A, and at a distance l_1 from the other.

FIG. 346.



Here the moment

$$P \, . \, \overline{BA} = \text{the moment } P_1 \, . \, \overline{BA_1},$$

i.e.

$$P\,(l + l_1) = P_1\, l_1,$$

and consequently the pressure on the point of support A is

$$P = \frac{P_1\, l_1}{l + l_1},$$

and, on the contrary, the pressure on the point of support B is

$$Q = P_1 - P = \frac{P_1\, l}{l + l_1}.$$

Since A and B are situated in a horizontal plane, we have

$$\alpha + \alpha_1 = 0,$$

and the angle β is not here $= 0$, but is a negative quantity $C\,B\,T_1$ to be determined.

We have here

$$\alpha = -\,\beta\, l + \frac{P\, l^2\, l_1 + \frac{1}{2}\,(P - P_1)\, l\, l_1^2}{W\,E} + \frac{P\, l^3}{3\; W\,E},$$

and also

$$\alpha_1 = -\,\beta\, l_1 + \frac{\frac{1}{2}\, P\, l\, l_1^2 + \frac{1}{3}\,(P - P_1)\, l_1^3}{W\,E},$$

and therefore their sum

$$\beta\,(l + l_1) - \frac{P}{6\; W\,E}(2\, l^3 + 6\, l^2\, l_1 + 6\, l\, l_1^2 + 2\, l_1^3)$$
$$+ \frac{P_1}{6\; W\,E}(3\, l\, l_1^2 + 2\, l_1^3) = 0,$$

or

$$6\,\beta\,(l + l_1)\; W\,E = P\,(2\, l^3 + 6\, l^2\, l_1 + 6\, l\, l_1^2 + 2\, l_1^3) - P_1\,(3\, l\, l_1^2 + 2\, l_1^3)$$
$$= [2\, l^3 + 6\, l^2\, l_1 + 6\, l\, l_1^2 + 2\, l_1^3 - (3\, l\, l_1 + 2\, l_1^2)\,(l + l_1)]\; P,$$

from which we deduce the angle of inclination at B

$$\beta = \frac{P\, l\,(2\, l^2 + 3\, l\, l_1 + l_1^2)}{6\,(l + l_1)\; W\,E} = \frac{P_1\, l\, l_1\,(2\, l^2 + 3\, l\, l_1 + l_1^2)}{6\,(l + l_1)^2\; W\,E}$$

and that at A

$$\alpha = \frac{P_1\, l\, l_1\,(l^2 + 3\, l\, l_1 + 2\, l_1^2)}{6\,(l + l_1)^2\; W\,E}.$$

If, for example, P_1 is suspended in the middle, we have

$$l_1 = l \text{ and } P = Q = \frac{P_1}{2},$$

and therefore

$$\beta = \frac{P\, l^2}{2\; W\,E} = \frac{P_1\, l^2}{4\; W\,E} \text{ (compare § 216).}$$

With the aid of the angle β, thus determined, all the relations of the flexure of the girder can be determined by the formulas found in what precedes.

The maximum value of the moment of flexure is for the point of application A_1, and it is

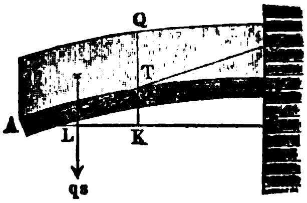
$$M = P\,l = Q\,l_1 = \frac{P_1\,l\,l_1}{l + l_1} = \frac{P_1}{l + l_1}\left[\left(\frac{l + l_1}{2}\right)^2 - \left(\frac{l - l_1}{2}\right)^2\right],$$

and it is a maximum for $l = l_1$, I.E., when the weight is hung in the middle, its value is then

$$M = \frac{P_1\,(l + l_1)}{4} = \tfrac{1}{2}\,P_1\,l.$$

§ 223. A uniformly loaded Girder.—If the load is uniformly distributed over the girder $A\ B$, Fig. 347, and if the unit of length bears a weight $= q$, or the whole girder, whose length is l, bears the load $Q = l\,q$, and a portion of the girder $A\ S = s$ the load $q\,s$, we must substitute, instead of the moments $\frac{1}{n} P\,s, \frac{2}{n} P\,s, \frac{3}{n} P\,s$, etc., the moments $\frac{1}{2} q\left(\frac{s}{n}\right)^2, \frac{1}{2} q\left(\frac{2\,s}{n}\right)^2, \frac{1}{2} q\left(\frac{3\,s}{n}\right)^2$ etc.; for the centres of gravity of the loads $q\left(\frac{s}{n}\right), q\left(\frac{2\,s}{n}\right), q\left(\frac{3\,s}{n}\right)$ etc., lie in the middle of $\frac{s}{n}, \frac{2\,s}{n}, \frac{3\,s}{n}$, etc., and their arms are $\frac{1}{2}\frac{s}{n}, \frac{1}{2}\frac{2\,s}{n}, \frac{1}{2}\frac{3\,s}{n}$, etc. In this way we find the angles of curvature of the elements of the arc

FIG. 347.



$$\delta_1 = \tfrac{1}{2} \cdot \frac{q\,s^3}{n^3\,W\,E},\ \delta_2 = \tfrac{1}{2} \cdot \frac{2^2 \cdot q\,s^3}{n^3\,W\,E},\ \delta_3 = \tfrac{1}{2} \cdot \frac{3^2 \cdot q\,s^3}{n^3\,W\,E}, \text{ etc.,}$$

and therefore the angle of curvature of $A\ S = s$ is

$$\phi = \tfrac{1}{2} \cdot \frac{q\,s^3}{n^3\,W\,E}\,(1^2 + 2^2 + 3^2 + \ldots + n^2) = \frac{q\,s^3}{2\,n^3\,W\,E} \cdot \frac{n^3}{3}$$

$$= \frac{q\,s^3}{6\,W\,E}, \text{ approximatively } = \frac{q\,x^3}{6\,W\,E}.$$

If $x = l$, we have the tangential angle $T\,A\,C = U\,T\,B$ of the end A

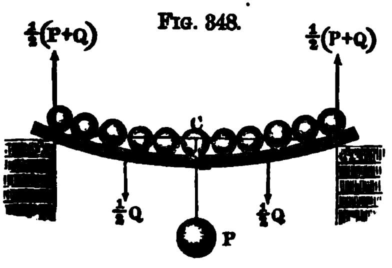
$$\beta = \frac{q\,l^3}{6\,W\,E} = \frac{Q\,l^2}{6\,W\,E},$$

and therefore for a point S, whose abscissa is $A\ K = x$,

$$\alpha = \beta - \phi = \frac{q}{6\,W\,E}\,(l^3 - x^3).$$

From the latter measure for the angle we find for an element of the ordinate
$$\frac{x}{m}a = \frac{x}{m} \cdot \frac{q}{6\,W\,E}(l^3 - x^3);$$
substituting instead of x^3 successively $\left(\frac{x}{m}\right)^3$, $\left(\frac{2\,x}{m}\right)^3$, $\left(\frac{3\,x}{m}\right)^3$, we obtain the required equation for the ordinate $K\,S = y$,
$$y = \frac{x}{m} \cdot \frac{q}{6\,W\,E}\left[m\,l^3 - \left(\frac{x}{m}\right)^3 . (1^3 + 2^3 + \ldots + m^3)\right]$$
$$= \frac{x}{m} \cdot \frac{q}{6\,W\,E}\left[m\,l^3 - \left(\frac{x}{m}\right)^3 . \frac{m^4}{4}\right], \text{i.e.}$$
$$y = \frac{q\,x}{6\,W\,E}\left(l^3 - \frac{x^3}{4}\right).$$
Assuming again $x = l$, we obtain the deflection
$$a = \frac{q\,l}{6\,W\,E} . \tfrac{3}{4}\,l^3 = \frac{q\,l^4}{8\,W\,E} = \frac{Q\,l^3}{8\,W\,E} = \tfrac{3}{8} . \frac{Q\,l^3}{3\,W\,E},$$
i.e., ⅜ of what it would be, if the load acted at the end of the girder. The ordinate of the middle of the girder is
$$y_1 = \frac{q\,l}{12\,W\,E}\left(l^3 - \frac{l^3}{32}\right) = \frac{31\,q\,l^4}{12\,.\,32\,W\,E},$$
hence the distance of this point below the horizontal line passing through B is
$$y_2 = a - y_1 = \frac{17\,q\,l^4}{12\,.\,32\,W\,E},$$
and therefore the mechanical effect corresponding to the deflection a or to the sinking (y_2) of the centre of gravity of the load $Q = l\,q$, when Q is gradually applied, is
$$L = \tfrac{1}{2}\,Q\,y_2 = \tfrac{1}{2}\,q\,l\,y_2 = \frac{17\,q^2\,l^5}{24\,.\,32\,.\,W\,E} = \frac{17\,Q^2\,l^3}{24\,.\,32\,.\,W\,E}$$
If the girder is acted upon simultaneously by a uniformly distributed load Q and a force P at the end, we have the deflection
$$a = \frac{P\,l^3}{3\,W\,E} + \frac{Q\,l^3}{8\,W\,E} = \left(\frac{P}{3} + \frac{Q}{8}\right)\frac{l^3}{W\,E}.$$
If the girder $A\,B\,A$, Fig. 348, is supported at both ends and carries not only the weight P applied at its centre, but also the load $Q = l\,q$ uniformly distributed over its length, we find the deflection $C\,B = a$ by substituting in the expression
$$a = \left(\frac{P}{3} + \frac{Q}{8}\right)\frac{l^3}{W\,E}$$
for the case represented in Fig. 347, instead of P the

FIG. 348.



pressure or reaction $\frac{P + Q}{2}$ at the extremity A, instead of Q the load $- \frac{Q}{2}$, which is equally distributed upon one-half $B\,A$, and instead of l half the length of the girder $\overline{B\,A} = \frac{1}{2}\,\overline{A\,A} = \frac{1}{2}\,l$.

In this manner we obtain

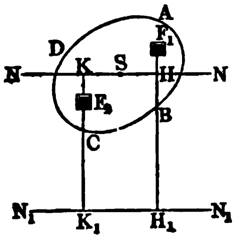
$$a = \left(\frac{P + Q}{6} - \frac{Q}{16}\right)\frac{l^3}{8\,W\,E} = (P + \tfrac{5}{8}\,Q)\frac{l^3}{48\,W\,E}$$

If $P = 0$, we have $a = \frac{5}{8} \cdot \frac{Q\,l^3}{48\,W\,E}$, that is, when the entire load is uniformly distributed upon a beam, supported at both ends, the deflection is but $\frac{5}{8}$ of what it would be, if the load was suspended at the centre of the girder.

The *weight* G of the beam has exactly the same influence upon the deflection as a load Q, which is equally distributed, and therefore enters in exactly the same manner into the calculation.

§ 224. Reduction of the Moment of Flexure.—If we know the moment of flexure $W_1\,E$ of a body $A\;B\;C\;D$, Fig. 349, in reference to an axis $N_1\,N_1$ without the centre of gravity, we can easily find this moment in reference to another axis $N\,N$, passing through the centre of gravity and parallel to the first. If the distance $H\,H_1 = K\,K_1$ between the two axes is $= d$, and if the distances of the elements of the surfaces F_1, F_2, etc., from the neutral axis $N\,N$ are $= z_1$, z_2, etc., we have their distances from the axis $N_1\,N_1$, $= d + z_1$, $d + z_2$ etc., and the moment of flexure is

FIG. 349.



$$\begin{aligned} W_1\,E &= [F_1\,(d + z_1)^2 + F_2\,(d + z_2)^2 + \ldots]\,E \\ &= [F_1\,(d^2 + 2\,d\,z_1 + z_1^2) + F_2\,(d^2 + 2\,d\,z_2 + z_2^2) + \ldots]\,E \\ &= [d^2\,(F_1 + F_2 + \ldots) + 2\,d\,(F_1\,z_1 + F_2\,z_2 + \ldots) \\ &\qquad + (F_1\,z_1^2 + F_2\,z_2^2 + \ldots)]\,E. \end{aligned}$$

But

$$F_1 + F_2 + \ldots$$

being the sum of all the elements is the cross-section F of the entire body, and

$$F_1\,z_1 + F_2\,z_2 + \ldots$$

being the sum of the statical moments in relation to an axis passing through the centre of gravity is $= 0$, and

$$(F_1\,z_1^2 + F_2\,z_2^2 + \ldots)\,E$$

is the moment of flexure $W E$ in relation to the neutral axis $N N$; consequently we have

$$W_1 E = (W + F d^2) E, \text{ or}$$
$$W_1 = W + F d^2$$

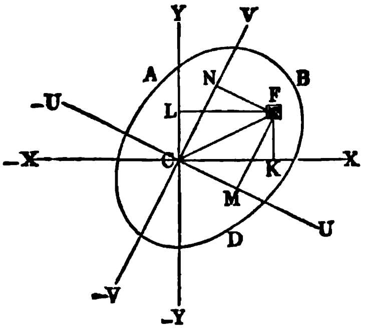
and inversely

$$W = W_1 - F d^2.$$

Therefore, *the measure W of the moment of flexure in reference to the neutral axis is equal to the measure W_1 of the moment of flexure in reference to a second parallel axis minus the product of the cross-section F and the square (d^2) of the distance between these axes.*

From this we see that, under any circumstances, the moment of flexure in relation to the neutral axis is always the smallest. The moment of flexure of many bodies in reference to some particular axis can often be found very easily, and we can employ it to determine, by the aid of the formula just found, the moment in reference to the neutral axis.

Fig. 350.



§ 225. Let $C K = x$ and $C L = y$, Fig. 350, be the coördinates of a point F, referred to a system of rectangular co-ordinates $\overline{X} X$, $\overline{Y} Y$, and let $C M = u$ and $C N = v$ be the co-ordinates of the same point, referred to another system of rectangular co-ordinates $\overline{U} U$, $\overline{V} V$, and, finally let $C F = r$ be the distance of the point F from the common origin C of the two systems of co-ordinates; according to the theorem of Pythagoras we have

$$x^2 + y^2 = u^2 + v^2 = r^2, \text{ and also}$$
$$F x^2 + F y^2 = F u^2 + F v^2 = F r^2.$$

If in this equation, instead of F, we substitute successively the elements F_1, F_2, F_3, etc., of the entire cross-section, and in like manner, instead of x, y, u and v, the corresponding co-ordinates x_1, x_2, x_3, etc., y_1, y_2, y_3, etc., u_1, u_2, u_3, etc., and v_1, v_2, v_3, etc., we obtain by addition the following formulas

$$F_1 x_1^2 + F_2 x_2^2 + \ldots + F_1 y_1^2 + F_2 y_2^2 + \ldots$$
$$= F_1 u_1^2 + F_2 u_2^2 + \ldots + F_1 v_1^2 + F_2 v_2^2 + \ldots$$
$$= F_1 r_1^2 + F_2 r_2^2 + \ldots,$$

and if we denote

$F_1 x_1^2 + F_2 x_2^2 + \ldots$ by $\Sigma\,(F x^2)$
$F_1 y_1^2 + F_2 y_2^2 + \ldots$ by $\Sigma\,(F y^2)$
$F_1 u_1^2 + F_2 u_2^2 + \ldots$ by $\Sigma\,(F u^2)$
$F_1 v_1^2 + F_2 v_2^2 + \ldots$ by $\Sigma\,(F v^2)$ and
$F_1 r_1^2 + F_2 r_2^2 + \ldots$ by $\Sigma\,(F r^2)$,

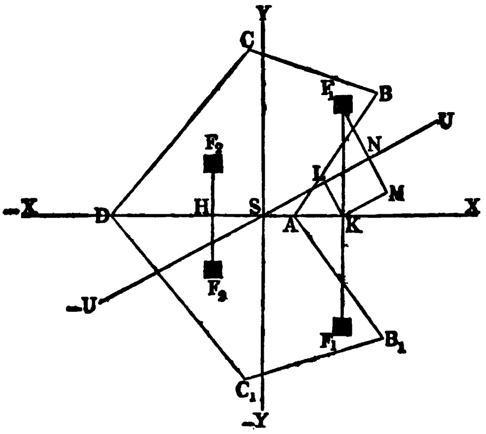
we have

$$\Sigma\,(F x^2) + \Sigma\,(F y^2) = \Sigma\,(F u^2) + \Sigma\,(F v^2) = \Sigma\,(F r^2).$$

Therefore the sum of the measures of the moment of flexure, in reference to the two axes X X and Y Y of one system of axes, is equal to the sum of the measures of the moments of flexure, in reference to the two axes of another system of axes, and equal to the measure of the moment of flexure, in reference to the origin, I.E. *equal to the sum of the products of the elements of the cross-section and the square of the distances from the axis C.*

If the cross-section $A\ C\ C_1$, Fig. 351, of a deflected body is a symmetrical figure, and if the axis $\overline{X}\,X$ at right angles to the plane of flexure is an axis of symmetry of the figure, there will be still another relation between the moments of flexure of the body. Let $S\,K = x$ and $K\,F_1 = y$ be the co-ordinates of an element of the surface F_1 in reference to the system of axes $\overline{X}\,X$ and $\overline{Y}\,Y$, and let $F_1 N = v$ be the distance of the same element from the axis $\overline{U}\,U$, which forms an angle $X\,S\,U = a$ with the first axis $\overline{X}\,X$, we have then

FIG. 351.



$$v = M F_1 - M N = M F_1 - K L$$
$$= K F_1 \cos. K F_1 M - S K \sin. K S L = y \cos. a - x \sin. a,$$

and therefore

$$v^2 = x^2 (\sin. a)^2 + y^2 (\cos. a)^2 - 2\,x\,y \sin. a \cos. a,$$
$$F_1 v^2 = (\sin. a)^2 F_1 x^2 + (\cos. a)^2 F_1 y^2 - \sin. 2\,a\,F_1\,x\,y, \text{ and}$$
$$\Sigma\,(F v^2) = (\sin. a)^2\,\Sigma\,(F x^2) + (\cos. a)^2\,\Sigma\,(F y^2) - \sin. 2\,a\,\Sigma\,(F x\,y).$$

In consequence of the symmetry of the figure, every element $F_1, F_2 \dots$ corresponds to another opposite element $F_1, F_2 \dots$, for which y, and consequently the entire product, is negative; hence the sum of the corresponding products for two such elements, and also the whole sum

$$\Sigma\,(F x y) = 0,$$

and therefore we have

$$\Sigma\,(F v^2) = (sin.\ a)^2\ \Sigma\,(F x^2) + (cos.\ a)^2\ \Sigma\,(F y^2),\ \text{or}$$
$$W = (sin.\ a)^2\ W_1 + (cos.\ a)^2\ W_2,$$

in which W denotes the measure of the moment of flexure in reference to any axis $\overline{U\,U}$, W_1 that in reference to the axis of symmetry $\overline{X\,X}$ and W_2 that in reference to the axis $\overline{Y\,Y}$ at right angles to the axis of symmetry, provided that the axes $\overline{U\,U}$ and $\overline{Y\,Y}$ as well as the axis of symmetry $\overline{X\,X}$ pass through the centre of gravity S of the figure.

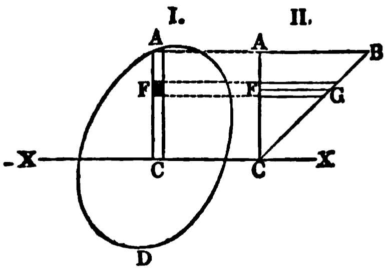
By the aid of foregoing formulas we can often find, from the known moments of flexure of a body in reference to a certain axis, its moment of flexure in reference to another axis.

§ 226. **Moment of Flexure of a Strip.**—In order to find the moment of flexure of a known cross-section $A\,B$, Fig. 352, I, of a body in reference to an axis $\overline{X\,X}$, let us imagine the cross-section divided by lines perpendicular to $\overline{X\,X}$ into small strips and every such strip as $C\,A$ to be divided again into rectangular elements F_1, F_2, F_3, etc. If z_1, z_2, z_3, etc. are the distances $(C\,F)$ of these elements from the axis $\overline{X\,X}$, we have the measure of the moment of such a strip

$$F_1\,z_1^2 + F_2\,z_2^2 + F_3\,z_3^2 + \dots$$
$$= F_1\,z_1 \,.\, z_1 + F_2\,z_2 \,.\, z_2 + F_3\,z_3 \,.\, z_3 + \dots$$

Now if we lay off in Fig. 352, II, $A\,B$ at right angles to and equal to $C\,A$, and join B and C by a straight line, it cuts off from the perpendiculars to $C\,A$, erected at the distances $(C\,F) = z_1, z_2, z_3$, etc., pieces of the same length $(F\,G) = z_1, z_2, z_3$, etc., and $F_1\,z_1$, $F_2\,z_2$, etc., can be regarded as the volumes of prisms, and $F_1\,z_1 \,.\, z_1$, $F_2\,z_2 \,.\, z_2$, etc., as their statical moments with reference to the

FIG. 352.



axis C. The prisms $F_1 z_1$, $F_2 z_2$, etc., however, form together a triangular prism, whose base is $A\ B\ C$, and whose height is the width of the strip $A\ C$ (I); the sum of the above statical moments is therefore equal to the moment of the prism $A\ B\ C$ in reference to the axis $\overline{X}\,X$. If we put the height $C\,A = z$ and the width of the prism $= b$, we have the volume of such a triangular prism

$$= \tfrac{1}{2}\, b\, z^2,$$

and since the distance of the centre of gravity from C is $\tfrac{2}{3}\, z$ (see § 109), we have the statical moment of the above prisms, and consequently the measure of the moment of flexure of the strip $C\,A$

$$W = \tfrac{1}{2}\, b\, z^2 \,.\, \tfrac{2}{3}\, z = \tfrac{1}{3}\, b\, z^3.$$

In order to find the moment of flexure of the entire cross-section $A\,D$, we have only to add together the moments of flexure of the strips, such as $C\,A$, into which the entire surface is decomposed by the perpendiculars to the axis $\overline{X}\,X$.

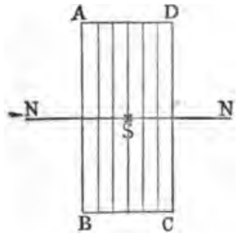
The most simple case is that of a rectangular cross-section $A\ B\ C\ D$, Fig. 353. The strips into which the surface is divided are here all of the same size and form together but a single strip, whose width $A\ D = b$ is that of the entire rectangle.. If the height $A\ B$ of this rectangle is $= h$, we have for the height of a strip

$$z = \tfrac{1}{2}\, h;$$

consequently the measure of the moment of flexure of half of this surface is

$$\tfrac{1}{3}\, b \left(\frac{h}{2}\right)^3 = \frac{b\, h^3}{24};$$

FIG. 353.



finally, the measure of the moment of the entire rectangle is

$$W = 2\, \frac{b\, h^3}{24} = \frac{b\, h^3}{12}.$$

§ 227. Moment of Flexure of a Girder, whose Form is that of a Parallelopipedon.

—From the foregoing we see that the *moment of flexure* of a *parallelopipedical girder* $W\,E = \frac{b\, h^3}{12}\, E$ *increases with the width* and *with the cube of the height of the girder.*

Substituting this value for $W\,E$ in the first formula

$$a = \frac{P\, l^3}{3\, W\, E} \text{ of § 217,}$$

we obtain the *deflection* of a *girder,* whose cross-section is *rectangular,* and which is *fixed* at *one end,*

$$a = 4 \cdot \frac{P\,l^3}{b\,h^3\,E}.$$

Substituting it in the second formula of the same paragraph

$$a = \frac{1}{48}\,\frac{P\,l^3}{W\,E},$$

we have for a *beam supported at both ends*

$$a = \frac{P\,l^3}{4\,b\,h^3\,E}.$$

Inversely, from the deflection a we obtain in the first case the modulus of elasticity $$E = \frac{4\,P\,l^3}{a\,b\,h^3},$$

and in the second $$E = \frac{P\,l^3}{4\,a\,b\,h^3}.$$

EXAMPLE—1) A wooden girder 10 feet = 120 inches long, 8 inches wide and 10 inches high is supported at both ends and carries a uniformly distributed load of Q = 10000 pounds; how much will it be bent?

The deflection is

$$a = \tfrac{5}{8}\,\frac{Q\,l^3}{4\,b\,h^3\,E} = \tfrac{5}{32} \cdot \frac{10000 \,.\, 120^3}{8 \,.\, 10^3 \,.\, E} = \frac{50000 \,.\, 12^3}{32 \,.\, 8\,E} = \frac{1350000}{4 \,.\, E}.$$

Substituting E = 1560000, we have $a = \dfrac{135}{4 \,.\, 156}$ = 0,216 inches.

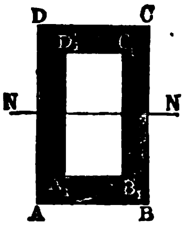
2) If a parallelopipedical cast-iron rod, supported at both ends, is 2 inches wide and ½ an inch thick, and is deflected ¼ of an inch by a weight P = 18 pounds placed upon it at its centre, the distance of the supports from each other being 5 feet, the modulus of elasticity is

$$E = \frac{P\,l^3}{4\,a\,b\,h^3} = \frac{18 \,.\, 60^3}{4 \,.\, \frac{1}{4} \,.\, 2 \,.\, (\frac{1}{2})^3} = \frac{18 \,.\, 60^3}{\frac{1}{4}} = 72 \,.\, 216000 = 15552000 \text{ pounds.}$$

§ 228. Hollow, Double-Webbed cr Tubular Girders.—

The moment of flexure of a hollow parallelopipedical girder $A\ B\ C\ D$, Fig. 354, is determined by subtracting from the moment of the whole cross-section the moment of the hollow portion. If $A\,B = b$ and $B\,C = h$ are the exterior and $A_1\,B_1 = b_1$ and $B_1\,C_1 = h_1$ the interior width and height, we have the measures of the moments of flexure of the surfaces $A\ C$ and $A_1\ C_1$

FIG. 354.



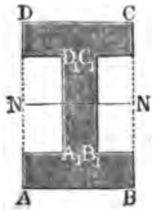
$$= \frac{b\,h^3}{12} \text{ and } \frac{b_1\,h_1^3}{12},$$

and consequently by subtraction the *measure of the moment of flexure of the tubular girder*

$$W = \frac{b\,h^3 - b_1\,h_1^3}{12}.$$

The moment of flexure of the *single-webbed girder* $A\ B\ C\ D$, Fig. 355, is determined in exactly the same manner. If $A\ B = b$ and $B\ C = h$ are the exterior height and width, and if $A\ B - A_1\ B_1 = b_1$ and $B_1\ C_1 = h_1$ are the sum of the widths and the height of the two cavities, we have by subtraction

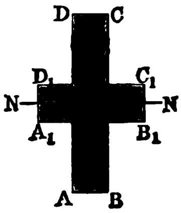
FIG. 355.



$$W = \frac{b\,h^3 - b_1\,h_1^3}{12}$$

The moment of flexure of the body $A\ B\ C\ D$, Fig. 356, the cross-section of which is a cross, is found in a similar manner. If $A\ B = b$ and $B\ C = h$ are the height and width of the central portion, and $A_1 B_1 - A\ B = b_1$ is the sum of the widths, and $B_1\ C_1 = h_1$ the height of the lateral portions, we obtain by addition the measure of the moment of flexure

FIG. 356.



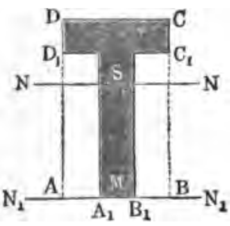
$$W = \frac{b\,h^3 + b_1\,h_1^3}{12}.$$

In the same manner we can determine the moments of flexure of many bodies which occur in practice. Thus for a body $A_1\ B_1\ C\ D$, Fig. 357, with a T-shaped cross-section, whose dimensions are

$A\ B = C\ D = b$,
$A\ B - A_1\ B_1 = A\ A_1 + B\ B_1 = b_1$,
$A\ D = B\ C = h$ and
$A\ D_1 = B\ C_1 = B\ C - C\ C_1 = h_1$,

the measure of the moment of flexure in reference to the lower edge $A_1\ B_1$ is = moment of the rectangle $A\ B\ C\ D$ minus moment of the rectangles $A_1\ D_1$ and $B_1\ C_1$, I.E.,

FIG. 357.



$$W_1 = \tfrac{1}{2} \cdot \frac{b\,(2\,h)^3}{12} - \tfrac{1}{2} \cdot \frac{b_1\,(2\,h_1)^3}{12} = \frac{b\,h^3 - b_1\,h_1^3}{3}.$$

These moments are found by assuming each of these rectangles to be the half of rectangles twice as high; for these the axis $N_1\ N_1$ is the neutral axis.

Now the surface $A_1\ C_1\ D = F = b\ h - b_1\ h_1$, and its statical moment is

$$F \,.\, e_1 = b\,h \,.\, \frac{h}{2} - b_1\,h_1 \,.\, \frac{h_1}{2} = \tfrac{1}{2}\,(b\,h^2 - b_1\,h_1^2);$$

consequently the lever arm is

$$M S = e_1 = \frac{b h^2 - b_1 h_1^2}{2 (b h - b_1 h_1)},$$

the product

$$F . e_1^2 = \tfrac{1}{4} (b h^2 - b_1 h_1^2)^2 : (b h - b_1 h_1)$$

and the measure of the moment of flexure of the body in reference to the neutral axis $N N$, passing through the centre of gravity S, is

$$W = W_1 - F . e_1^2 = \frac{b h^3 - b_1 h_1^3}{3} - \tfrac{1}{4} (b h^2 - b_1 h_1^2)^2 : (b h - b_1 h_1)$$

$$= \frac{4 (b h^3 - b_1 h_1^3) (b h - b_1 h_1) - 3 (b h^2 - b_1 h_1^2)^2}{12 (b h - b_1 h_1)}$$

$$= \frac{(b h^2 - b_1 h_1^2)^2 - 4 b h b_1 h_1 (h - h_1)^2}{12 (b h - b_1 h_1)}$$

It is also easy to perceive, that the high webbed and flanged girders have, for the same quantity of material, a greater moment of flexure than the wide and massive ones. Since this moment increases with the surface (F) and with the square (z^2) of the distance from the neutral axis, the same fibre is better able to resist the bending the farther it is removed from the neutral axis. If, for example, the height of a massive parallelopipedical girder is double the width b, the measure of moment of flexure is either

$$W = \frac{b . (2 b)^3}{12} = \tfrac{2}{3} b^4, \text{ or } = \frac{2 b . b^3}{12} = \tfrac{1}{6} b^4,$$

the first formula obtaining, when we place its greater dimension $2 b$ vertical, and the latter, when it is placed horizontal; in the first case the moment of flexure is four times as great as in the second. If, again, we replace the solid girder, whose cross-section is $b h$ by a double webbed one, in which the hollow is equal to the massive part of the cross-section $b_1 h_1 - b h$, or if $b_1 h_1 - b h = b h$, i.e., $b_1 h_1 = 2 b h$, or $b_1 = b \sqrt{2}$ and $h_1 = h \sqrt{2}$, the measure of the moment of flexure for the latter girder is

$$\frac{b_1 h_1^3 - b h^3}{12} = \frac{b \sqrt{2} (h \sqrt{2})^3 - b h^3}{12} = \tfrac{3}{12} b h^3$$

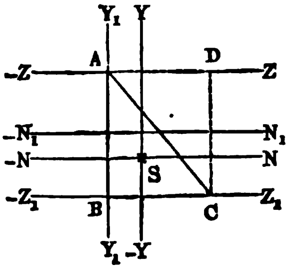
i.e., three times as great as for the first one.

§ 229. **Triangular Girders.**—The measure of the moment of flexure of a body with a triangular cross-section $A B C$, Fig. 358, can be found, in accordance with what has been stated in the last paragraphs, in the following manner.

The measure of the moment of flexure for the prism with a rectangular cross-section $A B C D$ is, when we retain the notations of the next to the last paragraph, $= \frac{b h^3}{12}$, and consequently that of its half with the triangular cross-section $A\ B\ C$ in reference to the central line $\overline{N_1}\ N_1$ is

$$W_1 = \tfrac{1}{2}\frac{b\,h^3}{12} = \frac{b\,h^3}{24}.$$

FIG. 358.



But the line of gravity $\overline{N}\,N$ of the triangle is at a distance $\tfrac{1}{6}\ A\ B = \tfrac{1}{6}\ h$ from the central line or line of gravity $\overline{N_1}\ N_1$ of the rectangle, and, therefore, according to § 224, the measure of the moment in reference to $\overline{N}\,N$ is

$$W = W_1 - \left(\frac{h}{6}\right)^2 F = \frac{b\,h^3}{24} - \frac{b\,h^3}{72}$$

$$= \frac{b\,h^3}{36} = \tfrac{1}{3}\cdot\frac{b\,h^3}{12}.$$

The measure of the moment of flexure W of a girder with a triangular cross-section is but one-third of the measure of the moment of flexure of a parallelopipedical one, the cross-section of which has the same base and altitude. But since the latter girder has but double the volume of the former, it follows, that for equal dimensions the moment of flexure of a triangular girder is but $\tfrac{2}{3}$ that of a rectangular one.

For the axis $\overline{Z_1}\ Z_1$ passing through the base $B\ C$, the measure of this moment is

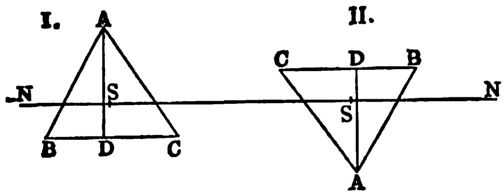
$$W_2 = W + \left(\frac{h}{3}\right)^2 . F = \frac{b\,h^3}{36} + \frac{b\,h^3}{18} = \frac{b\,h^3}{12},$$

and for the axis $\overline{Z}\ Z$, passing through the edge A,

$$W_3 = W + \left(\frac{2\,h}{3}\right)^2 . \frac{b\,h}{2} = \frac{b\,h^3}{36} + \frac{4\,b\,h^3}{18} = \frac{b\,h^3}{4}$$

These formulas do not require the cross-section to be a right-angled triangle. They hold good for any other triangle $A\ B\ C$, Fig. 359, whose base $B\ C$ is at right angles to the bending force P; for it can be decomposed into two right-angled triangles $A\,D\,B$ and $A\,C\,D$ whose bases $B\,D = b_1$ and $D\ C = b_2$ form together the base $B\ C = b$ of the triangle $A\ B\ C$, so that we have for this triangle

FIG. 359.



$$W = \frac{1}{36} b_1 h^3 + \frac{1}{36} b_2 h^3 = \frac{1}{36} (b_1 + b_2) h^3 = \frac{b h^3}{36}.$$

It is also of no importance whether the base $B\,C$ lies above or below the axis, I.E., whether it is placed as in I or II. The moment of flexure in both cases is

$$W E = \frac{b\,h^3}{36} E,$$

when the modulus of elasticity for extension is the same as that for compression. The same formulas can also be employed, when the cross-section is a rhomb $A\,B\,C\,D$, Fig. 360, with the horizontal diagonal $B\,D$. If $B\,D = b$ is the width and $A\,C = h$ the height, we have for the body with this cross-section

$$W = 2 \, . \frac{b}{12} \left(\frac{h}{2}\right)^3 = \frac{b\,h^3}{48} = \frac{1}{4} \frac{b\,h^3}{12},$$

I.E., one quarter of the measure of the moment of a girder with a rectangular cross-section of the same height and width. From this it follows, that for a double trapezoid $A\,B\,E\,D$, Fig. 361, the height of which is $A\,C = B\,D = h$, the exterior width $A\,B = C\,D = b$ and the interior width $E\,F = b_1$,

$$W = \frac{b\,h^3}{12} - (b - b_1) \frac{h^3}{48} = \frac{(3\,b + b_1)\,h^3}{48}.$$

FIG. 360.

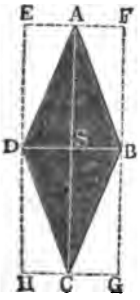


FIG. 361.

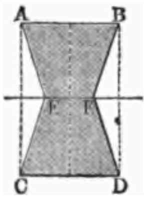
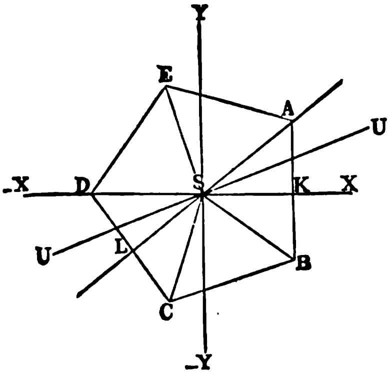


FIG. 362.



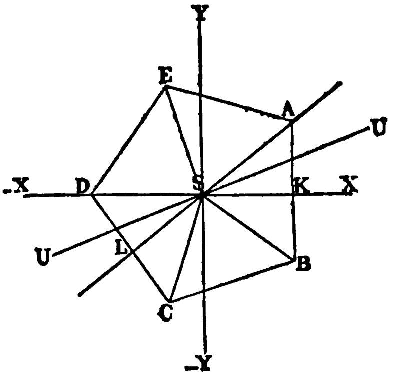
§ **230. Polygonal Girders.**—The foregoing theory can be applied to a body with a regular polygonal cross-section $A\,C\,E$, Fig. 362, whose neutral axis $\overline{X}\,X$ is at the same time an axis of symmetry. Since such a polygon can be resolved into triangles, having a common vertex S, the determination of its moment consists essentially in the calculation of the moment of flexure of one of those triangles $A\ S\ B$. If we denote the side $A\ B = B\ C = C\ D$ of the polygon or the base of one of the triangles composing it by s and the altitude $S\ K$ of the same by h, we have the measure of its moment of flexure in reference to the axis $\overline{X}\ X$

$$= \frac{1}{4} \cdot \frac{h\ s^3}{12} = \frac{h\ s^3}{48};$$

on the contrary, this moment in reference to a second axis $\overline{Y}\ Y$ is $= \frac{s\ h^3}{4}$, and consequently the sum of these two moments is

$$\frac{s h^3}{4} + \frac{h\ s^3}{48} = \frac{s\ h}{4}\left(h^2 + \frac{s^2}{12}\right).$$

FIG. 362 a.



This sum holds good (according to § 225) for every other triangle, and therefore, for a polygon of n sides, we have

$$W_1 + W_2 = \frac{n\ s\ h}{4}\left(h^2 + \frac{s^2}{12}\right) = \frac{F}{2}\left(h^2 + \frac{s^2}{12}\right),$$

when its area $n \cdot \frac{s\ h}{2}$, is denoted by F.

If we designate the angle $A\ S\ X$ by a, the measure of the moment in reference to the axis $A\ S\ L$ is

$$= W_1\ (sin.\ a)^2 + W_2\ (cos.\ a)^2;$$

but the latter is also equal to the measure of the moment W_1 in reference to $K\ S\ D$ or $\overline{X}\ X$, and therefore we have

$$W_1 = W_1\ (sin.\ a)^2 + W_2\ (cos.\ a)^2,$$

or $$W_1\ [1 - (sin.\ a)^2] = W_2\ (cos.\ a)^2,$$

i.e. $$W_1\ (cos.\ a)^2 = W_2\ (cos.\ a)^2,$$ and consequently

$$W_1 = W_2.$$

For an axis $\overline{U}\ U$, forming an arbitrary angle $X\ S\ U = \phi$ with the axis $\overline{X}\ X$ of symmetry, the measure of the moment is

$$W = W_1\ sin.^2\ \phi + W_2\ cos.^2\ \phi = W_1\ (sin.^2\ \phi + cos.^2\ \phi) = W_1.$$

Now if we substitute in the above equation

$$W_1 + W_2 = \frac{F}{2}\left(h^2 + \frac{s^2}{12}\right),\ W = W_1 = W_2,$$

we obtain for any arbitrary axis of a regular polygon the measure of the moment of flexure

$$W = W_1 = W_2 = \frac{F}{4}\left(h^2 + \frac{s^2}{12}\right),$$

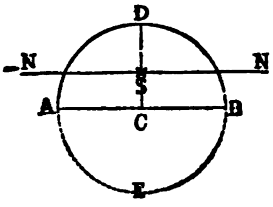
or, putting the radius of the polygon $SA = SB = r$ and therefore $h^2 = r^2 - \frac{s^2}{4}$,

$$W = \frac{F}{4}\left(r^2 - \frac{s^2}{6}\right).$$

§ **231. Cylindrical or Elliptical Girders.**—For the circle, considered as the polygon of an infinite number of infinitely small sides, $s = 0$, and therefore the measure of the moment of flexure of a cylinder is

$$W = \frac{F}{4} r^2 = \frac{\pi r^4}{4} = 0{,}7854\, r^4.$$

Fig. 363.



For a hollow cylinder or tube, whose exterior radius is r_1 and whose interior one is r_2, we have by subtraction

$$W = \frac{\pi(r_1^4 - r_2^4)}{4} = \frac{\pi(r_1^2 - r_2^2)(r_1^2 + r_2^2)}{4} = \frac{F(r_1^2 + r_2^2)}{4}$$

$$= \frac{F r^2}{2}\left[1 + \left(\frac{b}{2r}\right)^2\right],$$

in which $F = \pi(r_1^2 - r_2^2)$ denotes the area of the ring-shaped cross-section, $r = \frac{r_1 + r_2}{2}$ the mean radius and $b = r_1 - r_2$ the thickness of the wall of the tube. The horizontal diameter divides the entire circle DE, Fig. 363, into two semicircles ADB and AEB, and the measure of the moment for such a semicircle in reference to the diameter AB is

$$W_1 = \frac{1}{2}\frac{\pi r^4}{4} = \frac{\pi r^4}{8}.$$

But the distance of the centre of gravity S of the semicircle from the centre C of the circle is $CS = \frac{4r}{3\pi}$ (see § 113), and therefore the measure of the moment for the parallel axis $\overline{NN}$ is

$$W = W_1 - F.\overline{CS}^2 = W_1 - F.\left(\frac{4r}{3\pi}\right)^2$$

$$= \pi r^4\left(\frac{1}{8} - \frac{8}{9\pi^2}\right) = 0{,}1098\,.\,r^4,$$

while, on the contrary, for the semicircle, whose diameter is vertical,

$$W = \frac{\pi r^4}{8} = 0,3927\, r^4.$$

In reference to an axis $\overline{N}\, N$, which forms an angle $N S X = a$ with the axis of symmetry $C\, D$, Fig. 364, the measure of the moment of the *semicircle* is

$$W = \frac{\pi r^4}{8}\, sin.\, a^2 + \pi r^4 \left(\frac{1}{8} - \frac{8}{9\,\pi^2}\right) cos.^2\, a$$
$$= (0,3927\; sin.^2\, a + 0,1098\; cos.^2\, a)\, r^4.$$

FIG. 364.

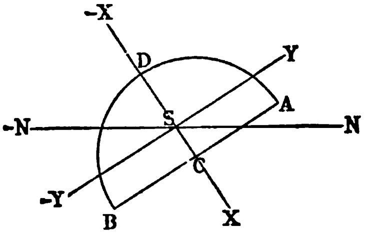
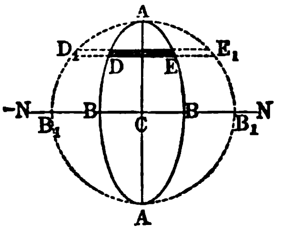


FIG. 365.



From the formula

$$W = \frac{\pi r^4}{4},$$

for the measure of the moment of flexure of the full circle, that of an *ellipse* $A\, B\, A\, B$, Fig. 365, is easily deduced. In consequence of the relation of the ellipse to the circle given in Art. 12 of the Introduction to the Calculus, when $A\, B_1\, A\, B_1$ represents a circle whose radius $C\, A$ is equal to the major semi-axis a of the ellipse, and when the other semi-axis $C\, B$ of the ellipse is represented by b, we have the ratio $\frac{D\, E}{D_1\, E_1}$ of the width $D\, E$ of an element of the ellipse to that $D_1\, E_1$ of a similarly placed and equally high element of the circle

$$= \frac{B\, B}{B_1\, B_1} = \frac{C B}{C\, B_1} = \frac{b}{a}.$$

But since the moment of flexure of such a strip increases with the simple width, the moment of a strip $D\, E$ of the ellipse is to that of the corresponding strip of the circle as b is to a, and consequently the measure of the moment of flexure of a body with an elliptical cross-section is equal $\frac{b}{a}$ times that of a body with a circular cross-section, I.E.

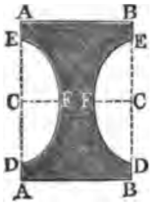
$$W = \frac{b}{a} \cdot \frac{\pi a^4}{4} = \frac{\pi a^3 b}{4}.$$

If this body contains also an *elliptical hollow*, the semi-axes of which are a_1 and b_1, we have for this body

$$W = \frac{\pi (a^3 b - a_1^3 b_1)}{4}.$$

If a body with a *rectangular cross-section* has an *elliptical hollow* around its axis, or, as is represented in Fig. 366, has an elliptical cavity on the side, we have the measure of its moment of flexure

FIG. 366.

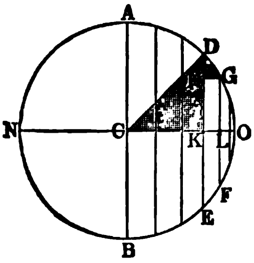


$$W = \frac{b h^3}{12} - \frac{\pi a_1^3 b_1}{4},$$

b and h denoting the length $A B$ and the height $A A = B B$ of the rectangular cross-section $A B B A$, and, on the contrary, a_1 and b_1 the semi-axes $C E$ and $C F$ of the semi-elliptical hollow $D F E$.

§ **232**. The measure W of the moment of flexure of a *cylinder* or a *segment of a cylinder* may be determined very simply in the following manner. We divide the quadrant $A D O$ of the segment of the cylinder $A O B N$, Fig. 367, into n equal parts, pass through the points of division vertical planes, such as $D E$, $F G$, etc. and determine the moment of flexure for each one of the slices $D E F G$, which we consider to be right parallelopipedons. The sum of the moments of these slices gives the moment of flexure of the semi-cylinder $A O B$, and by doubling this moment we obtain the moment of flexure of the entire cylinder. If r denotes the radius $C A = C O$ of the circular cross-section $A O B N$, a division $D G$ of the arc $= \frac{1}{n} \cdot \frac{\pi r}{2} = \frac{\pi r}{2 n}$, and in consequence of the similarity of the triangles $D G H$ and $C D K$, we have for the thickness $K L$ of the slice of the cylinder $D E F G = 2 D G L K$

FIG. 367.



$$K L = G H = \frac{K D}{C D} \cdot D G = \frac{K D}{C D} \cdot \frac{\pi r}{2 n} = \frac{\pi}{2 n} \cdot \overline{K D}.$$

Now according to the formula of § 226, the measure of the moment of flexure of the slice $D E F G$ is

$$= \frac{\overline{K L} \,.\, (2 \overline{K D})^3}{12} = \frac{8}{12} \cdot \frac{\pi}{2 n} \cdot \overline{K D}^4 = \frac{\pi}{3 n} \overline{K D}^4.$$

If we put the variable angle $A\ C\ D$, which determines the distance of the slice from the vertical diameter, $= \phi$, we obtain the ordinate or half-height of the slice, $D\ K = r\ cos.\ \phi$, and therefore the last measure of the moment of flexure can be put $= \frac{\pi r^4}{3\ n}\ (cos.\ \phi)^4$ $= \frac{\pi r^4}{3\ n}\ \frac{3 + 4\ cos.\ 2\ \phi + cos.\ 4\ \phi}{8}$, as $(cos.\ \phi)^4 = \frac{3 + 4\ cos.\ 2\ \phi + cos.\ 4\ \phi}{8}$ (see the "Ingenieur," page 157). In order to find the measure of the moment of flexure for the semi-cylinder, we must substitute in the factor $3 + 4\ cos.\ 2\ \phi + cos.\ 4\ \phi$, for ϕ successively the values $1 . \frac{\pi}{2\ n}$, $2 . \frac{\pi}{2\ n}$, $3 . \frac{\pi}{2\ n}$, to $n . \frac{\pi}{2\ n}$, then add the results found, and finally multiply by the common factor $\frac{\pi r^4}{2\ 4\ n}$. Now the number 3 added n times to itself gives $3\ n$, the sum of the cosines from 0 to π is $= 0$, since the cosines in the second quadrant $\frac{\pi}{2}$ to π are equal and opposite to the cosines in the first quadrant 0 to $\frac{\pi}{2}$, and the sum of the cosines in the third quadrant π to $\frac{3}{2}\ \pi$ cancel those in the fourth quadrant $\frac{3}{2}\ \pi$ to $2\ \pi$; therefore the measure of the moment of flexure of the semi-cylinder is

$$\frac{W}{2} = \frac{\pi r^4}{2 4 n} . 3\ n = \frac{\pi r^4}{8},$$

and that of the entire cylinder is

$$W = \frac{\pi r^4}{4} = 0{,}7854\ r^4, \text{ or}$$

$$W = \frac{\pi d^4}{64} = 0{,}04909\ d^4,$$

$d = 2\ r$ denoting the diameter of the cylinder.

(REMARK.)—If we employ the formulas of the Calculus, $d\ \phi$ denotes an element of the arc ϕ, and the element $D\ G = \frac{r\pi}{2 n} = r\ \delta\ \varphi$; hence the measure of the moment of the element $D\ E\ F\ G$ of the surface is

$$= \frac{2\ d\ \phi\ .\ r^4}{3}\ (cos.\ \phi)^4 = \frac{2\ r^4\ d\ \phi}{3} \left(\frac{3 + 4\ cos.\ 2\ \phi + cos.\ 4\ \phi}{8}\right)$$

$$= \frac{r^4}{12}(3 + 4\ cos.\ 2\ \phi + cos.\ 4\ \phi)\ d\ \phi = \frac{r^4}{12}(3\ d\phi + 4\ cos.\ 2\ \phi\ d\phi + cos.\ 4\ \phi\ \ d\phi)$$

$$= \frac{r^4}{12}\ [3\ d\ \phi + 2\ cos.\ 2\ \phi\ d\ (2\ \phi) + \tfrac{1}{4}\ cos.\ 4\ \phi\ d\ (4\ \phi)\ ],$$

and consequently that of the portion $A\ B\ E\ D$ of the cylinder is

$$W = \frac{r^4}{12}\left(3\int d\phi + 2\int \cos.\ 2\ \phi\ d\,(2\ \phi) + \frac{1}{4}\int \cos.\ 4\ \phi\ d\,(4\ \phi)\right), \text{ I.E.}$$

$$W = \frac{r^4}{12}\,(3\ \phi + 2\ \sin.\ 2\ \phi + \tfrac{1}{4}\ \sin.\ 4\ \phi).$$ (See *Introduction to the Calculus*, § 26, I.).

Substituting $\phi = \frac{\pi}{2}$, $\sin.\ 2\ \phi = \sin.\ \pi = 0$, and $\sin.\ 4\ \phi = \sin.\ 2\ \pi = 0$, and doubling the result obtained, we have the measure of the moment of flexure of the entire cylinder

$$W = \frac{r^4}{12} \cdot \frac{3\ \pi}{2} \cdot 2 = \frac{\pi\ r^4}{4}.$$

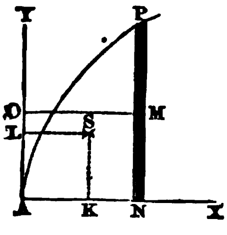
For the segment $D\ O\ E$, on the contrary, we have

$$\begin{aligned} W &= \frac{\pi\ r^4}{8} - (3\ \phi + 2\ \sin.\ 2\ \phi + \tfrac{1}{4}\ \sin.\ 4\ \phi)\,\frac{r^4}{12} \\ &= \left[\frac{\pi - 2\ \phi}{8} - \left(\frac{2\ \sin.\ 2\ \phi + \frac{1}{4}\ \sin.\ 4\ \phi}{12}\right)\right] r^4 \\ &= [6\ (\pi - 2\ \phi) - 8\ \sin.\ 2\ \phi - \sin.\ 4\ \phi]\,\frac{r^4}{48}. \end{aligned}$$

By simple subtraction we obtain, by means of the latter formula, the measure of the moment W of a board $D\ E\ F\ G$ of a finite thickness $K\ L$.

(§ 233.) **Beams with Curvilinear Cross-sections.**—The measure of the moment of flexure W of bodies with *regular curvilinear cross-sections* is determined most surely by the aid of the calculus. For this purpose we decompose such a surface $A\ N\ P$, Fig. 368, by ordinates into its elements, and we determine the moments of such an element in reference to the axis of abscissas $A\ X$ and also in reference to the axis of ordinates $A\ Y$.

FIG. 368.



If x is the abscissa $A\ N$ and y the ordinate $N\ P$, we have the area of an element

$$d\ F = y\ d\ x$$

(see *Introduction to the Calculus*, Art. 29) and therefore the measure of the moment of flexure in reference to the axis $A\ X$

$$d\ W_1 = \tfrac{1}{3}\ y^2 \,.\, d\ F = \tfrac{1}{3}\ y^3\ d\ x$$

(see § 226), and, on the contrary, that in reference to the axis $A\ Y$

$$d\ W_2 = x^2\ y\ d\ x,$$

since all points of the element are at the same distance x from $A\ Y$.

By integration we obtain for the whole surface $A\ N\ P = F$

$$W_1 = \tfrac{1}{3} \int y^3 \, d\, x$$

and

$$W_2 = \int x^2 \, y \, d\, x.$$

If we have determined (according to § 115) the centre of gravity of the surface $A\ N\ P$ and its co-ordinates $A\ K = u$ and $K\ S = v$, we find the measures of the moments of flexure in reference to the axes passing through the centre of gravity and parallel to the co-ordinate axes by putting

$$W_1 = \tfrac{1}{3} \int y^3 \, d\, x - v^2\, F$$

and

$$W_2 = \int x^2 \, y \, d\, x - u^2\, F.$$

E.G., for a *parabolic surface* $A\ N\ P$, whose equation is $y^2 = p\, x$, we have (*according to Art.* 29 *of the Introduction to the Calculus*)

$F = \tfrac{2}{3}\, x\, y$, and (*according to* § 115)

$u = \tfrac{3}{5}\, x$ and $v = \tfrac{3}{8}\, y$,

hence

$$v^2\, F = \left(\frac{3}{8}\right)^2 F\, y^2 = \left(\frac{3}{8}\right)^2 y^2 \cdot \frac{2}{3}\, x\, y = \frac{3}{32}\, x\, y^3$$

and

$$u^2\, F = \left(\frac{3}{5}\right)^2 F\, x^2 = \left(\frac{3}{5}\right)^2 x^2 \cdot \frac{2}{3}\, x\, y = \frac{6}{25}\, x^3\, y.$$

Since also from $y^2 = p\, x$, it follows, that $x = \dfrac{y^2}{p}$ and $d\, x = \dfrac{2\, y\, d\, y}{p}$, we have

$$\frac{1}{3}\int y^3\, d\, x = \frac{1}{3}\int y^3 \cdot \frac{2\, y\, d\, y}{p} = \frac{2}{3\, p}\int y^4\, d\, y = \frac{2\, y^5}{15\, p} = \frac{2}{15}\, y^3\, x$$
$$= \frac{1}{5} \cdot \frac{2}{3}\, x\, y \cdot y^2 = \frac{1}{5}\, F\, y^2$$

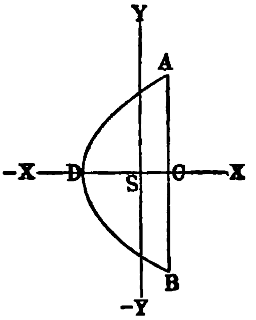
and

$$\int x^2\, y\, d\, x = \int \frac{y^4}{p^2} \cdot \frac{2\, y^2\, d\, y}{p} = \frac{2}{p^3}\int y^6\, d\, y = \frac{2\, y^7}{7\, p^3} = \frac{2}{7}\, x^3\, y$$
$$= \frac{3}{7} \cdot \frac{2}{3}\, x\, y \cdot x^2 = \frac{3}{7}\, F\, x^2.$$

Finally we obtain

$$W_1 = \frac{1}{5}\, F\, y^2 - \left(\frac{3}{8}\right)^2 F\, y^2 = \left(\frac{1}{5} - \frac{9}{64}\right) F\, y^2 = \frac{19}{320}\, F\, y^2 \text{ and}$$
$$W_2 = \frac{3}{7}\, F\, x^2 - \left(\frac{3}{5}\right)^2 F\, x^2 = \frac{12}{175}\, F\, x^2.$$

FIG. 369.



For a *symmetrical parabolic surface* $A\,D\,B$, Fig. 369, whose cord $A\,B = s$ and whose altitude $C\,D = h$, we can put the measure of the moment in reference to the axis of symmetry $\overline{X}\,X$

$$W_1 = \tfrac{1}{5}\,F\left(\frac{s}{2}\right)^2 = \frac{F\,s^2}{20} = \frac{s^3\,h}{30},$$

while, on the contrary, that in reference to the axis $\overline{Y}\,Y$ at right angles to it remains

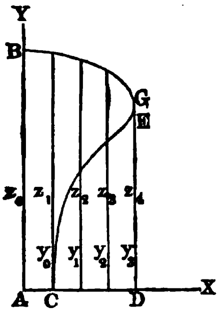
$$W_2 = \frac{12}{175}\,F\,h^2 = \frac{8}{175}\,h^3\,s.$$

§ **234. Curvilinear Cross-sections.**—If we are required to calculate the moment of flexure of a body, whose cross-section forms a compound or irregular figure, we must either divide this cross-section into parts, for which the measure W is already known, or we must decompose the same by vertical lines, calculate the measures of the moment of flexure of these strips (*according to* § 226), and, finally, add these values together, in doing which we can employ with advantage *Simpson's* or *Cotes'* rule.

If, E.G., $A\,B\,E\,C$, Fig 370, is such a figure or such a portion of the cross-section of a body and if its moment of flexure in reference to the axis $A\,X$ is to be determined, we calculate first the measure W_1 for the portion of surface $A\,B\,G\,D$ and then the measure W_2 for the part $C\,E\,D$; subtracting the latter from the former, we obtain the required moment

$$W = W_1 - W_2.$$

FIG. 370.



If the base $A\,D$ of the first part $= x$, and the altitudes of the same at equal distances from each other are z_0, z_1, z_2, z_3, z_4, we have the corresponding measure of the moment, according to Simpson's rule,

$$W_1 = \frac{1}{3}\cdot\frac{x}{12}\,(z_0^3 + 4\,z_1^3 + 2\,z_2^3 + 4\,z_3^3 + z_4^3).$$

If, on the contrary, the width $C\,D$ of the piece $C\,D\,E$ to be subtracted be $= x_1$ and the altitudes of the same are y_0, y_1, y_2, y_3, we have, according to Cotes' rule (*see Introduction to the Calculus, Art. 38*),

$$W_2 = \frac{1}{3}\cdot\frac{x_1}{8}\,(y_0^3 + 3\,y_1^3 + 3\,y_2^3 + y_3^3).$$

If *A X* does not pass through the centre of gravity *S* of the entire surface, we must reduce it by the well-known rule (§ 224) to the axis passing through *S*. In the same manner other parts of the cross-section, which lie below *A X* or alongside of *A Y*, may be treated. The centre of gravity *S* can be determined either according to § 124, or empirically by cutting a pattern of the section out of thin sheet iron or paper and laying it upon a sharp knife-edge. If we determine in this way two lines of gravity, their point of intersection gives the centre of gravity.

EXAMPLE.—*A B G E C*, in Fig. 370, is a portion of the cross-section of an iron rail, which can be considered as the difference of two surfaces *A B G D* and *C E D*. If the width of the first is $\frac{4}{3}$ inches and that of the second 1 inch, and if the heights of the first are

$$z_0 = 2{,}85;\ z_1 = 2{,}82;\ z_2 = 2{,}74;\ z_3 = 2{,}60;\ \text{and}\ z_4 = 2{,}30,$$

and those of the second

$$y_0 = 0{,}20;\ y_1 = 1{,}50;\ y_2 = 1{,}80\ \text{and}\ y_3 = 2{,}15,$$

we have for the measure of the moment of flexure of the first portion

$$W_1 = \frac{1}{3} \cdot \frac{4}{3} \cdot \frac{1}{12} \cdot [2{,}85^3 + 2{,}30^3 + 4 \,.\, (2{,}82^3 + 2{,}60^3) + 2 \,.\, 2{,}74^3]$$

$$= \frac{1}{27} \cdot (23{,}149 + 12{,}167 + 4 \,.\, 40{,}002 + 2 \,.\, 20{,}571)$$

$$= \frac{1}{27} \cdot 236{,}47 = 8{,}7584,$$

and, on the contrary, that of the second portion

$$W_2 = \frac{1}{3} \,.\, 1 \,.\, \frac{1}{8} \cdot [0{,}20^3 + 2{,}15^3 + 3\ (1{,}50^3 + 1{,}80^3)]$$

$$= \frac{1}{24} \cdot (0{,}0080 + 9{,}9384 + 27{,}6210) = \frac{37{,}5674}{24} = 1{,}5653,$$

consequently, the required measure for the entire surface *A B G E C* is

$$W = W_1 - W_2 = 8{,}7584 - 1{,}5653 = 7{,}1931.$$

REMARK.—We can also put

$$W = \frac{z}{12}\left(\frac{z}{4}\right)^2 (1 \,.\, 0^2 \,.\, y_0 + 4 \,.\, 1^2 \,.\, y_1 + 2 \,.\, 2^2 \,.\, y_2 + 4 \,.\, 3^2 \,.\, y_3 + 1 \,.\, 4^2 \,.\, y_4)$$

$$= \frac{z^3}{192} (4\, y_1 + 8\, y_2 + 36\, y_3 + 16\, y_4),$$

when y_0, y_1, y_2, y_3, y_4 denote the widths measured at the distances $\frac{0}{4} z, \frac{1}{4} z, \frac{2}{4} z, \frac{3}{4} z, \frac{4}{4} z$ from *A X*.

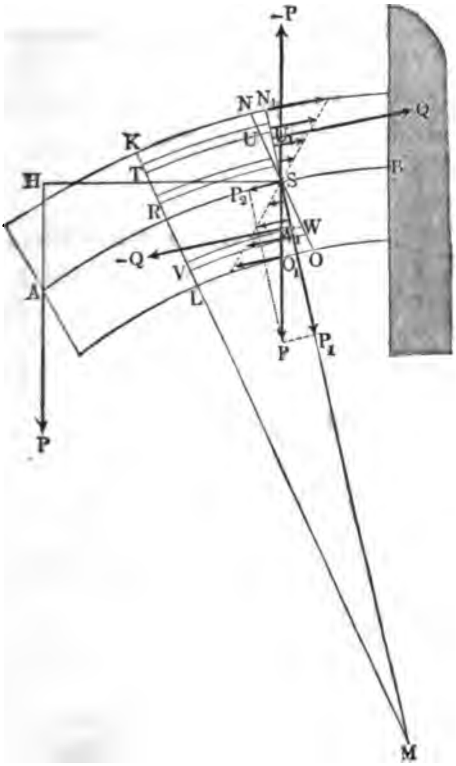
§ **235. Strength of Flexure.**—If we know the moment of flexure of a body *A K O B*, Fig. 371, fixed at one end *B* and at the other end *A* subjected to a force *P*, we can find the strain in every one of its cross-sections *N O*. If *S* denotes the strains per square inch at a distance *S N* = *e* from the neutral axis *S*, the strains at the distances $z_1, z_2, \ldots.$, are $S_1 = \frac{z_1}{e} S$, $S_2 = \frac{z_2}{e} S$, and their moments for the cross-sections F_1, F_2 . . . ., are

$$M_1 = F_1\, S_1\, z_1 = F_1\, z_1^2 \cdot \frac{S}{e},\ M_2 = F_2\, S_2\, z_2 = F_2\, z_2^2\, \frac{S}{e}, \text{ etc.},$$

and consequently the sum of the strains in the cross-section $N\,O$ is

$$M = M_1 + M_2 + \ldots = (F_1\, z_1^2 + F_2\, z_2^2 + \ldots)\, \frac{S}{e} = \frac{W\,S}{e}.$$

FIG. 871.



Now if x is the distance $S\,H$ of the cross-section $N\,O$ from the point of application A of the force P, we have also $M = P\,x$, and consequently

1) $P\,x = \frac{W\,S}{e}$, or

$$P\,x\,e = W\,S,$$

and the strain in the body at the distance e from the neutral axis is

2) $S = \frac{M\,e}{W} = \frac{P\,x\,e}{W}$.

The latter increases with x, and is therefore a maximum for $x = l$, I.E., at the fixed point B. In like manner it increases with e, and is therefore a maximum for the point most distant from the neutral axis.

If the body is nowhere to be stretched beyond the limit of elasticity, the maximum strain S should at most be equal to the modulus proof strength T, and consequently we must put

$$S = T = \frac{P\,l\,e}{W},$$

or

$$P\,l = \frac{W\,T}{e}$$

from which we obtain the *proof strength* of the girder $A\ K\ O\ B$

$$P = \frac{W\,T}{l\,e}.$$

In like manner we have for the *ultimate strength* or force necessary to break the body at B

$$P_1 = \frac{W\,K}{l\,e},$$

in which we must substitute for K the modulus of ultimate strength determined by experiment upon rupture. The fundamental formula $P\,x = \frac{W\,E}{r}$, found in § 215, can be obtained directly as follows.

If we denote by σ the extension $N\,N_1$ produced by the strain S, we have $S = \sigma\,E$, and substituting in the proportion

$$\frac{N\,N_1}{S\,N} = \frac{R\,S}{M\,R},$$

$\overline{N\,N_1} = \sigma$, $\overline{S\,N} = e$, $\overline{R\,S} = 1$, and $\overline{M\,R} = r$, the radius of curvature, we have $\frac{\sigma}{e} = \frac{1}{r}$ or $\sigma = \frac{e}{r}$; hence it follows, that

$$S = \frac{e}{r}\,E \text{ or } \frac{S}{e} = \frac{E}{r},$$

and therefore also

$$P\,x = \frac{W\,E}{r} = \frac{W\,S}{e}.$$

If in the formula $L = \frac{1}{6}\,\frac{P^2\,l^3}{W\,E}$ (§ 217) for the work done in bending the body $A\;K\;B$ we substitute the moment $P\,l = \frac{T\,W}{e}$ and the modulus of proof-strength $T = \sigma\,E$, we obtain

$$L = \frac{1}{6}\,\frac{T^2\,W^2}{e^2}\cdot\frac{l}{W\,E} = \frac{1}{2}\,\sigma^2\,E\,\frac{W\,L}{3\,e^2}.$$

But (according to § 206) $\frac{1}{2}\,\sigma^2\,E$ is the modulus of resilience A; therefore the work done in bending a body to the limit of elasticity is

$$L = A\cdot\frac{W\,l}{3\,e^2}.$$

If b is the greatest width of the body, we can imagine the whole cross-section F of the body to be divided in n equally wide strips, whose width is $\frac{b}{n}$, and whose altitudes are $z_1, z_2, z_3 \ldots$, and we can put

$$F = \frac{b}{n}\,(z_1 + z_2 + z_3 + \ldots) \text{ and}$$

$$W = \frac{b}{12\,n}\,(z_1^3 + z_2^3 + z_3^3 \ldots),$$

and therefore also

$$W\,l = \left(\frac{z_1^3 + z_2^3 + z_3^3 + \ldots}{z_1 + z_2 + z_3 + \ldots}\right)\frac{F\,l}{12}.$$

We can make $z_1 = \mu_1\,e$, $z_2 = \mu_2\,e$, $z_3 = \mu_3\,e$, μ_1, μ_2, μ_3 denoting numbers dependent upon the form of the cross-section, and therefore we have

$$\frac{W\,l}{e^2} = \left(\frac{\mu_1^3 + \mu_2^3 + \mu_3^3 + \ldots}{\mu_1 + \mu_2 + \mu_3 + \ldots}\right)\frac{F\,l}{12},$$

and consequently the mechanical effect

$$L = \frac{A}{3}\left(\frac{\mu_1^3 + \mu_2^3 + \mu_3^3 + \ldots}{\mu_1 + \mu_2 + \mu_3 + \ldots}\right)\frac{F\,l}{12}.$$

But $\frac{\mu_1^3 + \mu_2^3 + \mu_3^3}{\mu_1 + \mu_2 + \mu_3}$ is a coefficient ψ, dependent upon the form of the body alone, and $F\,l = V$ is the volume of the body; hence the work done $L = \frac{1}{36}\,\psi\,A\,V$ is not dependent upon the individual dimensions, but only upon the form of the cross-section and the volume of the body, which is bent. When the bodies are of the same nature and of similar cross-sections, the work done is proportional to the *volume* of the body.

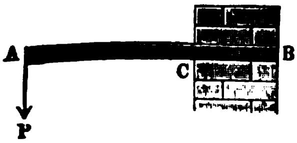
For the *work done in producing rupture* we must put

$$L_1 = B\,.\,\frac{W\,l}{3\,e^2},$$

B denoting the *modulus of fragility.*

§ 236. Formulas for the Strength of Bodies.—For a parallelopipedical girder $A\ C\ B$, Fig. 372, the length of which is l, the width b and the height h, we have $e = \frac{1}{2}\,h$, and, according to § 226, $W = \frac{b\,h^2}{12}$; hence $\frac{W}{e} = \frac{b\,h^2}{6}$, the proof strength of the girder is $P = \frac{b\,h^2}{l}\frac{T}{6}$, and its moment is $P\,l = b\,h^2\,.\,\frac{T}{6}$.

FIG. 372.



From this it follows, that the *mechanical effect* necessary to bend the girder to the limit of elasticity is

$$L = \frac{A}{3}\frac{W}{e}\frac{l}{e} = \frac{A}{3}\,.\,\frac{b\,h^2}{6}\frac{2\,l}{h} = \frac{1}{9}\,A\,b\,h\,l = \frac{1}{9}\,A\,V.$$

If the girder is *hollow*, and if its cross-section is shaped as is represented in Fig. 373 and Fig. 374, we have

$$\frac{W}{e} = \frac{b\,h^3 - b_1\,h_1^3}{12\,.\,\frac{1}{2}\,h} = \frac{b\,h^3 - b_1\,h_1^3}{6\,h}, \text{ whence}$$

$$P = \frac{b\,h^3 - b_1\,h_1^3}{6\,h\,l}\,T,$$

b and h being the exterior and b_1 and h_1 the interior width and height of the cross-section. For a body with a *rhombic cross-section*, such as Fig. 375, we have

FIG. 373.

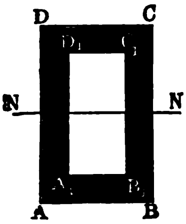


FIG. 374.

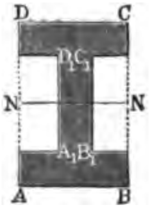


FIG. 375.

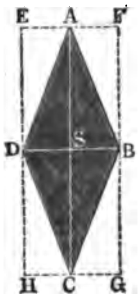
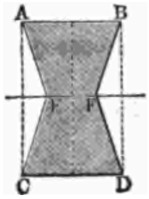


FIG. 376.



$$\frac{W}{e} = \frac{b\,h^3}{48\,.\,\frac{1}{2}\,h} = \frac{b\,h^2}{24}, \text{ and from this}$$

$$P = \frac{b\,h^2}{l}\,.\,\frac{T}{24} = \frac{1}{4}\,\frac{b\,h^2}{l}\,.\,\frac{T}{6},$$

I.E. $\frac{1}{4}$ as great as for a parallelopipedical girder of the same height $A\,C = h$ and width $B\,D = b$. For a girder, whose cross-section is a *double trapezoid*, such as is represented in Fig. 376, we have

$$\frac{W}{e} = \frac{(3\,b + b_1)\,h^3}{48\,.\,\frac{1}{2}\,h} = \frac{(3\,b + b_1)\,h^2}{24};$$

hence the moment of the proof strength is

$$P\,l = \frac{(3\,b + b_1)\,h^2}{4}\,.\,\frac{T}{6},$$

b denoting the upper and b_1 the central width and h the height of the cross-section.

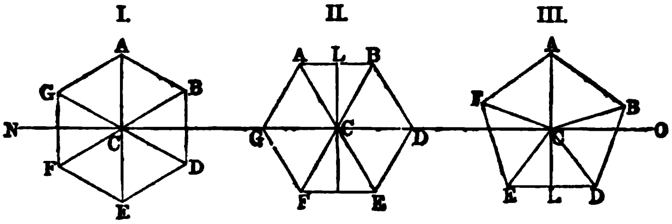
For a girder with a *regular 2n sided base*, such as $A\,D\,F$, Fig. 377, I and II, we have, if r denotes the exterior radius $C\,A$, s the length of the side $A\,B$, h the interior radius $C\,L$ and F the entire area of the cross-section,

$$W = \frac{F}{4}\,(r^2 - \tfrac{1}{6}\,s^2) = \frac{F}{4}\,(h^2 + \tfrac{1}{12}\,s^2) = \frac{F\,(r^2 + 2\,h^2)}{12}.$$

If the neutral axis $N O$, as in Fig. 377, I, passes through the middle of the opposite sides, $e = r$; and if, as in Fig. 377, II, it passes through the opposite corners,

$$e = h = \sqrt{r^2 - (\tfrac{1}{2} s)^2}.$$

FIG. 377.



Hence it follows, that in the first case

$P l = \frac{F (r^2 + 2 h^2)}{12 r} T$, and, on the contrary, in the second

$P_1 l = \frac{F (r^2 + 2 h^2)}{12 h} T$, while in both cases

$$F = \tfrac{1}{2} n s h = n h \sqrt{r^2 - h^2} = \tfrac{1}{2} n s \sqrt{r^2 - (\tfrac{1}{2} s)^2}.$$

The ratio $\frac{P_1}{P}$ of the proof strengths is $= \frac{r}{h}$.

If the number n of the sides of a polygon is uneven, as in Fig. 377, III, we must substitute $e = r$, and therefore we must employ the first formula only; provided always that the direction of the force coincides with that of the axis of symmetry.

For a square cross-section we have $s = 2 h = r \sqrt{2}$, $F = s^2$, and the moment of the proof load

$$P l = \frac{s^3}{6 \sqrt{2}} T = \frac{r^3}{3} T = 0{,}333\, r^3 T,$$

and, on the contrary,

$$P_1 l = \frac{s^3}{6} T = \frac{r^3 \sqrt{2}}{3} T = 0{,}471\, r^3 T.$$

For a hexagonal cross-section we have

$$s = r = \frac{2 h}{\sqrt{3}}, F = \frac{3 \sqrt{3}}{2} s^2 = 2{,}598\, s^2, \text{ and therefore}$$

$$P l = \frac{5 \sqrt{3}}{16} s^3 T = \frac{5 \sqrt{3}}{16} r^3 T = 0{,}541\, r^3 T, \text{ and}$$

$$P_1 l = \tfrac{5}{8} s^3 T = \tfrac{5}{8} r^3 T = 0{,}625\, r^3 T.$$

For a regular octagonal cross-section we have

$$s = r\sqrt{2 - \sqrt{2}},\ h = \frac{r}{2}\sqrt{2 + \sqrt{2}} \text{ and}$$

$$F = 4\,s\,h = 2\,\sqrt{2}\,.\,r^2 = \frac{2\,\sqrt{2}}{2 - \sqrt{2}}\,s^2;\ \text{hence}$$

$$P\,l = \frac{4\,(2\sqrt{2} + 1)}{3\sqrt{20 + 14\sqrt{2}}}\,s^3\,T = \left(\frac{2\,\sqrt{2} + 1}{6}\right)r^3\,T = 0{,}638\,r^3\,T,$$

and

$$P_1\,l = \frac{4\,(2\sqrt{2} + 1)}{3\sqrt{17 + 12\sqrt{2}}}\,s^3\,T = \frac{2\sqrt{2} + 1}{3\sqrt{2 + \sqrt{2}}}\,r^3\,T = 0{,}691\,r^3\,T.$$

For a *massive cylinder*, whose radius is r, we have

$$\frac{W}{e} = \frac{\pi\,r^4}{4\,r} = \frac{\pi\,r^3}{4},\ \text{and therefore}$$

$$P\,l = \frac{\pi}{4}\,r^3\,T = 0{,}785\,r^3\,T = \tfrac{1}{4}\,F\,r\,.\,T, \text{ and}$$

$$L = \frac{A}{3}\,.\,\frac{\pi\,r^2}{4}\,.\,\frac{l}{r} = \tfrac{1}{12}\,A\,.\,\pi\,r^2\,l = \tfrac{1}{12}\,A\,V.$$

But if the *cylinder* is *hollow*, we have, on the contrary,

$$P\,l = \frac{\pi}{4}\,\frac{(r_1^4 - r_2^4)}{r_1}\,T = \frac{1 + \left(\frac{b}{2\,r}\right)^2}{1 + \frac{b}{2\,r}}\,\frac{F\,r}{2}\,T \text{ (compare § 231),}$$

r_1 denoting the exterior, r_2 the interior and $r = \frac{r_1 + r_2}{2}$ the mean radius, $F = \pi\,(r_1^2 - r_2^2)$ the annular cross-section of the cylinder and $b = r_1 - r_2$ its width.

FIG. 378.

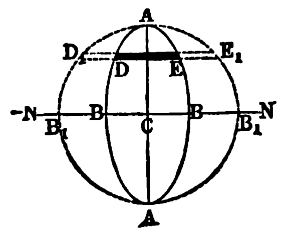
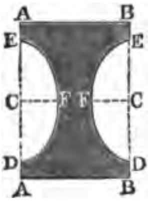


FIG. 379.



For a girder, whose cross-section is elliptical, as is represented in Fig. 378, when the direction of the semi-axis $C\,A = a$ is that of the force, and that of the semi-axis $C\,B = b$ coincides with the neutral axis, we have

$$P\,l = \frac{\pi\,a^2\,b}{4}\,T = \tfrac{1}{4}\,F\,a\,T.$$

Finally, for a parallelopipedical girder hollowed out on each side in the shape of a semi-ellipse, as is represented in Fig. 379, we have

$$P\,l = \frac{\frac{1}{12}\,b\,h^3 - \frac{1}{4}\,\pi\,b_1\,a_1^3}{\frac{1}{2}\,h}\,T = \frac{b\,h^3 - 3\,\pi\,b_1\,a_1^3}{6\,h}\,T$$

and, on the contrary, if the cross-sections of the hollows are parabolas,

$$P\,l = \frac{\frac{1}{12}\,b\,h^3 - \frac{8}{15}\,b_1\,a_1^3}{\frac{1}{2}\,h}\,T = \frac{5\,b\,h^3 - 32\,b_1\,a_1^3}{30\,h}\,T,$$

b denoting the exterior width, h the exterior height, b_1 the depth of the hollow and a_1 the height of the same.

§ 237. Difference in the Moduli of Proof Strength.—

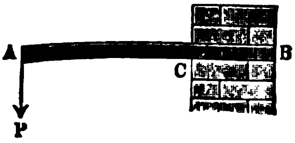
The formula $P = \frac{W\,T}{e\,l}$ for the proof load of a girder fixed at one end A, Fig. 380, holds good only, when the extension σ and the compression σ_1 of the body are equal to each other at the limit of elasticity; for under those circumstances only can the modulus of proof strength for extension

$$T = \sigma\,E$$

be equal to that of compression

$$T_1 = \sigma_1\,E.$$

FIG. 380.



For wrought iron this assumption seems to be nearly correct, and for wood approximately so, but these relations are entirely different in the case of *cast iron;* the latter has not only a much greater modulus of ultimate strength for crushing than for tearing, but also the compression σ_1 at the limit of elasticity, which can, however, be given only approximatively, is about twice as great as the extension σ, and consequently the modulus of proof strength T_1 for compression is twice as great as the modulus of proof strength T for extension.

In order to find the proof strength of cast iron or of any other body, for which there is a perceptible difference between σ and σ_1 or between T and T_1, we must first see which of the quotients $\frac{T}{e}$ and $\frac{T_1}{e_1}$ is the lesser, and substitute that instead of $\frac{T}{e}$ in the formula

$$P = \frac{W\,T}{e\,l}.$$

The other half of the beam, corresponding to the greater ratio $\left(\frac{T}{e} \text{ or } \frac{T_1}{e_1}\right)$, is of course not stretched to the limit of elasticity. In order to reduce this cross-section and consequently that of the whole body to a minimum and thus to economize as much material as possible, it is necessary, that both the halves of the girder shall be strained to the limit of elasticity. Therefore we must give the beam such a form and such a position that we will have

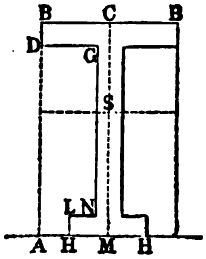
$$\frac{T}{e} = \frac{T_1}{e_1} \text{ or } \frac{e}{e_1} = \frac{T}{T_1} = \frac{\sigma}{\sigma_1},$$

I.E., that the ratio of the greatest distances e and e_1 of the fibres on the two sides from the neutral axis shall be equal to the ratio of the moduli of proof strength T and T_1 for compression and extension.

If, then, for cast iron we have $\frac{T_1}{T} = \frac{\sigma}{\sigma_1} = 2$ (see § 211), we must so fashion the cross-section of a cast iron girder that $\frac{e_1}{e}$ shall be as near as possible $= 2$. A triangular girder must be so placed, that the half with a *triangular cross-section* shall be compressed, and that with the trapezoidal cross-section shall be stretched. If we place one of the sides of the prism horizontal or at right angles to the force, we have $\frac{e_1}{e} = \frac{2}{1}$, while in the opposite position, we have $\frac{e_1}{e} = \frac{1}{2}$.

We can also give cast-iron girders, whose cross-section approach the shape of a T (as is represented in Fig. 381), such dimensions that the ratio $\frac{e_1}{e}$ shall be equal to 2.

FIG. 381.



Let the entire height of the beam be $A\,B = h$, the width of the upper flange be $B\,B = 2\,B\,C = b$, the height of the hollow on the side be

$$\overline{A\,D} = h_1 = \mu_1\,h,$$

the width of the same be

$$2\,\overline{D\,G} = b_1 = \nu_1\,b,$$

the height of the lower flange be

$$\overline{H\,L} = h_2 = \mu_2\,h$$

and its projection on both sides be

$$2\,L\,N = b_2 = \nu_2\,b,$$

then the distance of the centre of gravity s of the whole surface from the lower edge H is

$$M\,S = e_1 = \frac{1}{2}\,\frac{b\,h^2 - b_1\,h_1^2 + b_2\,h_2^2}{b\,h - b_1\,h_1 + b_2\,h_2}$$

$$= \frac{h}{2}\left(\frac{1 - \mu_1^2\,\nu_1 + \mu_2^2\,\nu_2}{1 - \mu_1\,\nu_1 + \mu_2\,\nu_2}\right)$$

(see § 105 and § 109). If we substitute $\frac{e_1}{e} = 2$ and $e + e_1 = h$, we have $e = \frac{1}{3}\,h$ and $e_1 = \frac{2}{3}\,h$, and therefore the equation of condition

$$\frac{2}{3} h = \frac{h}{2} \cdot \frac{1 - \mu_1^3 \nu_1 + \mu_2^3 \nu_2}{1 - \mu_1 \nu_1 + \mu_2 \nu_2},$$

which, when transformed, becomes

$$\mu_1 \nu_1 (4 - 3 \mu_1) - \mu_2 \nu_2 (4 - 3 \mu_2) = 1.$$

By the aid of this formula, when three of the ratios μ_1, ν_1, μ_2 and ν_2 of the dimensions are given, we can calculate the fourth. If we make $\mu_2 = 0$, we have the cross-section represented in Fig. 382, the moment of flexure of which has already been determined (§ 228), and for which we have $\mu_1 \nu_1 (4 - 3 \mu_1) = 1$.

REMARK.—Moll and Reuleaux (see their work, "Die Festigkeit der Materialen," Brunswick, 1853) recommend for the determination of the most advantageous cross-section the use of a balance, the beam of which forms a table. Patterns of the cross-section, cut out of sheet-iron, are placed upon it in such a manner that the neutral axis, determined by the ratio $\frac{e}{e_1} = \frac{\sigma}{\sigma_1}$, shall lie exactly above the centre of rotation of the beam. If the pattern has the most advantageous form, the beam will balance; if it does not, we must cause it to do so by cutting away portions from the side of the body, until the beam balances, when the pattern occupies the above position.

EXAMPLE 1.—If the cross-section of a cast-iron beam has the form of Fig. 381, and if the ratios of the heights are

$$\mu_1 = \frac{h_1}{h} = \frac{7}{8}, \mu_2 = 1 - \frac{7}{8} = \frac{1}{8},$$

we have for the ratios of the width the condition

$$\frac{7}{8}\left(4 - \frac{21}{8}\right) \nu_1 - \frac{1}{8}\left(4 - \frac{3}{8}\right) \nu_2 = 1, \text{ i.e.}$$

$$77 \nu_1 - 29 \nu_2 = 64.$$

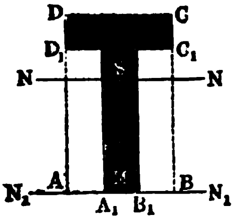
If the lower flange is omitted, then $\nu_2 = 0$, and we have

$$\nu_1 = \frac{b_1}{b} = \frac{64}{77} = 0{,}831,$$

and the thickness of the web proper is $b - b_1 = 0{,}169\, b$.

If, on the contrary, we make $\nu_2 = \frac{\nu_1}{6}$, we have $\left(77 - \frac{29}{6}\right) \nu_1 = 64$, and consequently $\nu_1 = 0{,}887$ and $\nu_2 = \frac{1}{6} \cdot 0{,}887 = 0{,}148$. For $h = 8$ inches and $b = 5\frac{1}{2}$ inches, h_1 is $= 7$ inches, $h_2 = 1$ inch, $b_1 = 5$ inches and $b_2 = \frac{4}{5}$ inch; so that the thickness of the upper and lower flange is 1 inch, and that of the vertical web but $\frac{1}{2}$ inch.

FIG. 382.



EXAMPLE 2.—For a girder with a T-shaped cross-section, Fig. 382, we have found (§ 228)

$$W = \frac{(b h^2 - b_1 h_1^2)^2 - 4 b b_1 h h_1 (h - h_1)^2}{12 (b h - b_1 h_1)},$$

in which we must put

$$e_1 = \frac{1}{2}\frac{b h^2 - b_1 h_1^2}{b h - b_1 h_1};$$

hence, if one end is fixed and the other loaded, we have

$$P l = \frac{(b h^2 - b_1 h_1^2)^2 - 4 b b_1 h h_1 (h - h_1)^2}{b h^2 - b_1 h_1^2} \frac{T_1}{6}.$$

If we put $h_1 = \mu_1 h$ and $b_1 = \nu_1 b$, we obtain

$$Pl = \frac{(1 - \mu_1^2 \nu_1)^2 - 4 u_1 \nu_1 (1 - \mu_1)^2}{1 - \mu_1^2 \nu_1} \frac{b h^2}{6} T_1,$$

and therefore if the beam is cast-iron and we substitute $\mu_1 = \frac{6}{7}$ and $\nu_1 = \frac{7}{8}$,

$$Pl = \frac{(\frac{5}{14})^2 - 3 (\frac{1}{7})^2}{\frac{5}{14}} \cdot \frac{b h^2}{6} T_1 = \frac{13}{70} \cdot \frac{b h^2}{6} T_1.$$

If, E.G., h is $= 10$ and $b = 8$ inches, and consequently

$h_1 = \frac{6}{7} . 10 = \frac{60}{7} = 8\frac{4}{7}, h - h_1 = 1\frac{3}{7}$ inches,
$b_1 = \frac{7}{8} . 8 = 7$ and $b - b_1 = 1$ inch,

we have

$$Pl = \frac{13}{70} \cdot \frac{8 . 100}{6} . T_1 = \frac{520}{21} T_1.$$

If we substitute $T_1 = 18700$ pounds, we have for the moment of the proof strength, which, for the sake of safety, we should put $= 150000$

$$Pl = \frac{520}{21} . 18700 = 463048 \text{ pounds}.$$

If this beam is 100 inches long, its safe load at the free end is

$$P = \frac{150000}{100} = 1500 \text{ pounds}.$$

If the girder is supported at both ends and carries the load in the middle, we have

$$P = 4 . 1500 = 6000 \text{ pounds}.$$

While in the first case the flange must be placed on top, in the latter it must be put at the bottom.

§ 238. Difference in the Moduli of Ultimate Strength.—

If we determine the moduli of elasticity and of proof strength by means of experiments on bending, making use of the formulas

$$E = \frac{P l r}{W} \text{ and } T = \frac{P l e}{W},$$

the values found for E and T generally agree very well with those given by direct experiments on extension and compression, when the formulas

$$E = \frac{P l}{\lambda F} \text{ and } T = \frac{P}{F} \text{ are employed.}$$

But this relation is entirely different for the modulus of ultimate strength. Since we cannot consider the modulus of elasticity E to be constant beyond the limits of elasticity (for it decreases, when the extension or compression increases), and since the modulus of elasticity for extension is no longer equal to that for compression, the strains in the superposed fibres are no longer proportional to their distances from the neutral axis, and consequently that axis no longer passes through the centre of gravity; the values of e and e_1 differ in that case essentially from what they are, when the limit of elasticity is not surpassed.

If W denotes the measure of the moment of flexure for the stretched half of the girder and E the mean modulus of elasticity of the same, and if W_1 denotes this measure for the compressed portion and E_1 the mean modulus of elasticity, we have for the moment of the bending force, when the bending becomes excessive,

$$P\,l = \frac{W E + W_1 E_1}{r},$$

and if we put, at least approximately, $\frac{K}{E} = \frac{e}{r}$ and $\frac{K_1}{E_1} = \frac{e_1}{r}$, K and K_1 denoting the moduli of ultimate strength for tearing and crushing, the moment of the force necessary to break it is

$$P\,l \text{ either } = \frac{K\,(W E + W_1 E_1)}{E\,e} \text{ or } = \frac{K_1\,(W E + W_1 E_1)}{E_1\,e_1}.$$

If we again denote the statical moment of the cross-section of the stretched portion of the body in reference to the neutral axis by M and that of the cross-section of the compressed portion of the body in reference to the same axis by M_1, we have the force on one side $= \frac{M E}{r}$ and on the other $= \frac{M_1 E_1}{r}$, and since the two forces must form a couple, $M E = M_1 E_1$. This equation serves to determine the neutral axis by means of the distances e and e_1.

For a girder with a rectangular cross-section we have

$$M = \frac{b\,e^2}{2} \text{ and } M_1 = \frac{b\,e_1^2}{2},$$

and therefore

$$E\,e^2 = E_1\,e_1^2.$$

From this we obtain

$$e_1 = e\sqrt{\frac{E}{E_1}}.$$

Substituting this value in the equation $e + e_1 = h$, we have

$$e = \frac{h\sqrt{E_1}}{\sqrt{E} + \sqrt{E_1}} \text{ and } e_1 = \frac{h\sqrt{E}}{\sqrt{E} + \sqrt{E_1}}.$$

The measures of the moments of flexure are in this case

$$W = \frac{b\,e^3}{3} \text{ and } W_1 = \frac{b\,e_1^3}{3},$$

and consequently we have

$$P\,l = \frac{b}{3\,r}(E\,e^3 + E_1\,e_1^3) = \frac{b\,h^3}{3\,r}\left(\frac{E\,E_1\sqrt{E_1} + E\,E_1\sqrt{E}}{(\sqrt{E} + \sqrt{E_1})^3}\right)$$

$$= \frac{b\,h^3}{3\,r}\;\frac{E\,E_1}{(\sqrt{E} + \sqrt{E_1})^2},$$

and therefore the moment necessary to produce rupture is

$$P\,l \text{ either} = \frac{K.\,b\,h^3}{3\,E\,e}\,.\,\frac{E\,E_1}{(\sqrt{E} + \sqrt{E_1})^2} = \frac{b\,h^2}{3}\,.\,K\,.\,\frac{\sqrt{E_1}}{\sqrt{E} + \sqrt{E_1}}$$

$$\text{or} = \frac{b\,h^2}{3}\,K_1\,.\,\frac{\sqrt{E}}{\sqrt{E} + \sqrt{E_1}}.$$

For $E = E_1$ we have, of course,

$$P\,l = \frac{b\,h^2}{6}\,K.$$

For *wood* and *wrought iron*, E is really about $= E_1$, and therefore we can write approximately

$$P\,l = \frac{b\,h^2}{6}\,K,$$

in which we must substitute for K the smaller value of the modulus of ultimate strength. For *cast iron*, E_1 is much greater than E, and therefore $P\,l$ approaches the value $\frac{b\,h^2}{3}\,K$, K being the modulus of rupture for extension. For wood we must substitute the mean value of the modulus of ultimate strength for crushing, $K_1 = 480$ kilograms $= 6800$ pounds, which value agrees very well with the results of the experiments of Eytelwein, Gerstner, etc.

In like manner, for a wrought iron girder we must substitute instead of K the modulus of ultimate strength for crushing $K_1 = 2200$ kilograms $= 31000$ pounds. While under the same circumstances wood and wrought iron break by crushing, cast iron breaks by tearing. If for the latter K were about $= K_1$, we would have to substitute for cast iron girders, in the above formulas, the modulus of ultimate strength of tearing, I.E., $K = 1300$ kilograms

$= 18500$ pounds; but, according to the results of many experiments, we must put

$$K = 3200 \text{ kilograms} = 45500 \text{ pounds},$$

I.E., about the mean value of the modulus of ultimate strength for tearing and of that for crushing.

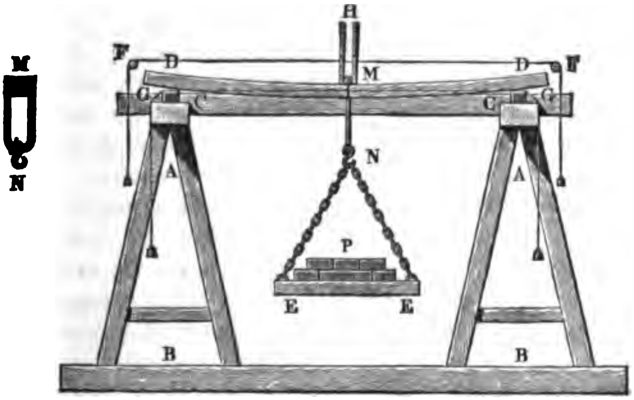
This great difference is caused not only by the difference of the moduli of elasticity E and E_1, but also by the granular texture of the cast iron, which precludes the supposition that the beam is composed of a bundle of rods.

Many different circumstances influence the elasticity, the proof strength and the ultimate strength of a body, so that notable differences occur in the results of experiment.

The wood, for example, near the heart and root of the tree is stronger than the sap wood and that near the top, and wood will resist a greater force, when the latter acts parallel to the yearly rings than when it acts at right angles to them; finally, the soil and position of the place where the wood grew, the state of humidity, the age, etc. influence the strength of wood. Finally, the deflection of a body, which has been loaded very long, is always a little greater than that produced, when the weight is first laid on.

§ **239. Experiments upon Flexure and Rupture.**—Experiments upon elasticity and strength were made by *Eytelwein* and *Gerstner* with the apparatus represented in Fig. 383. $A\ B$ and $A\ B$ are two trestles, upon which two iron bed-plates C and C are fastened, and $D\ D$ is the body to be experimented upon, which is

Fig. 383.

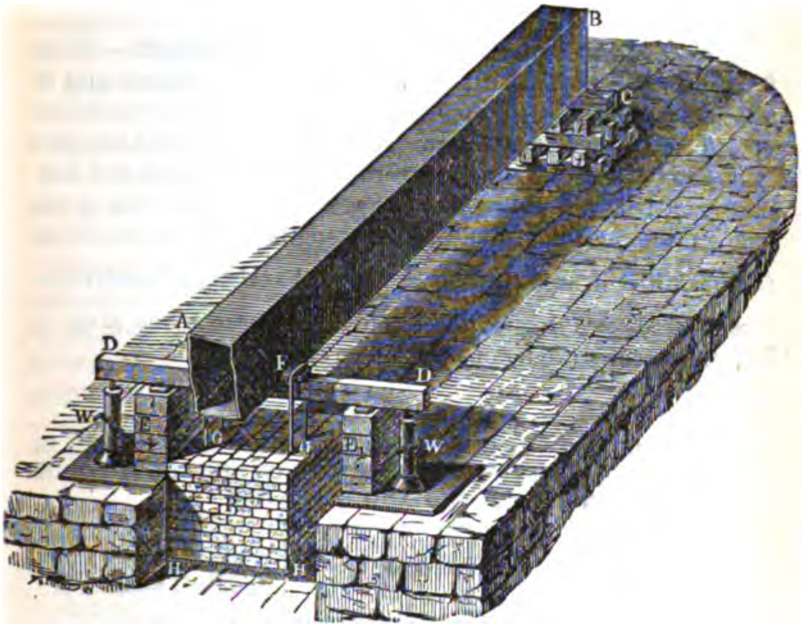


placed upon them. The weight P, which is to bend the body, is placed on a scale board $E\,E$, which is suspended to a stirrup $M\,N$, whose upper end is rounded and rests upon the centre M of the girder. In order to find the deflection produced by the weight, Eytelwein employed two horizontal strings $F\,F$ and $G\,G$ and a scale $M\,H$, placed upon the middle of the girder. Gerstner, on the contrary, employed a long sensitive one-armed lever, which rested upon the beam near its fulcrum and whose end indicated on a vertical scale the deflection of M in 15 times its real size. Lagerhjelm employed a pointer, which was moved by means of a string passing over a pulley, and which showed the deflection of the beam magnified upon a graduated circular dial. Others, as, E.G., Morin, made use of a cathometer to determine the deflection. The object observed was a point fastened in the centre of the girder. In the English experiments a long wedge was used to measure this deflection; it was inserted between the centre of the beam and a fixed support. In order that the accuracy of the measurement may not be affected by the yielding of the supports of the girder, it should rest during the experiments either upon stone foundations (Morin), or a long ruler should be placed a certain distance above the girder and fastened at its ends to the ends of the latter, but in such a manner that it cannot bend with the beam, and in each experiment the distance between the lower edge of the ruler and the centre of the deflected girder should be measured (Fairbairn).

The manner in which Stephenson, etc., determined the deflection and strength of tubular sheet iron girders, is shown with the principal details in Fig. 384. The tube $A\,B$ is 75 feet long (the front portion being omitted in the figure), is supported at both ends, as, E.G., in C, upon blocks of wood and its centre rests upon a beam $D\,D$, which is carried by two screw-jacks. An iron arm, the end F of which only can be seen in the figure, passes through the middle of the tubular girder near the bottom, and from this two stirrups G, G hang, to which the scale-board $H\,H$ to receive the weight P is suspended. Before the experiment and during the laying on of the weights, the entire load was supported by the beam $D\,D$; when the screw-jacks were lowered $D\,D$ sank and placed itself upon the supports E, E, while the centre of the tube $A\,F$, loaded with P, remained free and could assume a deflection corresponding to the force P. This deflection was measured by means of a wedge.

In order to avoid the use of very large weights in experimenting upon large girders, they are generally made to act upon the latter by means of a lever with unequal arms. With the same object in view, Hodgkinson caused the force of the lever to be applied not to the centre of a girder supported at both ends, but to one end of a girder, which was supported in the middle and the other end of which was fastened by a bolt to the foundation.

FIG. 384.



The results of experiments, made under very different circumstances and with very different kinds of materials, particularly of wood and iron, have shown the theory laid down in the foregoing pages to be correct in all important particulars. In regard to the rupture of parallelopipedical girders it was proved, that those of wood and wrought iron, under the same circumstances, gave way by crushing, and that in the case of cast iron the rupture began either by the exterior fibres being torn apart or by a wedge breaking out at the most compressed part (in the middle).

We can satisfy ourselves of the truth of the hypothesis, made in § 214, in regard to the behaviour of the fibres of a body, subjected to flexure, by making saw cuts upon the compressed side of parallelopipedical wooden rods and then filling them up with pieces of wood, by drawing a series of lines upon the side of a beam at right angles to its longitudinal axis, and finally by fastening two thin rods to the beam, one along the extended and the other along the compressed side.

§ 240. Moduli of Proof and Ultimate Strength.—In the following table the moduli of elasticity, of proof strength and of ultimate strength or of rupture, as determined by experiments upon bending and breaking are given. The first differ but little from those determined by the experiments on extension and compression; but, for the reasons given above (§ 238), this is not true of the modulus of ultimate strength. The upper of the two quantities in a parenthesis $\left\{ \right\}$ gives the value in English measures (pounds per square inch) and the lower one the same in French measures (kilograms per square centimeter).

TABLE

OF THE MODULI OF ELASTICITY, OF PROOF STRENGTH AND OF ULTIMATE STRENGTH OR OF RUPTURE OF DIFFERENT BODIES IN RELATION TO BENDING AND BREAKING.

Names of the Bodies.	Modulus of Elasticity E.	Modulus of Proof Strength T.	Modulus of Rupture or of Ultimate Strength K (K_1).
Wood of deciduous Trees	1280000 90000	3100 220	9240 650
Wood of evergreen Trees	2130000 150000	4300 300	12800 900
Cast Iron	17000000 1200000	10670 750	45500 3200
Wrought Iron	28400000 2000000	17000 1200	32700 2300
Limestone and Sandstone	—	—	1760 124
Clayslate	—	—	5000 350

In order to determine from the value in the foregoing table the load, which a girder can carry securely, we must introduce a factor of safety and substitute in the formulas for the proof strength already found for wood

either instead of T, $\frac{1}{3}\,T$ or instead of K, $\frac{1}{10}\,K$,

for cast iron

either instead of T, $\frac{1}{2}\,T$ or instead of K, $\frac{1}{5}\,K$,

and for wrought iron

either instead of T, $\frac{1}{2}\,T$ or instead of K, $\frac{1}{4}\,K$.

Consequently we can hereafter put for wood

$$T = 73 \text{ kilograms} = 1000 \text{ pounds},$$

for cast iron

$$T = 510 \text{ kilograms} = 7000 \text{ pounds}$$

and for wrought iron

$$T = 660 \text{ kilograms} = 9000 \text{ pounds}.$$

We cannot employ these values in calculating the dimensions of shafts and other parts of machines; for, on account of their constant motion and of the wearing away of the parts, a greater factor of safety must be introduced, which requires us to assume a smaller value for T.

If we substitute these values in the formulas

$$P\,l = b\,h^2\,\frac{T}{6} \text{ and } P\,l = \pi\,r^3\,\frac{T}{4} = \pi\,d^3\,\frac{T}{32}$$

for parallelopipedical and for cylindrical girders, we obtain the following *practical formulas:*

For wood

$$P\,l = 167\,b\,h^2 = 785\,r^3 = 98\,d^3 \text{ inch-pounds.}$$

For cast iron

$$P\,l = 1167\,b\,h^2 = 5500\,r^3 = 687\,d^3 \text{ inch-pounds.}$$

And for wrought iron the greatest value

$$P\,l = 1500\,b\,h^2 = 7070\,r^3 = 884\,d^3 \text{ inch-pounds.}$$

If with Morin, and in accordance with the practice in England, we put for cast iron

$$\text{instead of } T,\ \frac{K}{4} \text{ to } \frac{K}{5} = 750 \text{ kilograms},$$

and for wrought iron

$$\text{instead of } T,\ \frac{K}{5} = 600 \text{ kilograms},$$

we obtain for cast iron

$$P\,l = 1778\,b\,h^2 = 8376\,r^3 = 1047\,d^3 \text{ inch-pounds},$$

and for wrought iron the smaller value

$$P\,l = 1422\,b\,h^2 = 6700\,r^3 = 838\,d^3 \text{ inch-pounds.}$$

If the load Q is not applied at the end of the beam, but is equally distributed over the same, the arm of the load is no longer l, but $\frac{l}{2}$, and consequently, the moment being but half as great, we must put $\frac{Q\,l}{2} = \frac{W\,T}{e}$, or $Q\,l = 2\,.\,\frac{W\,T}{e}$.

If the girder is *supported at both ends* (Fig. 337) and the load P acts midway between the two points of support, whose distance from each other is $= l$, the force at each end is $= \frac{P}{2}$, its arm is $= \frac{l}{2}$ and its moment

$$\frac{P\,l}{4} = \frac{W\,T}{e} \text{ and } P\,l = 4\,\frac{W\,T}{e}.$$

Therefore, under the same circumstances, the girder bears twice as great a load in the second and four times as great a one in the third as in the first case.

If, finally, a girder *uniformly loaded* along its whole length is supported at both ends, it is in the first place bent upwards by a force $\frac{Q}{2}$, whose arm is $\frac{l}{2}$, and in the second place downwards by a force $\frac{Q}{2}$, whose point of application is the centre of gravity of one of the halves of the load, whose lever arm is therefore $\frac{l}{4}$ and whose moment is $\frac{Q\,l}{8}$. Consequently the moment with which one end of the girder is bent upwards is

$$= \frac{Q\,l}{4} - \frac{Q\,l}{8} = \frac{Q\,l}{8},$$

hence we have $Q\,l = 8\,\frac{W\,T}{e}$. The proof load of the girder is in this case 8 times as great as in the first one.

For a parallelopipedical girder we have in the first case

$$P\,l = b\,h^2\,\frac{T}{6}, \text{ in the second}$$

$$Q\,l = 2\,b\,h^2\,\frac{T}{6}, \text{ in the third}$$

$$P\,l = 4\,b\,h^2\,\frac{T}{6} \text{ and in the fourth}$$

$$Q\,l = 8\,b\,h^2\,\frac{T}{6},$$

b denoting the width and h the height of the rectangular cross-section.

EXAMPLE—1) What load can a girder of fir carry at its middle, when its width is $b = 7$ and its height $h = 9$ inches, and when the point of application of the load is 10 feet distant from the supports? Here we have $\frac{1}{2} l = 10 . 12 = 120$ inches, and therefore, according to the above formula,

$$P l = 4 . 167 \, b \, h^2 = 4 . 167 . 7 . 81,$$

and the required working load is

$$P = \frac{4676 . 81}{240} = 58{,}45 . 27 = 1578 \text{ pounds.}$$

2) A cylindrical stick of wood, firmly imbedded at one end in masonry, is required to bear a weight $Q = 10000$, uniformly distributed over its whole length $l = 5$ feet; what should be its diameter? We have here

$$Q l = 2 \frac{\pi r^3 T}{4} \doteq 2 . 785 . r^3,$$

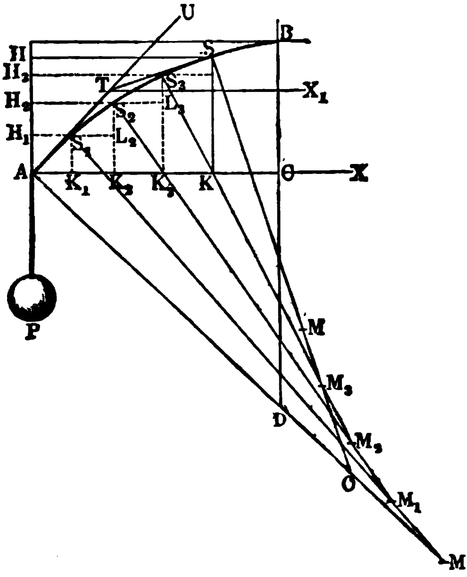
and consequently by inversion

$$r = \sqrt[3]{\frac{Q l}{1570}} = \sqrt[3]{\frac{10000 . 60}{1570}} = \sqrt[3]{382} = 7{,}26 \text{ inches,}$$

and the required diameter is $= 2 r = 14{,}52$ inches.

§ 241. Relative Deflection.—The bending of the moving parts of machines, such as the shafts, axles, etc., has often a very bad effect upon their working, either by giving rise to vibrations and concussions, or by preventing the different parts of the machine from engaging perfectly. We are therefore in certain cases required to determine the cross-sections of these parts of machines, not with reference to the modulus of proof strength, but to the deflection, by assuming the deflection to be a very small definite portion of the entire length of the body or part of the machine.

FIG. 385.



We have already found (§ 217) the deflection for a prismatic body $A \, S \, B$, Fig. 385, fixed at one end B and loaded at the other A, to be

$$B\,C = a = \frac{P\,l^3}{3\,W\,E},$$

and we can put its ratio to the length $A\,B$, which is given

$$\theta = \frac{a}{l} = \frac{P\,l^2}{3\,W\,E},$$

whence, by inversion,

$$P\,l^2 = 3\,\theta\,W\,E.$$

Hence we have for a *parallelopipedical girder*

$$P\,l^2 = 3\,\theta\,\frac{b\,h^3}{12}\,E = \frac{\theta\,b\,h^3\,E}{4},$$

and for a *cylindrical one*

$$P\,l^2 = 3\,\theta\,\frac{\pi\,r^4}{4}\,E = \frac{3}{4}\,\pi\,\theta\,r^4\,E.$$

Generally a relative deflection $\theta = \frac{a}{l} = \frac{1}{500}$ is admissible, and we can put

$$1)\;\; P\,l^2 = \frac{1}{2000}\,b\,h^3\,E = \frac{3\,\pi}{2000}\,r^4\,E.$$

If we substitute for wood the modulus of elasticity $E = 1600000$, we obtain

$$P\,l^2 = 800\,b\,h^3 = 7540\,r^4.$$

For cast iron we have $E = 15000000$ pounds, and therefore

$$P\,l^2 = 7500\,b\,h^3 = 70700\,r^4,$$

and for wrought iron $E = 22000000$ pounds and

$$P\,l^2 = 11000\,b\,h^3 = 103700\,r^4.$$

On the contrary, when the deflection reaches the limit of elasticity, we have (§ 235)

$$2)\;\; P\,l = \frac{W\,T}{e} \text{ or } P\,l^2 = \frac{W\,T\,l}{e},$$

and, therefore, equating the two values of $P\,l^2$, we obtain

$$\frac{W\,T\,l}{e} = 3\,\theta\,W\,E,$$

and consequently the ratio of the length l of the beam to the maximum distance e, when both the deflection and strain reach at the same time their limit values θ and T, is

$$\frac{l}{e} = \frac{3\,\theta\,E}{T} = \frac{3\,\theta}{\sigma},$$

hence for parallelopipedical bodies

$$\frac{l}{h} = \frac{3}{2}\,\frac{\theta}{\sigma}$$

and for cylindrical ones

$$\frac{l}{r} = \frac{3\,\theta}{\sigma} \text{ or } \frac{l}{d} = \tfrac{3}{2}\,\frac{\theta}{\sigma},$$

σ denoting the extension or compression at the limit of elasticity corresponding to the strain T.

If $\frac{l}{e} < \frac{3\,\theta}{\sigma}$, we obtain from the first formula the greater value for $P\,l$ and if, on the contrary, $\frac{l}{e} > \frac{3\,\theta}{\sigma}$, the second formula gives the greater value for the moment of the force. Therefore for a given moment of force ($P\,l$) the greater dimensions for the cross-section are given in the first case, where the length of the body is less than $l = \left(\frac{3\,\theta}{\sigma}\right) e$, by the formula

$$\frac{W\,T}{e} = P\,l$$

and in the second case, where $l > \left(\frac{3\,\theta}{\sigma}\right) e$, by the formula

$$3\,\theta\,W E = P\,l^2.$$

If we substitute in the ratio $\frac{l}{e} = \frac{3\,\theta}{\sigma}$ for the limit, $\theta = \frac{1}{500}$, we have for all materials $\frac{l}{e} = \frac{3}{500\,\sigma} = \frac{0{,}006}{\sigma}$, and, therefore, for wood, for which $\sigma = \frac{1}{600}$, $\frac{l}{e} = 0{,}006\,.\,600 = 3{,}6$, and more particularly for a prismatical beam of this material

$$\frac{l}{h} \text{ and } \frac{l}{d} = \frac{18}{10} = 1{,}8.$$

If we assume for cast and wrought iron $\sigma = \frac{1}{1500}$, we obtain for these substances

$$\frac{l}{e} = \frac{3\,.\,1500}{500} = 9 \text{ and therefore}$$

$$\frac{l}{h} \text{ or } \frac{l}{d} = \frac{9}{2} = 4{,}5.$$

The formula

$$P\,l^2 = \frac{b\,h^3}{2000}\,E = \frac{3\,\pi\,r^4\,E}{2000}\,.$$

is of course applicable to the normal case above, I.E., when the body is loaded at one end and fixed at the other. For a load equally distributed we must substitute (according to § 223), instead of P, $\tfrac{3}{8}\,Q$. If the body is supported at both ends and the load is suspended in the middle, we have, instead of P, $\frac{P}{2}$ and, instead of l, $\frac{l}{2}$, and therefore

$$P\,l^2 = 8\,.\,\frac{b\,h^3}{2000}\,E = 8\,.\,\frac{3\,\pi\,r^4\,E}{2000}.$$

If the girder is supported in the same manner and the load uniformly distributed, we must substitute for P, $\frac{5\,Q}{8}$.

EXAMPLE—1) What load placed upon the centre of a wooden beam, supported at both ends, will produce a relative deflection $\theta = \frac{1}{800}$, if its width is $b = 7$, its height $h = 9$ inches and the distance between the supports is $l = 20$ feet? Here we have

$$P = 8\,.\,\frac{800\,b\,h^3}{l^2} = \frac{6400\,.\,7\,.\,9^3}{(20\,.\,12)^2} = 7\,.\,9^2 = 567 \text{ pounds,}$$

while in the foregoing paragraph, under the assumption that the beam should be bent to the limit of elasticity, we found $P = 1578$ pounds.

2) How high and wide must we make a cast iron girder (the ratio of its dimensions being $\frac{h}{b} = 4$), which, when supported at both ends, will sustain a load $Q = 4000$ pounds, uniformly distributed over its length, which is 8 feet? Under the latter supposition, we have

$$\tfrac{5}{8}\,Q\,l^2 = 8\,.\,7500\,b\,h^3,$$

I.E.,

$$\tfrac{5}{8}\,.\,4000\,.\,8^2\,.\,12^2 = 8\,.\,7500\,\frac{h^4}{4} \text{ or } h^4 = 4^4\,.\,6,$$

consequently

$$h = 4\,\sqrt[4]{6} = 1{,}565\,.\,4 = 6{,}26 \text{ inches and}$$

$$b = \frac{h}{4} = 1{,}565 \text{ inches.}$$

According to the formulas of the foregoing paragraph, we would have

$$Q\,l = 8\,.\,1167\,b\,h^2, \text{ or } 4000\,.\,8\,.\,12 = 8\,.\,1167\,.\,\frac{h^3}{4},$$

whence the required height is

$$h = 4\,\sqrt[3]{\frac{3000}{1167}} = 4\,.\,1{,}37 = 5{,}48 \text{ inches,}$$

and the required width

$$b = \frac{h}{4} = 1{,}37 \text{ inches.}$$

§ 242. **Moments of Proof Load.**—From the formula

$$P\,l = b\,h^2\,\frac{T}{6}$$

for the *moment of the proof load of a parallelopipedical girder* we perceive that this moment increases with the *width* b and with the *square of the height* h, that *the proof load* itself

$$P = \frac{b\,h^2}{l}\,\frac{T}{6}$$

is inversely proportioned to the *length* (l) and that the height has a much greater influence than the width upon the solidity of such a girder. A girder, whose width is double that of another, will bear but twice as great a load as the latter, or as much as two such girders placed side by side. A girder, whose height is double that of another, bears, on the contrary, $(2^2) = 4$ times as much as the latter, when their widths are the same. For this reason we make the height of parallelopipedical girders greater than their width, or we place them on edge, or in such a position, that the smaller dimension shall be perpendicular to the direction of force P and that the greater dimension shall be parallel to it.

Since $b\,h$ expresses the cross-section F of the beam, we have also

$$P\,l = F\,h\,\frac{T}{6};$$

hence the moments of the proof load of bodies of equal cross-section, mass or weight *are proportional to their height.* If, for example, b and h are the width and height of one body and $\frac{b}{3}$ and $3\,h$ those of another body or $F = \frac{b}{3}\,3\,h = b\,h$ the area of both their cross-sections, the bodies have the same weight, when the other circumstances are the same, but the latter bears three times as great a load as the former.

If $b = h$, the cross-section of the beam is a *square*, and we can diminish the moment of proof load by placing the diagonal in a vertical position. In this case, W, as we know from § 230, remains unchanged and is $= \frac{b\,h^3}{12} = \frac{b^4}{12}$, while e becomes equal to the semi-diagonal $\frac{1}{2}\,b\,\sqrt{2} = b\,\sqrt{\frac{1}{2}}$. Therefore we have

$$P\,l = \frac{b^4}{12\,b\,\sqrt{\frac{1}{2}}}\,T = b^3\,\frac{T}{6}\,\sqrt{\tfrac{1}{2}} = 0{,}707\,b^3\,\frac{T}{6},$$

while, if it were laid on one of its sides, we would have $P\,l = b^3\,\frac{T}{6}$. See § 236.

The equations for parallelopipedical girders are analogous to those for girders with an *elliptical cross-section.* We have in the latter case (according to § 231) $W = \frac{\pi\,b\,a^3}{4}$ and $e = a$, the semi-axis a being supposed parallel and the semi-axis b perpendicular to the direction of the force or, as is generally the case, horizontal. Here we have for such a girder

$$P l = \frac{\pi b a^2}{4} T = F a \frac{T}{4},$$

the area of the elliptical cross-section being $F = \pi a b$. The moment of the proof load of this beam increases, therefore, with the area and with the height a of the cross-section.

If $b = a = r$, we have a *cylindrical girder*, whose radius is r, and the equation becomes

$$P l = \frac{\pi r^3}{4} T = F r \frac{T}{4}.$$

The moment of proof load of this body increases, therefore, with the product of the area of the cross-section and its radius.

If the cross-sections or weights are equal, the ratio of the moment of proof load of a body with an elliptical cross-section to that of one with a circular cross-section is $\frac{a}{r}$. Therefore, we should always prefer the elliptical to the cylindrical girder.

This holds good for all other forms of cross-section; the regular form (the square, the regular hexagon, the circle, etc.) has always, for the same area, a smaller moment of proof load than a form of greater height and less width.

Regular forms of cross-section should, therefore, be employed only for shafts and other bodies, revolving about their longitudinal axis, in which case during the rotation a continual change in the position of the dimension of the cross-section takes place, I.E., after one-quarter of a rotation the height becomes the width and the width the height.

§ 243. Cross-section of Wooden Girders.—If a cylindrical girder has the same cross-section $F = \pi r^2 = b^2$ as a parallelopipedical beam, whose height and width is $= b$, we have the ratio

$$\frac{b}{r} = \sqrt{\pi} = 1{,}77245,$$

and, on the contrary, the ratio between the moments of proof load M and M_1 (M_2) is in the first place, when the latter body is laid upon one of its sides,

$$\frac{M}{M_1} = \frac{r}{4} : \frac{b}{6} = \frac{3}{2} \frac{r}{b} = \frac{3}{2 \sqrt{\pi}} = 1{,}5 \,.\, 0{,}5642 = 0{,}8462,$$

and in the second place, when its diagonal is placed in a vertical position,

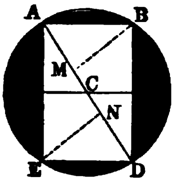
$$\frac{M}{M_2} = \frac{r}{4} : \frac{b\sqrt{2}}{12} = \frac{3}{\sqrt{2}\,\pi} = 3 \,.\, 0{,}3989 = 1{,}1967.$$

The moment of proof load of the cylinder (with circular base) is in the first place smaller and in the second place greater than that of a parallelopipedon with a square base.

Since wooden parallelopipedical girders are cut or sawed from the *round trunks of trees,* the question arises, what must be the ratio of the dimensions of the cross-section of such a beam, in order that it shall have the greatest moment of working load?

Let $A\ B\ D\ E$, Fig. 386, be the cross-section of the trunk of the tree, $A\ D = d$ its diameter and

FIG. 386.



$$A\,B = D\,E = b$$

the breadth and

$$A\,E = B\,D = h$$

the height of the beam; then we have

$$b^2 + h^2 = d^2, \text{ or}$$

$$h^2 = d^2 - b^2,$$

and the moment of proof load is

$$P\,l = \frac{T}{6} \,.\, b\,h^2 = \frac{T}{6}\,b\,(d^2 - b^2).$$

The problem now is to make

$$b\,(d^2 - b^2)$$

as great as possible. If we put, instead of b, $b \pm x$, x being very small, we obtain for the last expression

$$(b \pm x)\,d^2 - (b \pm x)^3 = b\,d^2 - b^3 \pm (d^2 - 3\,b^2)\,x - 3\,b\,x^2,$$

when x^3 is neglected. Now the difference of the two expressions is

$$y = \mp\,(d^2 - 3\,b^2)\,x + 3\,b\,x^2.$$

In order that the first value shall always be greater than the second, the difference

$$y = \mp\,(d^2 - 3\,b^2)\,x + 3\,b\,x^2$$

must be positive, whether we increase or diminish b by x. But this is only possible when $d^2 - 3\,b^2 = 0$; for this difference is then $= 3\,b\,x^2$ or positive, while, on the contrary, when $d^2 - 3\,b^2$ has a real positive or negative value, $3\,b\,x^2$ can be neglected, and the sign of the difference $\mp\,(d^2 - 3\,b^2)\,x$ varies with that of x. Therefore, putting $d^2 - 3\,b^2 = 0$, we obtain the required width

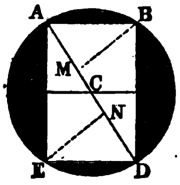
$$b = d\sqrt{\tfrac{1}{3}}, \text{ and the corresponding height}$$

$$h = \sqrt{d^2 - b^2} = d\sqrt{\tfrac{2}{3}};$$

the ratio of the height to the width is

$$\frac{h}{b} = \frac{\sqrt{2}}{\sqrt{1}} = 1{,}414 \text{ or about } \tfrac{7}{5}.$$

Fig. 387.



We should, therefore, cut the trunk of the tree in such a manner as to produce a beam, whose height is to its width as 7 is to 5. In order to find the cross-section corresponding to the greatest strength, we divide the diameter $A\,D$, Fig. 387, into three equal parts, erect in the points of division M and N the perpendiculars $M\,B$ and $N\,E$ and join the points B and E, where they cut the circumference, with the extremities A and D by straight lines. $A\,B\,D\,E$ is the cross-section of greatest resistance; for we have

$$A\,M : A\,B = A\,B : A\,D \text{ and } A\,N : A\,E = A\,E : A\,D,$$

and consequently

$$A\,B = b = \sqrt{A\,M\,.\,A\,D} = \sqrt{\tfrac{1}{3}\,d\,.\,d} = d\sqrt{\tfrac{1}{3}} \text{ and}$$

$$A\,E = h = \sqrt{A\,N\,.\,A\,D} = \sqrt{\tfrac{2}{3}\,d\,.\,d} = d\sqrt{\tfrac{2}{3}}, \text{ or}$$

$$\frac{h}{b} = \frac{\sqrt{2}}{1}, \text{ which is what was required.}$$

Remark 1. The moment of proof load of the trunk of the tree is

$$P\,l = \frac{\pi\,T}{4}\,.\,r^3,$$

and that of the beam of greatest resistance, cut from the same, is

$$P\,l = \frac{T}{6}\,d\,\sqrt{\tfrac{1}{3}}\,.\,\tfrac{2}{3}\,d^2 = \frac{T}{\sqrt{243}}\,d^3 = \frac{8\,T}{\sqrt{243}}\,r^3,$$

and consequently the beam loses by being cut

$$1 - \frac{8}{\sqrt{243}}\,.\,\frac{4}{\pi} = 1 - 0{,}65 = 0{,}35,$$

i.e. 35 per cent. of its proof strength. In order to reduce this loss, the beam is often made imperfectly four-sided, i.e. with the corners wanting. The moment of the proof load of a beam with a square cross-section, hewed from a tree of the same size is

$$P\,l = \frac{T}{6}\,.\,d\,\sqrt{\tfrac{1}{2}}\,.\,\frac{d^2}{2},$$

since the width is = height = $d\,\sqrt{\tfrac{1}{2}} = 0{,}707\,d$; the loss is

$$1 - \frac{8}{6\,.\,2\,\sqrt{2}}\,.\,\frac{4}{\pi} = 1 - \frac{8}{3\;\pi\;\sqrt{2}} \doteq 1 - 0{,}60 = 0{,}\,40,$$

i.e. 40 per cent.

(**Remark 2.**) In order to cut from a trunk of a tree a parallelopipedical beam, whose moment of flexure is a minimum, or for which $\theta = \frac{a}{l}$ is as small as possible (compare § 241), we must have

$$W = \frac{b\,h^3}{12} \text{ or } b\,h^3 = h^3\,\sqrt{d^2 - h^2}, \text{ or } (b\,h^3)^2 = h^6\,(d^2 - h^2)$$
$$= d^2\,h^6 - h^8$$

as great as possible. The first differential coefficient of the latter expression in reference to b is $6\,d^2\,h^5 - 8\,h^7$, which is equal to zero for $h^2 = \frac{3}{4}\,d^2$, I.E. for

$$h = d\sqrt{\tfrac{3}{4}} = \frac{d\ \sqrt{3}}{2} \text{ and}$$

$$b = \sqrt{d^2 - h^2} = \sqrt{\tfrac{1}{4}\,d^2} = \frac{d}{2}.$$

For these values the moment of flexure of the beam is a minimum (see Introduction to the Calculus, Art. 18).

Here we have $\frac{h}{b} = \frac{\sqrt{3}}{1} = 1{,}7321$, or about $\frac{7}{4}$, while above we found for the maximum of the moment of proof load $\frac{h}{b} = \frac{7}{5}$.

This condition corresponds to the construction in Fig. 387, when we make $A\,M = D\,N = \frac{1}{4}\,A\,D$.

§ 244. Hollow and Webbed Girders.—We have, according to § 228, for a *hollow parallelopipedical beam*

$$W = \frac{b\,h^3 - b_1\,h_1^3}{12},$$

and therefore the moment of proof load is

$$P\,l = \frac{W\,T}{e} = \frac{W\,T}{\frac{1}{2}\,h} = \left(\frac{b\,h^3 - b_1\,h_1^3}{h}\right)\frac{T}{6}.$$

If we put $\frac{h_1}{h} = \mu$ and $\frac{b_1}{b} = \nu$, we obtain

$$\frac{b\,h^3 - b_1\,h_1^3}{h} = b\,h^2\,(1 - \mu^3\,\nu),$$

and, since the cross-section of the body is

$$F = b\,h - b_1\,h_1 = b\,h\,(1 - \mu\,\nu),$$

$$P\,l = \left(\frac{1 - \mu^3\,\nu}{1 - \mu\,\nu}\right).\,F\,h\,.\,\frac{T}{6}.$$

Since $\frac{1 - \mu^3\,\nu}{1 - \mu\,\nu} = \frac{1 - \mu\,\nu + \mu\,\nu - \mu^3\,\nu}{1 - \mu\,\nu} = 1 + \frac{(1 - \mu^2)\,\mu\,\nu}{1 - \mu\,\nu}$ increases with ν, we obtain the maximum value of $P\,l$ for $\nu = 1$, and it is

$$1)\quad P\,l = \left[1 + \left(\frac{1 - \mu^2}{1 - \mu}\right)\mu\right]F\,h\,\frac{T}{6} = (1 + \mu + \mu^2)\,F\,h\,\frac{T}{6}.$$

If, on the contrary, we put $\mu = \nu$, we obtain

$$2)\quad P\,l = (1 + \mu^2)\,F\,h\,\frac{T}{6}.$$

In both cases we must make μ as great as possible, and therefore nearly $= 1$. If we wish the proof strength of the girder to be a maximum, we must make the webs as thin as possible. Hence we have for $\mu = 1$ in the first case

$$P\,l = 3\,F\,h\,\frac{T}{6} = F\,h\,\frac{T}{2},$$ and in the second case

$$P\,l = 2\,F\,h\,\frac{T}{6} = F\,h\,\frac{T}{3},$$ and, on the contrary, for $\mu = 0$,

$$P\,l = F\,h\,\frac{T}{6}$$

In all three cases the proof load of the girder, when the cross-section (F) or the weight is the same, increases with the height (h); but in the first case, where the girder consists of two flanges, it is a maximum; and in the second case, where it forms a parallelopipedical tube, it has a mean value; and in the third case, where it is composed of one or two webs, a minimum one.

If, for example, a massive girder, whose dimensions are b_1 and h_1, has the same cross-section or weight as the supposed tubular girder, we have

$$F = b_1\,h_1 = b\,h - b_1\,h_1, \text{ I.E. } 2\,b_1\,h_1 = b\,h \text{ or } \frac{b_1\,h_1}{b\,h} = \mu\,\nu = \tfrac{1}{2}.$$

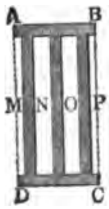
If we assume $\frac{b_1}{b} = \frac{h_1}{h}$, we have $\mu = \nu = \sqrt{\tfrac{1}{2}}$, and therefore the ratio of the proof loads of the two beams is

$$\frac{P}{P_1} = \frac{(1 - \mu^2\,\nu)}{1 - \mu\,\nu}\,\frac{h}{h_1} = \left(\frac{1 - \frac{1}{4}}{1 - \frac{1}{2}}\right)\sqrt{2} = \tfrac{3}{2}\,\sqrt{2} = 3\,\sqrt{\tfrac{1}{2}} = 2{,}12;$$

the tubular girder is therefore capable of carrying more than double the load that an equally heavy massive girder can, whose form is that of the hollow of the first girder.

The same relations also obtain for *I-shaped* girders, since (see § 228) the measure of the moment of flexure W is the same for both. These formulas can also be employed for bodies with *more than two webs*, as, E.G., bodies with the cross-section represented in Fig. 388, in which case b denotes the width of the upper and lower rib, h the entire height $A\,D = B\,C$, b_1 the sum of the widths and h_1 the height of the hollow spaces M, N, O, P.

FIG. 388.



The formulas for a *pipe or hollow cylinder* are analogous to those for a parallelopipedical beam. If r is the exterior and $r_1 = \mu\,r$ the interior radius, the moment of proof load of this body is

$$P\,l = \frac{\pi\,(r^4 - r_1^4)}{r}\,\frac{T}{4} = (1 - \mu^4)\,\pi\,r^3\,\frac{T}{4} = \left(\frac{1 - \mu^4}{1 - \mu^2}\right) F\,r\,\frac{T}{4}$$

$$= (1 + \mu^2)\,F\,r\,.\,\frac{T}{4}.$$

This value increases as $\mu = \frac{r_1}{r}$ approaches unity, and therefore as the wall of the pipe becomes thinner. If we put $\mu = 1$, we obtain the corresponding maximum moment of proof load

$$P\,l = 2\,F\,r\,\frac{T}{4} = F\,r\,\frac{T}{2}.$$

If we compare the proof load of this tube with that of a massive iron cylinder, whose radius $r_1 = \mu\,r = r\,\sqrt{\tfrac{1}{2}}$, we have then for the latter

$$P_1\,l = F\,r_1\,\frac{T}{4} = \mu\,F\,r\,\frac{T}{4} \text{ and}$$

$$\frac{P}{P_1} = \frac{1 + \mu^2}{\mu} = (1 + \tfrac{1}{2})\,\sqrt{2} = \frac{3}{2}\,\sqrt{2} = 2{,}12,$$

exactly what we found under the same suppositions for parallelopipedical girders.

We can see from the general equation

$$P\,l = \frac{W\,T}{e} = \frac{(F_1\,z_1^2 + F_2\,z_2^2 + \,.\,.)}{e}\,T = (F_1\,\mu_1^2 + F_2\,\mu_2^2 + \,.\,.)\,e\,T,$$

that the moment of proof load of a body increases as the distances $z_1 = \mu_1\,e$, $z_2 = \mu_2\,e$, etc., of the portions F_1, F_2, etc., of the cross-section from the neutral axis become greater. But since this distance can at most be $= e$, those girders will have the greatest moment of proof load, the different portions of whose cross-section are at one and the same distance (the maximum one) from the neutral axis. Such a beam consists of *two flanges*. Since the webs which unite the two flanges cannot satisfy the conditions of maximum moment of proof load, it is impossible to attain this maximum, and we must therefore content ourselves with increasing the proof strength of the girder by hollowing it out, by thinning it in the neighborhood of the neutral axis, or by adding flanges at the greatest possible distance from the same axis.

The thickness, which the web must possess in order to resist the shearing strain, will be determined in the following chapter.

REMARK.—Under the supposition that the proof strength increases and decreases with the ultimate strength, the English engineers increase the size of that portion of cast-iron girders, which is subject to compression; for that material resists compression best. On the contrary, they increase the dimensions of the compressed side of girders of wrought iron, as the latter resists extension best. If the girders are to be supported at both ends, their form must depend upon the substance of which they are made. If the beam is of cast iron, we make the bottom flange larger than the other; if of wrought iron, the upper flange, or the upper part of the girder is constructed of two flanges, united by vertical webs, as is represented in Fig. 388. The forms T and T, discussed in a previous paragraph (§ 237), are employed for cast iron.

EXAMPLE.—An oak girder 9 inches wide and 11 inches high, which has up to the present time shown sufficient strength, is to be replaced by a cast-iron girder, whose exterior width is 5 inches and whose height is 10 inches; how thick should it be made? If we put the double thickness of the metal $= x$, the width of the hollow is $= 5 - x$, and its height is $= 10 - x$, and consequently we have for the hollow girder

$$b_1 h_1^3 - b_2 h_2^3 = 5 . 10^3 - (5 - x)(10 - x)^3 = 2500\, x - 450\, x^2 + 35\, x^3 - x^4,$$

hence the moment of proof load is $P\,l = \frac{7000}{6 . 10}(2500\, x - 450\, x^2 + 35\, x^3 - x^4)$.

If the moment of proof load of the massive wooden beam is $P\,l = \frac{1000}{6} . 9 . 11^2 = \frac{1}{6} . 1089000$, we must put

$$700 . (2500\, x - 450\, x^2 + 35\, x^3 - x^4) = 1089000, \text{ or}$$
$$2500\, x - 450\, x^2 + 35\, x^3 - x^4 = 1556.$$

In the first place, x is approximatively $= \frac{1556}{2500} = 0,62$, for which, however, $x = 0,65$ should be put. From this we obtain $450\, x^2 = 450 . 0,4225 = 190,12$, $35\, x^3 = 9,61$, $x^4 = 0,18$, and finally

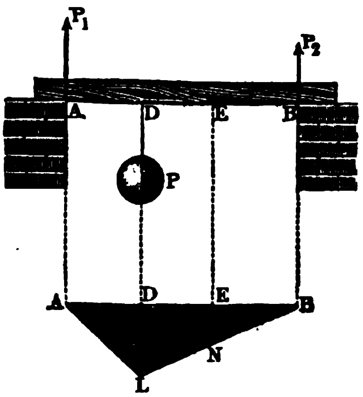
$$x = \frac{1556 + 190,12 - 7,56 + 0,18}{2500} = \frac{1738,7}{2500} = 0,695 \text{ inches},$$

and consequently the required thickness of metal is

$$\frac{x}{2} = 0,3475 \text{ inches.}$$

§ 245. Excentric Load.—If the force which acts upon a girder supported at both ends A and B, Fig. 389, is *not applied at the centre*, but at some intermediate point, situated at the distances $D\,A = l_1$ and $D\,B = l_2$ from the points of support, the proof load is greater than when the force is applied in the middle. Let us denote the forces, with which the points of support A and B react, by P_1 and P_2 and the entire length of the girder $A\,B = l_1 + l_2$ by l. Now, if we put the moment of P_1 in

FIG. 389.



reference to the point of support B equal to that of P in reference to the same point and in like manner the moment of P_2 in reference to A equal to that of P or $P_1\, l = P\, l_2$ and $P_2\, l = P\, l_1$, we obtain the reactions at the points of support

$$P_1 = \frac{l_2}{l}\, P \text{ and } P_2 = \frac{l_1}{l}\, P,$$

and consequently their moments in reference to the points of application

$$P_1\, l_1 = P_2\, l_2 = \frac{P\, l_1\, l_2}{l}.$$

For any other point E, whose distance $B\,E$ from the point of support B is $= x$, we have this moment

$$P_2 \,.\, \overline{B\,E} = \frac{P\, l_1\, x}{l}$$

smaller than that just found, and consequently at B we have the greatest deflection, and therefore we must determine the proof load in reference to this point alone, for which we have

$$\frac{P\, l_1\, l_2}{l} = \frac{W\,T}{e}.$$

If we substitute $l_1 = \frac{l}{2} - x$ and $l_2 = \frac{l}{2} + x$, we obtain the moment of the force

$$\frac{P\, l_1\, l_2}{l} = \frac{P\left(\frac{l}{2} - x\right)\left(\frac{l}{2} + x\right)}{l} = \frac{P\left(\frac{l^2}{4} - x^2\right)}{l};$$

hence the proof load is

$$P = \frac{l}{l_1\, l_2} \cdot \frac{W\,T}{e} = \frac{l\,W\,T}{\left(\frac{l^2}{4} - x^2\right) e}$$

and therefore greater or less as x is greater or less. For $x = \frac{l}{2}$, i.e., for $l_1 = 0$, in which case P is transferred to the point of support A, we have

$$P = \frac{l\,W\,T}{0\,.\,e} = \infty,$$

and on the contrary for $x = 0$, i.e. if the force P is applied at the centre, the proof load is a minimum and is

$$P = 4\,\frac{W\,T}{l\,e},$$

as we know already from § 240. A prismatical girder supported at both ends will sustain the smallest load, when the latter is applied at the centre, and more and more as the weight approaches the points of support.

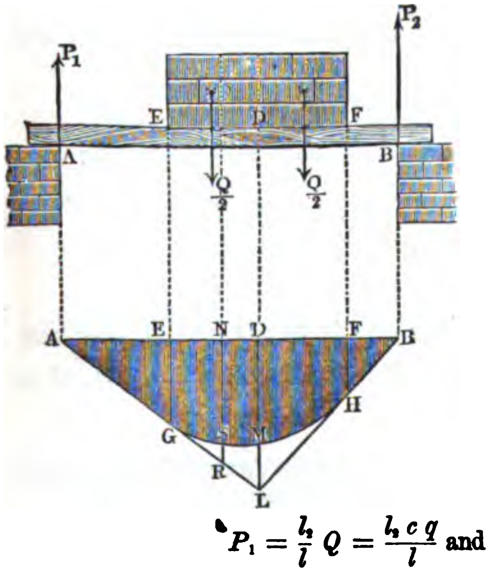
If we lay off as ordinates the moments of the force, which are inversely proportional to the radius of curvature and directly to the curvature itself, as ordinates at the different points upon the girder, we obtain a clear representation of the variation of the deflection at the different points upon the girder.

If, in the case just discussed, the moment of the force $\frac{Pl_1 l_2}{l}$ in D is represented by the ordinate $\overline{DL}$ and if from its extremity L the right lines LA and LB be drawn to the extremities of the abscissas $DA = l_1$ and $DB = l_2$, these lines will limit the different ordinates (as for example EN) representing the measures of the deflection for the different portions of the body; for since $\frac{EN}{EB} = \frac{DL}{DB}$, it follows that

$$\overline{EN} = \frac{EB}{DB} \cdot \overline{DL} = \frac{x}{l_2} \cdot \frac{Pl_1 l_2}{l} = \frac{Pl_1 x}{l},$$

as we had previously found.

FIG. 390.



Another case which often occurs in practice is, when the weight is *equally distributed* over a portion $EF = c$ of the entire length l of the girder AB, Fig. 390. Let us again denote the distances of the middle D of this weight from the points of support A and B by l_1 and l_2 and the reaction of the abutments by P_1 and P_2, then we have again

$$P_1 = \frac{l_2}{l} Q = \frac{l_2 c q}{l} \text{ and}$$

$$P_2 = \frac{l_1}{l} Q = \frac{l_1 c q}{l}.$$

If Q were not distributed, but if, on the contrary, the force was applied at D, the moment for D would be $= \frac{Q l_1 l_2}{l}$, and, representing the same by an ordinate $D\,L$, the moment for the other points of $A\,B$ will be cut off by the right lines $L\,A$ and $L\,B$. But, since for the points within $E\,F$ the forces P_1 and P_2 act in opposition to the weight placed upon it, the ordinates between $E\,G$ and $F\,H$ will be diminished. For the centre D of the loaded portion $E\,F$ the moment of half the weight

$$\frac{Q}{2}\cdot\frac{c}{4} = \overline{M\,L}$$

must be subtracted, and there remains, therefore, of the ordinate $\overline{D\,L} = \frac{Q\,l_1\,l_2}{l}$ only the portion

$$\overline{D\,M} = \overline{D\,L} - \overline{M\,L} = Q\left(\frac{l_1\,l_2}{l} - \frac{c}{8}\right).$$

For another point N, whose abscissa is $A\,N$, the moment is, on the contrary,

$$P_1\,.\,\overline{N\,A} - \overline{N\,E}\,.\,q\,.\,\frac{\overline{N\,E}}{2} = P_1\,x - \frac{(x - l_1 + \frac{1}{2}\,c)^2\,q}{2}$$

and if $P_1\,x$ is represented by the ordinate $\overline{N\,R}$ and $\frac{(x - l_1 + \frac{1}{2}\,c)^2\,q}{2}$ by the portion $\overline{S\,R}$, the ordinate $\overline{N\,S}$ will give the total moment

$$P_1\,x - \frac{(x - l_1 + \frac{1}{2}\,c)^2\,q}{2}.$$

This is of course very different for different values of x, I.E. for different points, but is a maximum for $x - l_1 + \frac{1}{2}\,c = \frac{P_1}{q}$, and then its value is

$$P_1\left(\frac{P_1}{q} + l_1 - \tfrac{1}{2}\,c\right) - \frac{P_1^2}{2\,q} = P_1\left(\frac{P_1}{2\,q} + l_1 - \tfrac{1}{2}\,c\right)$$

$$= P_1\left(l_1 - \frac{c}{2} + \frac{c\,l_2}{2\,l}\right) = P_1\,l_1\left(1 - \frac{c}{2\,l}\right) = \frac{Q\,l_1\,l_2}{l}\left(1 - \frac{c}{2\,l}\right).$$

Hence we must put the proof load of this girder

$$\frac{Q\,l_1\,l_2}{l}\left(1 - \frac{c}{2\,l}\right) = \frac{W\,T}{e}.$$

EXAMPLE.—What weight will a hollow parallelopipedical girder, made of ½ inch thick sheet iron, support, if its exterior height is 16 inches and its exterior width is 4 inches, when it is loaded uniformly along 5 feet of its length, the middle of the loaded portion being 8 and 4 feet distant from the points of support? Here we have

$$\frac{b\,h^3 - b_1\,h_1^3}{h} = \frac{4\,.\,16^3 - 3\,.\,15^3}{16} = 391{,}2$$

and

$$\frac{l_1\, l_2}{l}\left(1 - \frac{c}{2\,l}\right) = \frac{2}{3} \cdot 48 \left(1 - \frac{5}{24}\right) = \frac{32 \,.\, 19}{24} = \frac{76}{3},$$

and the weight required is therefore

$$Q = 391{,}2 \,.\, \frac{3}{76} \cdot \frac{T}{6} = \frac{195{,}6}{76} \,.\, 9000 = 23160 \text{ pounds.}$$

REMARK.—If the weight Q is not uniformly distributed over $E\,F$, but if half is applied at the extremity E and half at the extremity F, the line $G\,M\,H$ is then a right line, and the maximum moment is the ordinate $\overline{G\,E}$, for which

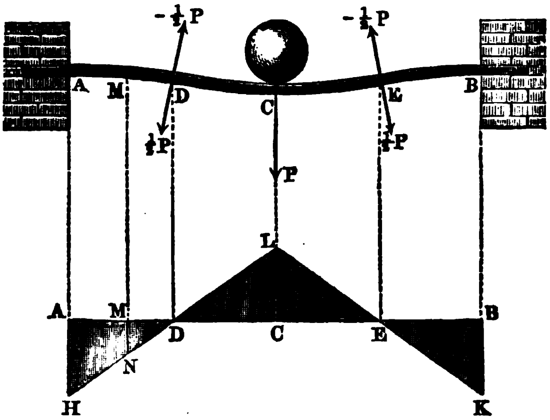
$$\frac{Q\,l_2}{l}\left(l_1 - \frac{c}{2}\right) = \frac{W\,T}{e}$$

l_1 denoting the greater distance $D\,A$ and l_2 the smaller distance $D\,B$ of the middle D from the two extremities A and B.

§ 246. Girders Fixed at Both Ends.

—If a beam $A\ B$, Fig. 391, is loaded in the centre C and *fixed at both ends*, it will be curved upwards at the centre, and at the points of support A and B downwards, and there will be formed at the centres D and E of the semi-girders $C\,A$ and $C\,B$ points of inflection, where there is no curvature or where the radius of curvature is infinitely great. One-half of the weight P is supported by $A\ D$ and the other half by $B\ E$, and we can therefore assume that both the quarters $A\ D$ and $B\ E$ of the beam are bent downwards at their ends D and E by $\frac{P}{2}$, and that, on the contrary, the half $D\ E$ of the girder is bent upwards at its ends D and E by $\left(-\frac{P}{2}\right)$. The arm of each of these

FIG. 391.



forces $A\,D = C\,D$, etc., is $= \frac{A\,B}{4} = \frac{l}{4}$; consequently their moment is

$$\frac{P}{2} \cdot \frac{l}{4} = \frac{P\,l}{8}, \text{ and therefore}$$

$$\frac{P\,l}{8} = \frac{W\,T}{e}; \text{ hence we can put the proof load}$$

$$P = \frac{8\,W\,T}{l\,e} = 2 \cdot \frac{4\,W\,T}{l\,e}.$$

Such a girder will bear twice as great a load as when it is simply supported at both ends.

If we make the ordinates $\overline{A\,H} = \overline{B\,K} = \overline{C\,L} = \frac{P\,l}{8}$, and draw the right lines $H\,L$ and $K\,L$, they will cut off ordinates $(\overline{M\,N})$ for every other point (M) upon the beam proportional to the moments of the force and to the deflection.

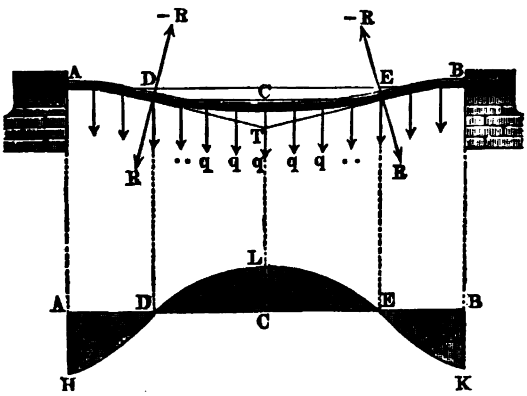
If in the formula, which we have found, we substitute the modulus of rupture K instead of the modulus of proof strength T, we obtain, of course, the force necessary to break the beam, which is

$$P = \frac{8\,W\,K}{l\,e}.$$

Since the curvature is the same in A, B and C, the rupture will take place at the same time in A, B and C.

If the position of the girder is the same and the load $Q = l\,q$ is uniformly distributed, the girder assumes, it is true, two curvatures upwards and two downwards, but the points of inflection

FIG. 392.



D and E, Fig. 392, do not lie at the centres of the semi-girders; for the deflecting forces R, R of the portions $A\ D$ and $B\ E$ are aided by the weight upon the latter, and, on the contrary, the action of the bending forces $-R$, $-R$ of the central piece D is diminished by this load. Let us put the length $AD = BE = l_1$, the length $CD = CE = l_2$ and the total length of the beam $l = 2(l_1 + l_2)$, and let us denote the weight upon AD or BE by $Q_1 = q\, l_1$, and that upon DE by $Q_2 = 2R = 2\, q\, l_2$. Now, since AD is bent downwards by R and Q_1, we have, according to § 216 and § 223, the angle of inclination to the horizon $EDT = DET = a$ at the point of inflection D

$$a = \frac{R\, l_1^2}{2\, WE} + \frac{Q_1\, l_1^2}{6\, WE},$$

and since CD is bent upwards by $(-R)$ and downwards by Q_2, we have for the same position D also

$$a = \frac{R\, l_2^2}{2\, WE} - \frac{Q_2\, l_2^2}{6\, WE}.$$

Equating the two values of a, we obtain the relation

$$3\, R\, (l_2^2 - l_1^2) = Q_1\, l_1^2 + Q_2\, l_2^2, \text{ or}$$

$$3\, q\, l_2\, (l_2^2 - l_1^2) = q\, (l_1^3 + l_2^3), \text{ i.e.,}$$

$$3\, l_2 \left[l_2^2 - \left(\frac{l}{2} - l_2 \right)^2 \right] = l_2^3 + \left(\frac{l}{2} - l_2 \right)^3.$$

Resolving this equation, we obtain

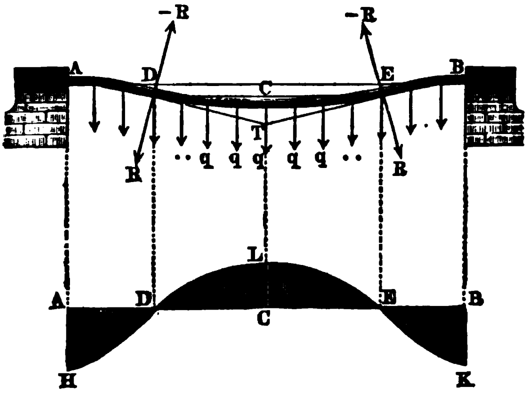
$$l_2 = \frac{l}{2} \sqrt{\tfrac{1}{3}} \text{ and } l_1 = \frac{l}{2} \left(1 - \sqrt{\tfrac{1}{3}}\right),$$

and, therefore, the moment of force in relation to the middle C is

$$M = R\, l_2 - \frac{R\, l_2}{2} = \frac{R\, l_2}{2} = \frac{q\, l_2^2}{2} = \frac{q\, l^2}{24} = \frac{Q\, l}{24},$$

and that in reference to the extremity A or B is

FIG. 398.



$$M_1 = R\, l_1 + \frac{Q_1\, l_1}{2} = q\, l_1\, l_2 + \frac{q\, l_1^2}{2} = q\, l_1 \left(l_2 + \frac{l_1}{2}\right)$$

$$= \frac{q\, l^2}{8}\left(1 - \sqrt{\tfrac{1}{3}}\right)\left(1 + \sqrt{\tfrac{1}{3}}\right)$$

$$= \frac{q\, l^2 \left(1 - \frac{1}{3}\right)}{8} = \frac{Q\, l}{12} = 2\, \frac{Q\, l}{24}.$$

The proof load of this beam is therefore

$$Q = 12 \,.\, \frac{W\, T}{l\, e} = \frac{3}{2} \,.\, \frac{8\, W\, T}{l\, e},$$

I.E., $\frac{3}{2}$ times as great as in the former case, where the weight acted at the centre C.

If we lay off $\frac{Q\, l}{12}$ as ordinate in A and B and also $\frac{Q\, l}{24}$ as ordinate in C, making $\overline{A\, H} = B\, K = \frac{Q\, l}{12}$ and $\overline{C\, L} = -\frac{Q\, l}{24}$, we obtain three points H, K and L of the curve $H\, D\, L\, E\, K$, which represents the variation of the deflection of the girder.

EXAMPLE.—How high can grain be piled in a grain house, when the floor rests on beams 25 feet long, 10 inches wide and 12 inches high, if the distance between two beams is = 3 feet and if a cubic foot of corn weighs 46,7 pounds? If we employ the last formula $Q\, l = 12 \,.\, 167 \,.\, b\, h^2$, we must put

$b = 10$, $h = 12$, $l = 25 \,.\, 12 = 300$, and consequently

$$Q = \frac{12 \,.\, 167 \,.\, 10 \,.\, 144}{300} = 9619 \text{ pounds.}$$

Now a parallelopipedical mass of grain 25 feet long, 3 feet wide and x feet high weighs $25 \,.\, 3 \,.\, x \,.\, 46{,}7$ pounds; if we substitute this value for Q, we obtain the required height of the mass

$$x = \frac{9619}{75 \,.\, 46{,}7} = 2{,}75 \text{ feet.}$$

§ **247. Beams Dissimilarly Supported.**—If a beam $A\, B\, C$, Fig. 394, is *fixed at one end A and supported at the other B* and if the load acts in the middle between A and B, we have, according to § 221, the reaction of the support B

$$P_1 = \frac{5}{16}\, P;$$

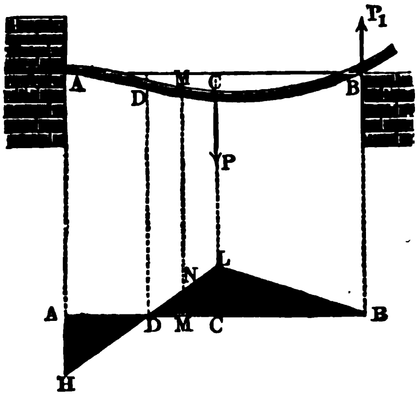
and therefore the moment of the force in reference to C

$$\overline{C\, L} = \frac{P_1\, l}{2} = \frac{5}{32}\, P\, l,$$

and, on the contrary, that in reference to A is

$$\overline{AH} = P\frac{l}{2} - P_1 l = P l\left(\frac{1}{2} - \frac{5}{16}\right) = \frac{3}{16} P l = \frac{6}{32} P l,$$

FIG. 394.



or greater, and consequently we can put the proof load

$$P = \frac{16}{3} \cdot \frac{W T}{l e}.$$

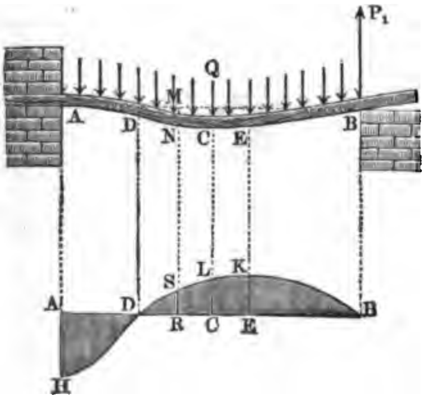
For an intermediate point M, at a distance $CM = x$ from the centre C, this moment is

$$\overline{MN} = P_1\left(\frac{l}{2} + x\right) - P x = P_1 \frac{l}{2} - (P - P_1)\, x.$$

If we assume $x = \frac{\frac{1}{2} P_1 l}{P - P_1} = \frac{5}{16 - 5} \cdot \frac{l}{2} = \frac{5}{22}\, l$, we obtain that point, for which the moment is equal to zero and the radius of curvature infinitely great. The variation of this moment and the deflection of the girder are represented by the ordinates of the right lines HL and LB, passing through the extremities of $\overline{AH} = \frac{6}{32} P l$ and of $\overline{CL} = \frac{5}{32} P l$.

If, finally, a girder AB, Fig. 395, supported in the same manner as the last, is uniformly loaded, as we have previously generally supposed, with a certain weight q upon the running foot of the girder, we can determine the reaction P_1 at the support B in the following manner. If the length of the beam is l, the entire load is $Q = l q$ and the moment of the force in reference to a point M, at a distance $BM = x$ from the point of support B, is

FIG. 395.



$$\overline{R\,S} = P_1\,x - \frac{q\,x^2}{2},$$

and consequently the angle of inclination

$$a = \frac{P_1\,(l^2 - x^2)}{2\,W\,E} - \frac{q\,(l^3 - x^3)}{6\,W\,E},$$

and (according to § 217 and § 223) the corresponding deflection is

$$y = M\,N = \frac{P_1\,(l^2\,x - \frac{1}{3}\,x^3)}{2\,W\,E} - \frac{q\,(l^3\,x - \frac{1}{4}\,x^4)}{6\,W\,E}.$$

But since *A* lies on the same level with *B*, the ordinate in *A*, I.E. for $x = l$, is $y = 0$, and we must put

$$3\,P_1\,.\,\tfrac{2}{3}\,l^3 = q\,.\,\tfrac{3}{4}\,l^4,$$

from which we obtain the reaction at *B*

$$P_1 = \tfrac{3}{8}\,q\,l = \tfrac{3}{8}\,Q.$$

If we substitute this value for P_1 in the expression for the moment, we obtain

$$\overline{R\,S} = \tfrac{3}{8}\,Q\,x - \frac{q\,x^2}{2} = \frac{q\,x}{2}\,(\tfrac{3}{4}\,l - x);$$ and therefore for $x = l$

$$\overline{A\,H} = -\,\frac{q\,l^2}{8} = -\,\frac{Q\,l}{8}.$$

For $x = B\,D = \frac{3}{4}\,l$ this moment is $= 0$, and for $x = B\,E = \frac{3}{8}\,l$ it is a maximum

$$\overline{E\,K} = \frac{9\,q\,l^2}{128} = \frac{9}{128}\,Q\,l.$$

Since $\frac{Q\,l}{8} = \frac{16}{128}\,Q\,l > \frac{9}{128}\,Q\,l$, the moment $\overline{A\,H}$ in reference to the fixed point *A* is greater than the moment $\overline{K\,E}$ in reference to the middle *E* of *B D*, and the proof load corresponding to the moment $\frac{Q\,l}{8}$ must therefore be determined, I.E. we must put

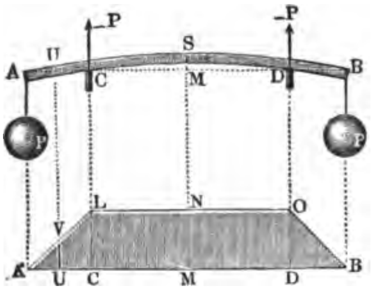
$$Q = 8\,\frac{W\,T}{l\,e},$$

in which case we assume, of course, that the modulus of proof strength for extension is the same as that for compression.

This proof load is $8\,.\,\frac{3}{16} = \frac{3}{2}$ times as great as it would be if the weight were concentrated in the middle.

§ 248. Girders Loaded at Intermediate Points.—If a girder *A B*, Fig. 396, loaded at both ends with equal weights *P*, *P*,

FIG. 396.



is supported at two points C and D, which are at the same distance $A\,C = B\,D = l_1$ from the ends, the reaction of each of these points of support is equal to the force P, and for a point M upon $C\,D$ the moment of flexure

$$\overline{CL} = \overline{DO} = \overline{MN} = P\,(x_1 - l_1) - P\,x_1 = -P\,l_1$$

is *constant*, and the form of neutral axis of $C\,D$ is therefore a *circle*, while, on the contrary, for a point U upon $A\ C$ this moment $U\,V = P\,x$ is *variable* and smaller than $P\,l_1$.

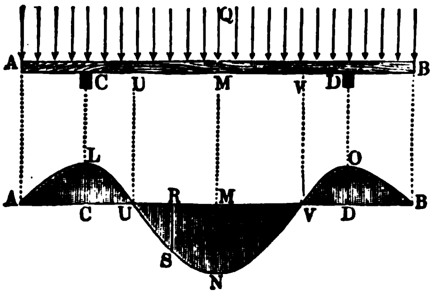
The radius of curvature of the middle piece $C\,D$ is $= r = \frac{W\,E}{P\,l_1}$, and the angle of inclination of the axis of the beam in C and D is consequently $\alpha_1 = \frac{l}{2\,r} = \frac{P\,l\,l_1}{2\,W\,E}$, l denoting the length of this middle piece. From this we obtain the deflection

$M\,S = a = \frac{(\frac{1}{2}\,l)^2}{2\,r} = \frac{l^2}{8\,r} = \frac{P\,l^2\,l_1}{8\,W\,E}$, as well as the deflection of $C\,A$

$$a_1 = \alpha_1\,l_1 + \frac{P\,l_1^3}{3\,W\,E} = \frac{P\,l\,l_1^2}{2\,W\,E} + \frac{P\,l_1^3}{3\,W\,E} = \frac{P\,l_1^2}{W\,E}\left(\frac{l}{2} + \frac{l_1}{3}\right).$$

The *moment of proof load* for this girder is $P\,l_1 = \frac{W\,T}{e}$.

FIG. 397.



If the same beam $A\ B$ is *uniformly* loaded, as is shown in Fig. 397, with q per running foot, under certain circumstances the moment of flexure for some points is positive, and for others negative, and therefore at two points U and V it is equal to zero.

For a point upon $A\ C$ and $B\ D$ this moment is $\frac{1}{2}\ q\ x^2$, and, on the contrary, for a point between C and the middle M, or between D and M, since the value of the reaction at C and D is $\frac{1}{2}\ Q = (\frac{1}{2}\ l + l_1)\ q$, it is $\overline{R\,S} = y = \frac{1}{2}$

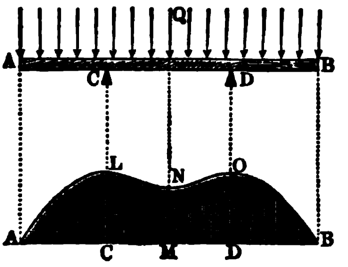
$(x + l_1)^2 q - (\frac{1}{2} l + l_1) x q = \frac{1}{2}(x^2 - l x + l_1^2) q$, and therefore $= 0$ for $x^2 - l x = - l_1^2$, I.E. for

$$\overline{C U} = x = \frac{l}{2} - \sqrt{\left(\frac{l}{2}\right)^2 - l_1^2} \text{ and for}$$

$$C V = x = \frac{l}{2} + \sqrt{\left(\frac{l}{2}\right)^2 - l_1^2},$$

which of course requires that $l_1 =< \frac{l}{2}$, I.E. $C A < C M$. Under any other circumstances the moment of flexure remains always positive, as is shown in Fig. 398.

FIG. 398.



The moment of flexure is a maximum or minimum for $x = \frac{l}{2}$ and is

$$\overline{M N} = - \frac{1}{2}\left[\left(\frac{l}{2}\right)^2 - l_1^2\right] q,$$

while the moment of flexure in C and D is $\overline{C L} = \overline{D O} = \frac{1}{2} q l_1^2$.

If, therefore, in the first case, Fig. 397, $\left(\frac{l}{2}\right)^2 - l_1^2 > l_1^2$ or $\left(\frac{l}{2}\right)^2 > 2 l_1^2$, I.E. $l > l_1 \sqrt{8}$, we have $\overline{M N} > \overline{C L}$, and since $q = \frac{Q}{l + 2 l_1}$, we must put the moment of proof load equal to

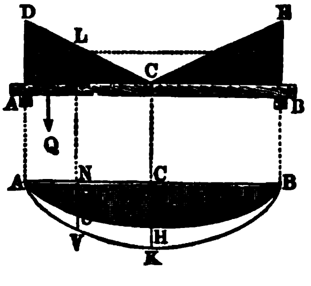
$$\left[\left(\frac{l}{2}\right)^2 - l_1^2\right] \frac{Q}{2(l + 2 l_1)} = \frac{W T}{e}, \text{ while, on the contrary, we have}$$

$$\frac{Q l_1^2}{2(l + 2 l_1)} = \frac{W T}{e}, \text{ when } l < l_1 \sqrt{8}.$$

§ 249. Girders not Uniformly Loaded.—If a beam $A B$, Fig. 399, is not *uniformly loaded*, but in such a manner that the load on the running foot increases towards the extremities of the girder regularly with the distance from its centre, the statical relations will be as follows.

If $l = A B = 2 C A = 2 C B$ is the length of the beam, measured between the points of support A and B, q the weight of the load per unit of surface of the cross-section and ρ the angle of inclination $A C D = B C E$

FIG. 399.



of the planes $C\,D$ and $C\,E$, which bound the load, we have the weight of the prism $A\ C\,D = B\ C\,E$ of the load, sustained by one point of support,

$$\frac{Q}{2} = \tfrac{1}{2}\,\overline{A\ C}\,.\,\overline{A\ D}\,.\,q = \tfrac{1}{2}\left(\frac{l}{2}\right)^2 tang.\,\rho\,.\,q = \tfrac{1}{8}\,q\,l^2\,tang.\,\rho,$$

and consequently the moment of this force in reference to a point N, at a distance $A\ N = x$ from A, is

$$y_1 = \frac{Q}{2}\,.\,x = \tfrac{1}{8}\,q\,l^2\,x\,tang.\,\rho.$$

The weight of the heavy prism above $A\,N = x$ is $q\left(\frac{A\,D + N\,L}{2}\right) A\,N$, and the centre of gravity of the same is at a distance $N\,O = \frac{2\,A\,D + N\,L}{A\,D + N\,L}\,.\,\frac{A\ N}{3}$ from N, and consequently the moment of this prism in reference to N is

$$y_2 = q\,(2\,A\,D + N\,L)\,\frac{\overline{A\ N}^2}{6} = q\left[l\,tang.\,\rho + \left(\frac{l}{2} - x\right) tang.\,\rho\right]\frac{x^2}{6}$$

$$= \frac{q\,x^2}{6}\,tang.\,\rho\,(\tfrac{3}{2}\,l - x),$$

and the entire moment of flexure for the girder at N is

$$\overline{N\,U} = y = y_1 - y_2 = \frac{q\,tang.\,\rho}{24}\,(3\,l^2 x - 6\,l\,x^2 + 4\,x^3)$$

$$= \frac{q\,x\,tang.\,\rho}{24}\,(3\,l^2 - 6\,l\,x + 4\,x^2) = \frac{q}{6}\left[\left(\frac{l}{2}\right)^3 - x_1^3\right] tang.\,\rho,$$

if we put $C\,N = x_1 = \frac{l}{2} - x$ or measure the abscissa x_1 from C.

This is a maximum for $x = \frac{l}{2}$ and equal to $\frac{q\,l^3}{48}\,tang.\,\rho$, and the *moment of proof load* of this girder is

$$\frac{q\,l^3}{48}\,tang.\,\rho,\ \text{I.E.,}\ \frac{Q\,l}{12} = \frac{W\,T}{e},$$

while for an uniformly loaded beam the moment of flexure is

$$\overline{N\,V} = y_0 = \frac{q\,l\,x}{2} - \frac{q\,x^2}{2} = \frac{q\,x}{2}\,(l - x) = \frac{q}{2}\left[\left(\frac{l}{2}\right)^2 - x_1^2\right]$$

$$= \frac{Q}{2\,l}\left[\left(\frac{l}{2}\right)^2 - x_1^2\right],$$

hence the moment of proof load is $\frac{Q\,l}{8} = \frac{W\,T}{e}$.

§ 250. **Girders Sustaining Two Loads.**—If a girder $A\ B$, Fig. 400, supported at both ends is loaded at a point C, which is at the distances $C\ A = l_1$ and $C\ B = l_2$ from the points of support

A and B, with a weight P and in addition carries a uniformly distributed load $Q = q\,l$, the reaction of points of support A and B are $R_1 = \frac{l_2 P}{l} + \frac{Q}{2}$ and $R_2 = \frac{l_1 P}{l} + \frac{Q}{2}$, and the moment of flexure at a point N, situated at a distance $A\,N = x$ from the point of support A, is

$$\overline{N\,V} = y = R_1\,x - \frac{q\,x^2}{2} = \left(R_1 - \frac{q\,x}{2}\right)x = \frac{q}{2}\left(\frac{2\,R_1}{q} - x\right)x.$$

FIG. 400.

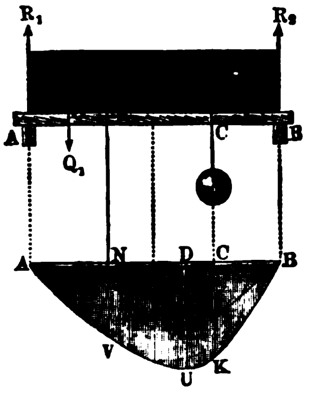
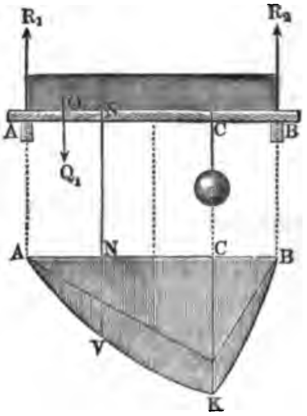


FIG. 401.



This moment is a maximum for

$$\frac{2\,R_1}{q} - x = x, \text{ i.e., for } x = \frac{R_1}{q}, \text{ and is then}$$

$$y = \overline{D\,U} = \frac{q}{2}\left(\frac{R_1}{q}\right)^2 = \frac{R_1^2}{2\,q} = \frac{1}{2\,q}\left(\frac{l_2\,P}{l} + \frac{Q}{2}\right)^2 = \frac{l}{2\,Q}\left(\frac{l_2}{l}P + \frac{Q}{2}\right)^2.$$

It is here assumed, that $C\,A > C\,B$, i.e., $l_1 > l_2$ and $x < l_1$. If $x \geqq l_1$ the maximum of the moment of flexure is at C (Fig. 401), and consequently

$$y = \overline{C\,K} = R_1\,l_1 - \frac{q\,l_1^2}{2} = \frac{l_1\,l_2}{l}P + \frac{Q\,l_1}{2} - \frac{Q\,l_1^2}{2\,l} = \left(P + \frac{Q}{2}\right)\frac{l_1\,l_2}{l}.$$

If we substitute

$$x = \frac{R_1}{q} = \left(\frac{l_2\,P}{l} + \frac{Q}{2}\right)\frac{l}{Q} = l_1, \text{ we obtain}$$

$$\frac{P}{Q} = \frac{l_1 - \frac{1}{2}\,l}{l_2} = \frac{2\,l_1 - l}{2\,l_2} = \frac{l_1 - l_2}{2\,l_2},$$

and the *moment of proof load of the girder*, when

$$\frac{P}{Q} < \frac{l_1 - l_2}{2\,l_2}, \text{ is}$$

$\left(\frac{P\,l_2}{l} + \frac{Q}{2}\right)^2 \frac{l}{2\,Q} = \frac{W\,T}{e}$, and, on the contrary, when

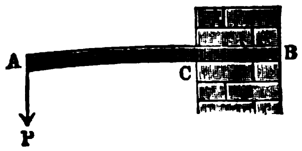
$\frac{P}{Q} \leqq \frac{l_1 - l_2}{2\,l_2}$, it is

$$\left(P + \frac{Q}{2}\right) \frac{l_1\,l_2}{l} = \frac{W\,T}{e}.$$

These formulas are specially applicable to cases, where the weight G of the beam is taken into consideration; here G must be substituted for Q.

§ **251. Cross-section of Rupture.**—In all the cases, which we have previously treated, we have assumed the body $A\,B$, Fig. 402, to be prismatical and, therefore, the moment of flexure $W\,E$ to be constant, hence we could conclude from the fundamental formula

FIG. 402.



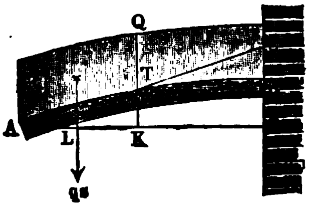
$$P\,x\,r = W\,E,$$

that the radius of curvature

$$r = \frac{W\,E}{P\,x}$$

was inversely, or the curvature itself directly, proportional to the moment $(P\,x)$ of the force P acting upon the body and that consequently the curvature becomes a maximum or a minimum at the same time that $P\,x$ does. If, therefore, the force P is constant, or if it increases with x (as, E.G., in the case represented in Fig. 403, where $Q = q\,x$), the curvature increases or diminishes with x and becomes with it a maximum and minimum. When, on the contrary, the cross-section F of the body is different in different points, then $W = \Sigma\,(F\,z^2)$ is also variable, the radius of curvature is proportional to the quotient $\frac{W}{P\,x}$ and the curvature itself to the expression $\frac{P\,x}{W}$. If we are required to find the points of greatest and least curvature, we have only to determine those, for which $\frac{P\,x}{W}$ is a maximum and a minimum.

FIG. 403.



In like manner, according to the formula

$$S = \frac{P x e}{W}$$

of § 235, the strain S in a body is proportional to the expression $\frac{P x e}{W}$, and becomes a maximum or a minimum simultaneously with it.

If the body is *prismatical*, $\frac{W}{e}$ is constant, and the maximum strain S is proportional to the moment $P x$ of the force only. If the *cross-section of the body varies*, $\frac{W}{e}$ is a variable quantity, and this strain is dependent upon this quotient also. In the first case the strain becomes a maximum with $P x$, E.G., when the beam is acted upon at one point by a force P and by a load $Q = q x$ uniformly distributed over a distance x, for $x = l$; in the second case this maximum cannot be determined unless we know how the cross-section varies. In order to find the point of maximum strain, it is necessary to determine by algebra the maximum of the expression $\frac{P x e}{W}$. In any case the part of the body where this maximum strain occurs is also that point at which, if the load is sufficient, the strain S first becomes equal to T and also to K, and, consequently, where the limit of elasticity will first be attained or where rupture will take place. This cross-section of the body corresponding to the maximum value of $\left(\frac{P x e}{W}\right)$ is therefore called the *section of rupture* (Fr. section de rupture, Ger. Brechungsquerschnitt) or also the *dangerous* (weak) section.

If the body has a *rectangular cross-section*, with the variable width u and the variable height v, we have

$$\frac{W}{e} = \frac{u v^2}{6},$$

and the section of rupture is determined by the maximum of $\frac{P x}{u v^2}$ or by the minimum of $\frac{u v^2}{P x}$.

For a body with an *elliptical cross-section*, whose variable semi-axes are u and v, we have

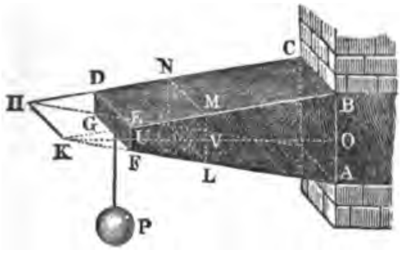
$$\frac{W}{e} = \frac{\pi u v^2}{4},$$

and we must therefore again determine the minimum value of $\frac{u\,v^2}{P\,x}$, when we wish to know the weakest point in the body.

When the weight is constant, P can be left out of consideration, and we have to determine only the minimum of $\frac{u\,v^2}{x}$. If, on the contrary, the weight $Q = q\,x$ is uniformly distributed upon the girder, we must determine the minimum of $\frac{u\,v^2}{x^2}$ in order to find the section of rupture.

§ **252.** If a body $A\,C\,D\,F$, Fig. 404, forms a *truncated wedge* or a horizontal prism with a trapezoidal base $A\,E\,B\,F$, whose constant width is $B\,C = D\,E = b$, and if the force P acts at the extremity $D\,F$ of the same, we have to find only the minimum of $\frac{v^2}{x}$ in order to determine the section of rupture. Putting the height $D\,G = E\,F$ of the end $= h$ and the height $K\,U$ of the truncated portion $H\,K\,U = c$, and assuming, as previously, that the section of rupture $L\,M\,N$ is at a distance $U\,V = x$ from the extremity $D\,E\,F$, we obtain the height of this section

FIG. 404.



$$M\,L = v = h + \frac{x}{c}\,h = h\left(1 + \frac{x}{c}\right),$$

and we have therefore but to determine the minimum of the expression

$$\frac{v^2}{x} = \frac{h^2}{x}\left(1 + \frac{x}{c}\right)^2 = h^2\left(\frac{1}{x} + \frac{2}{c} + \frac{x}{c^2}\right),$$

or, since h and c are determined, only that of $\frac{1}{x} + \frac{x}{c^2}$

If we assume $x = c$, the latter expression becomes $= \frac{2}{c}$; but if we make x a little (x_1) greater or less than c, we obtain

$$\frac{1}{x} = \frac{1}{c \pm x_1} = \frac{1}{c\left(1 \pm \frac{x_1}{c}\right)} = \frac{1}{c}\left(1 \mp \frac{x_1}{c} + \frac{x_1^2}{c^2}\right) \text{ and}$$

$$\frac{x}{c^2} = \frac{c \pm x_1}{c^2} = \frac{1}{c} \pm \frac{x_1}{c^2},$$

consequently

$$\frac{1}{x} + \frac{x}{c^2} = \frac{2}{c} + \frac{x_1^2}{c^3},$$

or in any case greater than $\frac{2}{c}$. Hence $x = c$ gives the minimum required, I.E. the section of rupture $L\,M\,N$ is at a distance from the end $D\,E\,F$ equal to the height $K\,U = c$ or to the distance of the truncated edge $H\,K$ from the same end $D\,E\,F$ in the other direction.

The height of this section of rupture is

$$v = h + \frac{h}{c} \cdot c = 2\,h,$$

and consequently the *proof load* is

$$P = \frac{b\,(2\,h)^2}{c} \cdot \frac{T}{6} = \frac{4\,b\,h^2}{c} \cdot \frac{T}{6}.$$

For a *parallelopipedical girder*, which has the same length $l = c$, the same width b and equal volume $V = b\,h\,l$, the height is

$$h_1 = \frac{h + 2\,h}{2} = \frac{3}{2}\,h,$$

and consequently the proof load is

$$P = \frac{b\,h_1^2}{c} \cdot \frac{T}{6} = \frac{9}{4}\,\frac{b\,h^2}{c} \cdot \frac{T}{6},$$

and such a girder bears, therefore, but $\frac{9}{16}$ as much as the wedged-shape body just treated. If the body is a *truncated pyramid*, the edges $A\,E$, $B\,D$, etc., when sufficiently prolonged, cut each other in a point, and if we designate the height of the truncated portion by c, we have

$$M\,N = u = b\left(1 + \frac{x}{c}\right) \text{ and } L\,M = v = h\left(1 + \frac{x}{c}\right).$$

and therefore the minimum of

$$\frac{u\,v^2}{x} = \frac{b\,h^2}{x}\left(1 + \frac{x}{c}\right)^3$$

or of

$$\frac{1}{x} + \frac{3\,x}{c^2} + \frac{x^2}{c^3}$$

must be determined, in order to find the section of rupture. By the differential calculus we obtain

$$x = \tfrac{1}{2}\,c,$$

and we can easily satisfy ourselves that this value is correct by first substituting $x = \frac{1}{2} c + x_1$ and then $x = \frac{1}{2} c - x_1$. In both cases we obtain a greater value than

$$\frac{2}{c} + \frac{3}{2c} + \frac{1}{4c} = \frac{15}{4c}, \text{ which is the value}$$

the expression

$$\frac{1}{x} + \frac{3x}{c^2} + \frac{x^2}{c^3} \text{ assumes for } x = \tfrac{1}{2} c.$$

The distance of the section of rupture from the end $D F$ is then equal to half the height c of the portion of the pyramid, which is cut off. The dimensions of this surface are

$$u = b\,(1 + \tfrac{1}{2}) = \tfrac{3}{2}\,b \text{ and } v = \tfrac{3}{2}\,h,$$

and, consequently, the required proof load of the beam is

$$P = \frac{\frac{3}{2}\,b\,(\frac{3}{2}\,h)^2}{\frac{1}{2}\,c}\,\frac{T}{6} = \frac{27}{4}\,\frac{b\,h^2}{c}\,\frac{T}{6}.$$

For a body, the form of which is a *truncated cone*, we have, when the radius of extremity is r and the height of the truncated portion is c, the radius of the section of rupture $r_1 = \frac{3}{2}\,r$, and therefore

$$P = \frac{27}{4} \cdot \frac{\pi\,r^3}{c} \cdot \frac{T}{4}.$$

§ 253. Bodies of Uniform Strength.—If a body is so bent, that the maximum strain S upon the extended and compressed side of the neutral axis is at all points the same, we have *a body of the strongest form*, or of *uniform strength* (Fr. corps d'égale résistance, Ger. Körper von gleichem Widerstande). By a certain force such a body is strained to the limit of elasticity in all its cross-section at the same time, and has, therefore, in each part a cross-section corresponding to its proof strength; it requires, therefore, when the other circumstances are the same, a smaller quantity of material than any other body of the same strength. Therefore, for the sake of economy and to avoid unnecessary weight, such forms are to be preferred in construction.

Since the greatest strain in a cross-section is determined by the expression

$$S = \frac{P\,x\,e}{W} \text{ (see § 251)},$$

a body of *uniform strength requires that* $\frac{P\,x\,e}{W}$ *shall be constant for all cross-sections of the body.*

If the force P is constant and applied at the end of the body, we have only to make

$$\frac{e\,x}{W} \text{ or } \frac{W}{e\,x}$$

constant, and when the force $Q = q\,x$ is uniformly distributed upon the girder.

$$\frac{e\,x^2}{W} \text{ or } \frac{W}{e\,x^2}$$

must be constant. For a girder with a *rectangular cross-section* (see § 251), whose dimensions are u and v, we must make in the first case $\frac{u\,v^2}{x}$, and in the second $\frac{u\,v^2}{x^2}$ constant.

If at another place at the distance l from the extremity the width is b and the height h, we must have consequently in the first case $\frac{u\,v^2}{x} = \frac{b\,h^2}{l}$, and in the second $\frac{u\,v^2}{x^2} = \frac{b\,h^2}{l^2}$

For the constant width $u = b$, we have in the first case

$$\frac{v^2}{x} = \frac{h^2}{l}, \text{ I.E.,}$$

$$\frac{v^2}{h^2} = \frac{x}{l} \text{ or } \frac{v}{h} = \sqrt{\frac{x}{l}}$$

Since the equation $\frac{v^2}{h^2} = \frac{x}{l}$ is that of a parabola (see § 35, Remark), the longitudinal profile $A\ B\ E$, Fig. 405, of such a body has the form of a parabola, whose vertex E coincides with the extremity or point of application of the load P.

FIG. 405.

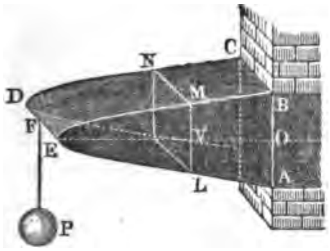
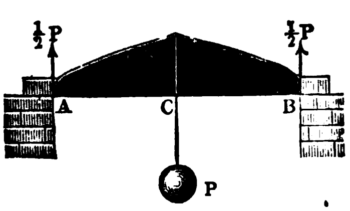


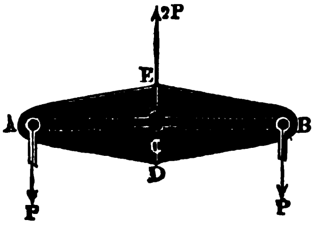
FIG. 406.



If a beam $A\ B$, Fig. 406, *whose width is constant*, is supported at both ends and sustains the load P in the middle, or if the beam

A B, Fig. 407, is supported in the middle and is acted upon at its ends *A* and *B* by two forces, which balance each other, its elevation must have the form of two parabolas united in the middle. As examples of the latter case, we may mention working beams, balance beams, etc. As the beam is weakened by the eyes, made for the shafts *A*, *B* and *C*, lateral or central ribs are added to it.

FIG. 407.



If the *height* $v = h$ is *constant*, we have

$$\frac{u}{x} = \frac{b}{l} \text{ or } \frac{u}{b} = \frac{x}{l},$$

and the width is proportional to the distance from the end; the horizontal projection of the beam *A C E*, Fig. 408, is a triangle *B C D* and the entire girder is a *wedge*, the vertical edge of which coincides with the direction of the force.

FIG. 408.

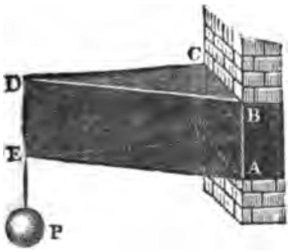
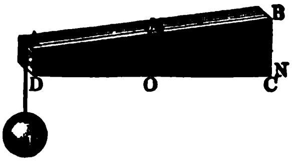


FIG. 409.



Instead of the parabolic girders, Fig. 405, we generally make use of girders, Fig. 409, with plane surfaces. In order to economize as much material as possible the girder is made in the middle *M* of the same height $M\,O = h_m = h\sqrt{\frac{1}{2}}$, as the parabolic girder would have been, and the limiting plane surface *C D* is made tangent to the corresponding parabolic surface. We have

$$\frac{B\,C}{M\,O} = \frac{3\,A\,M}{2\,A\,M} = \frac{3}{2} \text{ and } \frac{A\,D}{M\,O} = \frac{A\,M}{2\,A\,M} = \frac{1}{2}$$

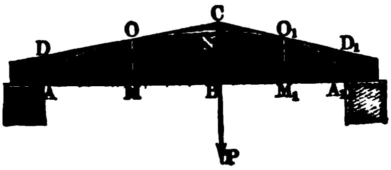
and consequently, if we denote the greater height *B C* by h_1 and the lesser one *A D* by h_2, we obtain

$$h_1 = \frac{3}{2}\,h_m = \frac{3}{2}\,h\sqrt{\frac{1}{2}} = 1{,}0607\,h \text{ and}$$
$$h_2 = \frac{1}{2}\,h_m = \frac{1}{2}\,h\sqrt{\frac{1}{2}} = 0{,}3536\,h,$$

for which we must determine the height $B N = h$ by means of the well-known formula $P l = b h^2 \frac{T}{6}$.

The volume of such a girder, whose faces are planes, is $\frac{b l (h_1 + h_2)}{2} = 0{,}7071\, b l h$, while that of the parabolic girder of equal strength is $= \frac{2}{3} b l h = 0{,}667\, b l h$, I.E., 5,7 per cent. smaller.

FIG. 410.



In like manner we can construct the girder $A N A_1$, Fig. 410, which is supported at its extremities A and A_1, of two portions, bounded by plane surfaces, which have a common height $B C = h_1 = 1{,}0607\, h$ at the point of application of the load, and at the extremities the altitude

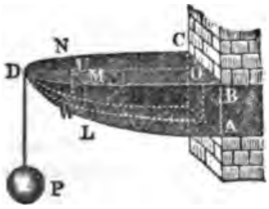
$$\overline{A D} = \overline{A_1 D_1} = h_0 = 0{,}3536\, h.$$

Here the altitude $B N = h$ must be determined by the formula

$$\frac{P l_1 l_2}{l} = \frac{b h^2 T}{6}$$

§ **254**. If the body $A B D$, Fig. 411, is to be made with all its cross-sections $L M N$, $A B C$, etc., *similar*, we must put

FIG. 411.



$$\frac{v}{h} = \frac{u}{b} \text{ and therefore}$$

$$\frac{u \,.\, u^2 h^2}{b^2 x} = \frac{b h^2}{l},$$

I.E., $$\frac{u^3}{b^3} = \frac{x}{l}, \text{ or } \frac{u}{b} = \frac{v}{h} = \sqrt[3]{\frac{x}{l}}.$$

The width and height are therefore proportional to the cube root of corresponding arms of the lever. When the distance from the end becomes eight-fold, the height and width are only doubled.

We can replace this body *by a truncated pyramid* $A C E G$, Fig. 412, at the middle of whose length the height is $M O = h_m = \sqrt[3]{\frac{1}{2}} \,.\, h = 0{,}7937\, h$ and the width $M N = b_m = \sqrt[3]{\frac{1}{2}} \,.\, b = 0{,}7937\, b$ and the strength of this body is *exactly the same* as that of the body just discussed. For the tangential angle of the curve $\frac{v}{h} = \sqrt[3]{\frac{x}{l}}$, or $v = \frac{h}{\sqrt[3]{l}} x^{\frac{1}{3}}$, we have, according to Art. 10 of the Introduction to the Calculus, $tang.\, a = \frac{h}{3\sqrt[3]{l}} x^{-1} = \frac{h}{3\sqrt[3]{l x^2}}$, therefore it follows, that for

$$\frac{x}{l} = \tfrac{1}{2},\ \tfrac{1}{2}\, l\ tang.\, a = \tfrac{1}{6}\, h \sqrt[3]{\left(\frac{l}{x}\right)^2} = \tfrac{1}{6}\, h \sqrt[3]{4} = \frac{h}{3} \sqrt[3]{\tfrac{1}{2}} = 0{,}2646\ h,$$

and in like manner we have for the curve

$$\frac{u}{b} = \sqrt[3]{\frac{x}{l}},\ tang.\, \beta = \frac{b}{3\sqrt[3]{l\, x^2}} \text{ and}$$

$$\tfrac{1}{2}\, l\ tang.\, \beta = \frac{b}{3} \sqrt[3]{\tfrac{1}{2}}.$$

From this we can calculate the dimensions of the base $A\ B\ C$

$A\ B = h_1 = h_m + \frac{1}{2}\, l\ tang.\, a = \frac{4}{3} \sqrt[3]{\frac{1}{2}} \,.\, h = 1{,}0583\ h$ and

$B\ C = b_1 = b_m + \frac{1}{2}\, l\ tang.\, \beta = \frac{4}{3} \sqrt[3]{\frac{1}{2}} \,.\, b = 1{,}0583\ b,$

and those of the smaller base $E\ F\ G$

FIG. 412.

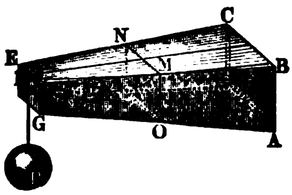
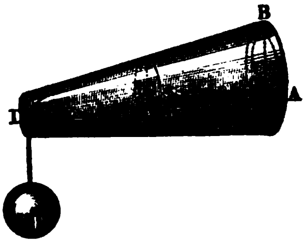


FIG. 413.



$F\ G = h_2 = h_m - \frac{1}{2}\, l\ tang.\, a = \frac{2}{3} \sqrt[3]{\frac{1}{2}} \,.\, h = 0{,}5291\ h$ and

$E\ F = b_2 = b_m - \frac{1}{2}\, l\ tang.\, \beta = \frac{2}{3} \sqrt[3]{\frac{1}{2}} \,.\, b = 0{,}5291\ b.$

We must of course put $P\, l = \frac{b\, h^2\, T}{6}$.

If we make the cross-section of the body of uniform strength *circular*, we have for the variable radius the equation

$$u = v = z = \sqrt[3]{\frac{x}{l}},$$

and if we replace this body by a truncated cone $A\ B\ E$, Fig. 413, its radii must be

$$M\ O = r_m = \sqrt[3]{\tfrac{1}{2}} \,.\, r = 0{,}7937\ r,\ C\ A = r_1 = 1{,}0583\ r \text{ and}$$
$$D\ E = r_2 = 0{,}5291\ r,$$

and the radius r of the base of the solid of uniform strength must be calculated according to the formula

$$P\, l = \frac{\pi\, r^3}{4}\, T.$$

If the girder is *uniformly loaded* and its *width is constant*, I.E. if $u = b$, we have

$$\frac{v^2}{h^2} = \frac{x^2}{l^2}, \text{ or}$$

$$\frac{v}{h} = \frac{x}{l},$$

and its form must be that of a wedge, whose elevation is a triangle $A\ B\ D$, Fig. 414.

FIG. 414.

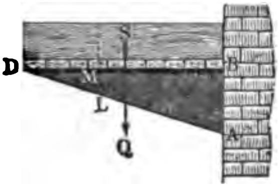
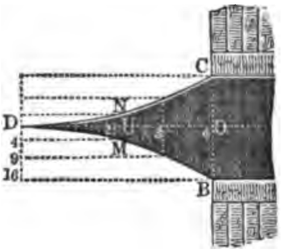


FIG. 415.

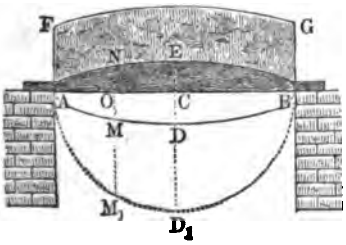


If the height is *constant*, we have $\frac{u}{b} = \frac{x^2}{l^2}$; hence the horizontal section of the girder is a surface limited by the two inverted arcs of a parabola $B\ D$ and $C\ D$, as is shown in Fig. 415.

If we again make the cross-sections similar, we have $\frac{u^3}{b^3} = \frac{v^3}{h^3} = \frac{x^2}{l^2}$, and the vertical and horizontal profiles are *cubic parabolas*, the cubes of the ordinates of which are proportional to the squares of the abscissas.

If a body $A\ E\ B$, Fig. 416, supported at both ends, is uniformly loaded with the weight q per running foot or upon its whole length $A\ B = l$ with $Q = q\ l$, we have the moment of the force at a point O, situated at the distance $A\ O = x$ from one of the supports A,

FIG. 416.



$$\frac{Q}{2} \cdot x - q\,x \cdot \frac{x}{2} = \frac{q}{2}\,(l\,x - x^2),$$

and, on the contrary, at the centre C

$$= \frac{Q}{2} \cdot \frac{l}{2} - \frac{Q}{2} \cdot \frac{l}{4} = \frac{Q\,l}{8} = \frac{q\,l^2}{8}.$$

Assuming the width b of the body to be constant, we have

$$b\, v^2 . \frac{T}{6} = \frac{q}{2}(l\, x - x^2) \text{ and}$$

$$b\, h^2 . \frac{T}{6} = \frac{q\, l^2}{8},$$

h denoting the height $C\,E$ of the body at the centre, and by division we obtain

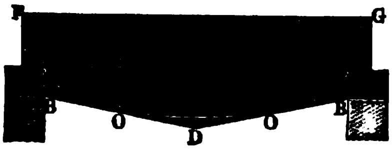
$$\frac{v^2}{h^2} = \frac{l x - x^2}{\frac{1}{4}\, l^2}, \text{ or}$$

$$v^2 = \left(\frac{h}{\frac{1}{2}\, l}\right)^2 (l\, x - x^2).$$

If $h = \frac{1}{2}\, l$, v^2 would be $= l\, x - x^2$, and therefore the longitudinal profile would be the circle $A\ D_1\ B$, described with the radius $\frac{1}{2}\, l$; but since $l\, x - x^2$ must be multiplied by $\left(\frac{h}{\frac{1}{2}\, l}\right)^2$ in order to obtain the square v^2 of the height $M\,O = N\,O$ at any point, the circle becomes an ellipse $A\ D\ B$ or $A\ E\ B$, whose semi-axes are $C\,A = a_1 = \frac{1}{2}\, l$ and $C\,D = C\,E = b_1 = h$.

We can replace this body by a girder $A\ A\ B\ D\ B$, Fig. 417, with *plane surfaces*, whose height at the distance $A\,M = \frac{1}{4}\, l$ from the points of support B and B is $M\,O = h_m$ $= \frac{h}{\frac{1}{2}\, l}\sqrt{\frac{1}{4}\, l^2 - \frac{1}{16}\, l^2} = \frac{1}{2}\sqrt{3} . h$. The angle of inclination α of the surface $B\,D$ to the axis $A\ C$ is given by the equation

FIG. 417.



$$tang.\, \alpha = \frac{h}{\frac{1}{2}\, l} . \frac{\frac{1}{2}\, l - x}{\sqrt{l x - x^2}} = \frac{2\, h}{l} . \frac{\frac{1}{4}\, l}{\sqrt{\frac{3}{16}\, l^2}} = \frac{2\, h}{l\sqrt{3}} = \frac{2}{3}\sqrt{3} . \frac{h}{l};$$

consequently we have $\frac{l}{4}\ tang.\, \alpha = \frac{1}{6}\sqrt{3} . h$ and the height of the body in the middle

$$C\,D = M\,O + \frac{l}{4}\, tang.\, \alpha = \frac{2}{3}\sqrt{3} . h = 1{,}1548\, h,$$

and, on the contrary, the height at the ends is

$$A\,B = M\,O - \frac{l}{4}\, tang.\, \alpha = \frac{1}{3}\sqrt{3} . h = 0{,}5774\, h.$$

(§ **255**.) The deflection of a body of uniform strength is, of course, under the same circumstances, greater than that of a prismatical girder. For the case, where the beam is fixed at one end and subjected to a stress P at the other, the deflection is found as follows. The well-known proportion $\frac{r}{e} = \frac{E}{T}$ gives us the formula $r = \frac{E\,e}{T}$, in which the radius of curvature is a function of the distance e. If we know the dependence of e and x upon each other, we obtain an equation between r and x, from which we can deduce (in the way explained in § 218) the equation of the co-ordinates of the *elastic curve*. If we assume the deflection to be small, we can again put the length of arc s equal to the abscissa x, and consequently equate the differentials $d\,s$ and $d\,x$; hence we can, as before, assume

$$r = -\frac{d\,x}{d\,a}.$$

From this we obtain

$$d\,x = -\frac{E}{T}\,e\,d\,a,$$

and, by integration, the tangential angle

$$a = -\frac{T}{E}\int\frac{d\,x}{e}.$$

For a girder with a rectangular cross-section $e = \frac{1}{2}\,v$, and therefore

$$a = -\frac{2\,T}{E}\int\frac{d\,x}{v}.$$

If the width is constant or $u = b$, we have

$$\frac{v^2}{h^2} = \frac{x}{l} \text{ (see § 253), and therefore}$$

$$v = h\sqrt{\frac{x}{l}} \text{ and}$$

$$a = -\frac{2\,T}{E}\cdot\frac{\sqrt{l}}{h}\int x^{-\frac{1}{2}}\,d\,x = -\frac{2\,T}{E}\cdot\frac{\sqrt{l}}{h}\cdot 2\sqrt{x} + Cons.,$$

or, since for $x = l$, $a = 0$ and consequently

$$Con. = \frac{2\,T}{E}\,\frac{\sqrt{l}}{h}\cdot 2\sqrt{l},$$

$$a = \frac{4\,T}{E}\,\frac{\sqrt{l}}{h}\,(\sqrt{l} - \sqrt{x}).$$

If we put $a = \frac{d\,y}{d\,x}$, we obtain

$$d\,y = \frac{4\,T}{E}\,\frac{\sqrt{l}}{h}\,(\sqrt{l} - \sqrt{x})\,d\,x,$$

and, therefore, the required equation of the co-ordinates is

$$y = \frac{4\,T}{E}\,\frac{\sqrt{l}}{h}\,(x\sqrt{l} - \tfrac{2}{3}\,x\sqrt{x}) = 4\,\frac{T}{E}\,\frac{\sqrt{l}}{h}\,(\sqrt{l} - \tfrac{2}{3}\sqrt{x})\,x.$$

For $x = l$, y becomes a; the deflection is then

$$a = \tfrac{4}{3} \frac{T l^2}{E h}.$$

But $P l = b h^2 . \frac{T}{6}$ or $T = \frac{6 P l}{b h^2}$, and, therefore, the deflection is given by the formula

$$a = \frac{8 P l^3}{E b h^3} = 2 . \frac{4 P l^3}{E b h^3},$$

I.E., it is twice as great as in the case of a parallelopipedical girder, whose height is h and whose width is b (compare § 227).

If the force acts at the middle of a girder, supported at both ends, we have only to substitute $\frac{P}{2}$ for P, and $\frac{l}{2}$ for l, and we obtain

$$a = \tfrac{1}{16} \cdot \frac{8 P l^3}{E b h^3},$$

I.E., it is 16 times smaller than when the force acts at the end.

For a body of uniform strength with a *triangular base*, as is represented in Fig. 408, the variable width is $u = \frac{x}{l} b$, and

$$P r x = \frac{u h^3}{12} E = \frac{b h^3 x}{12 l} E;$$

hence the radius of curvature $r = \frac{b h^3}{12 l} \cdot \frac{E}{P}$ is *constant*, the curve formed is a circle, and the corresponding deflection is

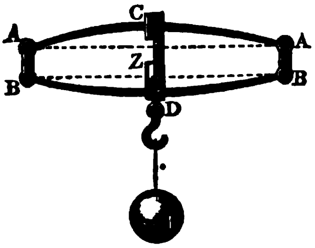
$$a = \frac{l^2}{2 r} = \frac{6 P l^3}{b h^3 E} = \tfrac{3}{2} \cdot \frac{4 P l^3}{b h^3 E},$$

I.E., $\frac{3}{2}$ times as great as for a parallelopipedical girder.

§ 256. Deflection of Metal Springs.—The most common examples of bodies of uniform strength, as well as of those which bend in a circle, are steel or other metal springs. The springs, of which the spring dynamometers are made, are of the finest steel and are from $\frac{1}{2}$ to 1 meter long, from 4 to 5 centimeters wide and in the middle from 8 to 21 millemeters thick. They form bodies of uniform strength, and their longitudinal profile is composed of two parabolas united in the middle (see § 253). In order to increase the action, the spring dynamometer is made of two such parabolic springs $A\ A$ and $B\ B$, Fig. 418, which are united at their ends A

by means of the links *A B*, *A B* (see Morin's Leçons de Mécanique Pratique, Résistance des Matériaux, No. 198). These dynamometers measure the force *P*, which is applied to the hook *D* in the middle of one of the springs, by the space described by the point *Z*, which is of course equal to the sum of the deflections of the two springs. But from what precedes we know that

FIG. 418.



$$a = \tfrac{1}{16} \cdot \frac{8\,P\,l^3}{b\,h^3\,E},$$

and consequently we have here

$$s = 2\,a = \frac{P\,l^3}{b\,h^3\,E},$$

and, therefore, the force

$$P = \left(\frac{b\,h^3\,E}{l^3}\right) s,$$

corresponding to the space *s* described by the pointer.

In experimenting with such an instrument, whose springs were of the following dimensions: $b = 0{,}05$, $h = 0{,}0211$, $l = 1{,}0$ meter, the space described by the pointer was $s = 9{,}7$ millemeter, when the load was $P = 1000$ kilograms; the coefficient of this dynameter was therefore

$$\frac{b\,h^3\,E}{l^3} = \frac{P}{s} = \frac{1000}{9{,}7} = 103{,}09,$$

and for other cases we must put

$$P = 103{,}09\ s \text{ kilograms},$$

when *s* is given in millimeters, or when the scale is divided into millimeters.

If, instead of parabolic springs, we employ triangular ones of uniform strength, we have

$$\frac{s}{2} = a = \tfrac{1}{16} \cdot \frac{6\,P\,l^3}{b\,h^3\,E}, \text{ and, therefore,}$$

$$P = \tfrac{4}{3}\left(\frac{b\,h^3\,E}{l^3}\right) s,$$

I.E., one-third greater than for a dynamometer with parabolic springs.

Wagon springs should unite great flexibility with great strength, while, on the contrary, it is not necessary to know the exact relation between *P* and *s*. For this reason, these springs are often formed of a number of simple springs laid upon one another.

If the compound spring is composed of n simple *parallelopipedical springs*, placed upon one another, we have, when the width is b, the thickness h and the length l, the deflection corresponding to the force P at the end A of the entire spring $a = \frac{4\,P\,l^3}{n\,E\,b\,h^3}$ and the proof load

$$P = n\,\frac{b\,h^2}{l}\,\frac{T}{6}, \text{ and therefore also}$$

$$a = \tfrac{2}{3}\,\frac{T}{E}\,\frac{l^2}{h} \text{ or } \frac{a}{l} = \tfrac{2}{3}\,\frac{T}{E}\,\frac{l}{h}.$$

If the entire spring $A\,C\,D$, Fig. 419, consists of n simple *triangular* springs, we have

$$a = \frac{6\,P\,l^3}{n\,E\,b\,h^3}, \text{ while } P = n\,\frac{b\,h^2}{l}\,\frac{T}{6}$$

remains unchanged, and therefore

$$a = \frac{T}{E}\,\frac{l^2}{h} \text{ or } \frac{a}{l} = \frac{T}{E}\,\frac{l}{h}.$$

Therefore, in both cases the measure $\frac{a}{l}$ of the flexibility increases with the ratios $\frac{T}{E}$ and $\frac{l}{h}$ and is the same as for a simple spring of n times the width $(n\,b)$.

FIG. 419.

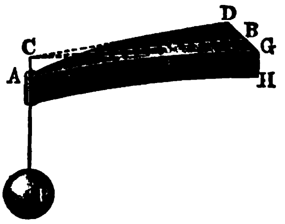
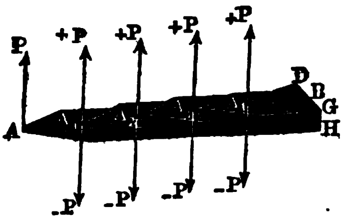


FIG. 420.



In order to economize material, we superpose springs of different lengths and construct them of such a shape, that by the action of the force P at the end A of the entire spring they are bent in arcs of circles of nearly or exactly the same radius. The force P bends the lowest triangular piece $A\,A$ of the the entire spring $A\,B\,H$, Fig. 420, whose length $= \frac{l}{n}$, in the arc of a circle, whose radius is $r = n\,\frac{b\,h^3}{12\,l}\cdot\frac{E}{P}$, and in order that the remaining parallelopipedical portion shall be bent in like manner, it is necessary that the same shall exert a pressure at A upon the succeeding spring, which shall be equal to the force P; for the moment of flexure of this spring is then equal to the moment $\frac{P l}{n}$ of a couple $(P, -P)$ whose arm is $\frac{l}{n}$. The relations of the flexure of the first spring repeat themselves in the second, which is $\frac{l}{n}$ shorter than it; it is bent in a circle whose radius $r = \frac{n b h^3}{12 l} \cdot \frac{E}{P}$, when its end $A_1 A_2$ is triangular and the other portion is parallelopipedical, and if it presses on the third spring with a force P. This is also the case for the third spring $A_2 G D$, etc., up to the last piece, which has no parallelopipedical portion, and which, by the action of the force P, is bent in a circle of the above radius r. The entire deflection of this compound spring is $a = \frac{l^2}{2 r} = \frac{6 P l^3}{n E b h^3}$, and the proof load is $P = n \frac{b h^2}{l} \frac{T}{6}$, hence

$$a = \frac{T}{E} \cdot \frac{l^2}{h}, \text{ or } \frac{a}{l} = \frac{T}{E} \frac{l}{h}.$$

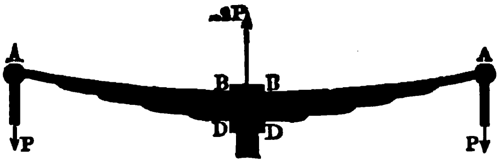
The relations of the flexure are here exactly the same as for a spring composed of single triangular springs; it can also easily be proved, that both sets of springs require the same amount of material.

It is not, however, necessary to make the ends of the springs exactly triangular; we can employ any other form of equal curvature, E.G., we can make them of the constant width b and then at the distance x from the end A the height must be

$$y = h \sqrt[3]{\frac{n x}{l}}$$

Such a double spring is represented in Fig. 421. Here the

FIG. 421.



total proof load is $2 P$; the length must not, however, be measured from the middle, but from the ends $B D$, $B D$ of the fastening.

REMARK.—The reader can consult upon the subject of wagon springs: F. Reuleux: Die Construction und Berechnung der für den Maschinenbau wichtigsten Federarten. Winterthur, 1857; also Redtenbacher: die Gesetze des Locomotivenbaues, Mannheim 1855, and Philips: Mémoire sur les ressorts en acier, etc., in the Annales des Mines, Tome I., 1852.

CHAPTER III.

THE ACTION OF THE SHEARING ELASTICITY IN THE BENDING AND TWISTING OF BODIES.

§ 257. The Shearing Force Parallel to the Neutral Axis.—In a body, which is subjected only to a tensile or compressive force, the bases *A B* and *C D* of an element *A B C D* of the body, Fig. 422, are only acted upon by the two opposite forces *P* and $-$ *P*, which balance each other, while the sides *A B* and *C D* remain free from the action of extraneous forces; for the neighboring elements of the body are subjected to the same axial strain as the supposed element *A B C D* itself. But the case is different when the body is bent; for on one side *A B* of the element *A B C D* a strain is produced which is opposite in direction to that upon the other side *C D* of the element, and in consequence of the cohesion

FIG. 422.

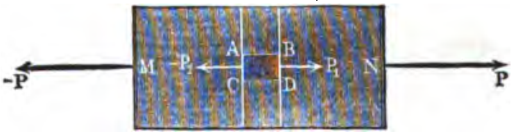
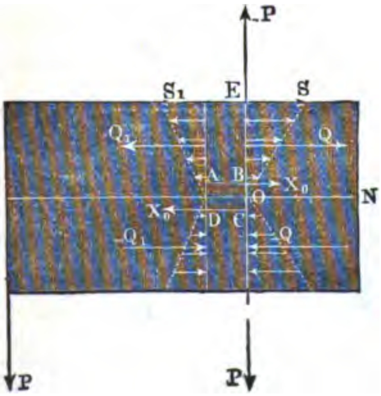


FIG. 423.



in $A\ B$ and $C\ D$, the element $A\ B\ C\ D$ is subjected to the action of a couple. This couple is a maximum for an element which lies in the neutral axis; for the element is here subjected on the side $A\ B$ to an extension, and on the side $C\ D$ to a compression.

If S is the strain upon a fibre at the distance e from the neutral axis, when the cross-section $= 1$, the strains upon the portions $F_1, F_2, F_3 \ldots$ of the entire cross-section, which are situated at the distances $z_1, z_2, z_3 \ldots$ from the neutral axis, are

$$\frac{F_1\, z_1}{e}\, S,\ \frac{F_2\, z_2}{e}\, S,\ \frac{F_3\, z_3}{e}\, S, \text{ etc.},$$

and the total strain in the cross-section $F_1 + F_2 + F_3 \ldots$ is

$$Q = \frac{S}{e}\,(F_1\, z_1 + F_2\, z_2 + \ldots) = \frac{S}{e}\, \Sigma\, (F z).$$

Now if $F_1 + F_2 + \ldots$ is the part of the cross-section on one side of the neutral axis, Q is the total strain on that side of the neutral axis. The strain on the other side is, according to the theory of the centre of gravity (compare § 215), equal in intensity to it, but opposite in direction.

Besides we have, according to § 235, $S = \frac{P\, x\, e}{W}$, or $\frac{S}{e} = \frac{P\, x}{W}$, whence also $Q = \frac{P\, x}{W}\,(F_1\, z_1 + F_2\, z_2 + \ldots)$.

In a cross-section, which is at a distance $A\ B = x_1$ from the first one, the strain is

$$Q_1 = \frac{P\,(x - x_1)}{W}\,(F_1\, z_1 + F_2\, z_2 + \ldots),$$

and therefore the total force with which the piece $A\ B\ E$ tends to slide upon $A\ B$ is

$$Q - Q_1 = \frac{P\, x_1}{W}\,(F_1\, z_1 + F_2\, z_2 + \ldots).$$

Now if b_0 is the width of the cross-section at the neutral axis, the shearing force along the unit of surface in this axis is

$$X_0 = \frac{Q - Q_1}{b_0\, x_1} = \frac{P}{b_0\, W}\,(F_1\, z_1 + F_2\, z_2 + \ldots) = \frac{P\, \Sigma\, (F z)}{b_0\ W}.$$

If, therefore, the girder is not to be ruptured by a sliding along the neutral axis, we must put $X_0 =$ the modulus of ultimate strength, and in order that it shall be as secure against rupture by shearing as against breaking across, it is necessary that X_0 shall be at most equal to the modulus of proof strength T, I.E. that

$$T = \frac{P}{b_0 W} \Sigma (F z), \text{ or } P = \frac{b_0 W T}{\Sigma (F z)}, \text{ and}$$

$$b_0 = \frac{P}{T W} \Sigma (F z).$$

$\Sigma (F z)$ is also $= F_1 s_1 = F_2 s_2$, when F_1 and F_2 denote the areas of the portions of the entire cross-section $F = F_1 + F_2$, lying on the opposite sides of the neutral axis, and s_1 and s_2 the distances of the centres of gravity of the two portions from that axis.

For a *rectangular girder*, whose cross-section $F = b h$, we have $\Sigma (F z) = F_1 s_1 = \frac{b h}{2} \cdot \frac{h}{4} = \frac{b h^2}{8}$, $W = \frac{b h^3}{12}$, and $b_0 = b$, whence $P = \frac{2}{3} b h T$ and $b_0 = b = \frac{3}{2} \frac{P}{T h}$.

For a *cylindrical girder*, whose cross-section is $F = \frac{\pi d^2}{4}$, we have, since the centre of gravity is situated at a distance $\frac{2}{3 \pi} d$ from the centre,

$$\Sigma (F z) = F_1 s_1 = \frac{\pi d^2}{8} \cdot \frac{2}{3 \pi} d = \frac{d^3}{12}, \text{ and, according to § 232,}$$

$$W = \frac{\pi d^4}{64}, \text{ and } b_0 = d, \text{ whence}$$

$$P = \frac{\pi d^5}{64 \cdot \frac{1}{12} d^3} T = \frac{3 \pi}{16} d^2 T, \text{ and}$$

$$d = 4 \sqrt{\frac{P}{3 \pi T}} = 1{,}303 \sqrt{\frac{P}{T}}.$$

In like manner for an *elliptical girder*, since $W = \frac{\pi a^3 b}{4}$, $F_1 s_1 = \frac{\pi a b}{2} \cdot \frac{2}{\pi} \cdot \frac{2}{3} a = \frac{2}{3} a^2 b$ and $b_0 = 2 b$, we have $P = \frac{3}{4} \pi a b T$, or $b = \frac{4}{3 \pi} \frac{P}{a T} = 0{,}4244 \frac{P}{a T}$.

Finally, for a *tubular parallelopipedical girder*, whose cross-section is $F = b h - b_1 h_1$ (Fig. 354, § 228), we have

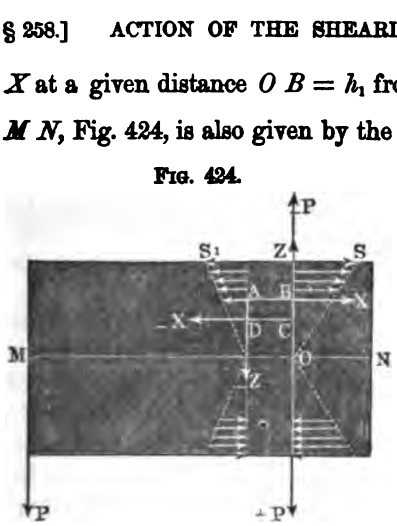
$$F_1 s_1 = \frac{b h^2 - b_1 h_1^2}{8}, \; W = \frac{b h^3 - b_1 h_1^3}{12} \text{ and } b_0 = b - b_1,$$

hence $$P = \frac{2}{3} \frac{(b - b_1)(b h^3 - b_1 h_1^3) T}{b h^2 - b_1 h_1^2}.$$

The shearing force X diminishes as the distance of the surface, in which it exists, from the neutral axis increases, and becomes finally null at the surface of the body, where the distance from the neutral axis is a maximum. The intensity of the shearing force

X at a given distance $O B = h_1$ from the neutral axis of the body $M N$, Fig. 424, is also given by the formula $X = \frac{P \Sigma (F z)}{b_0 W}$ found above, if instead of $\Sigma (F z)$ we substitute the sums of the products $F_1 z_1, F_2 z_2 \ldots$ on one side of $A B C D$, and instead of b_0 the width b_1 of the surface at the given distance h_1. The sums of the products $F_n z_n, F_{n+1} z_{n+1}$ for the other side is, however, equal to the sum of the products $F_1 z_1, F_2 z_2 \ldots$ since the products of the elements, situated on the opposite sides of the neutral axis within the distance $\pm h$, balance each other.

FIG. 424.



E.G. if the cross-section of a girder is rectangular, we have for the point situated midway between the neutral axis and the limiting surfaces, I.E., at the distance $\frac{h}{4}$ from the neutral axis

$$\Sigma (F z) = F_1 s_1 = \frac{b h}{4} \cdot \frac{3}{8} h = \frac{3}{32} b h^2,$$

and, therefore, the shearing force is

$$X = \frac{P \cdot \frac{3}{32} b h^2}{b \cdot \frac{b h^3}{12}} = \frac{9}{8} \frac{P}{b h},$$

while at the neutral axis its value is $X_0 = \frac{3}{2} \frac{P}{b h}$.

§ 258. The Shearing Force in the Plane of the Cross-section.—As the tensile and compressive forces of the ends of an element $A B C D$, Fig. 424, are in equilibrium, so also the shearing forces in this element, which form two couples, balance each other. Now if ξ is the length $A B$ and ζ the height $B C$ of the element, we have the shearing forces along $A B$ and $C D$, ξX and $- \xi X$, and the moment of the couple, formed by them, $\xi X . \zeta = \xi \zeta X$, and the shearing forces along $B C$ and $D A$ are ζZ and $- \zeta Z$, and the moment of the couple formed by the latter is $= \zeta Z . \xi = \xi \zeta Z$; now if equilibrium exists, we must have $\xi \zeta X = \xi \zeta Z$, I.E., that $X = Z$.

The formula $X = \frac{P \Sigma (F z)}{b W}$ is, therefore, also applicable to the determination of the *shearing force Z* along the *entire cross-section.* It is, E.G., in a girder with a rectangular cross-section, for an element in the neutral axis $= \frac{4}{3} \frac{P}{b h}$, and for one at a distance $\pm \frac{1}{4} h$ from the neutral axis $= \frac{9}{8} \frac{P}{b h}$, etc.

The sum of the shearing forces along the entire cross-section, must of course be equal to the force P, or, if several forces act at right angles to the axis of the beam, equal to the sum $\Sigma (P)$ of these forces. This can be proved as follows: if we divide the maximum distance e of the elements of the surface from the neutral axis into n equal parts, we can imagine the cross-section upon the corresponding side of the neutral axis to be composed of the strips $b_1 \frac{h}{n}$, $b_2 \frac{h}{n}$, $b_3 \frac{h}{n}$, etc., whose moments in reference to the neutral axis are

$$b_1 \left(\frac{h}{n}\right)^2, 2 b_2 \left(\frac{h}{n}\right)^2, 3 b_3 \left(\frac{h}{n}\right)^2, \text{ etc.,}$$

and the sum of the latter is

$$= \left(\frac{h}{n}\right)^2 (1 b_1 + 2 b_2 + 3 b_3 + 4 b_4 + \ldots).$$

In reference to the axis, which is at a distance $\frac{h}{n}$ from the neutral axis, the sum of these moments is

$$= \left(\frac{h}{n}\right)^2 (2 b_2 + 3 b_3 + 4 b_4 + \ldots),$$

in reference to the axis at the distance $2 \frac{h}{n}$, it is

$$= \left(\frac{h}{n}\right)^2 (3 b_3 + 4 b_4 + \ldots),$$

and therefore the sum of all these sums to the distance e is

$$= \left(\frac{h}{n}\right)^2 [b_1 + (2 + 2) b_2 + (3 + 3 + 3) b_3 + \ldots]$$

$$= \left(\frac{h}{n}\right)^2 (1^2 . b_1 + 2^2 . b_2 + 3^2 . b_3 + \ldots + n^2 b_n).$$

It follows that the sum of all the shearing forces along cross-section on one side of the neutral axis is

$$R_1 = X_1\, b_1 \left(\frac{h}{n}\right) + X_2\, b_2 \left(\frac{h}{n}\right) + X_3\, b_3 \left(\frac{h}{n}\right) + \ldots$$

$$= \frac{P}{W}\frac{h}{n} \text{ times the sum last found}$$

$$= \frac{P}{W}\left(\frac{h}{n}\right)^2 (1^2 . b_1 + 2^2 . b_2 + 3^2 . b_3 + \ldots + n^2 . b_n).$$

But the measure of the moment of flexure for this half of the cross-section is

$$W_1 = \Sigma\,(F z^2) = \frac{h}{n}\left[b_1 \left(\frac{h}{n}\right)^2 + b_2 \left(\frac{2\ h}{n}\right)^2 + b_3 \left(\frac{3\ h}{n}\right)^2 + \ldots\right]$$

$$= \left(\frac{h}{n}\right)^3 (1^2 . b_1 + 2^2 . b_2 + 3^2 . b_3 + \ldots + n^2 . b_n),$$

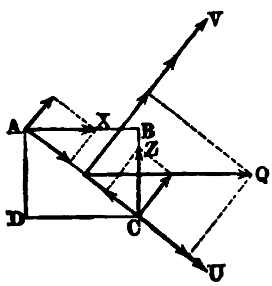
whence it follows, that the required shearing force along this surface is

$$R_1 = \frac{P\,W_1}{W}.$$

In like manner we find for the half of the cross-section, situated on the other side of the neutral axis, the shearing force $R_2 = \frac{P\,W_2}{W}$, and finally it follows that the shearing strain for the entire cross-section is $R = \frac{P\,(W_1 + W_2)}{W} = P$, since the measure W of the moment of flexure of the entire cross-section is equal to the sum $W_1 + W_2$ of measures of the moments of flexure of the two portions of it.

§ 259. **Maximum and Minimum Strain.**—If the strains in any section are known, the strain in any given cross-section can be found by employing the ordinary methods for the composition and decomposition of forces. In order to find the strains in an element $A\ C$, Fig. 425, of the surface, whose plane forms the variable angle $B\ A\ C = \psi$ with the longitudinal axis of the body, we decompose the tensions in the projections $A\ B$ and $B\ C$ of this element of the surface into two components, one of which acts in the plane of $A\ C$ and the other at right-angles to it, and we then combine the components in $A\ C$, so as to form a single shearing force, and the components, acting in a direction at right-angles to $A\ C$, so as to form a single tensile or compressive force. If the width of the elements $A\ B$, $B\ C$ and $A\ C$ of the surfaces is unity, we can put the shearing force along

FIG. 425.



$A\,B$, $= \overline{A\,B}\,.\,X$ and decompose it into its components $\overline{A\,B}\,.\,X$ $\cos.\,\psi$ and $\overline{A\,B}\,.\,X\sin.\,\psi$, and in like manner we can put the shearing force along $B\,C$, $= \overline{B\,C}\,.\,Z = \overline{B\,C}\,.\,X$ and decompose it into its components

$$-\,\overline{B\,C}\,.\,X\sin.\,\psi \text{ and } \overline{B\,C}\,.\,X\cos.\,\psi.$$

The components of the tensile force $\overline{B\,C}\,.\,Q = \overline{B\,C}\,.\,\frac{S\,z}{e}$, whose direction is perpendicular to $\overline{B\,C}$, on the contrary, are $\overline{B\,C}\,.\,Q\cos.\,\psi$ and $\overline{B\,C}\,.\,Q\sin.\,\psi$, and it follows that the entire shearing strain along $A\,C$ referred to the unit of surface is

$$U = (\overline{A\,B}\,.\,X\cos.\,\psi - \overline{B\,C}\,.\,X\sin.\,\psi + \overline{B\,C}\,.\,Q\cos.\,\psi) : A\,C,$$

and that the tensile strain at right-angles to $A\,C$ is for the unit of surface

$$V = (\overline{A\,B}\,.\,X\sin.\,\psi + \overline{B\,C}\,.\,X\cos.\,\psi + \overline{B\,C}\,.\,Q\cos.\,\psi) : A\,C.$$

But $\frac{A\,B}{A\,C} = \cos.\,\psi$ and $\frac{B\,C}{A\,C} = \sin.\,\psi$, whence it follows also that

$U = X\,(\cos.\,\psi)^2 - X\,(\sin.\,\psi)^2 + Q\sin.\,\psi\cos.\,\psi$ and

$U = 2\,X\sin.\,\psi\cos.\,\psi + Q\,(\sin.\,\psi)^2$, or, since

$(\cos.\,\psi)^2 - (\sin.\,\psi)^2 = \cos.\,2\,\psi$ and $2\sin.\,\psi\cos.\,\psi = \sin.\,2\,\psi$,

$$U = X\cos.\,2\,\psi + \tfrac{1}{2}\,Q\sin.\,2\,\psi = X\cos.\,2\,\psi + \frac{S\,z}{2\,e}\sin.\,2\,\psi \text{ and}$$

$$V = X\sin.\,2\,\psi + Q\,(\sin.\,\psi)^2 = X\sin.\,2\,\psi + \frac{S\,z}{2\,e}(1 - \cos.\,2\,\psi).$$

The strains in the surfaces $A\,D$ and $C\,D$, which together with the surfaces $A\,B$ and $C\,D$ fully limit the element $A\,B\,C\,D$, give, of course, equal and opposite shearing and tensile forces. On the contrary, for a similar element of the body upon the compressed side Q is negative, and therefore

$$U = X\cos.\,2\,\psi - \tfrac{1}{2}\,Q\sin.\,2\,\psi = X\cos.\,2\,\psi - \frac{S\,z}{2\,e}\sin.\,2\,\psi \text{ and}$$

$$V = X\sin.\,2\,\psi - \tfrac{1}{2}\,Q\,(1 - \cos.\,2\,\psi) = X\sin.\,2\,\psi - \frac{S\,z}{2\,e}(1 - \cos.\,2\,\psi).$$

In order now to find the values of the angle of inclination ψ, for which the shearing force U and the normal one V assume their maximum or minimum values, we substitute for ψ, $2\,\psi + \mu$, μ denoting a very small increment, and require that by it the corresponding values of U and V shall not be changed. For $U = X\cos.\,2\,\psi + \frac{1}{2}\,Q\sin.\,2\,\psi$, we obtain thus a second value

$$U_1 = X\cos.\,(2\,\psi + \mu) + \tfrac{1}{2}\,Q\sin.\,(2\,\psi + \mu)$$
$$= X\,(\cos.\,2\,\psi\cos.\,\mu - \sin.\,2\,\psi\sin.\,\mu) + \tfrac{1}{2}\,Q\,(\sin.\,2\,\psi\cos.\,\mu$$
$$+ \cos.\,2\,\psi\sin.\,\mu), \text{ or, since we can put } \cos.\,\mu = 1,$$

$U_1 = X \cos. 2\psi + \frac{1}{2} Q \sin. 2\psi - (X \sin. 2\psi - \frac{1}{2} Q \cos. 2\psi) \sin. \mu.$
Now if we put $U_1 = U$, we must have $X \sin. 2\psi - \frac{1}{2} Q \cos. 2\psi = 0$ and therefore

$$\sin. 2\psi = \frac{Q}{2X} \cos. 2\psi, \text{ I.E.,}$$

$$tang. 2\psi = \frac{Q}{2X} = \frac{Sz}{2Xe}.$$

From this it follows also that

$$\sin. 2\psi = \frac{Q}{\sqrt{Q^2 + 4X^2}} = \frac{Sz}{\sqrt{(Sz)^2 + (2Xe)^2}} \text{ and}$$

$$\cos. 2\psi = \frac{2Xe}{\sqrt{Q^2 + 4X^2}} = \frac{2Xe}{\sqrt{(Sz)^2 + (2Xe)^2}},$$

and that, finally, the required maximum value of the shearing force U is

$$U_m = \frac{2X^2 + \frac{1}{2}Q^2}{\sqrt{Q^2 + 4X^2}} = \sqrt{(\tfrac{1}{2}Q)^2 + X^2} = \sqrt{\left(\frac{Sz}{2e}\right)^2 + X^2}.$$

In the neutral axis Q is $= 0$, and therefore $U_m = X$ and $tang. 2\psi = 0$, I.E. $2\psi = 0$ and 180°, or $\psi = 0$ and 90°. For the most remote fibres, on the contrary, X is $= 0$ and $z = e$; therefore $U_m = \frac{Q}{2} = \frac{S}{2}$ and $tang. 2\psi = \infty$, or $2\psi = 90°$ and $\psi = 45$.

In passing from the neutral axis to the outmost fibre, the angles of inclination for the maximum strain change gradually from 0 and 90 degrees to 45 degrees, and the maximum strain varies from X_0 to $\frac{S}{2}$.

In order to be certain that this strain shall not become greater than the axial strain S, which is calculated by the aid of the formula $S = \frac{Pxe}{W}$ and is equal to the modulus of proof strength T, we must make X_0 at most $= S$, or rather

$$\frac{P \Sigma (Fz)}{b_0 W} < \frac{Pxe}{W}, \text{ I.E. } \frac{\Sigma (Fz)}{b_0} < xe.$$

If, then, in the formula $V = X \sin. 2\psi + \frac{Q}{2}(1 - \cos. 2\psi)$ we put $\psi + \mu$ instead of ψ and again make $\cos. \mu = 1$, we obtain

$$V_1 = X(\sin. 2\psi \cos. \mu + \cos. 2\psi \sin. \mu) + \frac{Q}{2}(1 - \cos. 2\psi \cos. \mu$$
$$+ \sin. 2\psi \sin. \mu) = X \sin. 2\psi + \frac{Q}{2}(1 - \cos. 2\psi)$$
$$+ \left(X \cos. 2\psi + \frac{Q}{2} \sin. 2\psi\right) \sin. \mu,$$

and in order that ψ shall cause V to become a maximum or a minimum, V_1 must be $= V$ or $X\,cos.\,2\,\psi + \frac{Q}{2}\,sin.\,2\,\psi = 0$, I.E.

$$tang.\,2\,\psi = -\frac{2\,X}{Q} = -\frac{2\,X e}{S\,z},\text{ as well as}$$

$$sin.\,2\,\psi = \mp \frac{2\,X}{\sqrt{Q^2+4\,X^2}}\text{ and } cos.\,2\,\psi = \pm \frac{Q}{\sqrt{Q^2+4\,X^2}}.$$

The corresponding *minimum* of V is

$$V_n = -\frac{2\,X^2}{\sqrt{Q^2+4\,X^2}} + \frac{Q}{2}\left(1 - \frac{Q}{\sqrt{Q^2+4\,X^2}}\right) = \frac{Q}{2} - \sqrt{\left(\frac{Q}{2}\right)^2 + X^2}$$
$$= \frac{S\,z}{2\,e} - \sqrt{\left(\frac{S\,z}{2\,e}\right)^2 + X^2},$$

and, on the contrary, its maximum is

$$V_m = \frac{2\,X^2}{\sqrt{Q^2+4\,X^2}} + \frac{Q}{2}\left(1 + \frac{Q}{\sqrt{Q^2+4\,X^2}}\right) = \frac{Q}{2} + \sqrt{\left(\frac{Q}{2}\right)^2 + X^2}$$
$$= \frac{S\,z}{2\,e} + \sqrt{\left(\frac{S\,z}{2\,e}\right)^2 + X^2}.$$

We must require the maximum V_m to be at most equal to the modulus of proof strength T or

$$\frac{S\,z}{2\,e} + \sqrt{\left(\frac{S\,z}{2\,e}\right)^2 + X^2} < T.$$

In the neutral axis Q is $= 0$, and therefore $tang.\,2\,\psi = -\infty$ or $2\,\psi = 270°$ and $\psi = 135$ or 45 degrees, and $V_n = -X_0$, on the contrary, $V_m = +X_0$. In the most distant fibre, on the contrary, X is $= 0$ and $Q = S$, and therefore $tang.\,2\,\psi = 0$ or $2\,\psi = 0$ or $180°$ and $\psi = 0$ or $90°$, and $V_n = 0$, on the contrary, $V_m = S$. In ordinary girders the maximum strain V_m increases gradually from $X_0 = \frac{P\,\Sigma\,(F\,z)}{b\,W}$ to $S = \frac{P\,x\,e}{W}$ as we pass from the neutral axis to the outmost fibre.

For a parallelopipedical girder we have $\Sigma\,(F\,z) = \frac{b\,h^2}{8}$, $W = \frac{b\,h^3}{12}$, $b_0 = b$ and $e = \frac{h}{2}$, and therefore the limit values are $X_0 = \frac{3}{2}\cdot\frac{P}{b\,h}$ and $S = \frac{6\,P\,x}{b\,h^2}$; but in general we have $X = \frac{P\left(\frac{h}{2} - z\right)\left(\frac{h}{2} + z\right)}{2\,W}$ $= \frac{6\,P}{b\,h^3}\left[\left(\frac{h}{2}\right)^2 - z^2\right]$ and $\frac{S\,z}{e} = \frac{12\,P\,x\,z}{b\,h^3}$, and therefore

$$V_m = \frac{6\,P\,x\,z}{b\,h^3} + \sqrt{\left(\frac{6\,P\,x\,z}{b\,h^3}\right)^2 + \left(\frac{6\,P}{b\,h^3}\right)^2 \left[\left(\frac{h}{2}\right)^2 - z^2\right]^2}$$

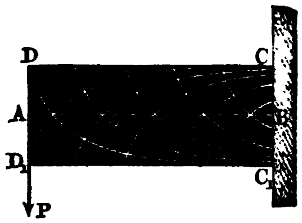
$$= \frac{6\,P}{b\,h^3}\left[x\,z + \sqrt{(x\,z)^2 + \left(\left(\frac{h}{2}\right)^2 - z^2\right)^2}\right], \text{ for example, for } z = \tfrac{1}{4}\,h,$$

$$V_m = \frac{3\,P}{2\,b\,h^2}\left[x + \sqrt{x^2 + (\tfrac{3}{4})^2\,h^2}\right], \text{ and for } x = 0,$$

$$V_m = \frac{9\,P}{8\,b\,h}, \text{ etc.}$$

If such a girder $A\,B$, Fig. 426, is fixed at one end B, the directions of the maximum and minimum normal forces V_m and V_n can be represented by two systems of lines, which cut the neutral axis at an angle of 45°, and the outer fibre and each other at an angle of 90°. The curves, which are concave downwards, correspond to the tensile forces, and those which are concave upwards to the compressive forces. The steeper end of any curve corresponds to the minimum and the flatter end, on the contrary, to the maximum forces. At the ends D and D_1 both these strains become equal to zero, while for the ends C and C_1 their values are the greatest.

FIG. 426.



§ 260. Influence of the Strength of Shearing upon the Proof Load of a Girder.

—The capability of a girder to support a certain load requires not only that the strain $S = \frac{P\,x\,e}{W}$ in the outermost fibre, but also that the shearing force $X_0 = \frac{P\,\Sigma\,(F\,z)}{b_0\,W}$ in the neutral axis shall not exceed the modulus of proof strength T. In the last chapter we have repeatedly given the moments which, in ordinary cases, we must substitute for $P\,x$ in the expression for S; we have, therefore, only to give the values, which we must substitute for the force P in the expression for X_0.

If the girder is fixed at one end and acted on by a force P at the other end, P can be directly employed in the formula $X_0 = \frac{P\,\Sigma\,(F\,z)}{b_0\,W}$. If the beam supports, in addition, a uniformly distributed load, whose intensity upon the unit of length is q, we must substitute for P in this expression $P + q\,x$ and $P + q\,l$, when we wish to determine the maximum value of X_0. If, on the contrary, the girder is supported at both ends and sustains at the distances l_1 and $l_2 = l - l_1$ from the points of support a load P, we must substitute for one portion of the beam $\frac{l_2}{l} P$, and for the other $\frac{l_1}{l} P$ instead of P in the formula for X_0, in order to find the shearing force in the neutral axis. If, on the contrary, this girder sustains an equally distributed load $q\,l$, each of the points of support bears $\frac{q\,l}{2}$, and the shearing force of the whole cross-section at any point at the distance x from the points of support is $P = q \left(\frac{l}{2} - x\right)$. The latter is $= 0$ in the middle, where $x = \frac{l}{2}$, becomes greater and greater towards the end, and at the point of support is $P = \frac{q\,l}{2}$.

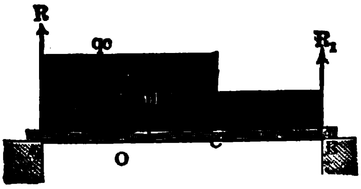
If a girder, supported at both ends, sustains a load, which is equally distributed over a part c of its total length, while the other portion $l - c$ is not loaded, the point of support of the first portion bears a part $q\,c \left(1 - \frac{c}{2\,l}\right)$ of the total load $q\,c$ and that of the second portion a load $\frac{q\,c^2}{2\,l}$, and the vertical shearing force at the distance x from the first point of support is

$$P = q\,c \left(1 - \frac{c}{2\,l}\right) - q\,x = q\left(c - \frac{c^2}{2\,l} - x\right).$$

The value of the latter becomes for $x = c$, $- \frac{q\,c^2}{2\,l}$, and this value remains the same for any distances $x > c$. If the load covers exactly one-half of the girder, I.E. if $c = \frac{l}{2}$, we have

$$P = q \left(\frac{3\,l}{8} - x\right) \text{ or for } x = \frac{l}{2},\ P = - \frac{q\,l}{8}.$$

FIG. 427.



If, finally, the girder $A\,B$, Fig. 427, bears a load $p\,l$ equally distributed over its entire length l and a load $q\,c$ equally distributed over the length $A\,C$ $= c$, the reactions of the points of support are

$$R_1 = \frac{p\,l}{2} + q\left(c - \frac{c^2}{2\,l}\right) \text{ and } R_2 = \frac{p\,l}{2} + \frac{q\,c^2}{2\,l},$$

whence it follows, that the vertical shearing force at the distance $A\,O = x$ from the point of support A is

$$P = \frac{p\,l}{2} + q\left(c - \frac{c^2}{2\,l}\right) - (p + q)\,x,$$

for $x = c$ the latter expression becomes $p\left(\frac{l}{2} - c\right) - \frac{q\,c^2}{2\,l}$, and for any distances $x > c$ it is

$$\frac{p\,l}{2} + \frac{q\,c^2}{2\,l} - p\,(l - x) = -\frac{p\,l}{2} + \frac{q\,c^2}{2\,l} + p\,x.$$

The vertical shearing force $P = p\left(\frac{l}{2} - c\right) - \frac{q\,c^2}{2\,l}$ in C is $= 0$ for $c^2 + \frac{2\,p}{q}\,l\,c = \frac{p}{q}\,l^2$, I.E., for

$$c = \left(-\frac{p}{q} + \sqrt{\left(\frac{p}{q}\right)^2 + \frac{p}{q}}\right)l.$$

If, in general, at a point of the girder the shearing force is $P = R - q\,x$, we have for the moment of flexure

$$M = R\,x - \frac{q\,x^2}{2} = \frac{q\,x}{2}\left(\frac{2\,R}{q} - x\right).$$

This, however, for $x = \frac{2\,R}{q} - x$, I.E., for $x = \frac{R}{q}$, is a maximum, in which case P becomes $= 0$; the moment of flexure of a girder becomes a maximum for the same point at which the vertical shearing force is $= 0$, and in the foregoing case c gives that length of the load $q\,c$, for which the moment

$$\left[\frac{p\,l}{2} + q\left(c - \frac{c^2}{2\,l}\right)\right]c - \frac{(p + q)\,c^2}{2}$$

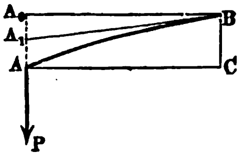
becomes a maximum, and it is then $= \frac{(p + q)\,c^2}{2}$.

These formulas are applicable to girders for bridges, where $q\,c$ denotes the intensity of the moving load.

The shearing force $X_0 = \frac{P\,\Sigma\,(F\,z)}{b_0\,W}$ must be specially considered in the case of bodies of *uniform strength*, the cross-section of which, according to what we have seen above (§ 253), might in some parts be infinitely small. For example, for the parabolic girder in Fig. 406, we have $X_0 = T = \frac{3}{2} \cdot \frac{\frac{1}{2}\,P}{b_0\,h_0}$, and therefore, the necessary cross-section at each end is $F_0 = b_0\,h_0 = \frac{3}{4}\,\frac{P}{T}$, in which T denotes the modulus of proof strength for shearing.

§ 261. Influence of the Elasticity of Shearing upon the Form of the Elastic Curve.—We have yet to determine what influence the elasticity of shearing has upon the form of the *elastic curve* or upon the form of the neutral axis of a loaded girder $A\ B$, Fig. 428. According to the formula $P = \iota\ F\ C$, in which C denotes the modulus of the elasticity of shearing and F the cross-section of the beam, the inclination the beam $A_1\ B$ produced by the shearing force is $\iota = \frac{X_0}{C}$, and, therefore, the corresponding deflection of the end A_1 of the girder, whose length $A_0\ B = l$, is

FIG. 428.



$$A_0\ A_1 = a_1 = \iota\ l = \frac{X_0\ l}{C} = \frac{P\ l\ \Sigma\ (F\ z)}{b_0\ W\ C}.$$

To this must be added the deflection $A_1\ A = a_2$, produced by the flexure of the beam, and which, according to § 217, is $a_2 = \frac{P\ l^3}{3\ W\ E}$; the total deflection of the girder is therefore

$$B\ C = A_0\ A = a = a_1 + a_2 = \frac{P\ l}{W}\left(\frac{\Sigma\ (Fz)}{b_0\ C} + \frac{l^2}{3\ E}\right).$$

For a parallelopipedical girder $b_0 = b$, $\Sigma\ (F\ z) = \frac{b\ h^2}{8}$ and $W = \frac{b\ h^3}{12}$, consequently

$$a = \frac{4\ P\ l^3}{b\ h^3\ E}\left[1 + \tfrac{3}{8}\,\frac{E}{C}\left(\frac{h}{l}\right)^2\right],$$

or, assuming $\frac{E}{C} = 3$,

$$a = \frac{4\ P\ l^3}{b\ h^3\ E}\left[1 + \tfrac{9}{8}\left(\frac{h}{l}\right)^2\right].$$

E.G., for $l = 10\ h$, we have $a = 1{,}01125\ .\ \frac{4\ P\ l^3}{b\ h^3\ E}$, if then the girder is ten times as long as thick, the deflection at the loaded end, due to the shearing force, is so small compared with that due to the flexure of the girder, that in most cases we can neglect it.

In order to determine the modulus of elasticity of a girder $A\ B$, we load it first with a small weight P at the greatest distance l, and afterwards with a large weight P_1 at a smaller distance l_1 from the point of support B, and we observe the corresponding deflections a and a_1 of the length l of the girder. Now we have

$$a = \frac{P\,l\,\Sigma\,(F\,z)}{b_0\,W\,C} + \frac{P\,l^3}{3\,W\,E} \text{ and}$$

$$a_1 = \frac{P_1\,l\,\Sigma\,(F\,z)}{b_0\,W\,C} + \frac{P_1\,l_1^3}{3\,W\,E} + \frac{P_1\,l_1^2\,(l - l_1)}{2\,W\,E}.$$

In order to eliminate C, divide the first equation by P and the second by P_1 and subtract the equations obtained from one another. Thus we obtain

$$\frac{a}{P} - \frac{a_1}{P_1} = \frac{1}{W\,E}\left(\frac{l^3 - l_1^3}{3} - \frac{l_1^2\,(l - l_1)}{2}\right) = \frac{1}{W\,E}\left(\frac{l^3}{3} - \frac{l\,l_1^2}{2} + \frac{l_1^3}{6}\right),$$

and therefore the modulus of elasticity for *tensile* and *compressive forces* is

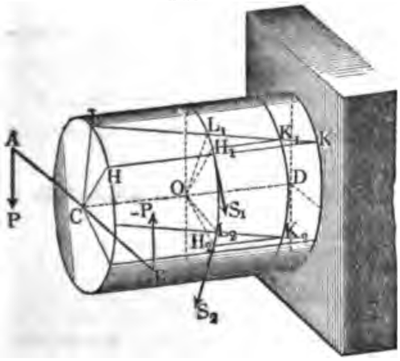
$$E = \frac{P\,P_1}{(a\,P_1 - a_1\,P)\,W}\left(\frac{l^3}{3} - \frac{l\,l_1^2}{2} + \frac{l_1^3}{6}\right).$$

With the aid of this expression and the formula for a, we determine the modulus of elasticity for shearing by the formula

$$C = \frac{P\,l}{b_0} \cdot \frac{3\,\Sigma\,(F\,z)\,E}{3\,W\,E\,a - P\,l^3}.$$

§ 262. Elasticity of Torsion.—In order to investigate the theory of the *twisting* or *torsion* of a body (see § 202), we can again begin with the case of a body $H\,C\,D\,L$, Fig. 429, fixed at one end, but, in order to avoid any complex change of form, we must assume that the free end is acted upon by a couple $(P, -P)$ whose plane $A\,H\,B$ coincides with the plane of rotation of the axis $C\,D$. Let us imagine the body to be composed of long fibres, such as $H\,K$, which, in consequence of the torsion, assume the form of a helix, by which $H\,K$ comes into the position $L\,K$ and the whole base is turned through an angle $H\,C\,L = a$. If the portions $H_1\,K_1$, $H_2\,K_2$, etc., of the fibres, whose lengths are unity and whose cross-sections are F_1, F_2 etc., undergo a lateral displacement through the distance $H_1\,L_1 = \sigma_1$, $H_2\,L_2 = \sigma_2$ etc., we can put, when the modulus of elasticity for shearing is C, the corresponding shearing forces $S_1 = \sigma_1\,F_1\,C$, $S_2 = \sigma_2\,F_2\,C$, etc. Now if the corresponding angle

FIG. 429.



of torsion is $H_1\,O\,L_1 = H_1\,O\,L_2 = \phi$ and if the distances of these fibres from the axis $C\,D$ of the body are $O\,H_1 = z_1$, $O\,H_2 = z_2$, we have $\sigma_1 = \phi\,z_1$, $\sigma_2 = \phi\,z_2 \ldots$; hence the strains are $S_1 = \phi\,C\,F_1\,z_1$, $S_2 = \phi\,C\,F_2\,z_2 \ldots$, and their moments are

$$S_1\,z_1 = \phi\,C\,F_1\,z_1^2,\ S_2\,z_2 = \phi\,C\,F_2\,z_2^2 \ldots$$

All the forces $S_1, S_2 \ldots$ of a cross-section $H_1\,O\,L_2$ must in any case *balance* the couple $(P, - P)$; if then a is the lever arm $A\,B$ of this couple or $P\,a$ its *moment*, we can put

$$\begin{aligned} P\,a &= S_1\,z_1 + S_2\,z_2 + \ldots = \phi\,C\,F_1\,z_1^2 + \phi\,C\,F_2\,z_2^2 + \ldots \\ &= \phi\,C\,(F_1\,z_1^2 + F_2\,z_2^2 + \ldots). \end{aligned}$$

Now if we designate the geometrical measure $F_1\,z_1^2 + F_2\,z_2^2 + \ldots.$ of the moment of torsion by W, we have $P\,a = \phi\,C\,W$.

But the angle of torsion for the entire length $C\,D = l$ of the body is $\alpha = \phi\,l$, therefore we can put

$$1)\ P\,a = \frac{\alpha\,C\,W}{l}, \text{ or } P\,a\,l = \alpha\,C\,W,$$

and the *angle of torsion*

$$2)\ \alpha = \frac{P\,a\,l}{C\,W}.$$

As we have done previously (§ 215), we can call $W\,C$ the *moment of torsion*, and consequently W the *measure of the moment of torsion*, and we can then assert, *that the moment of the force $P\,a$ increases directly as the angle of torsion and inversely as the length of the body.*

The work done in producing a torsion equal to the angle α is

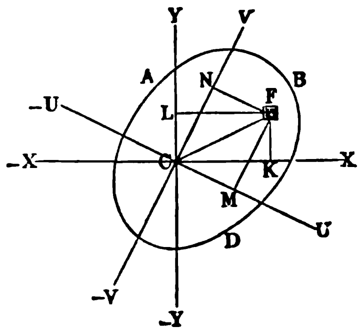
$$L = \frac{P}{2}\,.\,a\,\alpha = \frac{\alpha^2\,W\,C}{2\,l} = \frac{P^2\,a^2\,l}{2\,W\,C};$$

for the space described by the force P, which causes it, is $a\,\alpha$. These formulas hold good for prismatical bodies alone, for bodies with other forms we must substitute in them instead of the ratio $\frac{l}{W}$ a mean value of it.

§ 263. Moment of Torsion or Twisting Moment.—The measure $W = F_1\,z_1^2 + F_2\,z_2^2 + \ldots$ of the moment of torsion can easily be calculated, according to the rule explained in § 225, from the measure of the moment of flexure for the same cross-section. If, for example, W_1 is the measure of the moment of flexure of a surface $A\ B\ D$, Fig. 430, referred to an axis $\overline{X}\ X$ and W_2 the same in reference to an axis $\overline{Y}\ Y$ at right angles to the first, we have for the measure of the moment of torsion in reference to the intersection of the two axes

$$W = W_1 + W_2.$$

FIG. 430.



For a shaft with a *square cross-section* $A\ B\ D\ E$, Fig. 431, we have, when b denotes the length of the side $A\ B = D\ E$, according to § 226, the measure of the moment of flexure in reference to each axis $\overline{X}\ X$ and $\overline{Y}\ Y$

$$W_1 = W_2 = \frac{b\,h^3}{12} = \frac{b^4}{12},$$

and consequently the measure of the moment of torsion is

$$W = W_1 + W_2 = 2\,\frac{b^4}{12} = \frac{b^4}{6},$$

and the moment of the force

$$P\,a = \frac{a\ W\ C}{l} = \frac{a\,b^4\,C}{6\,l} = 0{,}1667\,\frac{a\,C\,b^4}{l}$$

For a shaft with a *rectangular cross-section* ($b\ h$) we would have, on the contrary,

$$P\,a = \frac{a\,b\,h\,(b^2 + h^2)}{12\,l}\,C = 0{,}0833\,\frac{a\,b\,h\,(b^2 + h^2)\,C}{l}.$$

FIG. 431.

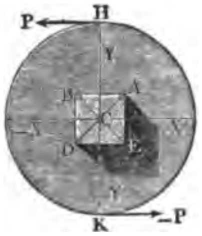
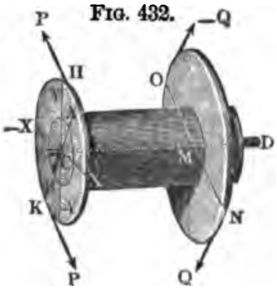


FIG. 432.



For a *cylindrical shaft* with circular cross-section $A\ B$, Fig. 432, whose radius is $C\ A = r$, the measure of the moment of flexure in reference to an axis $\overline{X}\ X$ or $\overline{Y}\ Y$ is (according to § 231)

$$W_1 = W_2 = \frac{\pi r^4}{4},$$

and therefore the measure of the moment of torsion in reference to the point C in that axis is

$$W = 2 W_1 = \frac{\pi r^4}{2}.$$

Now if the twisting couple $(P, - P)$ acts with an arm $H K = a$, or each of its components with an arm $C H = C K = \frac{a}{2}$, we have

$$P a = \frac{a W C}{l} = \frac{a \pi r^4 C}{2 l} = 1{,}5708 \frac{a r^4 C}{l}.$$

If the shaft is *hollow* and its radii are r_1 and r_2, we have the following formula:

$$P a = \frac{a \pi (r_1^4 - r_2^4) C}{2 l} = 1{,}5708\, a \frac{(r_1^4 - r_2^4) C}{l}$$

The torsion of a shaft $A B M$, Fig. 432, is generally produced by two couples $(P, - P)$, $(Q, - Q)$, which balance each other, and therefore, instead of l, we must substitute not the entire length of the shaft, but the distance between the planes in which the two couples act; it makes no difference, however, whether we make the moment of torsion equal to the moment of the couple $(P, - P)$ or to that of the couple $(Q, - Q)$. If we denote the arm $H K$ of the couple $(P, - P)$ by a, and the arm $N O$ of the other couple $(Q, - Q)$ by b, we have

$$P a = Q b = \frac{a W C}{l}.$$

The foregoing theory gives us for bodies limited by plane surface moments of torsion, which vary somewhat from the exact truth; for we suppose, in calculating them, that the bases of the prism subjected to the torsion remain plane surfaces, while, in reality, they become warped. According to the researches of Saint Venant, Werthheim, etc. (see the "Comptes rendus des séances de l'académie des sciences à Paris," T. 24 and T. 27, as well as "l'Ingénieur," Nos. 1 and 2, 1858; in German in the "Civilingenieur," 4 Vol., 1858), we have for a *square shaft*

$$P a = 0{,}841 \frac{a b^4 C}{6 l} = 0{,}1402 \frac{a b^4 C}{l},$$

in which b denotes the length of the side of the square cross-section.

For bodies, the dimensions of whose cross-sections differ very much from each other, these variations are greater; E.G., for a prismatical body with a rectangular cross-section, whose width is b and whose height is h, we have

$$W = W_1 + W_2 = \frac{b\,h^3}{12} + \frac{h\,b^3}{12} = \frac{b\,h\,(b^2 + h^2)}{12}, \text{ and therefore}$$

$$P\,a = \frac{a\,W\,C}{l} = \frac{a\,b\,h\,(b^2 + h^2)\,C}{12\,l}.$$

Now if this formula requires a correction, when $h = b$, in which case $P\,a = \frac{a\,b^4\,C}{6\,l}$, it is natural to expect that when b differs materially from h, in which case the surface of the sides will become more warped, it will no longer be sufficiently accurate. In fact, taking into consideration the warping of the surfaces, we find by means of the calculus

$$P\,a = \frac{a\,h^3\,b^3\,C}{3\,(b^2 + h^2)\,l},$$

and according to the later experiments of Werthheim, the mean value of the required coefficient of correction is $= 0{,}903$; consequently we must put

$$P\,a = 0{,}903\,\frac{a\,h^3\,b^3\,C}{3\,(b^2 + h^2)\,l} = 0{,}301\,\frac{a\,h^3\,b^3\,C}{(b^2 + h^2)\,l}.$$

If b is very small compared to h, we have

$$P\,a = 0{,}301\,\frac{a\,h\,b^3\,C}{l}.$$

If the angle of torsion is given in degrees, putting $a = \frac{a^\circ\,\pi}{180^\circ} = 0{,}017453\,a^\circ$, we obtain

1) for prismatic girders or shafts with a *circular cross-section*, the diameter of which is $d = 2\,r$

$$\begin{aligned} P\,a\,l &= \frac{a\,\pi\,r^4}{2}\,C = \frac{a\,\pi\,d^4}{32}\,C = \frac{a^\circ\,\pi^2\,r^4}{180^\circ\,.\,2}\,C = \frac{a^\circ\,\pi^2}{180^\circ}\,\frac{d^4}{32}\,C \\ &= 1{,}571\,a\,r^4\,C = 0{,}0982\,a\,d^4\,C = 0{,}02742\,a^\circ\,r^4\,C \\ &= 0{,}001714\,a^\circ\,d^4\,C; \end{aligned}$$

2) for prismatic girders, axles or shafts with a *square cross-section*, the length of whose side is b, when we neglect the coefficient of correction,

$$P\,a\,l = \frac{a\,b^4\,C}{6} = 0{,}1667\,a\,b^4\,C = \frac{a^\circ\,\pi\,b^4\,C}{1080^\circ} = 0{,}00291\,a^\circ\,b^4\,C.$$

Inversely we have

$$a = 0{,}637\,\frac{P\,a\,l}{r^4\,C} = 10{,}18\,\frac{P\,a\,l}{d^4\,C} = 6\,\frac{P\,a\,l}{b^4\,C}, \text{ and}$$

$$a^0 = 36{,}4\,\frac{P\,a\,l}{r^4\,C} = 583\,\frac{P\,a\,l}{d^4\,C} = 344\,\frac{P\,a\,l}{b^4\,C}.$$

The values for C must be taken from Table III. in § 213. Hence we have, E.G.,

1) For *cast iron,* $C = 2840000$, whence

$$P\,a\,l = 77900\,a^0\,r^4 = 4867\,a^0\,d^4 = 8264\,a^0\,b^4 \text{ and}$$

$$a^0 = 0{,}00001281^\circ\,\frac{P\,a\,l}{r^4} = 0{,}0002053^\circ\,\frac{P\,a\,l}{d^4} = 0{,}0001211^\circ\,\frac{P\,a\,l}{b^4}.$$

2) For *wrought iron,* $C = 9000000$, whence

$$P\,a\,l = 246780\,a^0\,r^4 = 15426\,a^0\,d^4 = 26190\,a^0\,b^4 \text{ and}$$

$$a^0 = 0{,}00000404^\circ\,\frac{P\,a\,l}{r^4} = 0{,}0000648^\circ\,\frac{P\,a\,l}{d^4} = 0{,}0000382^\circ\,\frac{P\,a\,l}{b^4}.$$

3) For *wood,* $C = 590000$,

$$P\,a\,l = 161800\,a^0\,r^4 = 1011\,a^0\,d^4 = 1712\,a^0\,b^4 \text{ and}$$

$$a^0 = 0{,}0000617^\circ\,\frac{P\,a\,l}{r^4} = 0{,}000988^\circ\,\frac{P\,a\,l}{d^4} = 0{,}000583^\circ\,\frac{P\,a\,l}{b^4}.$$

EXAMPLE—1) What moment of torsion can a square wrought-iron shaft 10 feet long and 5 inches thick withstand, without suffering the angle of torsion to become more than ¼ of a degree? Here, according to this table, we have

$$P\,a = 26190\,.\,\tfrac{1}{4}\,.\,\frac{625}{10\,.\,12} = 34102 \text{ inch-pounds} = 2842 \text{ foot-pounds.}$$

2) What is the amount of torsion sustained by a hollow cast-iron shaft, whose length is $l = 100$ inches and whose radii are $r_1 = 6$ inches and $r_2 = 4$ inches, when the moment of the force is $P\,a = 10000$ foot-pounds? Here

$$P\,a = 77900\,\frac{a^0\,(r_1^{\,4} - r_2^{\,4})}{l},$$

consequently

$$a^0 = \frac{P\,a\,l}{77900\,(r_1^{\,4} - r_2^{\,4})} = \frac{10000\,.\,12\,.\,100}{77900\,(6^2 + 4^2)\,(6^2 - 4^2)}$$

$$= \frac{120000}{779\,.\,52\,.\,20}$$

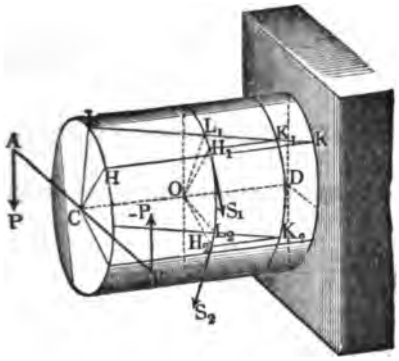
$$= \frac{1500}{10127} \text{ degrees} = 8{,}887 \text{ minutes} = 8 \text{ minutes } 53 \text{ seconds.}$$

§ **264**. **Resistance to Rupture by Torsion.**—If in a prism $C\,K\,L$, Fig. 433, twisted by a couple $(P, -P)$ the shearing force per unit of surface at a certain distance e from the axis $C\,D$ is $= S$, the shearing force at any other distance z_1 is $= \frac{z_1}{e} S$, and its moment is $= \frac{z_1^2}{e} S$, and for a cross-section F_1 it is

$$\frac{F_1 z_1^2}{e} S = \frac{S}{e} F_1 z_1^2;$$

in like manner the moments of the shearing forces of other cross-sections F_2, $F_3 \dots$, which are at the distances $z_2, z_3 \dots$ from the axis $C\,D$, are $\frac{S}{e} F_2 z_2^2$, $\frac{S}{e} F_3, z_3^2$, etc.; hence the ***total moment of torsion*** of the body is

FIG. 438.



$$P\,a = \frac{S}{e} F_1 z_1^2 + \frac{S}{e} F_2 z_2^2 + \frac{S}{e} F_3 z_3^2 + \dots$$

$$= \frac{S}{e} (F_1 z_1^2 + F_2 z_2^2 + \dots), \text{ i.e.}$$

1) $P\,a = \frac{S\,W}{e}$, or $P\,a\,e = S\,W$, and $\frac{W}{e} = \frac{P\,a}{S}$.

Substituting for S the ***modulus of proof strength*** T for shearing, and for e the greatest distance of the elements of the cross-section from the neutral axis, we obtain in the formula

2) $P\,a\,e = T\,W$ an equation for determining the dimensions of the cross-section, which the body must have if it is not to be strained at any point beyond the limit of elasticity. If, instead of the modulus of proof strength T, we substitute the modulus of rupture K for shearing, we obtain the moment $P_1\,a$, which will break the body by wrenching; it is

3) $P_1\,a = \frac{K\,W}{e}$

For a ***massive cylindrical shaft***, whose diameter $d = 2\,r$, we have

$$\frac{W}{e} = \frac{\pi\,r^4}{2\,r} = \frac{\pi\,r^3}{2}, \text{ and therefore}$$

$$P\,a = \frac{\pi\,r^3\,T}{2} = \frac{\pi\,d^3\,T}{16} = 0{,}1963\,d^3\,T, \text{ and also}$$

$$P_1 a = \frac{\pi\,r^3\,K}{2} = \frac{\pi\,d^3\,K}{16} = 0{,}1963\,d^3\,K.$$

If the shaft is *hollow* and the diameters are $d_1 = 2\,r_1$ and $d_2 = 2\,r_2$, in which case

$$\frac{W}{e} = \frac{\pi\,(r_1^4 - r_2^4)}{2\,r_1},\text{ we have, on the contrary,}$$

$$P\,a = \frac{\pi\,(r_1^4 - r_2^4)}{2\,r_1}\,T = \frac{\pi\,(d_1^4 - d_2^4)}{16\,d_1}\,T = \frac{F\,(d_1^2 + d_2^2)}{4\,d_1}\,T,$$

in which $F = \dfrac{\pi\,(d_1^2 - d_2^2)}{4}$ denotes the cross-section of the body.

For a prismatical body with a *square cross-section*, the length of whose side is b, we have

$$W = \frac{b^4}{6}\text{ and } e = \tfrac{1}{2}\,b\,\sqrt{2} = b\,\sqrt{\tfrac{1}{2}},\text{ whence}$$

$$\frac{W}{e} = \frac{b^3}{6\,\sqrt{\tfrac{1}{2}}} = \frac{b^3}{3\,\sqrt{2}}\text{ and } P\,a = \frac{b^3\,T}{3\,\sqrt{2}} = 0{,}2357\;b^3\;T.$$

If in the fundamental formula $P\,a = \phi\,C\,W$ of § 262 we substitute $\phi = \dfrac{\sigma}{e} = \dfrac{tang.\,\delta}{e}$, in which e denotes the distance of the most remote fibre from the axis of rotation $C\,D$ and δ the angle $H\,K\,L$, which this fibre has been turned from its original position by the torsion, we obtain

$$P\,a\,e = C\;W\,tang.\,\delta\text{; but we have also}$$
$$P\,a\,e = S\;W,\text{ hence}$$
$$S = C\,tang.\,\delta,\text{ and therefore}$$
$$T = C\,tang.\,\delta,\text{ or } tang.\,\delta = \frac{T}{C},$$

in which δ denotes the angle of displacement, when the strain has reached the limit of elasticity.

The mechanical effect, which is required to twist the shaft through an angle α, is, according to § 262, $L = \dfrac{P^2\,a^2\,l}{2\,W\,C}$, and therefore if we substitute $P\,a = \dfrac{S\,W}{e}$, we can put $L = \dfrac{S^2}{C}\,\dfrac{W\,l}{2\,e^2}$, in which S denotes the maximum strain.

At the limit of elasticity $S = T$; hence it follows that the mechanical effect necessary to twist the body to the limit of its elasticity is

$$L = \frac{T^2}{C}\cdot\frac{W\,l}{2\,e^2}.$$

For a prismatic body with a *circular cross-section* $W = \frac{\pi r^4}{2}$ and $e = r$, whence

$$L = \frac{T^2}{2\,C} \cdot \frac{\pi r^2 l}{2} = \frac{T^2}{4\,C} V,$$

and, on the contrary, when the cross-section is a square

$$W = \frac{b^4}{6} \text{ and } e^2 = \frac{b^2}{2}, \text{ and therefore}$$

$$L = \frac{T^2}{2\,C} \cdot \frac{b^4 l}{3\,b^2} = \frac{T^2}{6\,C} \cdot b^2 l = \frac{T^2}{6\,C} V.$$

Now $\frac{T^2}{2\,C} = \frac{\sigma\,C\,T}{2\,C} = \frac{\sigma\,T}{2}$ is the *modulus of resilience* for the limit of elasticity; hence we have for the cylinder $L = \frac{1}{2} A\,V$, and for the parallelopipedon $L = \frac{1}{3} A\,V$.

The *work done* in both cases is proportional to the volume of the body alone (compare § 206 and § 235).

We can also put for the mechanical effect necessary to rupture of the body by wrenching $L = \frac{1}{2} B\,V$ and $\frac{1}{3} B\,V$, in which B denotes the modulus of fragility for wrenching.

If we assume with General Morin for all substances

$$\frac{T}{C} = tang.\ \delta = 0{,}000667$$

or the angle of displacement $\delta = 2$ min. 18 sec., we obtain for *cast iron*

$$T = 200000\ .\ 0{,}000667 = 134 \text{ kilo.} = 1906 \text{ lbs.},$$

therefore, when we employ the French measures

$$P\,a = 26{,}3\ d^3 = 31{,}6\ b^3 \text{ kilogr. centimeters,}$$

and, on the contrary, when we employ the English measures

$$P\,a = 374\ d^3 = 449\ b^3 \text{ inch-pounds.}$$

Under the same conditions we have for *wrought iron*

$$T = 630000\ .\ 0{,}000667 = 420 \text{ kilo.} = 5974 \text{ lbs.},$$

and therefore

$$P\,a = 82{,}4\ d^3 = 99{,}2\ b^3 \text{ kilogram centimeters,}$$

or

$$P\,a = 1173\ d^3 = 1408\ b^3 \text{ inch-pounds.}$$

Likewise under the same conditions we have as a mean for *wood*

$$F = 41650\ .\ 0{,}000667 = 27{,}8 \text{ kilogr.} = 395 \text{ lbs.},$$

whence

$$P\,a = 5{,}46\ d^3 = 6{,}55\ b^3 \text{ kilogr. centimeters,}$$

or

$$P\,a = 77{,}5\ d^3 = 93{,}1\ b^3 \text{ inch-pounds.}$$

The coefficients of these formulas are correct only for bodies at rest or for shafts, which turn slowly and smoothly; for common shafts we give double security, I.E., we make the coefficients but half as great. When their motion is very quick and accompanied by concussions, we are obliged to make the coefficient but one-eighth of those given above.

EXAMPLE—1) The cast iron shaft of a turbine wheel exerts at the circumference of the cog-wheel upon it, which is 6 inches in diameter, a pressure of 4000 pounds. Required the thickness of the shaft. Here the moment of the force is $P a = 4000 . 6 = 24000$ inch-pounds, and consequently the diameter of the wheel, when we put $P a = \frac{374}{2} d^3$, is

$$d = \sqrt[3]{\frac{24000}{187}} = 5{,}04 \text{ inches.}$$

If the distance from the cog-wheel to the water-wheel is $l = 48$ inches, we have, according to the foregoing paragraph, the angle of torsion

$$= 0{,}0002058° \frac{24000 . 48}{5{,}04^4} = 0{,}367° = 22'.$$

2) A force $P = 600$ lbs. acts with a lever arm $a = 15$ feet $= 180$ inches upon a square fir shaft, while the load Q acts with an arm of 2 feet at a distance $l = 6$ feet $= 72$ inches in the direction of the axis; how thick should the shaft be made and what is the angle of torsion?

In order to have quadruple safety, we must put

$$P a = 600 . 180 = 108000 = \frac{98{,}1\, b^3}{4},$$

hence the width of the side is

$$b = \sqrt[3]{\frac{4 . 108000}{98{,}1}} = 16{,}68 \text{ inches,}$$

and the angle of torsion is

$$a° = 0{,}000583 \frac{108000 . 72}{(16{,}68)^4} = 0{,}0586 \text{ degrees} = 3\tfrac{1}{2} \text{ minutes.}$$

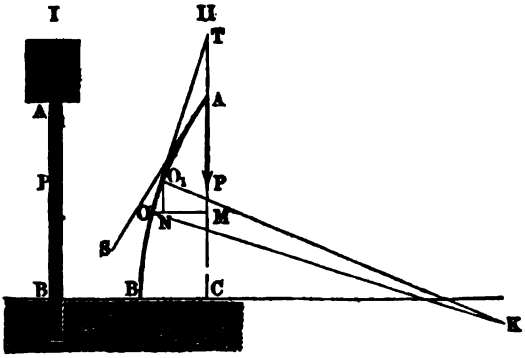
CHAPTER IV.

OF THE PROOF STRENGTH OF LONG COLUMNS OR THE RESISTANCE TO CRUSHING BY BENDING OR BREAKING ACROSS.

§ 265. Proof Strength of a Long Pillar Fixed at One End.—If a prismatic body A B (I), Fig. 434, is fastened at one end

B and acted upon at the other by a force *P*, whose direction is that of the longitudinal axis of the pillar, the relations of the flexure,

FIG. 434.



under these circumstances, are very different from what they are where the force acts, as we have seen in § 214, etc., at right angles to this axis. The neutral axis $A\ B$ (II) assumes in this case another form; for the lever arm of the force P is represented by the ordinate $M\ O = y$ and not by the abscissa $A\ M = x$, and its moment is not $P\ x$, but $P\ y$; consequently the radius of curvature $\overline{K\ O} = r$ is determined by the expression

$$r = \frac{W\,E}{P\,y},$$

while, according to § 215, for a bending force acting at right angles to the axis we must put

$$r = \frac{W\,E}{P\,x}.$$

At the point B, where the pillar is fastened, y becomes the deflection $B\ C = a$, the radius of curvature $r = \frac{W\,E}{P\,a}$ is a minimum and the curvature itself a maximum. On the contrary, at the point of application A, where $y = 0$, the radius of curvature is infinite and the curvature itself null.

If we denote by δ the arc, which measures the angle $O\ K\ O_1$ of curvature of the element $O\ O_1 = \sigma$ of the curve, we have $r = \frac{\sigma}{\delta}$, and therefore $P\,y\,\sigma = W\,E\,\delta$; and if $\beta°$ is the angle of inclination $O\ O_1\ N$ of the same to the axis $A\ C$, we can put the element $N\ O$ of the ordinate $= v = \sigma\,\beta$, and therefore

$P\,y\,v = W\,E\,\beta\,\delta$, and in like manner

$P\,\Sigma\,(y\,v) = W\,E\,\Sigma\,(\beta\,\delta)$.

In order to find the sum $\Sigma\,(y\,v)$ for the arc $A\ O$, let us substitute for y, $v, 2\,v, 3\,v \ldots n\,v$ in the above equation. Thus we obtain $\Sigma\,(y\,v) = v\,\Sigma\,(y) = v\,(v + 2\,v + 3\,v + \ldots + n\,v) = v\,\frac{n^2\,v}{2} = \frac{n^2\,v^2}{2}$, or since $n\,v = M\,O = y$,

$$\Sigma\,(y\,v) = \frac{y^2}{2} \text{ and } P\,\Sigma\,(y\,v) = \tfrac{1}{2}\,P\,y^2.$$

In like manner, to find $\Sigma\,(\beta\,\delta)$, we substitute for β successively $\beta, \beta + \delta, \beta + 2\,\delta \ldots \beta + n\,\delta$, and complete the summation as follows:

$$\begin{aligned}\Sigma\,(\beta\,\delta) &= \delta\,\Sigma\,(\beta) = \delta\,(\beta + \beta + \delta + \beta + 2\,\delta + \ldots + \beta + n\,\delta)\\ &= \delta\,[n\,\beta + (1 + 2 + 3 + \ldots + n)\,\delta]\\ &= \delta\left(n\,\beta + \frac{n^2\,\delta}{2}\right) = n\,\delta\left(\beta + \frac{n\,\delta}{2}\right).\end{aligned}$$

If the angle of inclination at A, $= \alpha$, we can put $\beta + n\,\delta = \alpha$, and therefore

$$\Sigma\,(\beta\,\delta) = (\alpha - \beta)\left(\beta + \frac{\alpha - \beta}{2}\right) = \tfrac{1}{2}\,(\alpha - \beta)\,(\alpha + \beta) = \tfrac{1}{2}\,(\alpha^2 - \beta^2),$$

whence

$$W{\cdot}E\,\Sigma\,(\beta\,\delta) = \tfrac{1}{2}\,W\,E\,(\alpha^2 - \beta^2), \text{ and finally}$$

$$P\,y^2 = W\,E\,(\alpha^2 - \beta^2).$$

For the end B, $y = a$ and $\beta = 0$, and therefore

$$P\,a^2 = W\,E\,\alpha^2 \text{ and}$$

$$P\,(a^2 - y^2) = W\,E\,\beta^2,$$

from this we obtain the tangential angle

$$1)\quad \beta = \sqrt{\frac{P\,(a^2 - y)^2}{W\,E}}.$$

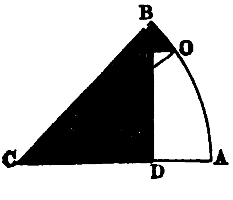
From β and the element $N\ O = v$ of the ordinate we obtain the element of the abscissa

$$N\,O = \xi = \frac{v}{\beta} = v\sqrt{\frac{W\,E}{P\,(a^2 - y^2)}} = \frac{v}{\sqrt{a^2 - y^2}}\sqrt{\frac{W\,E}{P}}, \text{ or}$$

$$\xi\sqrt{\frac{P}{W\,E}} = \frac{v}{\sqrt{a^2 - y^2}}.$$

If with the hypothenuse $C\ B = a$ of the right-angled triangle $B\ C\ D$, Fig. 435, whose altitude is $B\,D = y$ and whose base is $C\ D = \sqrt{a^2 - y^2}$, we describe an arc $A\ B$, we have for the element $B\ O = \psi$ the proportion

FIG. 435.



$$\frac{B\,O}{B\,N} = \frac{C\,B}{C\,D}, \text{ i.e. } \frac{\psi}{v} = \frac{a}{\sqrt{a^2 - y^2}},$$

whence

$$\frac{v}{\sqrt{a^2 - y}} = \frac{\psi}{a} \text{ and}$$

$$\xi \sqrt{\frac{P}{W E}} = \frac{\psi}{a}, \text{ as well as}$$

$$\sqrt{\frac{P}{W E}}\, \Sigma\, (\xi) = \frac{1}{a}\, \Sigma\, (\psi).$$

But $\Sigma\,(\xi)$ is the sum of all the elements of the abscissa and is $= x$, and $\Sigma\,(\psi)$ is the sum of all the elements of the arc $A\ B$ and is equal to the arc $A\ B$ itself; therefore we have also

$$x \sqrt{\frac{P}{W E}} = \frac{\text{arc } A\ B}{a} = sin.^{-1} \frac{y}{a}$$

The abscissa of the elastic curve $A\ B$, Fig. 434, II, is therefore

$$2)\ x = \sqrt{\frac{W E}{P}} \,.\, sin.^{-1} \frac{y}{a},$$

and its ordinate is

$$3)\ y = a\ sin. \left(x \sqrt{\frac{P}{W E}}\right).$$

If $x = A\ B = A\ C = l$, the length of the column, we have $y =$ the deflection $B\ C = a$; therefore

$$a = a\ sin. \left(l \sqrt{\frac{P}{W E}}\right), \text{ I.E., } sin. \left(l \sqrt{\frac{P}{W E}}\right) = 1,$$

whence

$l \sqrt{\frac{P}{W E}} = \frac{\pi}{2}$, from which we obtain the *bending force*

$$4)\ P = \left(\frac{\pi}{2 l}\right)^2 W E$$

Since this formula does not contain the deflection a, we can assume that the force P, determined by it, is capable of holding the body in equilibrium, however much the body may be bent. This peculiar circumstance is owing to the fact that the increase of the flexure is accompanied not only by an increase of resistance, but also by an increase of the lever arm a, and consequently of the moment $P\ a$ of the force.

The force necessary to rupture the pillar by *breaking it across*, is therefore

$$P = \left(\frac{\pi}{2 l}\right)^2 W E = 2{,}4674\ \frac{W E}{l^2}.$$

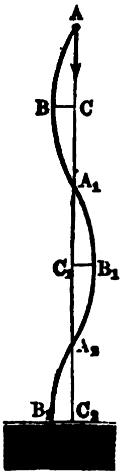
REMARK.—If we substitute in the formula $y = a\ sin. \left(x \sqrt{\frac{P}{W E}}\right)$, $P = \left(\frac{\pi}{2 l}\right)^2 W E$, we obtain the following equation of the elastic curve for this case of the action of a force

$$y = a \sin. \left(\frac{\pi x}{2 l}\right).$$

Substituting in this ..x =	0	l	$2l$	$3l$	$4l$	$5l$	$6l$, etc.,
we obtain............y =	0	a	0	$-a$	0	a	0, etc.

If, then, a column, whose length is l, is increased any amount in length, a force $P = \left(\frac{\pi}{2 l}\right)^2 W E$ will bend it in the shape of the serpentine line $A B A_1 B_1 A_2 \ldots$, Fig. 436, which is composed of a number of similar arcs $A B$ and is cut by the axis $A X$ at the distances $A A_1, A A_2, \ldots$, and at the distances $A C, A C_1, A C_2$, the curve is at its maximum distances $C B = a$, $C_1 B_1 = -a$, $C B_2 = a$ from this same axis.

Fig. 436.



§ 266. Parallelopipedical and Cylindrical Columns.—For a parallelopipedical column, the greater dimension of whose cross-section is b and the smaller one is h, we have $W = \frac{b h^3}{12}$ (see § 226), and consequently the force necessary to rupture the same by breaking it across is

$$P = \left(\frac{\pi}{2 l}\right)^2 \frac{b h^3 E}{12} = 0{,}2056 \frac{b h^3 E}{l^2}.$$

The resistance of a parallelopipedon to breaking across is directly proportional to the width b and to the cube (h^3) of the thickness or smaller dimension h of its cross-section and inversely proportional to the square (l^2) of the length.

For a *cylindrical pillar*, whose radius is r or whose diameter is d,

$$W = \frac{\pi r^4}{4} = \frac{\pi d^4}{64} \text{ (see § 231), consequently we have}$$

$$P = \left(\frac{\pi}{2 l}\right)^2 \cdot \frac{\pi r^4}{4} E = \frac{\pi^3}{16} \cdot \frac{r^4 E}{l^2} = \frac{\pi^3}{256} \cdot \frac{d^4 E}{l^2} = 1{,}9381 \cdot \frac{r^4 E}{l^2}$$
$$= 0{,}1211 \frac{d^4 E}{l^2}.$$

Therefore the (reacting) strength of a cylindrical column, by which it resists bending or breaking across, is directly proportional to the fourth power of its diameter and inversely proportional to the square of the length.

For a *hollow* column, whose radii are r and r_1, and whose diameters are d and $d_1 = \mu d$, we have

$$P = \frac{\pi^3}{16} \frac{(r^4 - r_1^4)\,E}{l^2} = \frac{\pi^3}{256} \frac{(d^4 - d_1^4)\,E}{l^2} \cdot$$

$$= \frac{\pi^3}{256} (1 - \mu^4) \frac{d^4 E}{l^2} = 0{,}1211\,(1 - \mu^4) \frac{d^4 F}{l^2}.$$

If the column $A\ B\ A$, Fig. 437, is not fixed at the lower end A, but only stands upon it, it will bend in a symmetrical curve, each half $B\ A$ and $B\ A_1$ having the form of the axis of a column fixed at one end (Fig. 434). The above formula can be applied directly to this case by substituting $\frac{l}{2}$ instead of l; l of course denotes the total length of the pillar. The proof load is therefore four times as great as in the first case, and it is

$$P = \left(\frac{\pi}{l}\right)^2 W E = \frac{\pi^2}{12} \frac{b\,h^3}{l^2} E = \frac{\pi^3}{64} \frac{d^4}{l^2} E.$$

This case of flexure occurs when, as is represented in Fig. 437,

FIG. 437

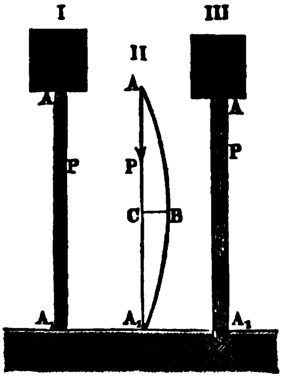
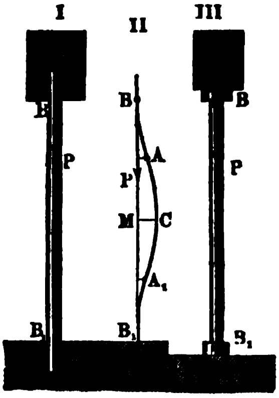


FIG. 438.



I. and III., the ends of the pillar are rounded or when they are movable around bolts. An example of the latter case is the *connecting rod* of a steam engine.

If a pillar is fixed at both ends, as is represented by $B\ A\ B_1$, Fig. 438, I. and III., its axis will be bent in a curve $B\ A\ C\ A_1\ B_1$, Fig. 438, II., with two points of inflection A and A_1, and in which the normal case of curvature is repeated four times, substituting, therefore, in the formula for the normal case $\frac{l}{4}$, instead of l, we obtain the proof load of such a pillar fixed at both ends

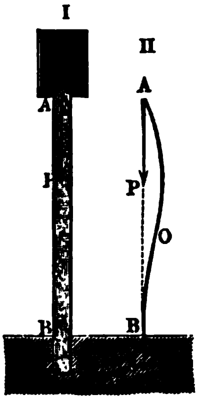
$$P = \left(\frac{2\,\pi}{l}\right)^2 W E = \frac{\pi^2}{3}\,\frac{b\,h^3}{l^2}\,E = \frac{\pi^3}{16}\,\frac{d^4}{l^2}\,E.$$

According to *Hodgkinson's* experiments, the proof load is only *twelve times* as great as in the normal case, while according to the above formula it would be *sixteen times* as great.

The principal example of this case of flexure is that of the *piston rod* of steam engines, etc.

If, finally, a column $A\ O\ B$, Fig. 439, is fixed at one end B and at the other prevented from sliding sideways, the proof load P is eight times as great as in the normal case, or

FIG. 439.



$$P = 8\left(\frac{\pi}{2\,l}\right)^2 W E = \frac{\pi^2}{6}\cdot\frac{b\,h^3\,E}{l^2} = \frac{\pi^3}{32}\,\frac{d^4\,E}{l^2}.$$

The force which is necessary to crush a column, whose cross-section is F and whose modulus of rupture is K, is given, according to § 205, by the simple formula $P = F\,K$.

If we put this force equal to the force

$$P = \left(\frac{\pi}{2\,l}\right)^2 W E$$

necessary to produce rupture by breaking across in the normal case, we obtain the equation

$$\frac{F\,l^2}{W} = \left(\frac{\pi}{2}\right)^2 \frac{E}{K}, \text{ or } l\sqrt{\frac{F}{W}} = \frac{\pi}{2}\sqrt{\frac{E}{K}}.$$

For a cylindrical pillar, whose thickness is d, in which case $\frac{F}{W} = \frac{16}{d^2}$, it follows that

$$\frac{l}{d} = \frac{\pi}{8}\sqrt{\frac{E}{K}} = 0{,}3927\,\sqrt{\frac{E}{K}}.$$

For *cast iron* $E = 17000000$ and $K = 104500$, hence

$$\sqrt{\frac{E}{K}} = \sqrt{162{,}68} = 12{,}8 \text{ and } \frac{l}{d} = 5.$$

For *wrought iron* $E = 28400000$ and $K = 31000$, hence

$$\sqrt{\frac{E}{K}} = \sqrt{916} = 30{,}3 \text{ and } \frac{l}{d} = 12.$$

Finally for *wood* we have as a mean

$E = 1664000$ and $K = 6770$, hence

$$\sqrt{\frac{E}{K}} = \sqrt{246} = 15,7 \text{ and } \frac{l}{d} = 6.$$

If a column is free at both ends, the values of $\frac{l}{d}$ are twice as great as those found above.

When the ratio of the length to the thickness is that just given, the resistance to breaking across is equal to that of crushing, and it is only when the pillars are longer than this, that the *resistance to breaking across* exceeds the resistance to crushing. In this case the dimensions of the cross-section are to be calculated by the above formula.

EXAMPLE—1) The working load of a cylindrical pine column 12 feet long and 11 inches thick, assuming 10 as a factor of safety, is

$$P = \frac{\pi^3}{64} \frac{d^4}{l^2} \frac{E}{10} = 0,4845 \left(\frac{11}{12}\right)^4 . 166400 = 80620 . 0,7061 = 56900.$$

2) How thick must such a column of cast iron be made, when its length is to be 20 feet and the load 10000 pounds? Here, if we put instead of E, $\frac{E}{10} = 1700000$, we have

$$d = \sqrt[4]{\frac{64 . P l^2}{\pi^3 . 1700000}} = \sqrt[4]{\frac{640000 . 240^2}{31 . 1700000}}$$

$$= \sqrt[4]{\frac{240^2}{82,84875}} = \sqrt{\frac{240}{9,074}} = 5,14 \text{ inches.}$$

According to the formula for the strength of crushing

$$d = \sqrt{\frac{4 P}{\pi K}},$$

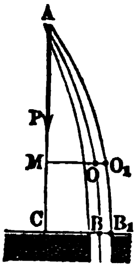
or, substituting $\frac{K}{10} = 10400$ pounds in the calculation, we have

$$d = \sqrt{\frac{4 . 10000}{\pi . 10400}} = \sqrt{\frac{400}{\pi . 104}} = \sqrt{\frac{50}{13 \pi}} = 1,106 \text{ inches.}$$

If the length of the pillar does not exceed 10 . 1,106 = 11,06 inches, the required thickness would then be but 1,106 inches.

(§ 267.) **Bodies of Uniform Resistance to Breaking Across.**—If a pillar $A B$, Fig. 440, fixed at one end, is so shaped, that in all its cross-section the strain is the same, a solid of uniform resistance is formed, which requires the minimum amount of material for its construction (see § 208 and § 253). The cross-section of such a body is certainly a maximum at the fixed end B, and it decreases gradually towards the end A. The law of this decrease is found as follows: denoting again by x and y the co-ordinates of a point O in the axis of the column, by a the tangential angle $M A O$ for this point, by W the measure of the moment of flexure, by z the radius $O O_1$ of the column at this point and by S the strain at the surface $A O_1 B_1$, which is therefore that at the point O_1 of the cross-section through O, we have

FIG. 440.



$$S = \frac{M z}{W} = \frac{P y z}{W} \text{ (see § 235) and}$$

$$M = P y = \frac{W E}{r} = - W E \frac{d\, tang.\, a}{d x},$$

(see § 218), whence

$$S = - E z \frac{d\, tang.\, a}{d x} \text{ or, since } tang.\, a = \frac{d y}{d x},$$

$$S\, d y = - E z\, tang.\, a\, d\, tang.\, a.$$

But, since for a circular cross-section $\frac{W}{z} = \frac{\pi z^3}{4}$,

$$S = P y \frac{z}{W} = \frac{4 P y}{\pi z^3}, \text{ or } \frac{\pi}{4} S z^3 = P y, \text{ and we have}$$

$$d y = \frac{\pi}{4} \frac{S}{P} d (z^3) = \frac{3 \pi}{4} \frac{S}{P} z^2 d z \text{ and } S d y = \frac{3 \pi}{4} \frac{S^2}{P} z^2 d z,$$

whence

$$\frac{3 \pi}{4} \frac{S^2}{P E} z\, d z = - tang.\, a\, d\, tang.\, a.$$

By integration we obtain

$$\tfrac{3}{4} \pi \frac{S^2}{P E} z^2 = Const. - tang.^2\, a,$$

and, if we denote the radius of the cross-section at B by r, we have

$$\tfrac{3}{4} \pi \frac{S^2}{P E} (r^2 - z^2) = tang.^2\, a, \text{ since } a = 0; \text{ hence}$$

$$tang.\, a = S \sqrt{\frac{3 \pi}{4 P E}} \cdot \sqrt{r^2 - z^2}.$$

Putting $tang.\, a = \frac{d y}{d x} = \frac{3}{4} \pi \frac{S}{P} \cdot \frac{z^2 d z}{d x}$, we obtain

$$\sqrt{\frac{3 \pi E}{4 P}} \cdot \frac{z^2 d z}{d x} = \sqrt{r^2 - z^2} \text{ and}$$

$$d x = \sqrt{\frac{3 \pi E}{4 P}} \cdot \frac{z^2 d z}{\sqrt{r^2 - z^2}} = r^2 \sqrt{\frac{3 \pi E}{4 P}} \cdot \frac{u^2 d u}{\sqrt{1 - u^2}},$$

when $\frac{z}{r}$ is denoted by u.

But

$$\frac{u^2}{1-u^2} = -\frac{1-u^2}{\sqrt{1-u^2}} + \frac{1}{\sqrt{1-u^2}} = -\sqrt{1-u^2} + \frac{1}{\sqrt{1-u^2}},$$

and therefore

$$\int \frac{u^2\,d\,u}{\sqrt{1-u^2}} = -\int \sqrt{1-u^2}\,.\,d\,u + \int \frac{d\,u}{\sqrt{1-u^2}}$$

$$= -\tfrac{1}{2}\,u\sqrt{1-u^2} + \tfrac{1}{2}\int \frac{d\,u}{\sqrt{1-u^2}}$$

$$= -\tfrac{1}{2}\,u\sqrt{1-u^2} + \tfrac{1}{2}\,\sin.^{-1} u.$$

(See the Introduction to the Calculus, Art. 27 and 26.)

Hence we have

$$x = \sqrt{\frac{3\,\pi\,E}{16\,P}}\left[r^2 \sin.^{-1}\frac{z}{r} - z\sqrt{r^2 - z^2}\right].$$

For $x = l$, $z = r$, the radius of cross-section of the base, for which $\sin.^{-1}\frac{z}{r} = \sin.^{-1} 1 = \frac{\pi}{2}$ and

$z\sqrt{r^2 - z^2} = 0$. Therefore it follows that

$l = \frac{\pi}{2}\,r^2\sqrt{\frac{3\,\pi\,E}{16\,P}}$ and that the proof load is

$$P = \left(\frac{\pi}{2\,l}\right)^2 \frac{3\,\pi\,r^4}{16}\,E = \tfrac{3}{4}\left(\frac{\pi}{2\,l}\right)^2 \frac{\pi\,r^4}{4}\,E,$$

that is, three-fourths of the proof load of a cylindrical pillar, whose radius is r (compare § 265). Consequently the radius of the base of a column of uniform strength is $= \sqrt[4]{\tfrac{4}{3}} = 1{,}075$ times the radius of a column of the same length whose proof strength is the same.

Comparing the abscissa x with the total length l of the column, we obtain

$\frac{x}{l} = \frac{\pi}{2}\left[\sin.^{-1} r - \frac{z}{r}\sqrt{1 - \left(\frac{z}{r}\right)^2}\,\right] = \frac{\pi}{2}$ times the area of the segment of a circle, whose radius $= 1$ and whose chord $= \frac{2\,z}{r}$.

If, then, we regard $\frac{2}{\pi}\frac{x}{l}$ as the area of the segment of a circle, we can determine, by means of a table of segments (see the Ingenieur, page 152), the corresponding angle ϕ at the centre, and from it we can calculate for a given abscissa x the corresponding radius of the cross-section $z = r\, sin.\, \frac{\phi}{2}$; E.G., for $x = \frac{1}{2}\, l$, $\frac{2\, x}{\pi\, l} = \frac{1}{\pi} = 0{,}3183$, and we find from the table of segments $\phi = 93°\, 49'$; hence the radius of the cross-section of the pillar is

$$z = r\, sin.\, 46°\, 50' = 0{,}729\, r.$$

To resist rupture by crushing, the radius of the cross-section of the pillar at the top must be $r_0 = \sqrt{\frac{P}{\pi\, T}}$, and this radius must always be employed for all points, where the formula for breaking across gives smaller values for z.

If the pillar stands with its base unretained, as is represented in Fig. 437, the calculation must be made in the same manner for one-half $\left(\frac{l}{2}\right)$ of it. The maximum radius r is, of course, that of the cross-section in the middle, and it corresponds to the formula

$$P = \tfrac{3}{4}\, (Z)^2 \left(\frac{\pi}{l}\right)^2 \frac{\pi\, r^3\, E}{4}.$$

§ **268. Hodgkinson's Experiments.**—The recent experiments of Mr. Hodgkinson upon the resistance of columns to breaking across (see Barlow's report in the "Philosophical Transactions," 1840) confirm, at least approximatively, the correctness of the formulas deduced in the foregoing pages. According to this experimenter the formula

$$P = \left(\frac{\pi}{2\, l}\right)^2 W\, E = \left(\frac{\pi}{2\, l}\right)^2 \frac{\pi\, d^4\, E}{64} = \left(\frac{\pi}{2\, l}\right)^2 \frac{b^4\, E}{12}$$

for prismatical columns with circular or square cross-sections is correct for wood when we introduce a particular value for E; but, on the contrary, it can be employed for wrought iron only when we substitute for d^4 the power $d^{3,55}$, and for cast iron it is sufficiently correct when d^4 and l^2 are replaced by the powers $d^{3,55}$ and $l^{1,7}$.

The chief results of Hodgkinson's experiments upon prismatic pillars with *circular* and *square cross-sections* are given in the following table. The coefficients given in it refer to the case when the pillars are cut off at both ends *at right angles* to their longitudinal axis and repose upon these bases. When the ends are rounded so that these extremities of the columns are not prevented from assuming any inclination, these coefficients are nearly *three times* as small. If, on the contrary, the column is fixed at one end and the other capable of turning, the coefficient is but *half as great* as in the first case. If, finally, one end of the pillar is fixed and the other capable of being turned and of sliding, the proof load is but one-tenth of that of the first case, where both ends are fixed.

TABLE OF THE FORCES NECESSARY TO RUPTURE COLUMNS BY BREAKING THEM ACROSS.

Name of the prismatic pillars.	Breaking stress.		
	English measure, tons.	French measure, kilograms.	Prussian measure, new pounds.
Cast-iron pillars with circular cross-section	$44,16 \frac{d^{3,55}}{l^{1,7}}$	$10900 \frac{d^{3,55}}{l^{1,7}}$	$94700 \frac{d^{3,55}}{l^{1,7}}$
Wrought-iron pillars with circular cross-section . .	$133,75 \frac{d^{3,55}}{l^2}$	$46140 \frac{d^{3,55}}{l^2}$	$284400 \frac{d^{3,55}}{l^2}$
Square pillars of dry Dantzic oak	$10,95 \frac{b^4}{l^2}$	$2480 \frac{b^4}{l^2}$	$23570 \frac{b^4}{l^2}$
Square pillars of dry fir . .	$7,81 \frac{b^4}{l^2}$	$1770 \frac{b^4}{l^2}$	$16840 \frac{b^4}{l^2}$

In the column for English measure d and b are given in inches, l in feet, and P in tons of 2240 pounds. In that for the French measures, on the contrary, d and b are given in centimetres, l in decimetres, and P in kilograms, and in the last column d and b are expressed in inches, l in feet, and P in new pounds.

Mr. Hodgkinson also found that cast-iron pillars, with round ends, were sooner crushed than broken across, when $l < 15\,d$, and when the ends were flat as long as l was $< 30\,d$. Dry wood possesses double as much strength as timber just felled. When employing this formula for calculating the working load of columns, we employ a coefficient of security of $\frac{1}{4}$ to $\frac{1}{12}$ or a factor of safety of from 4 to 12.

Hence, with sextuple security, we can put for *cast-iron pillars*, when d and l are given in inches,

$$P = \frac{44,16}{6} \,.\, 12^{1,7} \,.\, \frac{d^{3,55}}{l^{1,7}} = \frac{44,16}{6} \,.\, 68,3 \, \frac{d^{3,55}}{l^{1,7}} = 502,688 \, \frac{d^{3,55}}{l^{1,7}} \text{ tons,}$$

and $d = 0,0173\,(P\,l^{1,7})^{0,2817}$ inches.

For *wrought-iron pillars* we have, when we adopt the same coefficient of security,

$$P = 3210 \, \frac{d^{3,55}}{l^2} \text{ tons and}$$

$$d = 0,01028\,(P\,l^2)^{0,2817} \text{ inches.}$$

For *pillars of oak wood*, employing a coefficient of security of $\frac{1}{10}$, we have

$$P = 157,68 \left(\frac{b}{l}\right)^2 F = 157,68 \frac{b^4}{l^2} = 267,69 \frac{d^4}{l^2} \text{ tons},$$

$$b = 0,2822 \, (P\, l^2)^{\frac{1}{4}} \text{ and } d = 0,2472 \, (P\, l^2)^{\frac{1}{4}} \text{ inches.}$$

Finally, for *pillars of fir wood*, we have

$$P = 112,46 \left(\frac{b}{l}\right)^2 . F = 112,46 \frac{b^4}{l^2} = 190,92 \frac{d^4}{l^2},$$

$$b = 0,307 \, (P\, l^2)^{\frac{1}{4}} \text{ and } d = 0,269 \, (P l^2)^{\frac{1}{4}}.$$

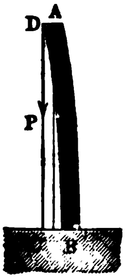
EXAMPLE.—For a cylindrical fir post, 11 inches thick and 12 . 12 = 144 inches long, fixed at both ends, the proof load is

$$P = 190,92 \left(\frac{121}{144}\right)^2 = 184,802 \text{ tons.}$$

If the ends of such a pillar are capable of moving freely, the proof load $P = \frac{1}{4} P = 44,934$ tons, while according to the theoretical formula we have $P_1 = 56900$ lbs. $= 25,402$ tons. (See Example 1 of § 266.)

§ 269. More Simple Determination of the Proof Load of Columns.—The foregoing formulas for the bending and breaking across of pillars are calculated upon the assumption that the force P is applied exactly at the end A of the longitudinal axis of the pillar. Now since in practice this is scarcely ever perfectly true, and since the action of the force ceases to be central as soon as the pillar bends, it is advisable, in determining the proof load of a beam, to take into consideration from the beginning the eccentricity of the point of application of the force. Assuming that the point of application D of the force P is at a distance $D\,A = c$ from the end A of the axis $A\,B$, Fig. 441, of the column and that the deflection $B\,C = a$ of the pillar is small, compared with c, we can consider the elastic curve formed by the axis of the pillar to be a circle, whose radius is $r = \frac{l^2}{2\,a}$. But now

FIG. 441.



$$P\,(a + c)\, r = W\,E, \text{ whence}$$

$$P\,(a + c)\, l^2 = 2\, W\,E\,a, \text{ as well as}$$

$$a = \frac{P\,l^2\,c}{2\,W\,E - P\,l^2} \text{ and}$$

$$a + c = \frac{2\,W\,E\,c}{2\,W\,E - P l^2}.$$

If F denotes the cross-section of the pillar and e half its thickness, measured in the plane $A\,B\,D$, the uniform strain produced in each cross-section by the force P is

$$S_1 = \frac{P}{F},$$

and the strain produced at the exterior surface by the moment $P\,(a + c)$ of the force is

$$S_2 = \frac{P\,(a + c)\,e}{W} = \frac{2\,P\,E\,c\,e}{2\,W\,E - P\,l^2},$$

and consequently the maximum strain in the pillar is

$$S = S_1 + S_2 = \frac{P}{F} + \frac{2\,P\,E\,c\,e}{2\,W\,E - P\,l^2} = \frac{P}{F}\left(1 + \frac{2\,E\,F\,c\,e}{2\,W\,E - P\,l^2}\right).$$

Putting $S =$ to the modulus proof strength T, we have

$$P\left(1 + \frac{2\,E\,F\,c\,e}{2\,W\,E - P\,l^2}\right) = F\,T, \text{ or}$$

$$P\,(2\,W\,E - P\,l^2 + 2\,E\,F\,c\,e) = (2\,W\,E - P\,l^2)\,F\,T.$$

Now if $P\,l^2$ is small compared with $(W + F\,c\,e)$, we can put

$$P = \frac{2\,W\,E\,F\,T}{2\,E\,(W + F\,c\,e) + F\,T\,l^2} = \frac{F\,T}{1 + \frac{F\,c\,e}{W} + \frac{F\,T}{2\,W\,E}\,l^2}, \text{ or}$$

$$P = \frac{F\,T}{\phi + \psi\,\frac{l^2}{d^2}},$$ in which ϕ and ψ are empirical numbers.

The civil engineer Love (see "Mémoire sur la Résistance du fer et de la fonte, etc.," Paris, 1852) deduced from the experiments of Hodgkinson the values $\phi = 0{,}45$ and $\psi = 0{,}00337$; hence we have

$$P = \chi\,F\,T = \frac{F\,T}{1{,}45 + 0{,}00337\left(\frac{l}{d}\right)^2},$$

from which the following table for the coefficient

$$\chi = \frac{1}{1{,}45 + 0{,}00337\left(\frac{l}{d}\right)^2}$$ has been calculated.

$\frac{l}{d} =$	10	20	30	40	50	60	70	80	90	100
$\chi =$	0,559	0,357	0,223	0,146	0,101	0,0735	0,0556	0,0435	0,0347	0,0285

These values of χ must be multiplied by the modulus of proof strength T for compression, when the modulus of proof strength for long pillars is to be determined for a given ratio of length.

General Morin gives, after Rondelet, the following table, which furnishes too great values for χ, when the pillars are of medium length.

$\frac{l}{d} =$	1	12	24	36	48	60	72
$\chi =$	1	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{24}$

EXAMPLE—1) What load can a pine post bear, whose length is 15 feet and whose thickness is 12 inches? According to the table upon page 404, the modulus of proof strength for a short pillar is $T = 2600$; but since the ratio of the length to the thickness is $\frac{l}{d} = \frac{15}{1}$, we have

$$\chi = \frac{1}{1,45 + 0,00337 \cdot 15^2} = \frac{1}{2,208} = 0,453,$$

whence we obtain the modulus of proof strength $\chi\, T = 0,453 \cdot 2600 =$ 1178 pounds; hence the proof strength of the pillar is

$$P = 1178 \frac{\pi d^2}{4} = 1178 \cdot 0,7854 \cdot 144 = 133000 \text{ pounds.}$$

If we employ a factor of safety 3, we can put

$$P = \frac{133000}{3} = 44300 \text{ pounds.}$$

2) How thick must a hollow cylindrical pillar of cast iron, 25 feet long, be made, when it stands vertical and is required to support a load $P = 100000$ pounds? Assuming the diameter d_1 of the hollow part to be three-fifths of the exterior diameter (d) of the pillar, we can substitute in the theoretical formula

$$P = \frac{\pi^3}{4} \cdot \frac{r^4}{l^2} E \text{ (§ 226),}$$

$$r^4 = \frac{d^4 - d_1{}^4}{16} = \frac{d^4}{16}[1 - (\tfrac{3}{5})^4] = 0,0544\, d^4, \text{ whence we obtain}$$

$$d = \sqrt[4]{\frac{4 P l^2}{0,0544\, \pi^3 E}}.$$

Substituting in this expression $P = 100000$, $l^2 = (25 \cdot 12)^2 = 90000$, $\pi^3 = 31$, and, instead of E,

$$\frac{E}{10} = \frac{14220000}{10} = 1422000,$$

we obtain the required thickness of the pillar

$$d = \sqrt[4]{\frac{400000 \cdot 90}{0,0544 \cdot 31 \cdot 1422}} = \sqrt[4]{\frac{6000000}{1,6864 \cdot 237}}$$

$$= \sqrt[4]{\frac{187500}{0,0527 \cdot 237}} = 11,07 \text{ inches.}$$

If we make $d = 11,25$ inches, we obtain $d_1 = 0,6 \cdot 11,25 = 6,75$ inches.

According to our last formula we have, when we assume

$$\frac{l}{d} = \frac{25}{1} = 25,$$

for the required cross-section of the pillar

$$F = \left[1,45 + 0,00337 \left(\frac{l}{d}\right)^2\right] \frac{P}{T} = \frac{3,556 \,.\, 100000}{T} = \frac{355600}{T},$$

and putting, according to § 212,

$$T = \frac{18700}{3} = 6200 \text{ pounds},$$

we obtain

$$F = \frac{355600}{6200} = 57,35, \text{ and therefore, since}$$

$$F = \frac{\pi}{4}(d^2 - d_1^2) = [1 - (\tfrac{3}{5})^2]\,\frac{\pi d^2}{4} = 0,16\,\pi\,d^2,$$

the required exterior diameter of the pillar

$$d = \sqrt{\frac{F}{0,16\,\pi}} = \sqrt{\frac{57,35}{0,16\,\pi}} = 10,68 \text{ inches.}$$

Assuming $d = 11$ inches, we obtain

$d_1 = 0.6\,d = 0,6 \,.\, 11 = 6,6$ inches.

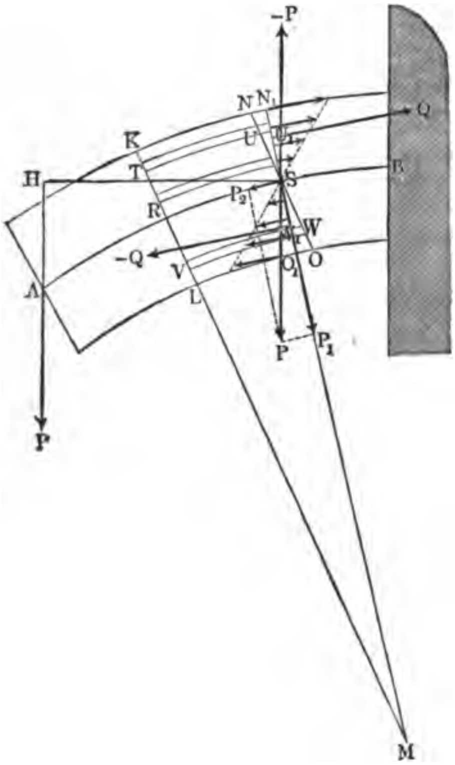
CHAPTER V.

COMBINED ELASTICITY AND STRENGTH.

§ 270. Combined Elasticity and Strength.—A body is often acted upon at the same time by two forces, E.G. a tensile and a bending one, etc., by which a double change of form is produced, as, E. G., an extension and a bending. We call the force with which a body resists this two-fold change of form its *combined elasticity* and *strength*, and we will proceed to investigate the most important cases of this kind.

Properly speaking, the case (§ 214) of the bending of a body $A\ K\ B\ O$, Fig. 442, is really one of combined strength; for the force $\overline{A\ P} = P$, which acts at the end A of the body, can be resolved into a couple $(P, -P)$ and a force $\overline{S\ P} = P$. The former, which alone we have previously considered, tends to bend the portion $A\ S$ of the body, and the latter tends to tear this piece from the remaining portion $S\ B$. The latter force can be resolved into two components

FIG. 442.



$$P_1 = P\ cos.\ a$$

and

$$P_2 = P\ sin.\ a$$

(§ 215), one of which acts at right angles to the direction of the fibres and the other in the direction of the axis of the fibres. The latter component combines with the strain in the fibres produced by the bending and increases the extensions upon the side of the tensile s'rains and decreases the compression upon the other side. The magnitude of the extension of each fibre

$$R\ S = K\ N,$$

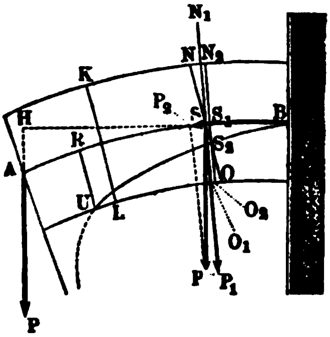
etc., whose length = 1, by the tensile force $P\ sin.\ a$ is (§ 204)

$$\sigma_1 = \frac{P\ sin.\ a}{F\ E},$$

F denoting the cross-section $N\ O$ of the body.

If at this distance from the line $N_1\ O_1$, Fig. 443, which determines the ends of the fibres, that have been extended by the bending, we draw a line $N_2\ O_2$ parallel to it, it will form the boundary of the fibres which have been submitted to both causes of change of length, and it will cut the original limit in a point S_2, which corresponds to the fibre, that is unchanged in length, and consequently gives the new or true position of the neutral axis. The distance $S\ S_2 = e_1$ of this neutral axis from the original one, which corresponds to the moment of flexure, is determined by the proportion

FIG. 448.



$$\frac{S\,S_2}{S\,S_1} = \frac{S\,N}{N\,N_1}, \text{ i.e., } \frac{e_1}{e} = \frac{\sigma_1}{\sigma},$$

whence $e_1 = \frac{e}{\sigma}\,\sigma_1$.

But we have also $\frac{\sigma}{e} = \frac{1}{r}$ (§ 235), hence

$$e_1 = r\,\sigma_1 = \frac{P\,r\,sin.\,a}{F\,E}.$$

The radius of curvature r_1 of the neutral axis determined in this more accurate manner is greater by the quantity (e_1) than that of the neutral axis previously considered; hence we have

$$r_1 = r + e_1 = r\,(1 + \sigma_1) = r\left(1 + \frac{P\,sin.\,a}{F\,E}\right).$$

The angle a, which the variable cross-section $N_1\,O_1$ or $N_2\,O_2$ forms with the direction of the force P, is equal to the tangential angle a (found in § 216); hence, as this angle is small, we can put

$$sin.\,a = a = \frac{P\,(l^2 - x^2)}{2\,W\,E},$$

or, since
$$r = \frac{W\,E}{P\,x} \text{ (§ 215)},$$

$$r\,sin.\,a = r\,a = \frac{l^2 - x^2}{2\,x}, \text{ from which we obtain}$$

$$e_1 = \frac{P\,(l^2 - x^2)}{2\,F\,E\,x}$$

Hence for the point B, where the beam is fixed and for which $x = l$, we have $e_1 = 0$, and for the point A at the other end, where $x = 0$, $e_1 = \frac{P\,l^2}{0} = \infty$; on the contrary, for $x = \frac{P\,(l^2 - x^2)}{2\,F\,E\,e}$ we have $e_1 = e$; consequently the neutral axis coincides at B with the original one, and in passing from B to A it separates more and more from it, until, finally, it reaches the concave side of the body, and, if prolonged beyond the body, at the end A it is at an infinite distance from the other axis.

The maximum extension produced by the flexure is

$$\sigma = \frac{P\,e\,x}{W\,E},$$

and that produced by the tensile force $P\, sin.\, a$ is

$$\sigma_1 = \frac{P\, sin.\, a}{F E},$$

hence the total extension is

$$N N_2 = N N_1 + N_1 N_2 = \frac{P}{E}\left(\frac{e\,x}{W} + \frac{sin.\, a}{F}\right),$$

and, if the latter has reached the limit of elasticity $\frac{T}{E}$, we can put

$$P\left(\frac{e\,x}{W} + \frac{sin.\, a}{F}\right) = T,$$

and the proof load is

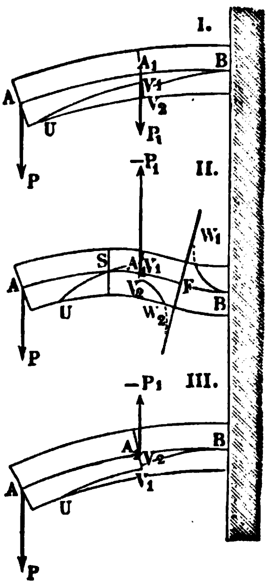
$$P = \frac{W\,T}{e\,x + \frac{W}{F}\, sin.\, a} = \frac{W\,T}{e\,x + \frac{P\,(l^2 - x^2)}{2\,F\,E}}.$$

For a moderate deflection, which is all these girders are generally exposed to, this value is a minimum for $x = l$, and it is

$$P = \frac{W\,T}{e\,l},$$

as we have already found.

FIG. 444.



REMARK.—If the girder, as, E.G., $A\,A_1\,B$, Fig. 444, I., II., III., is acted upon by two forces, two or even three displacements of the neutral axis from the centre of gravity may take place. If the two forces act in the same direction as represented in Fig. 444, I., this displacement on one side of the cross-section A_1 is determined by the formula

$$e_1 = \frac{P\,r\, sin.\, a}{F\,E},$$

and, on the contrary, on the other side by the formula

$$e_2 = \frac{(P + P_1)\,r\, sin.\, a}{F\,E}.$$

At the point of application A_1 this displacement changes from

$$\overline{A_1\,V_1} = e_1 = \frac{P\,r\, sin.\, a}{F\,E} \text{ to}$$

$$\overline{A_1\,V_2} = e_2 = \left(\frac{P + P}{P}\right) e_1,$$

when we pass from one side to the other, on the contrary, at the fixed point B, where $a = 0$, we have $e_2 = 0$.

If the two forces act in opposite directions and the moment

$$P_1 \,.\, \overline{A_1 B} = P_1 l_1$$

of the negative force is greater than the moment

$$P \,.\, \overline{A B} = P (l_1 + l)$$

of the positive one, in which case the girder is bent in two opposite directions, which meet in a point of inflection F, the neutral axis consists of three branches $U V_1$, $V_2 W_2$ and $W_1 B$ (Fig. 444, II.), which are not continuous, and the normals at the point of inflection F is an asymptote to the last two of these curves; for here $r = \infty$ and consequently

$$e_1 = \frac{P r \sin. a}{F E} = \infty.$$

If, although the forces act in opposite directions, we have $P(l + l_1) > P_1 l_1$, as represented in Fig. 444, III., the displacement of the neutral axis upon one side of A_1 is

$$\overline{A_1 V_1} = e_1 = \frac{P r \sin. a}{F E},$$

and that upon the other is

$$\overline{A_1 V_2} = e_2 = \frac{(P - P_1) r \sin\, a}{F E},$$

and at the cross-section through A_1 there is a break in the two branches $U V_1$ and $V_2 B$ of the neutral axis, the value of which is

$$\overline{V_1 V_2} = \frac{P_1 r \sin. a}{F E}.$$

§ 271. Eccentric Pull and Thrust.—If a column $A B$, Fig. 445 and 446, acted upon by a tensile or compressive force, whose direction, although parallel to, is not that of the longitudinal axis of the body, the combined elasticity and strength will come into play. This eccentric force can, as we know, be replaced by a force P in the direction of the axis, and a couple $(P, - P)$, whose lever arm c is the distance $C A$ of the point of application of the force P from the axis of the body, and whose moment is therefore $= P c$. The force $A P = P$ in the line of the axis produces in all the fibres the constant strain $S_1 = \frac{P}{F}$, in which F denotes the cross-section of the body; the

FIG. 445.

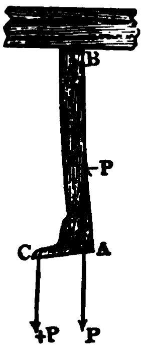
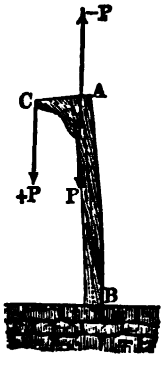


FIG. 446.



couple, on the contrary, bends the body in a curve, whose radius is determined by the well-known formula (§ 215) $P x r = W E$, in which we must substitute for the moment of the force the moment $P c$ of the couple. Consequently $r = \frac{W E}{P c}$ is constant, when W or the cross-section F is constant, and therefore the curve formed by the axis of the body is an *arc of a circle.*

If e is the maximum distance of the fibres from the neutral axis passing through the cross-section of the body, we have the maximum strain produced in the body by the couple

$$S_2 = \frac{P c e}{W},$$

and hence the total strain is

$$S = S_1 + S_2 = \frac{P}{F} + \frac{P c e}{W},$$

consequently, when we put this equal to the modulus of proof strength T, or assume that the most remote fibre is strained to the limit of elasticity, we obtain

$$T = \frac{P}{F} + \frac{P c e}{W} = \left(1 + \frac{F c e}{W}\right) \frac{P}{F}.$$

Hence the *proof load of the pillar* is

$$P = \frac{F T}{1 + \frac{F c e}{W}},$$

E.G., for one with *a rectangular cross-section,* the dimensions of which are b and h,

$$P = \frac{F T}{1 + \frac{6 c}{h}},$$

and for one with a *circular cross-section,* whose radius is r,

$$P = \frac{F T}{1 + \frac{4 c}{r}}.$$

From this we see that the strength of a body is tried much more severely by an eccentric pull or thrust than by an equal one acting in the direction of the longitudinal axis of the body.

If the column is prevented from bending by a support upon the side, as, E.G., $B A C$, Fig. 447, represents, P remains of course $= F T$.

If the force acts at the periphery of a parallelopipedical pillar $A B$, Fig. 448, and at the distance $c = \frac{h}{2}$ from the axis, we have

$$P = \frac{F\,T}{1 + 3} = \tfrac{1}{4}\,F\,T;$$

and the proof load is but one-fourth of what it would be if the weight were applied in the prolongation of the axis of the body (Fig. 449).

Fig. 447.

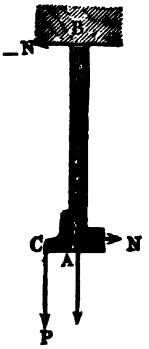
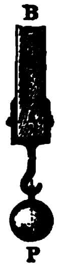


Fig. 448.

Fig. 449.



For a cylindrical pillar, with a force acting at the circumference, we have $c = r$, and consequently

$$P = \frac{F\,T}{1 + 4} = \tfrac{1}{5}\,F\,T,$$

i.e., but one-fifth what it would be if its point of application was in the axis of the body.

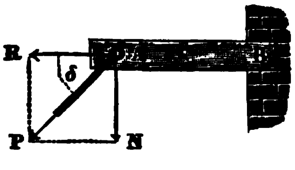
These formulas can be applied to rupture by extension, compression and breaking across; it is only necessary for each species of separation to substitute a different coefficient of ultimate strength, or put

$$P = \frac{F\,K}{1 + \frac{F\,c\,e}{W}} = \frac{F}{\frac{1}{K_1} + \frac{F\,c\,e'}{W\,K_2}},$$

in which K_1 denotes the modulus of rupture by compression (or extension) and K_2 that for breaking across.

§ **272. Oblique Pull or Thrust.**—The theory of combined elasticity and strength is particularly applicable to the case, where the direction of the force P forms an acute angle $R\,A\,P = \delta$ with the axis of the beam $A\,B$, Fig. 450. One of the two components $R = P \cos.\,\delta$ acts as a tensile force and the other $P \sin.\,\delta$ as a bending one upon the body, and the strain

Fig. 450.



$$S_1 = \frac{P \cos.\,\delta}{F},$$

produced in the whole cross-section by the first component combines with the strain

$$S_2 = \frac{P \sin.\,\delta\,.\,l\,e}{W\,e},$$

produced by the moment $P\,l \sin.\,\delta$ of the second component in the outside fibres, and causes the strain

$$T = S = S_1 + S_2 = \frac{P \cos. \delta}{F} + \frac{P\, l\, e \sin. \delta}{W},$$

or more simply

$$T = P\left(\frac{\cos. \delta}{F} + \frac{l\, e \sin. \delta}{W}\right).$$

Hence the required proof load is

$$P = \frac{F\, T}{\cos. \delta + \frac{F\, l\, e}{W} \sin. \delta},$$

or, inversely, the required cross-section is

$$F = \frac{P}{T}\left(\cos. \delta + \frac{F\, l\, e}{W} \sin. \delta\right).$$

Or, if we substitute a modulus of proof strength T_1 for bending different from that (T) for extension we have

$$F = P\left(\frac{\cos. \delta}{T} + \frac{F\, l\, e}{W\, T_1} \sin. \delta\right).$$

For a *parallelopipedical girder* we have

$\frac{F\, e}{W} = \frac{6}{h}$, and consequently

$$F = P\left(\frac{\cos. \delta}{T} + \frac{6\, l}{h\, T_1} \sin. \delta\right),$$

and for a *cylindrical one*

$\frac{F\, e}{W} = \frac{4}{r}$, whence

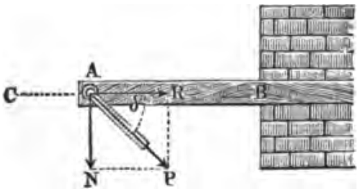
$$F = P\left(\frac{\cos. \delta}{T} + \frac{4\, l}{r\, T_1} \sin. \delta\right).$$

The same formula holds good for the case represented in Fig. 451, in which the first component R produces compression in the girder. If here again δ denotes the angle, which the direction of the force P makes with the axis of the girder, the values of the components are

$$R = P \cos. \delta \text{ and}$$
$$N = P \sin. \delta.$$

In order to find the proof load of the girder, we must combine the strain produced by R

FIG. 451.



$$S_1 = \frac{P \cos. \delta}{F}$$

with the greatest strain

$$S_2 = \frac{P\, l\, e\, sin.\, \delta}{W}$$

produced by the bending, and then we must substitute in the formula

$$T = P\left(\frac{cos.\, \delta}{F} + \frac{l\, e\, sin.\, \delta}{W}\right) \text{ or}$$

$$F = \frac{P}{T}\left(cos.\, \delta + \frac{F\, l\, e}{W}\, sin.\, \delta\right)$$

just found, instead of T, not the modulus of proof strength for extension, but that for compression.

In both the cases treated above the displacement of the neutral layer of fibres from the centre of gravity is

$$e_1 = \frac{\sigma_1}{\sigma_2}\, e = \frac{S_1}{S_2}\, e = \frac{W\, cotg.\, \delta}{F\, e\, x},$$

which, E.G., for parallelopipedical beams, becomes

$$e_1 = \frac{h\, cotg.\, \delta}{6\, x}.$$

It is also easy to see that by the combination of the maximum extension or compression with the extension or compression of the fibres, which is equally distributed over the entire cross-section of the body, there is produced an extension or compression

$$\sigma_1 \pm \sigma_2 = \frac{S_1 \pm S_2}{E} = \frac{P}{E}\left(\frac{cos.\, \delta}{F} \pm \frac{l\, e\, sin.\, \delta}{W}\right).$$

If we introduce the modulus of proof strength T and for the sake of security employ for wood and iron only $\frac{T}{3}$, we obtain

1) for *wood in both cases*

$$P = \frac{780\, F}{cos.\, \delta + \frac{6\, l}{h}\, sin.\, \delta} = \frac{780\, F}{cos.\, \delta + \frac{4\, l}{r}\, sin.\, \delta},$$

2) for *cast iron*, in the first case (Fig. 450)

$$P = \frac{3640\, F}{cos.\, \delta + \frac{6\, l}{h}\, sin.\, \delta} = \frac{3640\, F}{cos.\, \delta + \frac{4\, l}{r}\, sin.\, \delta},$$

and in the second case (Fig. 451)

$$P = \frac{9360\, F}{cos.\, \delta + \frac{6\, l}{h}\, sin.\, \delta} = \frac{9360\, F}{cos.\, \delta + \frac{4\, l}{r}\, sin.\, \delta}.$$

§ 273. The case just treated occurs often in practice. If, E.G., a weight P is hung from a girder $A\ B$, Fig. 452, which is inclined to the horizon, we have, when the angle of inclination of the direction of the axis is $P\ A\ R = \delta$, the tensile force $R = P\ cos.\ \delta$ and the bending force $N = P\ sin.\ \delta$, and therefore

$$P = \frac{F\ T}{cos.\ \delta + \frac{6\ l}{h}\ sin.\ \delta}.$$

FIG. 452.

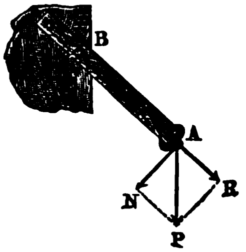
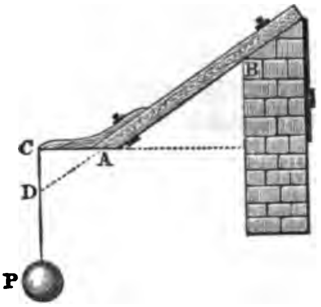


FIG. 453.



If, as is represented in Fig. 453, not only the direction of the stress P is inclined to the axis of the body, but also its point of application lies without it, in calculating the proof load we must consider the point of application transported to D in the prolongation of the axis $A\ B$ of the girder, I.E. we must substitute in place of the length $B\ A = l$ the length $B\ D = B\ A + A\ D = l + \frac{c}{sin.\ \delta}$, in which the horizontal distance $C\ A$ is denoted by c, and the angle $C\ D\ A$, formed by the axis of the girder with the vertical, is represented by δ.

In like manner, for the pillar $A\ B$, Fig. 454, which is inclined at an angle δ to the vertical, we have the proof load

$$P = \frac{F\ T}{cos.\ \delta + \frac{6\ l}{h}\ sin.\ \delta} = \frac{F\ T}{cos.\ \delta + \frac{4\ l}{r}\ sin.\ \delta},$$

in which we must substitute the modulus of proof strength for compression, while in the former case we should employ that for extension.

If a loaded girder $A\ A$, Fig. 455, is not freely supported, but wedged between two walls, a decomposition of the forces takes place into components producing compression and into components producing a flexure. If the terminal surfaces A, A of this beam form an angle δ with its cross-section, and if a force P acts in the middle B of the girder, the reactions of the walls upon the ends of the girder are Q and Q, and these forces are inclined at an angle δ to the horizon and give a resultant $\overline{CP} = - P$, which balances the force P.

FIG. 454.

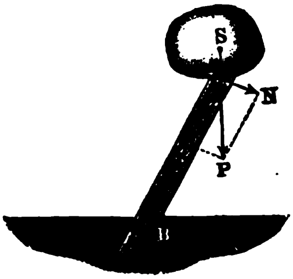
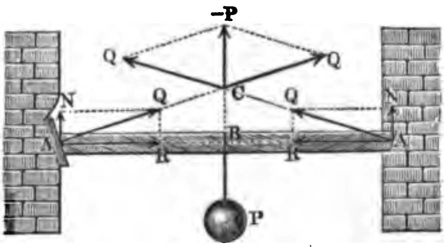


FIG. 455.



Hence

$$P = 2\, Q \cos. A\, C\, P = 2\, Q \sin. \delta,$$

or inversely

$$Q = \frac{P}{2 \sin. \delta}.$$

The reactions of the walls can be decomposed into a compressive force in the direction of the axis of the girder

$$R = Q \cos. \delta = \frac{P}{2} \frac{\cos. \delta}{\sin. \delta} = \tfrac{1}{2} P \operatorname{cotg.} \delta$$

and into a force

$$N = Q \sin. \delta = \frac{P}{2},$$

which is perpendicular to the latter and produces a bending; consequently we have

$$T = S = S_1 + S_2 = \frac{R}{F} + \frac{N . \frac{1}{2} l e}{W},$$

i.e.

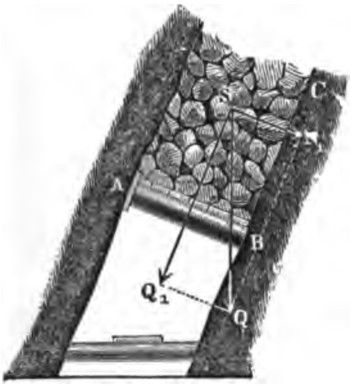
$$T = \frac{P \operatorname{cotg.} \delta}{2 F} + \frac{P l e}{4 W},$$

and the proof load of the girder is

$$P = \frac{2 F T}{\operatorname{cotg.} \delta + \frac{1}{2} \frac{F l e}{W}}.$$

The condition of affairs is the same, when an inclined prop $A\, B$, Fig. 456, carries a load which has been dumped upon it. But here Q can be resolved into a force Q_1 at right angles to the axis of the prop and into a force N_1 at right angles to the side (in miners' language, the floor). Neglecting, for greater safety, the friction of the loose masses of stone upon the floor and denoting the angle formed by the terminal surfaces of the prop with its cross-section by δ, and the inclination of the floor $B\,C$ to the horizon by β, we obtain $Q_1 = Q \sin.\,\beta$ and

FIG. 456.



$$= \frac{2\,F\,T}{cotg.\,\delta + \frac{1}{4}\,\frac{F\,l\,e}{W}}$$

(see § 240), and therefore

$$Q = \frac{2\,F\,T}{\left(cotg.\,\delta + \frac{1}{4}\,\frac{F\,l\,e}{W}\right)\sin.\,\beta}.$$

EXAMPLE—1) What must be the dimensions of the cross-section of the inclined girder $A\,B$, Fig. 452, which is made of pine and is 9 feet long and whose direction forms an angle of 60° with the horizon, when it bears at the extremity A a weight $P = 6000$ pounds? The formula

$$P = \frac{F\,T}{\cos.\,\delta + \frac{6\,l}{h}\sin.\,\delta}$$

gives, when we substitute $P = 6000$ pounds, $T = 780$, $\delta = 90° - 60° = 30°$ and $l = 9\,.\,12 = 108$ inches, and assume $\frac{b}{h} = \frac{5}{7}$,

$$F = b\,h = \tfrac{5}{7}\,h^2 = \frac{6000}{780}\left(\cos.\,30° + \frac{6\,.\,108}{h}\sin.\,30°\right), \text{ I.E.}$$

$$h^2 = 10{,}77\left(0{,}866 + \frac{648\,.\,0{,}500}{h}\right) = 9{,}33 + \frac{3489}{h}.$$

Approximatively, we have

$$h = \sqrt[3]{3489} = 15{,}17,$$

more accurately

$$h = \sqrt[3]{3489 + 9{,}33\,.\,15{,}17} = \sqrt[3]{3631} = 15{,}37 \text{ inches},$$

and consequently

$$b = \tfrac{5}{7}\,h = 10{,}98 \text{ inches}.$$

2) At what distance from each other must two 12 inches thick collars $A\,B$ of a so-called overhand stoping $A\,B\,C$, Fig. 456, be laid, when the gob is piled 60 feet high upon it in a vein 4 feet thick, dipping at 70°, if we assume that the weight of the gob is 65 pounds per cubic foot? Denoting the required distance by x, we have the weight upon each collar

$Q = 4 \,.\, 60 \,.\, 65\, x = 15600\, x$, and consequently the pressure upon each collar is

$$Q_1 = Q\, sin.\, 70^\circ = 15600\, x\, sin.\, 70^\circ = 15600 \,.\, 0{,}9397\, x = 14659\, x \text{ lbs.}$$

If the ends $A\ A$ of the collar form an angle of 70° with the axis, or if $\delta = 20^\circ$, we have

$$14659\, x = \frac{2\, F\, T}{cotg.\, 20^\circ + \frac{2\, l}{d}} = \frac{2 \,.\, 113{,}1 \,.\, 780}{2{,}747 + \frac{2 \,.\, 48}{12}} = \frac{176436}{10{,}747},$$

and therefore

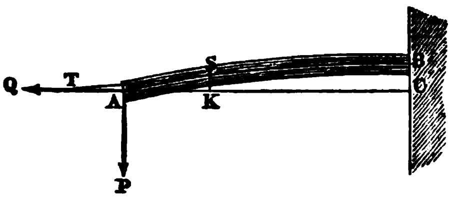
$$x = \frac{176436}{10{,}747 \,.\, 14659} = 1{,}12 \text{ feet} = 13{,}44 \text{ inches.}$$

The required distance between the two collars is therefore

$$x - d = 1{,}44 \text{ inches.}$$

(§ 274.) Flexure of Girders Subjected to a Tensile Force.—The *normal proof load P* of a girder $A\ B$, Fig. 457, is diminished by the application of a small force in the direction of the axis only when the girder is short. If, on the contrary, the length of the girder and the tensile force exceed certain limits, the moment of the latter acts in the opposite direction to the moment of the bending stress, thus diminishing the deflection of the body and *increasing* its proof load.

FIG. 457.



If we put again the co-ordinates of the elastic curve $A\ S\ B$, Fig. 457, formed by the axis of the girder, $A\ K = x$ and $K\ S = y$, we have the moment of the forces in reference to a point S in the axis

$$P\, x - Q\, y,$$

we can therefore write (according to § 215)

$$(P\, x - Q\, y)\, r = W\, E,$$

substituting

$$r = -\frac{d\, x}{d\, a},$$

in which a denotes the tangential angle $S\ T\ K$, and denoting, in order to simplify the expression, $\sqrt{\frac{P}{W\, E}}$ by p, and $\sqrt{\frac{Q}{W\, E}}$ by q, we obtain the equation

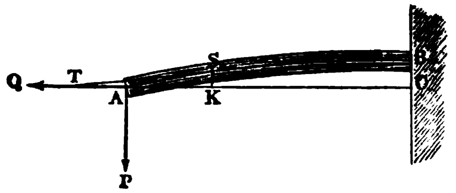
$$d\, a = -\frac{d\, x}{r} = -\frac{(P\, x - Q\, y)\, d\, x}{W\, E} = -(p^2\, x - q^2\, y)\, d\, x.$$

Now making

$$1)\ y = \frac{p^2 x}{q^2} - (m\,\varepsilon^{qx} + n\,\varepsilon^{-qx}),$$

in which m and n denote constants, to be determined, and ε the base of the Naperian system of logarithms (see Introduction to the Calculus, Art. 19), we obtain

FIG. 458.



$$2)\ a = \frac{dy}{dx} = \frac{p^2}{q^2} - (m\,\varepsilon^{qx} - n\,\varepsilon^{-qx})\,q,$$

and since the differential of the last equation, viz.,

$$d\,a = -\,(m\,\varepsilon^{qx} + n\,\varepsilon^{-qx})\cdot q^2\,d\,x,$$

when substituted in equation 1), gives the above fundamental formula

$$d\,a = \left(y - \frac{p^2 x}{q^2}\right) q^2\,d\,x = -\,(p^2 x - q^2 y)\,d\,x,$$

the correctness of the above expression for y is proved.

Since for $x = 0$ we have $y = 0$, we obtain by substituting these values in 1) the following equation

$$0 = 0 - (m\,\varepsilon^0 + n\,\varepsilon^0),\ \text{I.E.,}$$
$$m + n = 0,$$

and since for $x = l$, $a = 0$, we obtain by substituting these values in 2) the equation

$$0 = \frac{p^2}{q^2} - (m\,\varepsilon^{ql} - n\,\varepsilon^{-ql})\,q,$$

and substituting the value $n = -\,m$ taken from the foregoing equation, we have

$$0 = \frac{p^2}{q^2} - m\,q\,(\varepsilon^{ql} + \varepsilon^{-ql}),$$

whence

$$m = -\,n = \frac{p^2}{q^3\,(\varepsilon^{ql} + \varepsilon^{-ql})}$$

and the moment of the forces is

$$P\,x - Q\,y = Q\,m\,(\varepsilon^{qx} - \varepsilon^{-qx})$$
$$= \frac{P}{q}\left(\frac{\varepsilon^{qx} - \varepsilon^{-qx}}{\varepsilon^{ql} + \varepsilon^{-ql}}\right).$$

The latter is certainly a maximum for the fixed point B, the co-ordinates of which are $x = A\,C = l$ and $y = B\,C = a$, and then its value is

$$P\,l - Q\,a = \frac{P}{q}\left(\frac{\varepsilon^{ql} - \varepsilon^{-ql}}{\varepsilon^{ql} + \varepsilon^{-ql}}\right).$$

If $q\,l$ is a proper fraction, that is, if the *girder is short* and the *force in the direction of the axis is small,* we can put

$$\varepsilon^{ql} = 1 + q\,l + \frac{q^2\,l^2}{2} + \frac{q^3\,l^3}{6} + \ldots,$$

and also

$$\varepsilon^{-ql} = 1 - q\,l + \frac{q^2\,l^2}{2} - \frac{q^3\,l^3}{6} + \ldots,$$

hence we have the moment of the forces

$$P\,l - Q\,a = \frac{P\,l\,(1 + \frac{1}{6}\,q^2\,l^2)}{1 + \frac{1}{2}\,q^2\,l^2} = P\,l\,(1 + \tfrac{1}{6}\,q^2\,l^2)\,(1 - \tfrac{1}{2}\,q^2\,l^2)$$

$$= P\,l\,(1 - \tfrac{1}{3}\,q^2\,l^2) = P\,l\left(1 - \frac{Q\,l^2}{3\,W\,E}\right).$$

If, on the contrary, the force Q is so great that $q\,l$ becomes at least $= 2$, we can then neglect

$$\varepsilon^{-ql} = \frac{1}{\varepsilon^{ql}},$$

when it occurs with ε^{ql}, and therefore we can put

$$\frac{\varepsilon^{ql} - \varepsilon^{-ql}}{\varepsilon^{ql} + \varepsilon^{-ql}} = \frac{\varepsilon^{ql}}{\varepsilon^{ql}} = 1,$$

so that the moment of the forces becomes simply

$$P\,l - Q\,a = \frac{P}{q} = P\sqrt{\frac{W\,E}{Q}}.$$

(§ 275.) **Proof Load of a Girder Subjected to a Tensile Force.**—By the aid of the moments of the forces P and Q, found in the foregoing paragraph, we can determine by the method, which we have so often employed, the proof load of the girder.

The force Q produces a tension per unit of surface

$$S_1 = \frac{Q}{F}$$

in the direction of the axis of the body, and the moment $P\,l - Q\,a$ of the two forces P and Q produces a tension in the fibres at the maximum distance e from the neutral axis, which is

$$S_2 = \frac{(P\,l - Q\,a)\,e}{W},$$

hence the total tension is

$$S = S_1 + S_2 = \frac{Q}{F} + \frac{(P\,l - Q\,a)\,e}{W}.$$

When the latter reaches the limit of elasticity, $S = T$, and we can put

$$T = \frac{Q}{F} + \frac{(P\,l - Q\,a)\,e}{W}.$$

If the modulus of proof strength T_1 for compression is different from that T for extension, we have

$$T_1 = -\frac{Q}{F} + \frac{(P\,l - Q\,a)\,e}{W},$$

in which e denotes the maximum distance of the compressed fibres from the neutral axis. In both cases we must substitute

$$P\,l - Q\,a = \frac{P}{q}\left(\frac{\varepsilon^{q\,l} - \varepsilon^{-q\,l}}{\varepsilon^{q\,l} + \varepsilon^{-q\,l}}\right),$$

so that the required proof load of the body becomes either

$$P = \left(\frac{\varepsilon^{q\,l} + \varepsilon^{-q\,l}}{\varepsilon^{q\,l} - \varepsilon^{-q\,l}}\right)\left(1 - \frac{Q}{F\,T}\right)\frac{W\,T\,q}{e},$$

or

$$P = \left(\frac{\varepsilon^{q\,l} + \varepsilon^{-q\,l}}{\varepsilon^{q\,l} - \varepsilon^{-q\,l}}\right)\left(1 + \frac{Q}{F\,T_1}\right)\frac{W\,T_1\,q}{e}.$$

For a *small tensile force* Q we can put

$$P\,l - Q\,a = P\,l\left(1 - \frac{Q\,l^2}{3\,W\,E}\right),$$

so that, when we take into consideration the extension only, we have

$$P = \frac{(F\,T - Q)\,W}{\left(1 - \frac{Q\,l^2}{3\,W\,E}\right)F\,l\,e} = \left(1 + \frac{Q\,l^2}{3\,W\,E}\right)\left(1 - \frac{Q}{F\,T}\right)\frac{W\,T}{l\,e}.$$

Without the tensile force Q the proof load of the body would be

$$P_1 = \frac{W\,T}{l\,e},$$

hence we have the ratio

$$\frac{P}{P_1} = \left(1 + \frac{Q\,l^2}{3\,W\,E}\right)\left(1 - \frac{Q}{F\,T}\right),$$

from which it is easy to see, that the proof load is increased or diminished by Q, as $\frac{Q}{F\,T}$ is greater or less than $\frac{Q\,l^2}{3\,W\,E}$, I.E., as $\frac{3\,W}{F\,l^2}$ is greater or less than $\frac{T}{E}$.

When the *tensile force is great*, in which case we can put

$$P\,l - Q\,a = P\sqrt{\frac{W\,E}{Q}},$$

we have the proof load

$$P = \left(1 - \frac{Q}{F\,T}\right)\sqrt{\frac{Q\,W}{E}} \cdot \frac{T}{e}.$$

This expression becomes a maximum with the expression $\sqrt{Q} - \frac{\sqrt{Q^3}}{F\,T}$. By differentiating the latter and putting the differential equation obtained equal to zero, we obtain

$$Q = \frac{F\,T}{3}.$$

This maximum value is

$$P = \tfrac{2}{3}\sqrt{\frac{F\,W\,T}{3\,E}} \cdot \frac{T}{e},$$

and the ratio of the latter to the proof load P_1 of a girder, which is not subjected to a tensile force, is

$$\frac{P}{P_1} = \tfrac{2}{3}\,l\sqrt{\frac{F\,T}{3\,W\,E}} = \tfrac{2}{3}\,l\sqrt{\frac{\sigma\,F}{3\,W}}.$$

For a parallelopipedical beam, whose height is h and whose width is b, we have $F = b\,h$, $W = \frac{b\,h^2}{12}$ and $e = \frac{1}{2}\,h$, whence

$$\frac{P}{P_1} = \frac{4\,l}{3\,h}\sqrt{\frac{T}{E}} = \frac{4\,l}{3\,h}\sqrt{\sigma}.$$

If the beam is of wood,

$$\sigma = \frac{T}{E} = \frac{1}{600},$$

and therefore

$$\frac{P}{P_1} = \tfrac{4}{3}\sqrt{\frac{1}{600}} \cdot \frac{l}{h} = 0{,}0544\,\frac{l}{h},$$

E.G., for $\frac{l}{h} = 30$, $P = 1{,}632\,P_1$;

the girder carries nearly *two-thirds more* than when it is not subjected to a tensile force.

For $\frac{l}{h} = \frac{10000}{544} = 18{,}4$, $P_1 = P$, and for values of $\frac{l}{h}$ smaller than 18,4, P_1 is smaller than P, and the proof load P of the beam is diminished by the stress Q.

§ 276. Torsion Combined with a Tensile or Compressive Force.—If a column $A\,B$, Fig. 459, is acted upon at the same time by a *force* Q, whose direction is that of its axis, and by a *couple* $(P, -P)$, which tends to twist it, both the elasticity of torsion and that of extension (or compression) come into play. The result of the combination of these two elasticities may be investigated as follows: If the strain per unit of surface produced by the force Q is $S_1 = \frac{Q}{F}$ and that produced by the moment of torsion at the distance e from the longitudinal axis of the body is $S_2 = \frac{P\,a\,e}{W}$, we can assume, that a parallelopipedical element

FIG. 459.

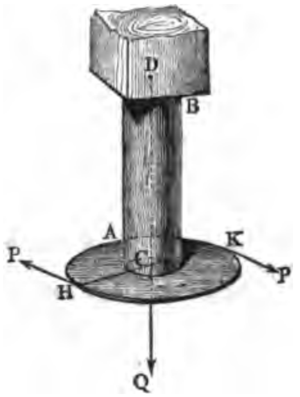
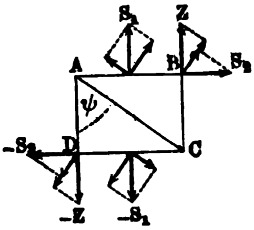


FIG. 460.



$A\,B\,C\,D$, Fig. 460, of the body, is acted upon by the normal forces $\overline{A\,B}\,.\,S_1$ and $-\,\overline{C\,D}\,.\,S_2$ upon $A\,B$ and $C\,D$ and by the couple $(\overline{A\,B}\,.\,S_2, -\,\overline{C\,D}\,.\,S_2)$ along $A\,B$ and $C\,D$ and by the opposite couple $(\overline{B\,C}\,.\,Z, -\,\overline{A\,D}\,.\,Z)$ along $B\,C$ and $A\,D$. If the diagonal plane $A\,C$ forms an angle ψ with the axis of the body or with the direction of the strain S_1, the components of the forces S_1, S_2 and Z upon one side of $A\,C$ are

$$\overline{A\,B}\,.\,S_1\,sin.\,\psi,\ \overline{A\,B}\,.\,S_2\,cos.\,\psi, \text{ and } \overline{B\,C}\,.\,Z\,sin.\,\psi,$$

and consequently the total normal force upon $A\,C$ is

$$\overline{A\,C}\,.\,S = \overline{A\,B}\,.\,S_1\,sin.\,\psi + \overline{A\,B}\,.\,S_2\,cos.\,\psi + \overline{B\,C}\,.\,Z\,sin.\,\psi,$$

or, since the moment of $(\overline{B\,C}\,.\,Z, -\,\overline{A\,D}\,.\,Z)$ is equal to the moment of $(\overline{A\,B}\,.\,S_2, -\,\overline{C\,D}\,.\,S_2)$, I.E.

$$A\,B\,.\,B\,C\,.\,Z = B\,C\,.\,A\,B\,.\,S_2 \text{ or } Z = S_2,$$

$$\overline{A\,C}\,.\,S = \overline{A\,B}\,.\,S_1\,sin.\,\psi + (\overline{A\,B}\,cos.\,\psi + \overline{B\,C}\,sin.\,\psi)\,S_2,$$

so that, finally, the normal strain upon the unit of surface of $A\,C$ is

$$S = \frac{A\,B}{A\,C} \cdot S_1\, sin.\,\psi + \left(\frac{A\,B}{A\,C}\, cos.\,\psi + \frac{B\,C}{A\,C}\, sin.\,\psi\right) S_2.$$

But $\frac{A\,B}{A\,C} = sin.\,\psi$ and $\frac{B\,C}{A\,C} = cos.\,\psi$, whence

$$S = S_1\,(sin.\,\psi^2) + 2\,S_2\, sin.\,\psi\; cos.\,\psi = S_1\,(sin.\,\psi)^2 + S_2\, sin.\,2\,\psi$$
$$= S_1 \left(\frac{1 - cos.\,2\,\psi}{2}\right) + S_2\, sin.\,2\,\psi \text{ (compare § 259).}$$

This equation gives a *maximum value* for S, when $tang.\,2\,\psi = -\frac{2\,S_2}{S_1}$ or $sin.\,2\,\psi = \frac{2\,S_2}{\sqrt{S_1^2 + (2\,S_2)^2}}$ and $cos.\,2\,\psi = -\frac{S_1}{\sqrt{S_1^2 + (2\,S_2)^2}}$, and this maximum value is

$$S_m = \frac{S_1}{2}\left(1 + \frac{S_1}{\sqrt{S_1^2 + (2\,S_2)^2}}\right) + \frac{2\,S_2^2}{\sqrt{S_1^2 + (2\,S_2)^2}}.$$
$$= \frac{S_1}{2} + \sqrt{\left(\frac{S_1}{2}\right)^2 + S_2^2}.$$

Substituting the above values for S_1 and S_2 in this equation, we obtain the required maximum strain

$$S_m = \frac{Q}{2\,F} + \sqrt{\left(\frac{Q}{2\,F}\right)^2 + \left(\frac{P\,a\,e}{W}\right)^2}.$$

Now, since the body should resist with safety the actions of these forces P and Q, we must put S_m = to the modulus of proof strength T or

$$\frac{Q}{2\,F} + \sqrt{\left(\frac{Q}{2\,F}\right)^2 + \left(\frac{P\,a\,e}{W}\right)^2} = T,$$

from which we obtain the equation of condition

$$\left(\frac{P\,a\,e}{W}\right)^2 = T^2 - \frac{Q\,T}{F}.$$

The allowable moment of torsion is therefore

$$1)\; P\,a = \frac{W}{e}\sqrt{T^2 - \frac{Q\,T}{F}},$$

and the allowable force in the direction of the axis is

$$2)\; Q = F\,T - \frac{F}{T}\left(\frac{P\,a\,e}{W}\right)^2$$

In order to find the dimensions of the cross-section corresponding to the forces P and Q, we put

$$\frac{W}{e} = \frac{P\,a}{\sqrt{T^2 - \frac{Q\,T}{F}}},$$

when the force producing torsion is the greater, and, on the contrary,

$$F = \frac{Q}{T - \frac{1}{T}\left(\frac{P\,a\,e}{W}\right)^2}$$

when that in the direction of the axis is the greater.

For a *parallelopipedical column,* whose dimensions are b and h, we have

$$F = b\,h,\ W = (b^2 + h^2)\,\frac{b\,h}{12} \text{ and } e = \tfrac{1}{2}\sqrt{b^2 + h^2}, \text{ consequently}$$

$$\frac{W}{e} = \frac{b\,h}{6}\sqrt{b^2 + h^2} = \frac{P\,a}{\sqrt{T^2 - \frac{Q\,T}{b\,h}}} = \frac{P\,a}{T}\left(1 - \frac{Q}{b\,h\,T}\right)^{-\frac{1}{2}} \text{ and}$$

$$F = b\,h = \frac{Q}{T - \frac{36}{(b^2+h^2)\,T}\left(\frac{P\,a}{b\,h}\right)^2} = \frac{Q}{T}\left[1 - \left(\frac{6\,P\,a}{\sqrt{b^2+h^2}\cdot b\,h\,T}\right)^2\right]^{-1}.$$

If we know the ratio $\nu = \frac{b}{h}$ of the dimensions, we can calculate the dimensions themselves by means of this formula.

For a *pillar with a square base* $b = h$, and therefore

$$\frac{h^3\sqrt{2}}{6} = \frac{P\,a}{T}\left(1 - \frac{Q}{h^2\,T}\right)^{-\frac{1}{2}},$$

$$h = b = \sqrt[3]{\frac{6\sqrt{\frac{1}{2}}\,P\,a}{T}}\left(1 - \frac{Q}{h^2\,T}\right)^{-\frac{1}{2}},$$

$$h^2 = \frac{Q}{T}\left[1 - \tfrac{1}{2}\left(\frac{6\,P\,a}{h^3\,T}\right)^2\right]^{-1} \text{ and}$$

$$h = b = \sqrt{\frac{Q}{T}}\left[1 - \tfrac{1}{2}\left(\frac{6\,P\,a}{h^3\,T}\right)^2\right]^{-\frac{1}{2}}.$$

For a cylindrical pillar or shaft we have

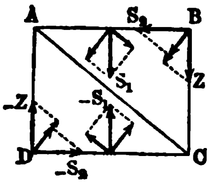
$$F = \pi\,r^2,\ W = \frac{\pi\,r^4}{2}, \text{ and } e = r, \text{ whence}$$

$$\frac{\pi\,r^3}{2} = \frac{P\,a}{\sqrt{T^2 - \frac{Q\,T}{\pi\,r^2}}} \text{ and } r = \sqrt{\frac{2\,P\,a}{\pi\,T}}\left(1 - \frac{Q}{\pi\,r^2\,T}\right)^{-\frac{1}{2}}, \text{ as well as}$$

$$\pi\,r^2 = \frac{Q}{T - \frac{1}{T}\left(\frac{2\,P\,a}{\pi\,r^3}\right)^2} \text{ and } r = \sqrt{\frac{Q}{\pi\,T}}\left[1 - \left(\frac{2\,P\,a}{\pi\,r^3\,T}\right)^2\right]^{-\frac{1}{2}}.$$

If the force Q in the direction of the axis is a compressive one, the formulas found above still hold good; f not only the direction of the force S_1 (1 461) is opposite, but also the forces S_2 and Z can be assumed to act in the opposite direction, when we wish to obtain the maximum resultant S_m.

FIG. 461.



EXAMPLE.—If a vertical wooden shaft weighing 10000 pounds is subjected to a moment of torsion $P a = 72000$, the required radius, assuming $T = 400$ pounds, is

$$r = \sqrt[3]{\frac{2 P a}{\pi T}}\left(1 - \frac{Q}{\pi r^2 T}\right)^{-\frac{1}{3}} = \sqrt[3]{\frac{0,6366 \,.\, 72000}{400}}\left(1 - \frac{10000}{400 \pi r^2}\right)^{-\frac{1}{3}}$$

$$= \sqrt[3]{0,6366 \,.\, 180}\left(1 - \frac{7,958}{r^2}\right)^{-\frac{1}{3}}.$$

Approximatively, we have

$$r = \sqrt[3]{114,6} = 4,85, \text{ whence}$$

$$\frac{7,958}{r^2} = \frac{7,958}{23,52} = 0,3383, \text{ and}$$

$$\left(1 - \frac{7,958}{r^2}\right)^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{0,6617}} = 1,071,$$

so that the required radius is, more accurately,

$$r = 4,85 \,.\, 1,071 = 5,194 \text{ inches},$$

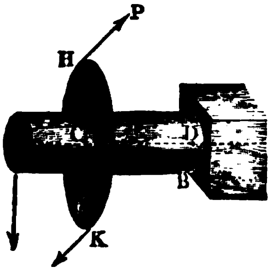
and consequently the diameter of the shaft is

$$d = 10,39 \text{ inches}.$$

§ 277. Flexure and Torsion Combined.

—Cases often occur where a girder or shaft is acted upon at the same time by a bending force and a twisting couple. Horizontal shafts are generally submitted to both of these actions. In order to investigate the relations of the combined action of these two forces, let us imagine a prismatic body $A\ B\ C\ D$, Fig. 462, fixed at one end $B\ D$, to be acted upon at the other end by a bending force Q and at the same time by a twisting couple $(P, -P)$. If l_1 is the length $A\ C$ of the shaft, W_1 the measure of the moment of flexure and e_1 the maximum distance of an element of the cross-sec-

FIG. 462.



tion from the neutral axis, we have the maximum strain produced in the direction of the axis by the force Q

$$S_1 = \frac{Q\,l_1\,e_1}{W_1} \text{ (compare § 235).}$$

If, on the contrary, a denotes the lever arm $H\,K$ of the couple $(P, -P)$, W the measure of the moment of torsion and e the greatest distance of any element of the cross-section from the axis $C\,D$ of the body, we can put the maximum shearing strain produced by the couple

$$S_2 = \frac{P\,a\,e}{W}.$$

Now here, as we can easily understand, the strain $S_1 = \frac{Q\,l_1\,e}{W_1}$ takes the place of the absolute strain $S_1 = \frac{Q}{F}$ of the foregoing paragraph, and therefore we can put for the maximum strain in the whole body $A\ B\ C\ D$, Fig. 462,

$$S_m = \frac{S_1}{2} + \sqrt{\left(\frac{S_1}{2}\right)^2 + S_2^2}, \text{ or}$$

$$T = \frac{Q_1\,l_1\,e_1}{2\,W_1} + \sqrt{\left(\frac{Q_1\,l_1\,e_1}{2\,W_1}\right)^2 + \left(\frac{P\,a\,e}{W}\right)^2},$$

from which we obtain the equation of condition

$$\left(\frac{P\,a\,e}{W}\right)^2 = T^2 - \frac{Q\,l_1\,e_1\,T}{W_1}.$$

The allowable moment of torsion is therefore

$$1)\ P\,a = \frac{W}{e}\sqrt{T^2 - \frac{Q\,l_1\,e_1\,T}{W_1}} = \frac{W\,T}{e}\sqrt{1 - \frac{Q\,l_1\,e_1}{W_1\,T}},$$

and the bending force is

$$2)\ Q = \frac{W_1}{l_1\,e_1\,T}\left[T^2 - \left(\frac{P\,a\,e}{W}\right)^2\right], \text{ from which we obtain either}$$

$$\frac{W}{e} = \frac{P\,a}{\sqrt{T^2 - \frac{Q\,l_1\,e_1\,T}{W_1}}}, \text{ or}$$

$$\frac{W_1}{e_1} = \frac{Q\,l_1}{T - \frac{1}{T}\left(\frac{P\,a\,e}{W}\right)^2}.$$

For a *square shaft*

$$\frac{W}{e} = \frac{h^3\sqrt{2}}{6} \text{ and } \frac{W_1}{e_1} = \frac{h^3}{6}, \text{ whence}$$

$$h^3 = \frac{6\sqrt{\frac{1}{2}}\,P a}{T}\left(1 - \frac{6\,Q\,l_1}{h^3\,T}\right)^{-\frac{1}{2}}, \text{ and}$$

$$h = \sqrt[3]{\frac{6\sqrt{\frac{1}{2}}\,P\,a}{T}}\left(1 - \frac{6\,Q\,l_1}{h^3\,T}\right)^{-\frac{1}{6}},$$

as well as

$$h^3 = \frac{6\,Q\,l_1}{T}\left[1 - \left(\frac{6\sqrt{\frac{1}{2}}\,P\,a}{h^3\,T}\right)^2\right]^{-1} \text{and}$$

$$h = \sqrt[3]{\frac{6\,Q\,l_1}{T}}\left[1 - \tfrac{1}{2}\left(\frac{6\,P\,a}{h^3\,T}\right)^2\right]^{-\frac{1}{3}};$$

while, on the contrary, for a *cylindrical shaft*,

$$\frac{W}{e} = \frac{\pi\,r^3}{2} \text{ and } \frac{W_1}{e_1} = \frac{\pi\,r^3}{4}; \text{ hence we can put}$$

$$r^3 = \frac{2}{\pi}\,\frac{P a}{T}\left(1 - \frac{4\,Q\,l_1}{\pi\,r^3\,T}\right)^{-\frac{1}{2}}, \text{ and}$$

$$r = \sqrt[3]{\frac{2}{\pi}\,\frac{P a}{T}}\left(1 - \frac{4\,Q\,l_1}{\pi\,r^3\,T}\right)^{-\frac{1}{6}},$$

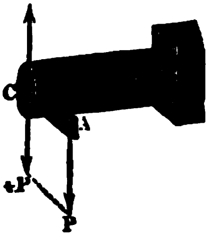
as well as

$$r^3 = \frac{4}{\pi}\,\frac{Q\,l_1}{T}\left[1 - \left(\frac{2\,P\,a}{\pi\,r^3\,T}\right)^2\right]^{-1}, \text{and}$$

$$r = \sqrt[3]{\frac{4}{\pi}\,\frac{Q\,l_1}{T}}\left[1 - \left(\frac{2\,P\,a}{\pi\,r^3\,T}\right)^2\right]^{-\frac{1}{3}}.$$

Very often it is not a couple, but a force P, acting eccentrically to the axis, which produces the torsion in the body $B\,C\,D$, Fig. 463.

FIG. 463.

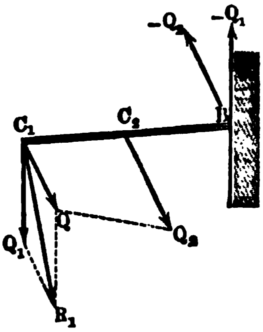


Since such a force can be decomposed into an equal central force $\overline{C\,P} = +\,P$ and into a couple $(P, -\,P)$, whose lever arm is the distance $C\,A$ between the axis $C\,D$ of the body and the line of application of the force P, we have here a case of combined strength, although there is no other force Q; for the twisting produced by the couple $(P, -\,P)$, combines with the bending produced by the axial force $+\,P$. The above formulas can be employed directly for determining the thickness of such a body, when we substitute in them $P\,l = Q\,l_1$.

If, in addition to the eccentric force P, there is another Q, whose moment is $Q\,l_1$, we must substitute instead of $P\,l$, $P\,l + Q\,l_1$.

§ 278. Bending Forces in Different Planes.—If a girder or shaft $B\ C$, Fig. 464, is acted upon by two bending forces Q_1 and Q_2, whose directions $C_1\ Q_1$ and $C_2\ Q_2$, although at right angles to the axis $C_1\ B$ of the body, are not parallel to each other, the portion $C_2\ B$ of the body will be bent by two couples $(Q_1, -Q_1)$ and $(Q_2, -Q_2)$, the resultant of which must be found, when we wish to determine the nature and magnitude of the bending. If l_1 and l_2 denote the arms of the forces Q_1 and Q_2 in reference to the fixed point B, $Q_1\ l_1$ and $Q_2\ l_2$ are their moments, and if a is the angle formed by the directions of the forces, when passing through the same point, we have, according to § 95, the moment of the resulting couple

FIG. 464.



$$R\,c = \sqrt{(Q_1\,l_1)^2 + (Q_2\,l_2)^2 + 2\,(Q_1\,l_1)\,(Q_2\,l_2)\,cos.\ a},$$

and for the angle β, which the plane of this couple makes with that of the couple $(Q_1, -Q_1)$,

$$sin.\ \beta = \frac{Q_2\,l_2}{R\,c}.$$

In order to find the intensity and the plane of this couple $(R, -R)$, we can reduce the force Q_2 from C_2 to C_1, combine the reduced force $Q = \frac{Q_2\,l_2}{l_1}$ by means of the parallelogram of forces with the force Q_1 and thus determine the resultant R_1; the product $R_1\,l_1 = R\,c$ is the value of the moment of the resulting couple and the angle $Q_1\,C_1\,R$ is the angle β, which the plane of this couple forms with that of the couple $(Q_1, -Q_1)$. This plane is of course that in which the body is bent, and by the aid of the moment $R_1\,l_1 = R\,c$, just found, we obtain the maximum strain in the body

$$S = \frac{R\,c\,e}{W},$$

or, putting this equal to the modulus of proof strength T, we have

$$\frac{T\,W}{e} = \sqrt{(Q_1\,l_1)^2 + (Q_2\,l_2)^2 + 2\,(Q_1\,l_1)\,(Q_2\,l_2)\,cos.\ a.}$$

If a twisting couple $(P, -P)$, whose moment is $P\,a$, also acts upon this body $A\ B$, the maximum strain becomes

$$S_m = T = \frac{R\,c\,e_1}{2\,W_1} + \sqrt{\left(\frac{R\,c\,e_1}{2\,W_1}\right)^2 + \left(\frac{P\,a\,e}{W}\right)^2},$$

in which W_1 denotes the measure of the moment of flexure, W that of torsion, e_1 the greatest distance of any element of the body from the neutral axis and e that of any element from the longitudinal axis of the body at D.

From the above we obtain

$$\left(\frac{P\,a\,e}{W}\right)^2 = T^2 - \frac{R\,c\,e_1\,T}{W}$$

$$= T^2 - [(Q_1\,l_1)^2 + (Q_2\,l_2)^2 + 2\,(Q_1\,l_1)\,(Q_2\,l_2)\,cos.\,a]\,\frac{e_1\,T}{W_1}.$$

By the aid of the formulas of the foregoing paragraph the required dimensions of the cross-section of the body can be found by substituting in them instead of $Q\,l$ the sum $Q_1\,l_1 + Q_2\,l_2$.

If only one bending force Q_1 acts upon the body and if at the same time it is acted upon by a single twisting force P instead of a couple $(P, - P)$, this force P can be resolved into a twisting couple $(P, - P)$ and a force P acting upon the axis, so that instead of $Q_2\,l_2$ we must substitute in the latter formula $P\,l$.

FINAL REMARK.—Although there is no portion of mechanics which has been the subject of so many experiments as the elasticity and strength of bodies, yet much remains to be investigated and many points are still uncertain. Experiments upon this subject have been made by Ardant, Banks, Barlow, Bevan, Brix, Busson, Burg, Duleau, Ebbels, Eytelwein, Finchan, Gerstner, Girard, Gauthey, Fairbairn and Hodgkinson, Lagerjhelm, Musschenbrock, Morveau, Navier, Rennie, Rondelet, Tredgold, Wertheim, etc. The older experiments are discussed at length in Eytelwein's "Handbuch der Statik fester Körper," Vol. II., and also in Gerstner's "Handbuch der Mechanik," Vol. I. A copious treatise on this subject by v. Burg is given in the 19th and 20th volumes of the Jahrbücher des Polytechn. Instituts zu Wien. Theories which differ somewhat from those given in this work are also to be found in this treatise. The experiments of Brix and Lagerjhelm have already been mentioned (page 394). New and very varied experiments upon the reacting strength of different kinds of stone by Brix are reported in the 32d year (1853) of the transactions of the "Verein zur Beförderung des Gewerbefleiszes in Preussen." A simple theory of flexure by Brix is to be found in the treatise "Elementare Berechnung des Widerstandes prismatischer Körper gegen die Biegung," which is printed separately from the transactions of the Preussischen Gewerbevereins. Wertheim's latest experiments upon elasticity have already been mentioned (page 396). An abstract of Hodgkinson's experiments is to be found in Moseley's "Mechanical Principles of Engineering and Architecture." Hodgkinson's principal work, the title of which is "Experimental Researches on the strength and other properties of cast iron, etc.," was published by John Weale in 1846. A French translation of it by Pirel appeared in Tome IX., 1855, of the "Annales des Ponts et Chaussées," and an abstract of it by Couche in Tome XX., 1855, of the "Annales des Mines." Tredgold has published a treatise upon the strength of cast iron and other metals. The following works are also recommended for study. Poncelet's ".Introduction à la Mécanique Industrielle," Part I., Navier's Résumé des Leçons sur l'application de la Mécanique, Part I., translated into German by Westphal under the title "Mechanik der Baukunst," to which work Poncelet has made some additions in his theory of the resistance of rigid bodies (see his Manual of Applied Mechanics, Vol. II., translated into German by Schnuse). We would also recommend particularly the "Resistance des Materiaux" (Leçons de Mécanique Pratique), by A. Morin, which has been much used in preparing this work. We may mention further the "Theorie der Holz-und Eisenconstructionen mit besonderer Rücksicht auf das Bauwesen," by George Rebhan, Vienna, 1856, the work of Moll and Reuleaux (already quoted in page 469) upon "die Festigeit der Materialien," a "Memoire sur la Resistance du Fer et de la Fonte, par G. H. Love, Paris, 1852," as well as Tate's work upon the strength of materials as applied to tubular bridges, etc. The theory of combined elasticity and strength was first treated by the author in "der Zeitschrift für das gesammte Ingenieurwesen (dem Ingenieur), by Bornemann, etc., Vol. I. In the first volume of the new series of this magazine (Civilingenieur, 1854) the graphic representation of the relative strength is treated by Mr. Bornemann, and the results of the experiments made by Bornemann and by Lemarle are also given.

The theory of elasticity and strength will be treated of again when we discuss the theory of oscillation and of impact.

Mr. Fairbairn's Useful Information for Engineers, I. and II. Series, gives the results of many experiments upon the strength of wrought iron of different forms, as well as upon stone, glass, etc. From a theoretical point of view, we can particularly recommend, "Leçons sur la theorie mathematique de l'élasticité des corps solides," par Lamé, "A Manual of Applied Mechanics," by W. J. Rankine, the "Cours de Mécanique appliquée," I. Partie, by Bresse, and the "Théorie de la résistance et de la flexion plane des solides," par Belanger. The treatise of Laissle and Schüblen, "Ueber den Bau der Brückenträger," is a fair exponent of the state of science upon this question, when it was written, and is therefore to be recommended. Rühlmann's "Grundzüge der Mechanik," 3. Auflage (1860), contains also a treatise upon the resistance of materials worth reading.

The "Civilingenieur" and the "Zeitschrift des deutschen Ingenieurvereins" contain several valuable treatises upon the theory of elasticity and strength, particularly those by Grashof, Schwedler, Winkler, etc., as well as several good translations from the French and English of Barlow, Bouniceau, Fairbairn, Love, etc. The results of many experiments by Fairbairn, Karmarsch, Schönemann, Völkers, etc., are also given in these journals.

FIFTH SECTION.

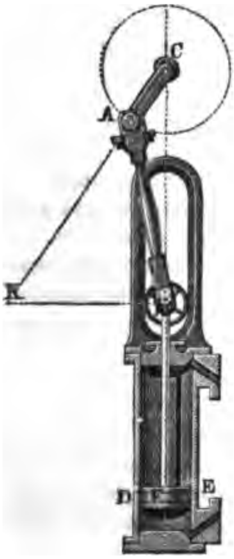
DYNAMICS OF RIGID BODIES.

CHAPTER I.

THEORY OF THE MOMENT OF INERTIA.

§ 279. **Kinds of Motion.**—The motion of a rigid body is either one *of translation*, or *of rotation*, or a *combination of the two.* In the motion of translation (Fr. mouvement de translation; Ger. fortschreitende or progressive Bewegung) the spaces described simultaneously by the different parts of the body are parallel and equal to each other; in the motion of rotation (Fr. mouvement de rotation; Ger. drehende or rotirende Bewegung), on the contrary, the parts of the body describe concentric arcs of circles about a certain line, called the axis of rotation (Fr. axe de rotation; Ger. Umdrehungsaxe). Every compound motion can be considered as a motion of rotation around a *movable axis.* The latter is either *variable* or *constant.* The piston $D\ E$ and the piston-rod $B\ F$ of a pump or steam engine, Fig. 465, have a motion of translation, and the crank $A\ C$ has a motion of rotation. The connecting rod $A\ B$ has a compound motion; for one of its extremities B has a motion of translation, while the other A has a motion of rotation. The axis of rotation of a cylinder, which is rolling, is con-

FIG. 465.



stant, while that of the connecting rod $A\,B$ is variable; for its position is determined by the intersection M of the perpendicular $B\,K$ to direction $C\,B$ of the axis of the piston-rod and of the prolongation of the crank $C\,A$ (see § 101).

§ 280. Rectilinear Motion.—The laws of motion of a material point, discussed in § 82 and § 98, are directly applicable to a *rectilinear motion of translation.* The elements of the mass M_1, M_2, M_3, etc., of a body, moving with the acceleration p, resist the motion, by virtue of their inertia, with the forces $M_1\,p$, $M_2\,p$, $M_3\,p$, etc. (§ 54), and since the motions of all these elements take place in parallel lines, the directions of these forces are also parallel; the *resultant* of all these forces due to the inertia is equal to the sum

$$M_1 p + M_2 p + M_3 p + \ldots = (M_1 + M_2 + M_3 + \ldots) p = Mp,$$

when M denotes the mass of the whole body, and the point of application of the resultant coincides with the *centre of gravity.* In order to set in motion a body, whose mass is M and whose weight is $G = M\,g$ and which in other respects is free to move, we require a force

$$P = M p = \frac{G\,p}{g},$$

whose direction must pass through the centre of gravity S of the body.

If, in consequence of the action of the force P, the velocity c is changed to the velocity v while the space s is described, the ***energy stored*** by the mass is (§ 72)

$$P\,s = \left(\frac{v^2 - c^2}{2}\right) M = \left(\frac{v^2 - c^2}{2\,g}\right) G = (h - k)\,G.$$

EXAMPLE.—The motion of the piston and piston-rod of a pump, steam-engine, blowing-machine, etc., is variable; at the beginning and end of its stroke the velocity is $= 0$, and near the middle of it it is a maximum. If the weight of the piston and piston-rod $= G$, and if the maximum velocity at the middle of its stroke $= v$, the energy stored by them in the first half of the stroke and restored in the second half is

$$L = \frac{v^2}{2g}\,G.$$

If $G = 800$ pounds and $v = 5$ feet, we have

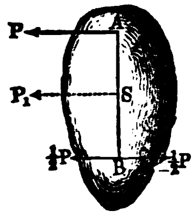
$$L = 0{,}0155 \,.\, 5^2 \,.\, 800 = 310 \text{ foot-pounds.}$$

Now if half the stroke of the piston is $s = 4$ feet, we have the mean force, which is necessary to produce the acceleration of the piston in the first half of the stroke and which the piston exerts in the second half, when it is retarded,

$$P = \frac{L}{s} = \frac{v^2}{2\,g\,s} \cdot G = \frac{310}{4} = 77\tfrac{1}{2} \text{ pounds.}$$

§ 281. Motion of Rotation.—If the motive force P of a body $A\,B$, Fig. 466, does not pass through its centre of gravity S, the body turns around that point, and at the same time moves forward exactly as if the force acted directly at the point S, as can be shown in the following manner. Let us let fall from the centre of gravity S a perpendicular $S\,A$ upon the direction of the force and continue it in the other direction until the prolongation $S\,B$ is equal to the perpendicular $S\,A$, and let us suppose that two forces $+\frac{1}{2}\,P$ and $-\frac{1}{2}\,P$, which balance each other and are parallel to P, are applied at B. The force $+\frac{1}{2}\,P$ combines with half the force P acting in A and gives rise to the resultant

FIG. 466.



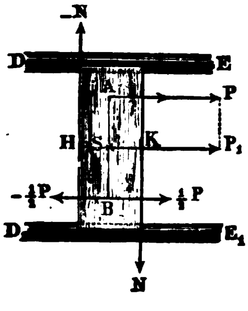
$$P_1 = \tfrac{1}{2}\,P + \tfrac{1}{2}\,P = P$$

applied at the centre of gravity, while, on the contrary, the force $-\frac{1}{2}\,P$ forms with the other half ($\frac{1}{2}\,P$) of the force P applied in A a couple; hence the force P, applied eccentrically, is equivalent to a *force* $P_1 = P$, which is applied at *the centre of gravity*, and which moves this point and with it the body, and to a couple ($\frac{1}{2}\,P$, $-\frac{1}{2}\,P$), which causes the body to *turn around its centre of gravity* S without producing a pressure upon it. The statical moment of this couple is

$$= \tfrac{1}{2}\,P\,.\,\overline{S\,A} + \tfrac{1}{2}\,P\,.\,\overline{S\,B} = P\,.\,\overline{S\,A} = P\,a,$$

or equal to the *statical moment* of the force P applied in A in reference to the centre of gravity S; the resulting rotation would therefore be the same if the centre of gravity S were fixed and P alone were acting.

FIG. 467.



If a body $A\,B$, Fig. 467, is compelled, by means of guides $D\,E$, $D_1\,E_1$, to assume a motion of translation, the eccentric force $\overline{A\,P} = P$ produces the same effect upon the motion of the body as an equal force acting at the centre of gravity, and the couple ($\frac{1}{2}\,P$, $-\frac{1}{2}\,P$) is counteracted by the guides. If a is the eccentricity $S\,A$ of the force P, or the distance of its direction from the centre of gravity S of the body, and if b denotes the distance $H\,K$

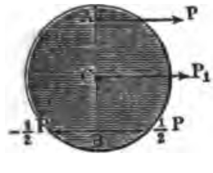
between the perpendiculars to the guides at the diagonally opposite points F and G and $(N, - N)$ the couple, with which the body acts on the guides, we have, by equating the moment of the couples $(\frac{1}{2} P, - \frac{1}{2} P)$ and $(N, - N)$,

$$N b = P a, \text{ and therefore}$$

$$N = \frac{a}{b} P.$$

If, finally, the body $A\ B$, Fig. 468, is prevented from moving forward by the fixed axis C, the eccentric force $\overline{A\ P} = P$ produces the same effect upon the rotation of the body about this axis C as a couple $(\frac{1}{2} P, - \frac{1}{2} P)$ with the arm $2\ C\ A = 2\ C\ B = 2\ a$, or with the moment $\frac{1}{2} P . 2\ a = P\ a$; for the remaining central force $\overline{C\ P_1} = P_1 = P$ is counteracted by the bearings of the axis (compare § 130).

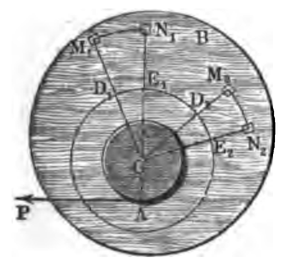
FIG. 468.



§ 282. **Moment of Inertia.**—During the rotation of a body $A\ B$, Fig. 469, about a fixed axis C, all points M_1, M_2, etc., of it describe equal angles at the centre $M_1\ C\ N_1 = M_2\ C\ N_2$, etc., $= \phi°$, which, when the radii $C\ D_1 = C\ D_2$, etc., = one (1) are equal, correspond to the same arc

$$D_1\ E_1 = D_2\ E_2, \text{ etc.}, = \phi = \frac{\phi°}{180°}\ \pi.$$

FIG. 469.



Since the velocity is determined by the quotient of the element ϕ of the space and the corresponding element τ of the time, the *angular velocity* (Fr. vitesse angulaire, Ger. Winkelgeschwindigkeit), I.E. the velocity of those points of the body which are situated at a distance equal to the unit of length (E.G. a foot) from the axis of rotation, is therefore one and the same for the whole body, and its value is

$$\omega = \frac{\phi}{\tau}$$

and in like manner the *angular acceleration*, or the acceleration of the rotating body at the distance = unity from the axis of rotation, is the same for the whole body, and its value is

$$\kappa = \frac{\omega}{\tau},$$

ω denoting the increase of angular velocity in the element of time τ.

In order to find the spaces s_1, s_2, etc., the velocities v_1, v_2, etc., and the accelerations p_1, p_2, etc., of the points M_1, M_2, etc., of the body, which are situated at the distances $C\,M_1 = r_1$, $C\,M_2 = r_2$, etc., from the axis of rotation C, we must multiply the angular space ϕ, the angular velocity ω, and the angular acceleration p by r_1, r_2, etc.; thus we obtain

$$s_1 = \phi\, r_1,\ s_2 = \phi\, r_2, \text{ etc.,}$$
$$v_1 = \omega\, r_1,\ v_2 = \omega\, r_2, \text{ etc., and}$$
$$p_1 = \kappa\, r_1,\ p_2 = \kappa\, r_2 \text{ etc.}$$

If the whole mass M of the body is composed of the parts M_1, M_2, etc., which are at distances equal to the radii r_1, r_2, etc., from the axis of rotation C, the forces with which these elements of the mass resist the rotation are

$$P_1 = M_1 p_1 = \kappa\, M_1 r_1,\ P_2 = M_2 p_2 = \kappa\, M_2 r_2, \text{ etc.,}$$

and their moments are

$$P_1 r_1 = \kappa\, M_1 r_1^2,\ P_2 r_2 = \kappa\, M_2 r_2^2, \text{ etc.,}$$

and the *moment* necessary to cause the body to rotate with the *angular acceleration* κ is

$$P\,a = \kappa\, M_1 r_1^2, + \kappa\, M_2 r_2^2 + \ldots$$
$$= \kappa\,(M_1 r_1^2 + M_2 r_2^2 + M_3 r_3^2 + \ldots).$$

In like manner (according to § 84) the energy stored by the elements M_1, M_2, etc., while they acquire the velocities v_1, v_2, etc., is

$$A_1 = \tfrac{1}{2}\, M_1 v_1^2 = \tfrac{1}{2}\, \omega^2 M_1 r_1^2,$$
$$A_2 = \tfrac{1}{2}\, M_2 v_2^2 = \tfrac{1}{2}\, \omega^2 M_2 r_2^2, \text{ etc.,}$$

and therefore the *work done* in communicating to the whole body the *angular velocity* ω is

$$A = A_1 + A_2 + \ldots$$
$$= \tfrac{1}{2}\, \omega^2 (M_1 r_1^2 + M_2 r_2^2 + M_3 r_3^2 + \ldots).$$

The force of and the energy stored by a body in rotation depends principally upon the sum of the products $M_1 r_1^2 + M_2 r_2^2 + M_3 r_3^2 + \ldots$ of the different elements M_1, M_2, etc., of the mass and of the squares of the distances r_1, r_2, etc., from the axis of revolution. This sum is called the *moment of inertia* (Fr. moment d'inertie, Ger. Trägheits-, Drehungs- or Massenmoment), and we will hereafter denote it by $M\,r^2$ or W. Hence the moment of the force, by which the mass $M = M_1 + M_2 + \ldots$, whose moment of inertia is

$$W = M\,r^2 = M_1 r_1^2 + M_2 r_2^2 + \ldots,$$

has imparted to it the angular acceleration κ, is

$$1)\ P\,a = \kappa\,M\,r^2 = \kappa\,W,$$

and, on the contrary, the work done in putting the mass M in rotation with the angular velocity ω is

$$2)\ P\,s = \tfrac{1}{2}\,\omega^2\,M\,r^2 = \tfrac{1}{2}\,\omega^2\,W.$$

If the initial angular velocity of the mass was ε, the work done in increasing it to ω is

$$P\,s = \tfrac{1}{2}\,\omega^2\,W - \tfrac{1}{2}\,\varepsilon^2\,W = \tfrac{1}{2}\,(\omega^2 - \varepsilon^2)\,W.$$

We can also determine from the work done and the initial velocity ε the final velocity ω; it is

$$\omega = \sqrt{\varepsilon^2 + \frac{2\,P\,s}{W}}.$$

Example.—If the body $A\,B$, Fig. 469, movable about a fixed axis C and in the beginning at rest, possesses a moment of inertia of 50 foot-pounds, and if it is set in rotation, by means of a rope passing round a pulley, by a force $P = 20$ pounds, which describes the space $s = 5$ feet, the angular velocity produced is

$$\omega = \sqrt{\frac{2\,P\,s}{W}} = \sqrt{\frac{2\,.\,20\,.\,5}{50}} = \sqrt{4} = 2 \text{ feet},$$

i.e., every point at the distance of a foot from the axis of rotation describes, after this work has been done, 2 feet in each second. The time of one revolution is

$$t = \frac{2\,\pi}{\omega} = 3{,}1416 \text{ seconds},$$

and the number of revolutions in a minute is

$$u = \frac{60}{t} = \frac{60}{3{,}1416} = 19{,}1.$$

If the angular velocity $\omega = 2$ feet, just found, is transformed into a velocity $\varepsilon = \frac{3}{4}$ foot, the work performed by the body is

$P_1\,s_1 = [2^2 - (\frac{3}{4})^2]\,.\,\frac{50}{2} = (4 - \frac{9}{16})\,.\,25 = \frac{55}{16}\,.\,25 = 85{,}93$ foot-pounds, e.g., it has lifted a weight of 10 pounds 8,593 feet high.

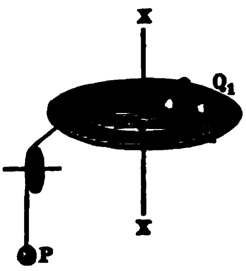
§ **283. Reduction of the Mass.**—If the angular velocities of two masses M_1 and M_2 are the same, if, e.g., they belong to the same rotating body, their living forces are to each other as their moments of inertia $W_1 = M_1\,r_1^2$ and $W_2 = M_2\,r_2^2$, and if the latter are equal, both masses have the same living force. Two masses have, then, equal influence upon the state of motion of a rotating body, and one can be replaced by the other, without causing a change in that state, when their moments of inertia $M_1\,r_1^2$ and $M_2\,r_2^2$ are equal, or when the masses themselves are to each other inversely as the square of their distances from the axis of rotation.

With the aid of the formula $M_1 r_1^2 = M_2 r_2^2$ we can reduce a mass from one distance to another, I.E. we can find a mass M_2, which at the distance r_2 has the same influence on the state of motion of the rotating body as the given mass M_1 at the distance r_1, and this mass is

$$M_2 = \frac{M_1 r_1^2}{r_2^2} = \frac{W_1}{r_2^2},$$

I.E., *the mass reduced to the distance r_2 is equal to the moment of inertia of the mass divided by the square of that distance.*

FIG. 470.



Two weights Q and Q_1, fixed upon a disc $A\ C\ B$, Fig. 470, at the distances $C\,B = b$ and $C\,B_1 = a$ from the axis of rotation $X\,X$, have the same influence upon the movement of the disc in consequence of their inertia, when $Q_1\, a^2 = Q\, b^2$ or $Q_1 = \frac{Q\, b^2}{a^2}$. If, therefore, a force P, whose arm is $C\,A = C\,B_1 = a$, causes a body, whose weight is Q and whose distance from the axis of rotation is $C\,B = b$, to rotate, we must reduce the latter to the arm a of the force P and put instead of Q,

$$Q_1 = \frac{Q\, b^2}{a^2},$$

and the mass moved by P is

$$M = \left(P + \frac{Q\, b^2}{a^2}\right) : g,$$

consequently the acceleration of the weight P is

$$p = \frac{\text{Force}}{\text{Mass}} = \frac{P}{P + Q\frac{b^2}{a^2}} \cdot g = \frac{P\, a^2}{P\, a^2 + Q\, b^2} \cdot g,$$

and the angular acceleration is

$$\kappa = \frac{p}{a} = \frac{P\, a}{P\, a^2 + Q\, b^2} \cdot g.$$

EXAMPLE.—If the weight of the rotating mass is $Q = 360$ pounds, its distance from the axis of rotation is $b = 2{,}5$ feet, the weight acting as moving force is $P = 24$ pounds and its arm is $a = 1{,}5$ feet, the mass accelerated by P is

$$M = \left[P + \left(\frac{2{,}5}{1{,}5}\right)^2 Q\right] : g = 0{,}031\left(24 + \frac{25}{9} \,.\, 360\right) = 0{,}031 \,.\, 1024$$
$$= 31{,}74 \text{ pounds},$$

and the acceleration of the weight is

$$p = \frac{24}{31{,}74} = 0{,}756 \text{ feet},$$

on the contrary, that of the mass Q is

$$q = \frac{b}{a} \cdot p = \frac{5}{3} p = \frac{5 \; . \; 0,756}{3} = 1,26 \text{ feet,}$$

and the angular acceleration is

$$\kappa = \frac{p}{a} = 0,504.$$

After four seconds the angular velocity is

$$\omega = 0,504 \; . \; 4 = 2,016 \text{ feet,}$$

and the corresponding space described is

$$\tfrac{1}{2} \omega t = \frac{2,016 \; . \; 4}{2} = 4,032 \text{ feet,}$$

hence the angle of rotation is

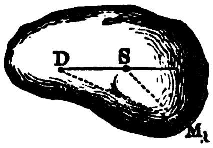
$$\phi^{\circ} = \frac{4,032}{\pi} \; . \; 180^{\circ} = 1,2834 \; . \; 180^{\circ} = 231^{\circ} \; 1'$$

and the space described by the weight P is

$$s = \frac{p\,t^2}{2} = \frac{0,756 \; . \; 4^2}{2} = 6,048 \text{ feet.}$$

§ 284. **Reduction of the Moments of Inertia.**—If the moment of inertia of a body or of a system of bodies in reference to an axis passing through the centre of gravity S of the body is known, the moment of inertia in reference to any other axis, parallel to the former, can easily be determined. Let S, Fig. 471, be the first axis of rotation, which passes through the centre of gravity, and D the other axis of rotation, for which the moment of inertia is to be determined; let $S\,D = d$ be the distance between the two axes and $S\,N_1 = x_1$ and $N_1\,M_1 = y_1$ the rectangular co-ordinates of an element M_1 of the mass of the whole body. The moment of inertia of this element in reference to D will be

FIG. 471.



$$= M_1 \, . \, \overline{D\,M_1}^2 = M_1\,(\overline{D\,N_1}^2 + \overline{N_1\,M_1}^2) = M_1\,[(d + x_1)^2 + y_1^2]$$

and in reference to S

$$= M_1 \, . \, \overline{S\,M_1}^2 = M_1\,(\overline{S\,N_1}^2 + \overline{N_1\,M_1}^2) = M_1\,(x_1^2 + y_1^2),$$

and, therefore, the difference of these moments is

$$= M_1\,(d^2 + 2\,d\,x_1 + x_1^2 + y_1^2) - M_1\,(x_1^2 + y_1^2) = M_1\,d^2 + 2\,M_1\,d\,x_1.$$

For another element of the mass it is

$$= M_2\,d^2 + 2\,M_2\,d\,x_2,$$

for a third it is

$$= M_3\,d^2 + 2\,M_3\,d\,x_3,$$

and, therefore, the moment of all the elements together is

$$= (M_1 + M_2 + M_3 + \ldots)\,d^2 + 2\,d\,(M_1\,x_1 + M_2\,x_2 + M_3\,x_3 + \ldots).$$

But $M_1 + M_2 + \ldots$ is the sum M of all the masses and $M_1 x_1 + M_2 x_2 + M_3 x_3$ is the sum $M x$ of the statical moments; hence it follows that the difference between the moment of inertia W_1 of the whole body in reference to the axis D and its moment of inertia W in reference to S is

$$W_1 - W = M d^2 + 2 d M x.$$

But since the sum of the statical moments of all the elements upon one side of every plane passing through the centre of gravity is equal to that of the moment of those on the other, the algebraical sum of all the moments is $= 0$, and we have $M x = 0$, and consequently

$$W_1 - W = M d^2,$$

i.e. $$W_1 = W + M d^2.$$

The moment of inertia of a body in reference to an eccentric axis is equal to the moment of inertia in reference to a parallel axis passing through the centre of gravity plus the product of the mass of the body by the square of the distance of the two axes from each other.

We see from this that of all the moments of inertia in reference to a set of parallel axes that one is the least, whose axis is a line of gravity of the body.

§ **285. Radius of Gyration.**—It is very important to determine the moment of inertia for various geometrical bodies; for the values thus deduced are frequently employed in the different calculations in mechanics. If the bodies, as we will hereafter suppose, are homogeneous, the different portions M_1, M_2, etc., of the mass, are proportional to the corresponding portions V_1, V_2, etc., of the volume, and the measure of the moment of inertia, or as it is generally called, the moment of inertia, can be replaced by the sum of the products of the portions of the volume and the square of their distances from the axis of rotation. In this sense we can also determine the moment of inertia of lines and surfaces. If we imagine the entire mass of a body concentrated in one point, we can determine the distance of the same from the axis, if we suppose that the moment of inertia of the mass, which is thus concentrated, is the same as it was, when distributed through the whole space. This is called the *radius of gyration* (Fr. rayon d'inertie, Ger. Drehungs- or Trägheitshalbmesser). If W is the moment of inertia, M the mass and k the radius of gyration, we have $M k^2 = W$, and therefore

$$k = \sqrt{\frac{W}{M}}.$$

We must also remember that this radius does not give a definite point, but only a circle, in whose circumference the mass can be distributed arbitrarily.

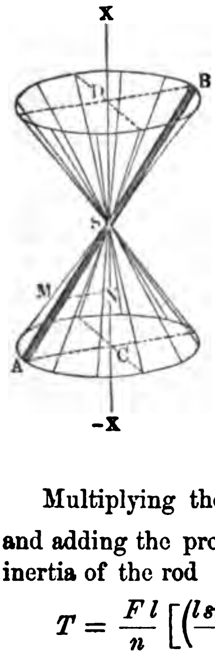
If in the formula $W_1 = W + M d^2$ we substitute $W = M k^2$ and $W_1 = M k_1^2$, we obtain

$$k_1^2 = k^2 + d^2,$$

I.E., the square of the radius of gyration in relation to any axis is equal to the square of the radius of gyration in reference to the line of gravity parallel to that axis plus the square of the distance of the two axes from each other.

§ **286. Moment of Inertia of a Rod.**—The moment of inertia of a *rod* $A\ B$, Fig. 472, which revolves about an axis $\overline{X}\ X$ passing through its middle S, is determined in the following manner. Let the cross-section of the rod be $= F$ and half its length be $= l$, and the angle, which its axis makes with the axis of rotation, I.E. $A\ S\ X$, be $= a$. Let us divide the half length of the rod into n parts, the contents of each of which are $\frac{F\,l}{n}$; the distances of the different portions of it from the centre S are $\frac{l}{n}, \frac{2\,l}{n}, \frac{3\,l}{n}$, etc., hence their distances from the axis of $\overline{X}\ X$, such as $M\ N$, are $= \frac{l}{n}\ sin.\ a$, $\frac{2\,l}{n}\ sin.\ a$, $\frac{3\,l}{n}\ sin.\ a$, etc., and the squares of the latter are $= \left(\frac{l\ sin.\ a}{n}\right)^2$, $4\left(\frac{l\ sin.\ a}{n}\right)^2$, $9\left(\frac{l\ sin.\ a}{n}\right)^2$ etc.

FIG. 472.



Multiplying these squares by the contents $\frac{F\,l}{n}$ of an element and adding the products thus obtained, we obtain the moment of inertia of the rod

$$T = \frac{F\,l}{n}\left[\left(\frac{l\,sin.\,a}{n}\right)^2 + 4\left(\frac{l\,sin.\,a}{n}\right)^2 + 9\left(\frac{l\,sin.\,a}{n}\right)^2 + \dots\right]$$

$$= \frac{F\,l^3\,sin.\,a^2}{n^3}\,(1^2 + 2^2 + 3^2 + \dots + n^2),$$

but since $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n^3}{3}$, we have

$$W = \frac{F\, l^3\, sin.^2\, a}{3}.$$

Now since $F\, l$ is the volume of the half rod, which we treat as the mass M of the body; we have

$$W = \tfrac{1}{3}\, M\, l^2\, sin.^2\, a.$$

The distance of one end of the rod from the axis $\overline{X}\, X$ is

$$A\ C = B\ D = a = l\, sin.\, a,$$

and, therefore, we have more simply

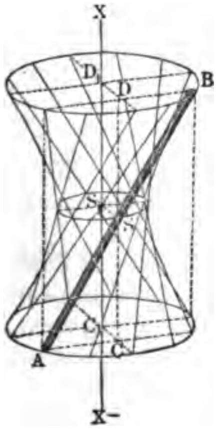
$$W = \tfrac{1}{3}\, M\, a^2,$$

which formula applies to the entire rod, when we understand by M the mass of the whole rod.

The moment of inertia of a mass M_1 at the end A of the rod is $M_1\, a^2$; if, therefore, we make $M_1 = \frac{1}{3}\, M$, M_1 has the same *moment of inertia* as the rod. Hence, so far as the moment of inertia is concerned, it makes no difference whether the mass is equally distributed along the rod, or whether one-third of it is concentrated at the end A. If we put $W = M\, k^2$, we obtain $k^2 = \frac{1}{3}\, a^2$, and, therefore, the *radius of gyration* of the rod is

$$k = a \sqrt{\tfrac{1}{3}} = 0{,}5773\, .\, a.$$

FIG. 473.



If the rod is at right angles to the axis of rotation $a = l$, and consequently

$$W = \tfrac{1}{3}\, M\, l^2.$$

If, finally, the rod does not lie in the same plane as the axis of rotation, if the shortest distance between the axis of rotation and the axis of the rod is

$$S\, S_1 = C\, C_1 = D\, D_1 = d,$$

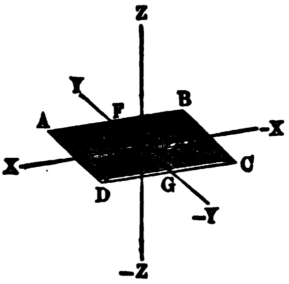
and if the normal distances $A\ C = B\ D$ of the ends A and B of the rod from the axis $C\, D$, passing through the centre of gravity S of the rod and parallel to $C_1\, D_1$ is a, we have (*according to* § 284) the moment of inertia of the rod

$$W_1 = W + \tfrac{1}{3}\, M\, a^2 = M\, (d^2 + \tfrac{1}{3}\, a^2).$$

§ **287. Rectangle and Parallelopipedon.**—The moments of inertia of *plane surfaces* are found in exactly the same way as their moments of flexure $W = F_1\, z_1^2 + F_2\, z_2^2 + \ldots$ We can, consequently, employ here the values of W, found in the last section for various surfaces, as their moments of inertia W.

For the rectangle $A\ B\ C\ D$, Fig. 474, the moment of inertia in reference to the axis $\overline{X\ X}$, which runs parallel to one side and through the middle S of the figure, is, according to § 226,

FIG. 474.



$$W = \frac{b\,h^2}{12},$$

b denoting the width $A\ B = C\ D$ parallel to the axis of rotation and h the length $A\ D = B\ C$ of the surface. But the area of this surface can be regarded as the mass M, and therefore we have

$$W = \frac{M\,h^2}{12} = \frac{M}{3}\left(\frac{h}{2}\right)^2,$$

i.e. equal to that of one-third of this mass concentrated at the distance $\overline{S\ F} = \overline{S\ G} = \frac{h}{2}$ from the axis of rotation.

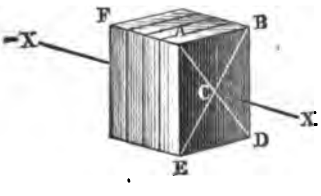
If this rectangle turns upon an axis $Z\ \overline{Z}$, which is at right angles to its plane and which at the same time passes through the middle S of the figure, we have, according to § 225,

$$W = \frac{M\,h^2}{12} + \frac{M\,b^2}{12} = \frac{M\,(h^2 + b^2)}{12} = \frac{M}{3}\left[\left(\frac{h}{2}\right)^2 + \left(\frac{b}{2}\right)^2\right]$$
$$= \frac{M}{3}\left(\frac{d}{2}\right)^2,$$

d designating the diagonal $\overline{A\ C} = \overline{B\ D}$ of the rectangle. We can imagine here also one-third of the whole mass to be concentrated at one of the corners $A, B \ldots$

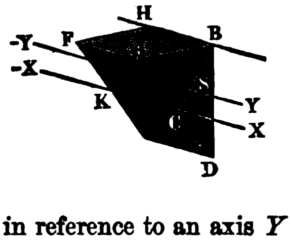
Since *a regular parallelopipedon* $B\ E\ F$, Fig. 475, can be decomposed by parallel planes into equal rectangular slices, this formula is applicable, when the axis of rotation passes through the centres of two opposite surfaces. It follows also that the moment of inertia of the parallelopipedon is equal to the moment of inertia of one-third of its mass applied at one of the corners A.

FIG. 475.



§ **288. Prism and Cylinder.**—By the aid of the formula for the moment of inertia of a parallelopipedon, we can also calculate that of a *triangular prism.* The diagonal plane $A\ D\ F$ divides the parallelopipedon into two equal triangular prisms, whose bases $A\ B\ D$, Fig. 476, are right-angled triangles. The moment of inertia for a rotation about an axis $\overline{X}\ X$, passing through the middles C and K of the hypothenuses, is $= \frac{1}{12}\ M\ d^2$. Now if we employ the rule given in § 284, we obtain the moment of inertia in reference to an axis $Y\ \overline{Y}$ passing through the centres of gravity S and S_1

FIG. 476.



$$W = \tfrac{1}{12}\ M\ d^2 - M\,.\,\overline{C\ S^2} = M\left(\frac{d^2}{12} - (\tfrac{1}{3}\ \overline{C\ B})^2\right)$$
$$= M\left[\frac{d^2}{12} - \left(\frac{d}{6}\right)^2\right],$$

I.E.

$$W = \tfrac{1}{18}\ M\ d^2,$$

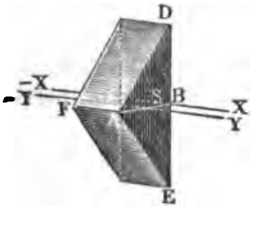
and it follows also that the moment of inertia in reference to the edge $B\ H$ is

$$W_1 = W + M\,.\,\overline{S\ B^2} = \tfrac{1}{18}\ M\ d^2 + M\,(\tfrac{1}{3}\ d)^2 = \tfrac{3}{18}\ M\ d^2$$
$$= \tfrac{1}{6}\ M\ d^2,$$

d denoting the hypothenuse $A\ D$ of the triangular base.

For a *prism* $A\ D\ F\ E$, Fig. 477, whose bases are *isosceles triangles,* the moment of inertia in reference to an axis $X\ \overline{X}$, joining the centres of gravity of the bases, is $W_1 = \frac{1}{6}\ M\ d^2$, d denoting the side $A\ D = A\ E$ of one of the bases; for this surface can be divided by the perpendicular $A\ B$ into two right-angled triangles. Now if the altitude $A\ B$ of the isosceles triangles, which form the bases, is $= h$, we have the moment of inertia of this prism in reference to the axis $Y\ \overline{Y}$ passing through the centres of gravity of the bases

FIG. 477.



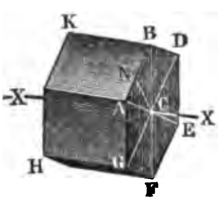
$$W = \tfrac{1}{6}\ M\ d^2 - M\left(\frac{h}{3}\right)^2 = M\,(\tfrac{1}{6}\ d^2 - \tfrac{1}{9}\ h^2)$$
$$= \tfrac{1}{3}\ M\,(\tfrac{1}{2}\ d^2 - \tfrac{1}{3}\ h^2),$$

and, finally, the moment of inertia in reference to the edge, passing through the points *A* and *F* of the bases, is

$$W_1 = W + M\left(\tfrac{2}{3}h\right)^2 = M\left(\frac{d^2}{6} - \frac{h^2}{9} + \frac{4h^2}{9}\right)$$
$$= \tfrac{1}{3}M\left(\tfrac{1}{2}d^2 + h^2\right).$$

By the aid of the latter formula, we can calculate the moment of inertia of a *regular right prism A D F K*, Fig. 478, which revolves about its geometrical axis. Let $C A = C B = r$ be the radius of base or of one of the triangles composing the base, h the altitude $C N$ of one of these triangles $A C B$, and M the mass of the whole prism, then, according to the last formula, when we substitute r for d, we have

FIG. 478.



$$W = \tfrac{1}{3}M\left(\frac{r^2}{2} + h^2\right).$$

The regular prism becomes a cylinder, when h becomes equal to r, and the moment of inertia of the cylinder in reference to its geometrical axis is

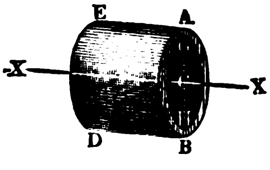
$$W = \tfrac{1}{3}M\left(\frac{r^2}{2} + r^2\right) = \tfrac{1}{2}M r^2.$$

The moment of inertia of a cylinder is equal to the moment of inertia of half the mass of the cylinder concentrated upon its circumference, or equal to the moment of inertia of the whole mass at the distance

$$k = r\sqrt{\tfrac{1}{2}} = 0{,}7071 \,.\, r.$$

If the *cylinder A B D E,* Fig. 479, is *hollow,* we must subtract the moment of inertia of the hollow space from that of the solid cylinder. Let l denote the length, r the radius $C A$ of the exterior and r_2 that $C G$ of the interior cylinder, then we have, according to the above formula, for the moment of inertia of the hollow cylinder

FIG. 479.



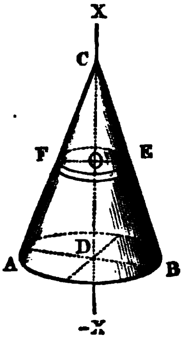
$$W = \tfrac{1}{2}(M_1 r_1^2 - M_2 r_2^2) = \tfrac{1}{2}\pi(r_1^2 \,.\, r_1^2 - r_2^2 \,.\, r_2^2)\,l = \tfrac{1}{2}\pi(r_1^4 - r_2^4)\,l$$
$$= \tfrac{1}{2}\pi(r_1^2 - r_2^2)(r_1^2 + r_2^2)\,l = \tfrac{1}{2}M(r_1^2 + r_2^2);$$

for the volume of the body, which may also be considered as its mass, is $= \pi(r_1^2 - r_2^2)\,l$. If r denotes the mean diameter $\frac{r_1 + r_2}{2}$ and b the width $r_1 - r_2$ of the annular surface, we have

$$W = M\left(r_2 + \frac{b^2}{4}\right)$$

§ 289. Cone and Pyramid.—With the aid of the formula for the moment of inertia of a cylinder we can calculate those of a *right cone* and of a *pyramid.* Let $A\ C\ B$, Fig. 480, be a *cone* turning upon its geometrical axis and let $r = D\ A = D\ B$ be the radius of its base and $h = C\ D$ its altitude, which coincides with the axis. If by passing planes through it, parallel to the base and at equal distances from each other, we divide it into n slices, we obtain n discs, whose radii are

FIG. 480.



$$\frac{r}{n}, 2\frac{r}{n}, 3\frac{r}{n} \ldots n\frac{r}{n}$$

and whose common height is $\frac{h}{n}$; the volumes of these slices are

$$\pi\left(\frac{r}{n}\right)^2 \cdot \frac{h}{n}, \pi\left(\frac{2\,r}{n}\right)^2 \cdot \frac{h}{n}, \pi\left(\frac{3\,r}{n}\right)^2 \cdot \frac{h}{n}, \ldots \text{etc.},$$

and consequently their moments of inertia are

$$\pi\left(\frac{r}{n}\right)^4 \cdot \frac{h}{2\,n}, \pi\left(\frac{2\,r}{n}\right)^4 \cdot \frac{h}{2\,n}, \pi\left(\frac{3\,r}{n}\right)^4 \cdot \frac{h}{2\,n}, \text{etc.}$$

The sum of these values gives the *moment of inertia of the entire cone*

$$W = \frac{\pi\,r^4\,h}{2\,n^4}\,(1^4 + 2^4 + 3^4 + \ldots + n^4),$$

i.e., since $1^4 + 2^4 + 3^4 + \ldots + n^4 = \frac{n^5}{5}$ and the mass of the cone is $M = \frac{\pi\,r^2\,h}{3}$,

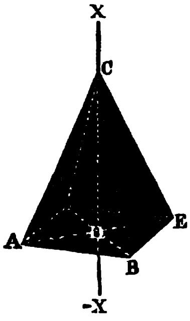
$$W = \frac{\pi\,r^4\,h}{10} = \frac{3}{10} \cdot \frac{\pi\,r^2\,h}{3} \cdot r^2 = \frac{3}{10}\,M\,r^2.$$

In like manner we have under the same circumstances for a *right pyramid* $A\ C\ E$, Fig. 481, whose base is a rectangle,

$$W = \tfrac{1}{5}\,M d^2,$$

in which formula d denotes the half $D\ A$ of the diagonal of the base.

FIG. 481.



We obtain, by subtracting one moment of inertia from another, the moment of inertia of a frustum of a cone ($A\ B\ E\ F$, Fig. 480) in reference to its geometrical axis $X\ \overline{X}$.

If we denote the radii $D\ A$ and $O\ F$ by r_1 and r_2 and the altitudes $C\,D$ and $C\,O$ by h_1 and h_2, we have

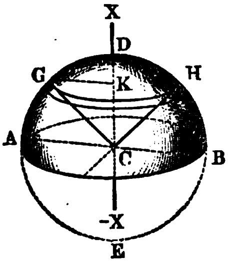
$$W = \frac{\pi}{10}(r_1^4 h_1 - r_2^4 h_2) = \frac{\pi h_1}{10 r_1}(r_1^5 - r_2^5),$$

or, since the mass is

$$M = \frac{\pi}{3}(r_1^2 h_1 - r_2^2 h_2) = \frac{\pi h_1}{3 r_1}(r_1^3 - r_2^3),$$

$$W = \frac{3}{10} M \left(\frac{r_1^5 - r_2^5}{r_1^3 - r_2^3}\right).$$

FIG. 482.



§ 290. Sphere.—In the same manner the moment of inertia of a sphere, revolving upon one of its diameters $D E = 2 r$, is determined. Let us divide the hemisphere $A D B$, Fig. 482, by planes parallel to its base $A C B$, into n equally thick slices, such as $G K H$, etc., and let us determine their moments. The square of the radius $G K$ of one of these slices is

$$\overline{GK}^2 = \overline{CG}^2 - \overline{CK}^2 = r^2 - \overline{CK}^2,$$

and, therefore, its moment of inertia is

$$= \tfrac{1}{2}\pi \,.\, \frac{r}{n}(r^2 - \overline{CK}^2)^2$$

$$= \frac{\pi r}{2 n}(r^4 - 2 r^2 \,.\, \overline{CK}^2 + \overline{CK}^4).$$

Substituting successively for $C K$, $\frac{r}{n}$, $\frac{2 r}{n}$, $\frac{3 r}{n}$, etc., to $\frac{n r}{n}$ and adding the results, we obtain the moment of inertia of the hemisphere

$$W = \frac{\pi r}{2 n}\left[n \,.\, r^4 - 2 r^2 \left(\frac{r}{n}\right)^2 (1^2 + 2^2 + \ldots + n^2) + \left(\frac{r}{n}\right)^4 (1^4 + 2^4 + \ldots + n^4)\right]$$

$$= \frac{\pi r}{2 n}\left[n r^4 - \frac{2 r^4}{n^2} \,.\, \frac{n^3}{3} + \left(\frac{r}{n}\right)^4 \,.\, \frac{n^5}{5}\right],$$

i.e., $$W = \frac{\pi r^5}{2}(1 - \tfrac{2}{3} + \tfrac{1}{5}) = \frac{4 \pi r^5}{15}.$$

Now since the contents of a hemisphere are $M = \frac{2}{3}\pi r^3$, we can put

$$W = \tfrac{2}{5} \,.\, \tfrac{2}{3}\pi r^3 \,.\, r^2 = \tfrac{2}{5} M r^2,$$

and if we consider M as the mass of *the whole sphere*, the formula still holds good.

The radius of gyration is

$$k = r\sqrt{\tfrac{2}{5}} = 0{,}6324 \,.\, r;$$

two-fifths of the mass of the sphere, at a distance equal to the radius of the sphere from the axis of rotation, has the same moment of inertia as the entire sphere. The formula

$$W = \tfrac{2}{5} M r^2$$

holds good also for any *spheroid* whose equatorial radius is $= r$. (See § 123.)

If the sphere revolves about another axis at the distance d from the centre, we must put the moment of inertia

$$W = M\,(d^2 + \tfrac{2}{5}\,r^2).$$

§ **291. Cylinder and Cone.**—The moment of inertia of a circle $A\ B\ D\ E$, Fig. 483, in reference to an axis passing through its centre C and at right angles to the plane of the circle, since all points are at a distance $C\,A = r$ from the axis, is

$$W = M\,r^2,$$

and consequently that in reference to a diameter $\overline{X}\,X$ or $\overline{Y}\,Y$ (compare § 231) is

$$W_1 = \tfrac{1}{2}\,W = \tfrac{1}{2}\,M\,r^2.$$

On the contrary, the moment of inertia of a *circular disc* $A\ B\ D\ E$, Fig. 483, which revolves about its diameter $B\ E$, is found to be, like the moment of flexure of a cylinder,

$$= \frac{\pi\,r^4}{4} = \frac{M\,r^2}{4},$$

consequently the radius of gyration of this surface is

$$k = r\,\sqrt{\tfrac{1}{4}} = \tfrac{1}{2}\,r,$$

I.E., half the radius of the circle.

FIG. 483.

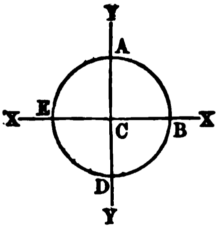
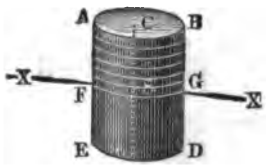


FIG. 484.



From this we can calculate the moment of inertia of a *cylinder* $A\ B\ D\ E$, Fig. 484, which revolves around its diameter $F\ G$, which passes through its centre of gravity S. Let l be the half height $A\ F$ and r the radius $C\,A = C\,B$ of the cylinder, then the volume of one half of it is $= \pi\,r^2\,l$, and if we pass through it planes parallel to the base and at equal distances from each other, we decompose this body into n equal parts, each of which is $= \frac{\pi\,r^2\,l}{n}$ and the first of which is at a distance $\frac{l}{n}$, the second at a distance $\frac{2\,l}{n}$, the third at a distance $\frac{3\,l}{n}$, etc., from the centre of gravity S. By means of the formula in § 284, we obtain the moments of inertia of these discs or slices

$$\frac{\pi r^2 l}{n}\left[\tfrac{1}{4} r^2 + \left(\frac{l}{n}\right)^2\right], \frac{\pi r^2 l}{n}\left[\tfrac{1}{4} r^2 + \left(\frac{2l}{n}\right)^2\right],$$
$$\frac{\pi r^2 l}{n}\left[\tfrac{1}{4} r^2 + \left(\frac{3l}{n}\right)^2\right], \text{ etc.,}$$

whose sum is the *moment of inertia*

$$W = \frac{\pi r^2 l}{n}\left[\frac{n r^2}{4} + \left(\frac{l}{n}\right)^2 (1^2 + 2^2 + 3^2 + \ldots + n^2)\right]$$
$$= \pi r^2 l \left(\frac{r^2}{4} + \frac{l^2}{n^3} \cdot \frac{n^3}{3}\right) = M\left(\frac{r^2}{4} + \frac{l^2}{3}\right)$$

of half the cylinder. This formula holds good for the *whole cylinder,* when M denotes its mass.

The moment of inertia of a *right prism* $A\ B\ D$, Fig. 485, in reference to a transverse axis $\overline{X}\ X$ passing through the centre of gravity S is determined in a similar way. Let k be the radius of gyration of the base $A\ B$ of the prism in reference to an axis $\overline{N}\ N$, passing through the centre of gravity C of the base and parallel to $\overline{X}\ X$, and let l denote the half length or height $C\ S = D\ S$ of the prism; we have the required moment of inertia in reference to the axis $\overline{X}\ X$

$$W = M\left(k^2 + \tfrac{1}{3}\, l^2\right).$$

FIG. 485.

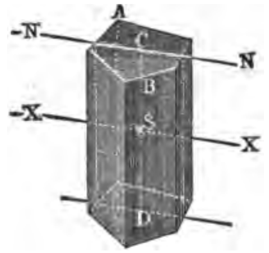
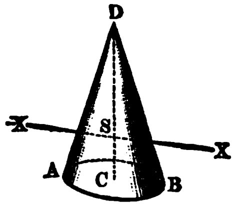


FIG. 486.



In like manner we find for the *right cone* $A\ B\ D$, Fig. 486, whose axis of rotation passes through its centre of gravity at right angles to its geometrical axis $C\ D$,

$$W = \tfrac{3}{20}\, M\left(r^2 + \frac{h^2}{4}\right).$$

§ **292. Segments.**—The moment of inertia of a paraboloid of revolution $B\ A\ D$, Fig. 487, which revolves around its axis of revolution $A\ C$, is determined in a similar manner to that of a sphere. If the radius of the base is $C\ B = C\ D = a$, and the altitude $C\ A = h$, and if we divide the body into slices of the height $\frac{h}{n}$, we have their contents

$$= \frac{h}{n}\pi\,.\,\frac{1}{n}a^2,\ \frac{h}{n}\pi\,.\,\frac{2}{n}a^2,\ \frac{h}{n}\pi\,.\,\frac{3}{n}a^2, \text{ etc.,}$$

for the squares of the radii are as the altitudes or distances from the vertex A. From this we obtain the moments of inertia of the successive disc-shaped elements of the body, which are

$$= \frac{h}{n}\cdot\frac{\pi}{2}\cdot\frac{a^4}{n^2},\ \frac{h}{n}\cdot\frac{\pi}{2}\cdot\frac{4\,a^4}{n^2},\ \frac{h}{n}\cdot\frac{\pi}{2}\cdot\frac{9\,a^4}{n^2}, \text{ etc.,}$$

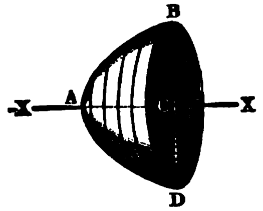
and consequently the *moment of inertia of the whole paraboloid* is

$$W = \frac{\pi\,a^4\,h}{2\,n^3}\,(1^2 + 2^2 + 3^2 + \ldots + n^2) = \frac{\pi\,a^4\,h}{2\,n^3}\cdot\frac{n^3}{3} = \frac{\pi\,a^4\,h}{6}$$

$$= \frac{\pi\,a^2\,h}{2}\cdot\frac{a^2}{3} = \tfrac{1}{3}\,M\,a^2;$$

for the volume of this body is $M = \dfrac{\pi\,a^2\,h}{2}$.

FIG. 487.



This formula may be applied to a *low segment of a sphere.*

If the altitude h of such a segment is not very small compared with a, we have for the moment of inertia of one of its slices

$$W_1 = \frac{\pi\,h}{2\,n}\,.\,a^4 = \frac{\pi\,h}{2\,n}\,.\,h^2\,(2\,r - h)^2$$

$$= \frac{\pi\,h}{2\,n}\,.\,(4\,r^2\,h^2 - 4\,r\,h^3 + h^4),$$

in which r denotes the radius of the sphere.

Now if we substitute for h successively the values $\frac{h}{n}, \frac{2\,h}{n}, \frac{3\,h}{n}$, etc., we obtain the moment of inertia of the segment of the sphere

$$W = \frac{\pi\,h}{2\,n}\left[4\,r^2\left(\frac{h}{n}\right)^3.\,\frac{n^3}{3} - 4\,r\left(\frac{h}{n}\right)^4.\,\frac{n^4}{4} + \left(\frac{h}{n}\right)^5.\,\frac{n^5}{5}\right]$$

$$= \frac{\pi\,h^3}{30}\,(20\,r^2 - 15\,r\,h + 3\,h^2).$$

The volume or the mass of the segment of the sphere is

$$M = \pi\,h^2\,(r - \tfrac{1}{3}\,h),$$

and therefore

$$W = \pi\,h^2\,(r - \tfrac{1}{3}\,h)\,.\,\frac{2\,h}{3}\left(r - \tfrac{5}{12}\,h + \tfrac{1}{90}\cdot\frac{h^2}{r - \frac{1}{3}\,h}\right)$$

$$= \tfrac{2}{3}\,M\,h\left(r - \tfrac{5}{12}\,h + \tfrac{1}{90}\cdot\frac{h^2}{r - \frac{1}{3}\,h}\right);$$

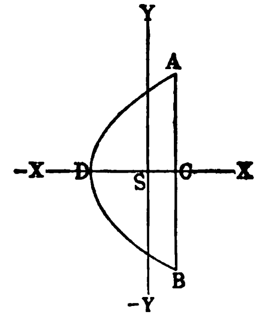
generally it is sufficiently correct to put

$$W = \tfrac{2}{3}\,M\,h\,(r - \tfrac{5}{12}\,h) = \tfrac{1}{2}\,M\,(a^2 + \tfrac{1}{6}\,h^2).$$

This formula is applicable to the *bob of a pendulum.*

§ **293. Parabola and Ellipse.**—For the surface $A\ B\ D$, Fig. 488, of a parabola, if, instead of the surface F, we substitute the mass M or change F into M, and if we denote the chord $A\ B$ by s and the height of the arc $C\ D$ by h, we have (according to § 233) the moment of inertia in reference to the geometrical axis $\overline{X}\ X$ of this surface

FIG. 488.



$$W_1 = \frac{M s^2}{20},$$

and that in reference to the axis $\overline{Y}\ Y$, passing through the centre of gravity S at right angles to $\overline{X}\ X$, is

$$W_2 = \tfrac{12}{175} M h^2.$$

Hence the moment of inertia in reference to an axis, passing *through S at right angles* to the surface of the parabola, is

$$W = W_1 + W_2 = M\left(\frac{s^2}{20} + \tfrac{12}{175} h^2\right) = \tfrac{1}{5} M\left[\left(\frac{s}{2}\right)^2 + \tfrac{12}{35} h^2\right].$$

For such an axis, passing through the vertex D of the parabola, the moment is, since $D\ S = \frac{3}{5} h$ (§ 115),

$$W_3 = W + M\left(\tfrac{3}{5} h\right)^2 = \tfrac{1}{5} M\left[\left(\frac{s}{2}\right)^2 + \tfrac{15}{7} h^2\right],$$

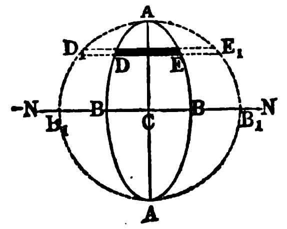
and, on the contrary, the moment in reference to an axis passing through the centre C of the chord is

$$W_4 = W + M\left(\tfrac{2}{5} h\right)^2 = \tfrac{1}{5} M\left[\left(\frac{s}{2}\right)^2 + \tfrac{8}{7} h^2\right].$$

This formula is also applicable to a *prism* whose bases are parabolas, E.G. a working-beam, which consists of two such prisms oscillating about an axis passing through their middle C.

The moment of inertia of an *ellipse* $A\ B\ A\ B$, Fig. 489, whose semi-axes are $C\ A = a$ and $C\ B = b$, in reference to the axis $B\ B$, is (according to § 231)

FIG. 489.



$$W_1 = \frac{\pi a^3 b}{4} = \frac{M a^2}{4},$$

and that in reference to the axis $A\ A$ is

$$W_2 = \frac{\pi a b^3}{4} = \frac{M b^2}{4};$$

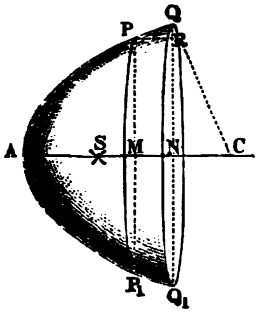
hence the moment of inertia in reference to an axis, passing through the centre C at *right angles to the plane of the figure*, is

$$W = W_1 + W_2 = \tfrac{1}{4} M (a^2 + b^2).$$

(**§ 294.**) **Surfaces and Solids of Revolution.**—The moments of inertia of *surfaces and solids of revolution* can be determined with the aid of the Calculus by means of the following formulas.

1) If a zone or belt $P\ Q\ Q_1\ P_1$, Fig. 490, whose radius is $M\ P = y$ and whose width is $P\ Q = d\ s$, is caused to revolve around its geometrical axis $A\ C$, we have (according to § 125) its area

FIG. 490.



$$d\ O = 2\ \pi\ y\ d\ s,$$

and its moment of inertia is

$$y^2\ d\ O = 2\ \pi\ y^3\ d\ s;$$

hence the moment of inertia of the whole *surface of revolution* $A\ P\ P_1$ in reference to its axis $A\ C$ is

$$W = 2\ \pi \int y^3\ d\ s.$$

2) For a slice $P\ Q\ Q_1\ P_1$, whose volume is $d\ V = \pi\ y^2\ d\ x$, the moment of inertia in reference to the axis $A\ C$ is (according to § 288)

$$\frac{d\ V . y^2}{2} = \frac{\pi y^4\ d\ x}{2},$$

and consequently the moment of inertia for the whole solid of revolution $A\ P\ P_1$ is

$$W = \frac{\pi}{2} \int y^4\ d\ x.$$

If $A\ P$ is an arc of a circle, in which case the surface generated by its revolution is *a spherical cup or zone*, we have

$$y^2 = 2\ r\ x - x^2 \text{ and } y\ d\ s = r\ d\ x,$$

and consequently the *moment of inertia of this zone* is

$$W = 2\ \pi \int (2\ r\ x - x^2)\ r\ d\ x = 2\ \pi\ r \left(2\ r \int x\ d\ x - \int x^2\ d\ x\right)$$

$$= 2\ \pi\ r \left(r\ x^2 - \frac{x^3}{3}\right),$$

or, if we substitute h for the altitude $A\ M = x$, we have

$$W = 2\ \pi\ r\ h^2 \left(r - \frac{h}{3}\right) = M\ h \left(r - \frac{h}{3}\right),$$

since the area or mass of the *zone* is $M = 2\ \pi\ r\ h$.

For the entire *surface of the sphere* $h = 2\,r$, and therefore

$$W = \tfrac{2}{3}\,M\,r^2.$$

If, on the contrary, $A\,P$ is the arc of an ellipse, and consequently the solid of revolution $A\,P\,P_1$ generated by the rotation of the plane surface $A\,P\,M$ a segment of an *ellipsoid of revolution*, we will have

$$y^2 = \frac{b^2}{a^2}(2\,a\,x - x^2),$$

and therefore its moment of inertia in reference to the axis $A\;C$ is

$$W = \frac{\pi}{2}\cdot\frac{b^4}{a^4}\int(2\,a\,x - x^2)^2\,d\,x$$

$$= \frac{\pi\,b^4}{2\,a^4}\int(4\,a^2\,x^2 - 4\,a\,x^3 + x^4)\,d\,x$$

$$= \frac{\pi\,b^4}{2\,a^4}\left(\tfrac{4}{3}\,a^2\,x^3 - a\,x^4 + \frac{x^5}{5}\right),$$

E.G. for the entire *ellipsoid*, in which case $x = 2\,a$,

$$W = \tfrac{8}{15}\,\pi\,b^4\,a = \tfrac{2}{5}\,.\,\tfrac{4}{3}\,\pi\,a\,b^2\,.\,b^2 = \tfrac{2}{5}\,M\,b^2\,;$$

for the contents of this body are expressed by $\frac{b^2}{a^2}\,.\,\frac{4}{3}\,\pi\,a^3 = \frac{4}{3}\,\pi\,a\,b^2$ (compare § 123).

3) If the belt $P\,Q\,Q_1\,P_1$ revolves about an axis passing through A at right angles to its geometrical axis $A\;C$, we have (see § 284 and § 291) its moment of inertia

$$= d\,O\,(x^2 + \tfrac{1}{2}\,y^2) = 2\,\pi\,(x^2 + \tfrac{1}{2}\,y^2)\,y\,d\,s,$$

and, therefore, the moment of inertia of the whole zone $A\,P\,P_1$ is

$$W = \pi\int(2\,x^2 + y^2)\,y\,d\,s.$$

4) If the entire disc $P\,Q\,Q_1\,P_1$ revolves around this same axis passing through A, its moment of inertia is

$$d\,V\,(x^2 + \tfrac{1}{4}\,y^2) = \pi\,y^2\,(x^2 + \tfrac{1}{4}\,y^2)\,d\,x,$$

and, therefore, that of the entire body $A\,P\,P_1$ is

$$W = \pi\int(x^2 + \tfrac{1}{4}\,y^2)\,y^2\,d\,x.$$

For a paraboloid of revolution (see § 292), we have, when we denote its altitude $A\,M$ by h and the radius of its base $M\,P$ by a,

$$\frac{y^2}{a^2} = \frac{x}{h},$$

and consequently the moment of inertia in reference to the axis of ordinates passing through A is

$$W = \frac{\pi\,a^2}{h}\int\left(x^2 + \tfrac{1}{4}\,\frac{a^2\,x}{h}\right)x\,d\,x = \frac{\pi\,a^2}{h}\left(\tfrac{1}{4}\,x^4 + \tfrac{1}{12}\,\frac{a^2\,x^3}{h}\right),$$

or, when we substitute $x = h$,

$$W = \tfrac{1}{4}\,\pi\,a^2\,h\,(h^2 + \tfrac{1}{3}\,a^2) = \tfrac{1}{2}\,M\,(h^2 + \tfrac{1}{3}\,a^2),$$

since the volume of this body is $= \frac{1}{2}\,\pi\,a^2\,h$ (comp. § 124).

Hence we have the moment of inertia of this body in reference to an axis, passing through the *centre of gravity S at right angles to A C*

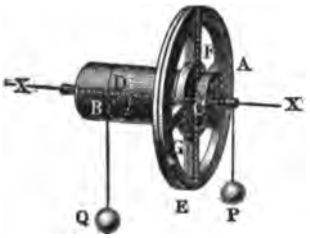
$$W_1 = \tfrac{1}{2}\,M\,(h^2 + \tfrac{1}{3}\,a^2) - (\tfrac{2}{3})^2\,M\,h^2 = \tfrac{1}{6}\,M\,(a^2 + \tfrac{1}{3}\,h^2).$$

§ 295. Accelerated Rotation of a Wheel and Axle.—

The most frequent applications of the theory of the moment of inertia are to machines and instruments; for rotary motions around a fixed axis are very common in them. Since throughout this work we shall meet with very many applications of this theory, we shall treat here but a few simple cases.

If two weights P and Q act by means of two perfectly flexible strings upon the ***wheel and axle*** $A\,C\,D\,B$, Fig. 491, if their arms are $C\,A = a$ and $D\,B = b$ and if the journals are so small that the friction can be neglected, the machine is in equilibrium, when the statical moments $P\,.\,\overline{C\,A}$, and $Q\,.\,\overline{D\,B}$, are equal to each other, or when $P\,a = Q\,b$. If, on the contrary, the moment of the weight P is greater than that of Q, or $P\,a > Q\,b$, P will fall and Q will rise; on the contrary, if $P\,a < Q\,b$, P will rise and Q will fall. Let us therefore seek the relations of the motions in this case, taking, E.G., $P\,a > Q\,b$. The force, which acts with the arm b and corresponds to the weight Q, produces a force $\frac{Q\,b}{a}$, whose arm is a and which acts in opposition to the force corresponding to the weight P, so that the motive force in action at A is $P - \frac{Q\,b}{a}$. The mass $\frac{Q}{g}$, reduced from the arm b to the arm a, is $\frac{Q\,b^2}{g\,a^2}$, hence the mass moved by the force $P - \frac{Q\,b}{a}$ is

FIG. 491.



$$M = \left(P + \frac{Q\,b^2}{a^2}\right) : g,$$

or, if the moment of inertia of the wheel and axle is $W = \frac{G\,k^2}{g}$ and therefore the mass of the same reduced to A is $= \frac{G\,k^2}{g\,a^2}$, we have more accurately

$$M = \left(P + \frac{Q\,b^2}{a^2} + \frac{G\,k^2}{a^2}\right) : g = (P\,a^2 + Q\,b^2 + G\,k^2) : g\,a^2.$$

Hence the acceleration of P or of the circumference of the wheel is

$$p = \frac{\text{motive force}}{\text{mass}} = \frac{P - \frac{Q\,b}{a}}{P\,a^2 + Q\,b^2 + G\,k^2} \cdot g\,a^2$$

$$= \frac{P\,a - Q\,b}{P\,a^2 + Q\,b^2 + G\,k^2} \cdot g\,a;$$

hence the acceleration of the rising weight Q or of the circumference of the axle is

$$q = \frac{b}{a}\,p = \frac{P\,a - Q\,b}{P\,a^2 + Q\,b^2 + G\,k^2} \cdot g\,b.$$

The tension of the cord, to which P is attached, is

$$S = P - \frac{P\,p}{g} = P\left(1 - \frac{p}{g}\right) \text{ (see § 76)},$$

and that of the cord, to which Q is attached, is

$$S_1 = Q + \frac{Q\,q}{g} = Q\left(1 + \frac{q}{g}\right),$$

and, therefore, the pressure on the bearings is

$$S + S_1 = P + Q - \frac{P\,p}{g} + \frac{Q\,q}{g} = P + Q - \frac{(P\,a - Q\,b)^2}{P\,a^2 + Q\,b^2 + G\,k^2}.$$

The pressure on the bearing of a wheel and axle, when in rotation, is consequently less than when it is standing still.

From the accelerations p and q the other relations of the motion can be found; after t seconds the velocity of P is

$$v = p\,t$$

and that of Q is

$$v_1 = q\,t;$$

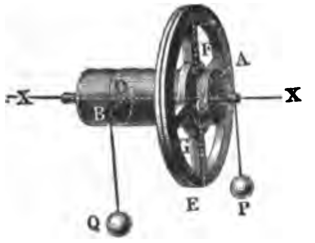
the space described by P is

$$s = \tfrac{1}{2}\,p\,t^2$$

and that by Q,

$$s_1 = \tfrac{1}{2}\,q\,t^2.$$

FIG. 492.



EXAMPLE.—Let the weight upon the wheel, Fig. 492, be $P = 60$ pounds and that on the axle, $Q = 160$ pounds; let the arm of the former be $C\,A = a = 20$ inches and that of the latter $D\,B = b = 6$ inches, and let the axle be composed of a massive cylinder, weighing 10 pounds, and the wheel of two rings, one weighing 40 pounds and the other 12 pounds, and of four arms, weighing together 15 pounds; finally, let the radii of the large ring *A E* be = 20 and 19 inches and those of the smaller one *F G* be = 8 and 6 inches. Required the conditions of motion of this machine. The motive force at the circumference of the wheel is

$$P - \frac{b}{a}\, Q = 60 - \tfrac{6}{20} \,.\, 160 = 60 - 48 = 12 \text{ pounds},$$

and the moment of inertia of the machine, when we disregard the masses of the ropes and journals, is equal to the moment of inertia of the axle, which is

$$= \frac{W b^2}{2} = \frac{10 \,.\, 6^2}{2} = 180,$$

plus the moment of the smaller ring, which is

$$= \frac{R_1\, (r_1^2 + r_2^2)}{2} = \frac{12 \,.\, (8^2 + 6^2)}{2} = 600,$$

plus the moment of the larger ring, which is

$$= \frac{R_2\, (r_3^2 + r_4^2)}{2} = \frac{40 \,.\, (20^2 + 19^2)}{2} = 15220,$$

plus the moment of the arms, which is, approximately,

$$= \frac{A\,(r_4^3 - r_1^3)}{3\,(r_4 - r_1)} = \frac{A\,(r_1^2 + r_1\, r_4 + r_4^2)}{3} = \frac{15 \,.\, (19^2 + 19 \,.\, 8 + 8^2)}{3} = 2885;$$

hence, by addition, we obtain

$$G k^2 = 180 + 600 + 15220 + 2885 = 18885,$$

or, taking the foot as the unit of measure,

$$= \frac{18885}{144} = 131{,}14.$$

The whole mass, reduced to the radius of the wheel, is

$$M = \left(P + \frac{Q b^2 + G k^2}{a^2}\right) : g = \left[60 + 160 \left(\frac{6}{20}\right)^2 + \frac{18885}{20^2}\right] : g$$

$$= \left(60 + 160 \,.\, 0{,}09 + \frac{18885}{400}\right) . \, 0{,}031$$

$$= (60 + 14{,}4 + 47{,}21) \,.\, 0{,}031 = 121{,}61 \,.\, 0{,}031 = 3{,}76991 \text{ pounds.}$$

Hence we have the acceleration of the weight *P*, or that of the circumference of the wheel,

$$p = \frac{P - \frac{b}{a}\, Q}{\frac{P a^2 + Q b^2 + G k^2}{a^2}} \,.\, g = \frac{12}{3{,}76991} = 3{,}183 \text{ feet};$$

and, on the contrary, that of *Q* is

$$q = \frac{b}{a}\, p = \tfrac{6}{20} \,.\, 3{,}183 = 0{,}955 \text{ feet};$$

the tension on the rope to which *P* is hung is

$$S = \left(1 - \frac{p}{g}\right) . \, P = \left(1 - \frac{3{,}183}{32{,}20}\right) . \, 60 = (1 - 0{,}099) \,.\, 60 = 54{,}06 \text{ pounds},$$

and that of the rope supporting *Q* is

$$S_1 = \left(1 + \frac{q}{g}\right) . \, Q = (1 + 0{,}955 \,.\, 0{,}031) \,.\, 160 = 1{,}03 \,.\, 160 = 164{,}8 \text{ pounds};$$

consequently the pressure on the bearings is $S + S_1 = 54{,}06 + 164{,}8 = 218{,}86$ lbs., or, if we include the weight of the machine, it is = 218,86

+ 77 = 295,86 pounds. At the end of 10 seconds P has attained the velocity $v = p\,t = 3{,}183 \,.\, 10 = 31{,}83$ feet, and has described the space $s = \frac{v\,t}{2} = 31{,}83 \,.\, 5 = 159{,}2$ feet, and Q has been raised up $s_1 = \frac{b}{a}\,s = 0{,}3 \,.\, 159{,}2 = 47{,}76$ feet.

§ **296.** The weight P, which imparts to the weight Q the acceleration

$$q = \frac{P\,a\,b - Q\,b^2}{P\,a^2 + Q\,b^2 + G\,k^2} \cdot g,$$

can be replaced by another P_1, without changing the acceleration of Q, when the arm of the latter is a_1, in which case we have

$$\frac{P_1\,a_1 - Q\,b}{P_1\,a_1^2 + Q\,b^2 + G\,k^2} = \frac{P\,a - Q\,b}{P\,a^2 + Q\,b^2 + G\,k^2}.$$

If we designate the quantity $\frac{P\,a^2 + Q\,b^2 + G\,k^2}{P\,a - Q\,b}$ by c, we obtain

$$a_1^2 - c\,a_1 = -\frac{Q\,b\,(b + c) + G\,k^2}{P_1},$$

and the required arm of the lever

$$a_1 = \tfrac{1}{2}\,c \pm \sqrt{\left(\frac{c}{2}\right)^2 - \frac{Q\,b\,(b + c)\,G\,k^2}{P_1}}.$$

We find, also, by the differential calculus that the greatest acceleration is imparted to Q by P, when the arm of the latter corresponds to the equation $P\,a^2 - 2\,Q\,a\,b = Q\,b^2 + G\,k^2$, or when

$$a = \frac{b\,Q}{P} + \sqrt{\left(\frac{b\,Q}{P}\right)^2 + \frac{Q\,b^2 + G\,k^2}{P}}.$$

The foregoing formulas become very complicated when we take into consideration the friction of the journals and the rigidity of the ropes. If we denote the resistance due to both of these, reduced to a radius r, by F, we must substitute, instead of the motive force $P - \frac{b}{a}\,Q$, the expression $P - \frac{Q\,b + F\,r}{a}$, and then we have the acceleration of Q

$$q = \frac{(P\,a - F\,r)\,b - Q\,b^2}{P\,a^2 + Q\,b^2 + G\,k^2} \cdot g$$

and

$$a = \frac{Q\,b + F\,r}{P} + \sqrt{\left(\frac{Q\,b + F\,r}{P}\right)^2 + \frac{Q\,b^2 + G\,k^2}{P}}.$$

EXAMPLE—1) If the weights $P = 30$ pounds and $Q = 80$ pounds act with the arms $a = 2$ feet and $b = \frac{1}{3}$ foot upon a wheel and axle, and if the moment of inertia of this machine is $G\,k^2 = 60$, the acceleration of the rising weight Q will be

$$q = \frac{80 \,.\, 2 \,.\, \frac{1}{2} - 80 \,.\, (\frac{1}{2})^2}{30 \,.\, 2^2 + 80 \,.\, (\frac{1}{2})^2 + 60} \cdot g = \frac{80 - 20}{120 + 20 + 60} \,.\, 32{,}2 = \frac{32{,}2}{20}$$

$= 1{,}61$ feet.

Now if we wish to produce the same acceleration with a weight $P_1 =$ 45 pounds, the arm of P_1 must be

$$a_1 = \frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - \frac{80 \,.\, \frac{1}{2}\,(\frac{1}{2} + c) + 60}{45}},$$

but

$$c = \frac{200}{60 - 40} = 10,$$

hence

$$a_1 = 5 \pm \sqrt{25 - \frac{32}{3}} = 5 \pm \tfrac{1}{3} \,.\, 11{,}358 = 5 \pm 3{,}786$$

$= 8{,}786$ or $1{,}214$ feet.

2) The acceleration of Q is a maximum when the arm of the force or radius of the wheel is

$$a = \frac{\frac{1}{2} \,.\, 80}{30} + \sqrt{\left(\frac{40}{30}\right)^2 + \frac{20 + 60}{30}} = \frac{4}{3} + \sqrt{\frac{16}{9} + \frac{24}{9}} = \frac{4 + \sqrt{40}}{3}\cdot$$

$= 3{,}4415$ feet,

and this maximum acceleration is then

$$q = \left(\frac{30 \,.\, 1{,}7207 - 20}{30 \,.\, (3{,}4415)^2 + 80}\right) g = \frac{31{,}621}{435{,}32} \cdot g = 2{,}339 \text{ feet.}$$

3) If the moment of the friction and of the rigidity of the ropes be $F r = 8$, we must substitute, instead of $Q\, b$, $Q\, b + F r = 40 + 8 = 48$, whence it follows that

$$a = \frac{48}{30} + \sqrt{\left(\frac{48}{30}\right)^2 + \frac{8}{3}} = 1{,}6 + \sqrt{5{,}227} = 3{,}886 \text{ feet,}$$

and that the corresponding maximum acceleration is

$$q = \frac{30 \,.\, 1{,}943 - 8 \,.\, \frac{1}{2} - 20}{30 \,.\, (3{,}886)^2 + 80} \cdot g = \frac{34{,}29}{533} \,.\, 32{,}2 = 2{,}07 \text{ feet.}$$

§ **297. Atwood's Machine.**—The formulas for the wheel and axle found in § 295 are applicable to the simple fixed pulley; for if we put $b = a$, the wheel and axle becomes a *fixed pulley*. Retaining the same notations that we employed in the foregoing paragraphs, we have the acceleration with which P sinks and Q rises

$$p = q = \frac{(P - Q)\, a^2}{(P + Q)\, a^2 + G\, k^2} \cdot g,$$

or, taking the friction into consideration,

$$p = q = \frac{(P - Q)\, a^2 - F a r}{(P + Q)\, a^2 + G\, k^2} \cdot g.$$

In order to diminish the friction, the axle C of the pulley $A\, B$, Fig. 493, is placed upon the friction-wheels $D\, E\, F$ and $D_1\, E_1\, F_1$. Now if the moment of inertia of these wheels is $G_1\, k_1^2$ and their radius is $D\,E = D_1\,E_1 = a_1$, we have, when F designates the friction reduced to the circumference of the axle C,

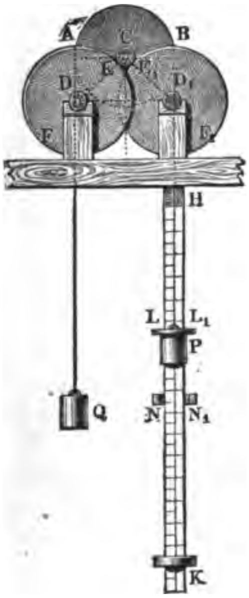
$$p = q = \frac{(P - Q)\,a^2 - F\,a\,r}{(P + Q)\,a^2 + G\,k^2 + G_1\,\frac{k_1^2\,r^2}{a_1^2}} \cdot g;$$

for the moment of inertia of these friction rollers, reduced to their circumference or that of the axle of the wheel, is $= \frac{G_1\,k_1^2}{a_1^2}$.

Inversely we have the acceleration of gravity

$$g = \frac{(P + Q)\,a^2 + G\,k^2 + G_1\,\frac{k_1^2\,r^2}{a_1^2}}{(P - Q)\,a^2 - F\,a\,r} \cdot p.$$

FIG. 493.



When the difference $P - Q$, of the two weights is small, the acceleration p is small and the motion is consequently very slow; hence the resistance opposed to the weights by the air is unimportant, and the acceleration of gravity can be determined with a certain degree of accuracy by means of such an apparatus, while the determination of it by observations upon a body falling freely is impossible. Experiments of this kind were first made by an Englishman named Atwood (see Atwood's treatise on Rectilinear and Rotary Motion), and for this reason the apparatus is known as Atwood's Machine. The scale $H\,K$, along which the weight P falls, serves to measure the distance fallen through. From the spaces fallen through and the corresponding time t we obtain

$$p = \frac{2\,.\,s}{t^2},$$

but if during the fall we remove the motive force by causing the weight $L\,L$, which is made in the shape of a ring and is equal to the force, to be caught by the fixed ring $N\,N_1$, the remainder of the space s_1, through which the weight P falls, will be described uniformly, and the velocity, which is determined by the time t_1 (which can be observed by means of a good watch), is

$$v = \frac{s_1}{t_1},$$

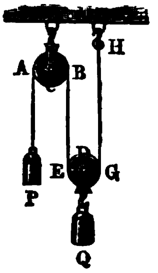
and the acceleration is

$$p = \frac{v}{t} = \frac{s_1}{t\,t_1}.$$

If we make $t_1 = t = 1$, we obtain directly by the experiment $p = s_1$. Substituting this value of p in the above-mentioned formula, we obtain the acceleration g of gravity.

§ 298. Accelerated Motion of a System of Pulleys or Tackle.—The accelerations of the weights P and Q, which are supported by a system composed of a *fixed pulley* $A\,B$, and a *loose pulley* $E\,G$, Fig. 494, are found in the following manner. Let the weight of the pulleys $A\,B$ and $E\,G$ be $= G$ and G_1, their moments of inertia $G\,k^2$ and $G_1\,k_1^2$, their radii $C\,A = a$ and $D\,E = a_1$ and their masses reduced to the circumference $M = \frac{G}{g}\cdot\frac{k^2}{a^2}$ and $M_1 = \frac{G_1}{g}\cdot\frac{k_1^2}{a_1^2}$. If the weight P sinks a certain distance s, $Q + G_1$ rises $\frac{1}{2}\,s$ (§ 164), the work done is therefore $P\,s - (Q + G_1)\,\frac{s}{2}$. Now if in sinking the weight P has acquired the velocity v, then the velocity $\frac{v}{2}$ is communicated to $Q + G_1$, the velocity of the pulley $A\,B$ at the circumference is v and the pulley $E\,G$ acquires at its circumference the velocity $\frac{v}{2}$; for in rolling motion the motions of translation and of rotation are equal to each other. The sum of the *living forces*, corresponding to the masses and velocities, is

FIG. 494.



$$\frac{P}{g}\cdot v^2 + \frac{Q + G_1}{g}\cdot\left(\frac{v}{2}\right)^2 + \frac{G\,k^2}{g\,a^2}\cdot v^2 + \frac{G_1\,k_1^2}{g\,a_1^2}\cdot\left(\frac{v}{2}\right)^2,$$

putting the half of it equal to the work done, we obtain the equation

$$\left(P - \frac{(Q + G_1)}{2}\right)s = \left(P + \frac{Q + G_1}{4} + \frac{G\,k^2}{a^2} + \frac{G_1\,k_1^2}{4\,a_1^2}\right)\frac{v^2}{2\,g}.$$

Hence the velocity corresponding to the space s, described by P, is

$$v = \sqrt{\frac{2\,g\,s\left(P - \frac{Q + G_1}{2}\right)}{P + \frac{Q + G_1}{4} + \frac{G\,k^2}{a^2} + \frac{G_1\,k_1^2}{4\,a_1^2}}}$$

For the acceleration p we have $p\,s = \frac{v^2}{2}$, and therefore

$$p = \left(\frac{P - \frac{Q + G_1}{2}}{P + \frac{Q + G_1}{4} + \frac{G k^2}{a^2} + \frac{G_1\, k_1^2}{4\, a_1^2}}\right) g$$

The acceleration of $Q + G_1$ is $p_1 = \frac{p}{2}$, and the rotary acceleration of G_1 is also the same. The tension on the rope $B\,E$, which unites the two pulleys, is

$$S = P - \left(P + \frac{G\,k^2}{a^2}\right)\frac{p}{g};$$

for the force $\left(P + \frac{G\,k^2}{a^2}\right)\frac{p}{g}$ is expended in producing the acceleration of P and G; the tension on the rope $G\,H$, which is fastened at one end, is, on the contrary,

$$S_1 = S - \frac{G_1\,k_1^2}{a_1^2} \cdot \frac{p}{2\,g};$$

for the pulley $E\,G$ is set in rotation by the difference $S - S_1$ of the tensions on the rope.

EXAMPLE.—The weights $P = 40$ pounds and $Q = 66$ pounds hang upon the system of pulleys or tackle represented in Fig. 494, and each of the pulleys weighs 6 pounds; required the acceleration of each of the weights.

The motive force is

$$P - \frac{Q + G_1}{2} = 40 - \frac{66 + 6}{2} = 4 \text{ pounds.}$$

The masses of these pulleys, reduced to their circumferences, are

$$\frac{G\,k^2}{g\,a^2} = \frac{G_1\,k_1^2}{g\,a_1^2} = \frac{G}{2\,g} = \frac{6}{2\,g} = \frac{3}{g} \text{ (§ 288)},$$

and the total mass is

$$= \left(P + \frac{Q + G_1}{4} + \frac{G\,k^2}{a^2} + \frac{G_1\,k_1^2}{4\,a_1^2}\right) : g = (40 + \tfrac{72}{4} + 3 + \tfrac{3}{4}) : g = \frac{247}{4\,g},$$

hence the acceleration of the sinking weight is

$$p = \frac{4}{247} \cdot 4\,g = \frac{16\,.\,g}{247} = \frac{16\,.\,32{,}2}{247} = \frac{515{,}2}{247} = 2{,}086 \text{ feet,}$$

and that of the rising weight is

$$p_1 = \frac{p}{2} = 1{,}043 \text{ feet}$$

The tension of the rope $B\,E$ is

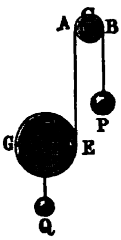
$$S = P - \left(P + \frac{G}{2}\right)\frac{p}{g} = 40 - 43\,.\,\frac{2{,}086}{32{,}2} = 40 - 2{,}785 = 37{,}215 \text{ pounds.}$$

and that of the rope $G\,H$ is

$$S_1 = S - \frac{G}{2} \cdot \frac{p}{2\,g} = 37{,}215 - 3\,.\,\frac{1{,}043}{32{,}2} = 37{,}118 \text{ pounds.}$$

§ **299**. The motion is more complicated, when the pulley $E\ G$, Fig. 495, hangs only upon a cord wound around it. Let us suppose that P sinks with the acceleration p, and that Q rises with the acceleration q, then the acceleration of the motion at the circumference of the loose pulley is

FIG. 495.



$$q_1 = p - q \text{ (§ 45).}$$

Now if we put the tension of the cord $A\ E$, $= S$, we obtain

$$P - S = \left(P + \frac{G k^2}{a^2}\right)\frac{p}{g}$$

and

$$S - (Q + G_1) = (Q + G_1)\frac{q}{g};$$

for, according to § 281, we can assume, that S acts at the centre of gravity D of $E\ G$. Finally we have

$$S = \frac{G_1 k_1^2}{a_1^2} \cdot \frac{q_1}{g},$$

since we can assume that the centre of gravity D is fixed and that the pulley is put in rotation by S.

The last three formulas give the accelerations

$$p = \frac{P - S}{P + \frac{G k^2}{a^2}} g,\ q = \left(\frac{S - (Q + G_1)}{Q + G_1}\right) g \text{ and } q_1 = \frac{S a_1^2}{G_1 k_1^2} g;$$

substituting all three in the equation $q_1 = p - q$, we obtain

$$\frac{S a_1^2}{G_1 k_1^2} g = \frac{P - S}{P + \frac{G k^2}{a^2}} g - \frac{S - (Q + G_1)}{Q + G_1} g,$$

whence it follows that the tension of the rope is

$$S = \frac{2 P a^2 + G k^2}{\left(\frac{a_1^2}{G_1 k_1^2} + \frac{1}{Q + G_1}\right)(P a^2 + G k)^2 + a^2}.$$

From this value of S we find by the application of the above formula the accelerations of the weights P and Q.

If we neglect the mass G of the fixed pulley and put $Q = 0$, we obtain simply

$$S = \frac{2 P a^2 . G_1 k_1^2}{P(a_1^2 + k_1^2) a^2 + G a^2 k_1^2} = \frac{2 P G_1 k_1^2}{G_1 k_1^2 + P(a_1^2 + k_1^2)}.$$

If the end of the cord $A\ E$, instead of passing over the pulley, is fixed, we have the acceleration $p = 0$, and therefore $q_1 = -q$, and the tension

$$S = \frac{(Q + G_1)\, G_1\, k_1^2}{(Q + G_1)\, a_1^2 + G_1\, k_1^2};$$

for $Q = 0$, we have

$$S = \frac{G_1\, k_1^2}{a_1^2 + k_1^2}.$$

If the rolling body G_1 is a massive cylinder, we have

$$\frac{G_1\, k_1^2}{a_1^2} = \tfrac{1}{2}\, G_1,$$

and the tension in the first case is

$$S = \frac{2\, P\, G_1}{3\, P + G_1},$$

and in the second

$$S = \frac{G_1}{3}.$$

If in the first case the weight P must rise, we have p negative and $S > P$, I.E.,

$$2\, P\, G_1\, k_1^2 > P\, G_1\, k_1^2 + P^2\, (a_1^2 + k_1^2),$$

or simply

$$\frac{G_1}{P} > 1 + \frac{a_1^2}{k_1^2};$$

in order that G_1 shall sink it is necessary that $S < G_1$, or that

$$\frac{G_1}{P} > 1 - \frac{a_1^2}{k_1^2}.$$

EXAMPLE.—If the rope $G\,H$ of the system of pulleys in the example of § 298, Fig. 494, suddenly breaks, the rope B will be, for an instant at least, stretched by a force

$$S = \frac{2\, P + \frac{G\, k^2}{a^2}}{\left(\frac{a_1^2}{G_1\, k_1^2} + \frac{1}{Q + G_1}\right)\left(P + \frac{G\, k^2}{a^2}\right) + 1} = \frac{2\,.\,40 + 3}{(\frac{1}{3} + \frac{1}{72})\,(40 + 3) + 1}$$

$$= \frac{83\,.\,72}{25\,.\,43 + 72} = \frac{5976}{1147} = 5{,}210 \text{ pounds.}$$

Hence the acceleration of the sinking weight P is

$$p = \left(\frac{P - S}{P + \frac{G\, k^2}{a^2}}\right) g = \left(\frac{40 - 5{,}210}{40 + 3}\right).\, 32{,}2 = \frac{34{,}79}{43}\,.\, 32{,}2 = 26{,}05,$$

and that of the sinking pulley is

$$q = \left(\frac{Q + G_1 - S}{Q + G_1}\right) g = \left(\frac{72 - 5{,}210}{72}\right).\, 32{,}2 = \frac{66{,}79}{72}\,.\, 32{,}2 = 29{,}87 \text{ feet,}$$

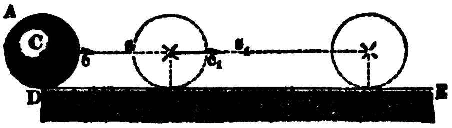
and the acceleration of rotation of this pulley is

$$q_1 = \frac{S\, a_1^2}{G_1\, k_1^2}\,.\, g = \frac{5{,}210}{3}\,.\, 32{,}2 = 55{,}92 \text{ feet.}$$

§ 300. Rolling Motion of a Body on a Horizontal Plane.

—If a round body $A\ C\ D$, Fig. 496, is pushed forward with a certain initial velocity c upon the horizontal path $D\ E$, it will, in consequence of the friction upon this path, assume a motion of rotation, the velocity of which will gradually increase; its acceleration p is determined by the formula

FIG. 496.



$$p = \frac{\text{Force}}{\text{Mass}} = \frac{\phi\, G\, a^2}{M\, k^2} = \frac{\phi\, a^2}{k^2}\, g,$$

in which ϕ denotes the coefficient of friction, $G = M g$ the weight, $\phi\ G$ the friction, $M\, k^2$ the moment of inertia and a the radius $C\, D$ of rotation of the body. The velocity of rotation at the distance $C\ D$ from the axis c, engendered by this acceleration in the time t, is

$$v = p\, t = \phi \frac{a^2}{k^2} g\, t.$$

On the contrary, the forward motion of the body suffers a retardation q, which is determined by the formula

$$q = \frac{\text{Resistance}}{\text{Mass}} = \frac{\phi\, G}{M} = \phi\, g,$$

hence the velocity of this motion after t seconds is

$$v_1 = c - q\, t = c - \phi\, g\, t.$$

Now if we put $v_1 = v$, or

$$\phi \frac{a^2}{k^2} g\, t = c - \phi\, g\, t,$$

we obtain the time after which the velocity of rotation becomes equal to that of translation and the *rolling of the body* begins. This time is

$$t = \frac{c}{\left(1 + \frac{a^2}{k^2}\right)\phi g} = \frac{k^2}{a^2 + k^2} \cdot \frac{c}{\phi\, g}.$$

At the end of this time the common velocity is

$$c_1 = \frac{a^2}{k^2}\, \phi\, g\, t = \frac{a^2\, c}{a^2 + k^2},$$

and the space described by the centre C of the body is

$$s = \left(\frac{c + c_1}{2}\right) t = \frac{2\, a^2 + k^2}{a^2 + k^2}\, \frac{c}{2} \cdot \frac{k^2}{a^2 + k^2} \cdot \frac{c}{\phi\, g} = \frac{(2\, a^2 + k^2)\, k^2}{(a^2 + k^2)^2} \cdot \frac{c^2}{2\, \phi\, g}.$$

If the coefficient of rolling friction was = 0, the body $A\ C$ would roll on forever with the constant velocity $c_1 = \frac{a^2\,c}{a^2 + k^2}$ upon the horizontal plane without coming to rest; but since the rolling friction $\frac{f\,G}{a}$ constantly opposes this motion (see § 192), the body, after describing a certain space s_1, will come to rest. At the end of this space the work $\frac{f\,G\,s_1}{a}$ of this friction has consumed the whole of the energy

$$\frac{G\,c_1^2}{2\,g} + \frac{G\,k^2}{a^2} \cdot \frac{c_1^2}{2\,g} = \left(\frac{a^2 + k^2}{a^2}\right) \frac{G\,c_1^2}{2\,g}$$

stored by the mass of the body, and therefore we can put

$$\frac{f\,G\,s_1}{a} = \left(\frac{a^2 + k^2}{a^2}\right) \frac{G\,c_1^2}{2\,g};$$

hence the space

$$s_1 = \frac{a^2 + k^2}{f\,a} \cdot \frac{c_1^2}{2\,g} = \frac{a^2}{f\,(a^2 + k^2)}\,\frac{c^2}{2\,g}$$

is described in the time

$$t_1 = \frac{2\ s_1}{c_1} = \frac{a^2 + k^2}{f\,a} \cdot \frac{c_1}{g} = \frac{a\,c}{f\,g}.$$

For a rolling ball $\frac{k^2}{a^2} = \frac{2}{5}$, and for a cylinder $\frac{k^2}{a^2} = \frac{1}{2}$ (see § 290).

In the latter case $t = \frac{1}{3}\,\frac{c}{\phi\,g}$, $c_1 = \frac{2}{3}\,c$, $s = \frac{5}{9}\,\frac{c^2}{2\,\phi\,g}$ and $s_1 = \frac{2}{3}\,\frac{a}{f}\,\frac{c^2}{2\,g}$.

CHAPTER II.

THE CENTRIFUGAL FORCE OF RIGID BODIES.

§ 301. **The Normal Force.**—The force of inertia manifests itself not only when *the velocity of a moving body changes,* but also when *there is a change in the direction of the motion;* for a body, by virtue of its inertia, moves uniformly and in a straight line (see § 55). The action of inertia, when the direction changes continually, i.e. when the motion of a body takes place in a curved line, and particularly in a circle, will be the subject discussed in this chapter.

If a material point moves in a curved line, it is at every point subjected to an acceleration, which causes it to deviate from its former direction. This acceleration has already been treated of in phoronomics under the name of the *normal acceleration.* Let the radius of curvature of the path of the moving body be $= r$ and its velocity v, then the normal acceleration is

$$p = \frac{v^2}{r} \text{ (§ 42).}$$

Now if the mass of the point $= M$, the acceleration corresponds to a force

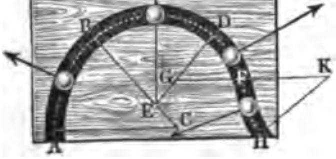
$$P = M p = \frac{M v^2}{r},$$

which we must consider as the original cause of the continued change of the direction of motion of the point. If the point is acted upon by no other (tangential) force than the normal one, its velocity will be constant and $= c$, and therefore the *normal force*

$$P = \frac{M c^2}{r}$$

is dependent only upon the curvature or radius of curvature, i.e. smaller for a smaller curvature or for a greater radius of curvature, and greater for a greater curvature or for a smaller radius of curvature. When the radius of curvature is doubled, the normal force is but one-half as great as before. If a material point M, Fig. 497, is obliged to pass over a horizontal plane in a curved line $A\,B\,D\,F\,H$, if we neglect the friction, the point will have in all points the same velocity and the pressure against the side wall in every position will be equal to the normal force. While the point describes the arc $A\ B$ this pressure is $= \frac{M c^2}{C A}$; while it describes $B\,D$ it is $= \frac{M c^2}{E B}$; for the arc $D\,F$ it is $= \frac{M c^2}{G D}$ and

FIG. 497.



for the arc $F H$, $= \frac{M c^2}{K F}$, $C A$, $E B$, $G D$ and $K F$ denoting the radii of curvature of the portions $A B$, $B D$, $D F$ and $F H$ of the path.

§ 302. Centripetal and Centrifugal Forces.—If a material point or body moves in a circle, the normal force acts radially inwards, and for this reason it is called the *centripetal force* (Fr. force centripède, Ger. Centripetal- or Annäherungskraft), and the force in the opposite direction, I.E. radially outwards, with which the body through its inertia resists the former force, has received the name of the *centrifugal force* (Fr. force centrifuge, Ger. Centrifugal-, Flieh- or Schwungkraft). The centripetal force is the one which acts upon the body inwards, and the centrifugal force is the resistance of the body, which acts in the opposite direction. In the revolution of the planets around the sun, the attraction of the sun is the centripetal force; if the moving body is compelled to describe a circle by a guide, such as is represented in Fig. 497, the guide acts by its resistance as the centripetal force and opposes the centrifugal force of the body. If, finally, the revolving body is connected by means of a string or rod with the centre of rotation, then it is the elasticity of the rod, which puts itself in equilibrium with the centrifugal force of the body and acts as the centripetal force.

If G is the weight, and therefore $M = \frac{G}{g}$ the mass of the revolving body, r the radius of the circle, in which the revolution takes place, and v the velocity of revolution, we have, according to the last paragraph, for the centrifugal force

$$P = \frac{M v^2}{r} = \frac{G v^2}{g r} = 2 \cdot \frac{v^2}{2 g} \cdot \frac{G}{r},$$

or

$$P : G = 2 \cdot \frac{v^2}{2 g} : r,$$

I.E., *the centrifugal force is to the weight of the body as double the height due to the velocity is to the radius of rotation.*

If the *motion is uniform,* which is always the case when no other force (tangential force) besides the centripetal force acts upon the body, we can then express velocity $v = c$ in terms of the duration t of a revolution by putting $c = \frac{\text{space}}{\text{time}} = \frac{2 \pi r}{t}$, and the expression for the centrifugal force becomes

$$P = \left(\frac{2\pi r}{t}\right)^2 \frac{M}{r} = \frac{4\pi^2}{t^2} . M r = \frac{4\pi^2}{g t^2} . G r.$$

Since $4\pi^2 = 39,4784$, and in feet $\frac{1}{g} = 0,031$, we have, in a more convenient form for calculation, the value of the centrifugal force

$$P = \frac{39,4784}{t^2} . M r = 1,2238 . \frac{G r}{t^2} \text{ pounds.}$$

The number u of revolutions per minute is often given, in which case, substituting for t, $\frac{60}{u}$, we have

$$P = \frac{39,4784}{3600} u^2 M r = 0,010966 \, u^2 M r = 0,0003399 \, u^2 G r \text{ pounds.}$$

We have also $P = 4,0243 \frac{G r}{t^2} = 0,001118 \; u^2 \; G \; r$ kilograms.

Since $\frac{2\pi}{t}$ is the angular velocity ω, we can also write

$$P = \omega^2 . M r.$$

Hence it follows *that for equal times of revolution*, I.E. *for the same number of revolutions in a given time or for the same angular velocities, the centrifugal force increases as the product of the mass and the radius of gyration; and if the other circumstances are the same, it is inversely proportional to the square of the time of revolution, or directly proportional to the square of the number of revolutions and to the square of the angular velocity.*

EXAMPLE—1) If a body, weighing 50 pounds, describes a circle of 3 feet radius 400 times in a minute, the centrifugal force is $P = 0,0003399 . 400^2 . 50 . 3 = 3,399 . 16 . 50 . 3 = 339,9 . 24 = 8158$ pounds.

If this body is connected with the axis by a hemp rope, the modulus of ultimate strength of which is (§ 212) 7000 lbs., we should put $8158 = 7000 . F$, and therefore the cross-section of rope should be $F = \frac{8158}{7000} = 1,165$ square inches, and its diameter should be

$$d = \sqrt{\frac{4 F}{\pi}} = 0,5642 . \sqrt{4,660} = 0,5642 . 2,159 = 1,22 \text{ inches.}$$

In order to have triple security, we must make $d = 1,22 \sqrt{3} = 1,22 . 1,732 = 2,11$ inches.

2) From the radius of the earth $r = 20\frac{3}{4}$ million feet, and the time of

revolution or length of day $t = 24$ hours $= 24 \,.\, 60 \,.\, 60 = 86400$ seconds, we obtain for the centrifugal force of body upon the earth at the equator

$$P = 1{,}2238 \,.\, \frac{20750000 \; G}{86400^2} = \frac{2539}{864^2} \,.\, G = \frac{1}{290} \,.\, G,$$

but if the day were 17 times as short, or $\frac{24}{17} = 1\text{h}.\, 24'\, 42''$, this force would be $17^2 = 289$ times as great, and the centrifugal force would be nearly equal to the weight G of the body. At the equator, in that case, the centrifugal force would be equal to the force of gravity, and the body would neither fall nor rise.

3) The centrifugal force arising from the revolution of the moon around the earth is counteracted by the attraction of the latter. If G is the weight of the moon and r is its distance from the earth, and t the time of revolution around the latter, the centrifugal force of this body is

$$= 1{,}2238 \,.\, \frac{G\, r}{t^2}.$$

Now let a be the radius of the earth, and let us assume that the force of gravity at different distances from its centre is inversely proportional to the nth power of this distance; we have the weight of the moon or the attraction of the earth

$$= G \left(\frac{a}{r}\right)^n,$$

and putting both forces equal to each other

$$\left(\frac{a}{r}\right)^n = 1{,}2238 \,.\, \frac{r}{t^2}.$$

But $\frac{a}{r} = \frac{1}{60}$, $r = 1251$ million feet, $t = 27$ days 7 hours 42 minutes $=$ 39342 minutes $= 39342 \,.\, 60 = 2360520$ seconds, whence

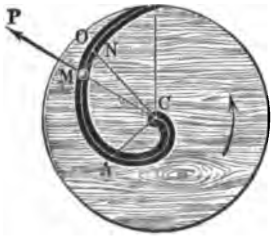
$$\left(\frac{1}{60}\right)^n = \frac{1{,}2238 \,.\, 1251}{393{,}4^2 \,.\, 36} = \frac{1}{3600} = \left(\frac{1}{60}\right)^2;$$

hence $n = 2$, I.E. the attraction of the earth (or gravity) is inversely proportional to the square of the distance from its centre.

§ 303. Mechanical Effect of the Centrifugal Force.—

If the path $C\,A\,B$, Fig, 498, in which the body M moves, is not at rest, *but turning upon an axis C*, it imparts to the body a *centrifugal force P*, by virtue of which it either *gives out or absorbs a certain amount of mechanical effect.* The former occurs when, in moving in its path, it departs from, and the latter when it approaches the axis of rotation C. Let M be the mass of the body, ω the constant angular velocity with which the path, E.G. a *top* (Fr. sabot, Ger. Kreisel), turns around its axis C, and let z de-

FIG. 498.



note the variable distance CM of the body, which is moving in the path CAB; we have the centrifugal force of the body

$$P = \omega^2 M z,$$

and the work done by this force, while the body describes an element MO of its path and the radius CM is increased by an amount $NO = \zeta$, is

$$P\zeta = \omega^2 M z \,.\, \zeta.$$

Let us imagine the radius z to be composed of n parts, each $= \zeta$, then if we put $z = n\,\zeta$ and assume that the body begins to move at the centre of rotation C, we obtain the work done by the centrifugal force of the body, while the body is describing the space CAM, during which time the distance of the body is gradually increasing from 0 to z. By substituting successively in the last equation, instead of z, the values $\zeta, 2\,\zeta, 3\,\zeta, \ldots n\,\zeta$, and then adding the values thus found, we obtain this mechanical effect

$$A = \omega^2 M \zeta (\zeta + 2\,\zeta + 3\,\zeta + \ldots + n\,\zeta) = \omega^2 M \zeta^2 (1 + 2 + 3 + \ldots + n),$$

or, since $1 + 2 + 3 + \ldots + n$, when the number of members is great, $= \frac{n^2}{2}$, we can write

$$A = \omega^2 M \zeta^2 \frac{n^2}{2} = \tfrac{1}{2}\,\omega^2 M z^2.$$

Now the velocity of rotation of the top at the distance $CM = z$ from its axis is

$$v = \omega\, z,$$

hence we can write more simply

$$A = \tfrac{1}{2}\, M v^2 = \frac{v^2}{2\,g}\, G,$$

when we substitute, instead of the mass of the body, the weight $G = M g$.

If the body begins its motion, not at C, but at any other point A without the axis of rotation, and at a distance $CA = z_1$ from C, where the velocity of rotation is

$$v_1 = \omega\, z_1,$$

the work $\frac{1}{2}\,\omega^2 M z_1^2$ done by the centrifugal force while the body is passing from C to A must be omitted, and we have the work done by the centrifugal force while the body passes from A to M

$$A = \tfrac{1}{2}\,\omega^2 M z^2 - \tfrac{1}{2}\,\omega^2 M z_1^2 = \tfrac{1}{2}\,\omega^2 M (z^2 - z_1^2)$$
$$= \tfrac{1}{2}\, M (v^2 - v_1^2) = \left(\frac{v^2 - v_1^2}{2\,g}\right) G.$$

If a body moves in a rigid path or groove, which revolves about a fixed axis, the *vis viva* of this body is increased or diminished by the product of the mass (M) and the difference of the heights due to the velocities of revolution $\left(\frac{v^2}{2g} \text{ and } \frac{v_1^2}{2g}\right)$ at the two ends A and M of the path. The increase takes place when the motion is from within outward, and the decrease when the motion is from without inward.

§ **304.** If a body begins its path $A\ M\ B$ upon a top $A\ B\ C$, Fig. 499, at A with a relative velocity c_1, and leaves the top at B with the relative velocity c_2, and if the velocities of rotation of the top in A and B are v_1 and v_2, the energy stored by the body in describing the path $A\ M\ B$, supposing no other force to act upon it, is

$$A = \frac{c_2^2 - c_1^2}{2g} G = \frac{v_2^2 - v_1^2}{2g} G,$$

and therefore

$$c_2^2 - c_1^2 = v_2^2 - v_1^2,$$

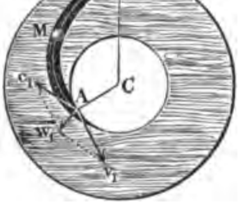
or

$$c_2^2 = c_1^2 + v_2^2 - v_1^2,$$

and consequently the velocity of exit is

$$c_2 = \sqrt{c_1^2 + v_2^2 - v_1^2} = \sqrt{c_1^2 + \omega^2 (r_2^2 - r_1^2)},$$

FIG. 499.



ω denoting the angular velocity of the top and r_2 and r_1 the distances $C\ A$ and $C\ B$ of the points (A and B) of entrance and exit from the axis of rotation C.

The relative velocity of exit c_1 is determined in like manner, when the body enters at B upon the top with the relative velocity c_2 and moves upon it from without inwards. It is then

$$c_1 = \sqrt{c_2^2 - (v_2^2 - v_1^2)} = \sqrt{c_2^2 - \omega^2 (r_2^2 - r_1^2)}.$$

Since the body in describing the path $A\ M\ B$ has, besides its relative velocity (c) in the path, also the velocity of rotation v of the path, it must be introduced at A with an absolute velocity $\overline{A\ w_1} = w_1$, which is determined in intensity and direction by the diagonal of the parallelogram constructed with c_1 and v_1, and the body leaves at B with an absolute velocity $\overline{B\ w_2} = w_2$, determined by the diagonal of the parallelogram $B\ c_2\ w_2\ v_2$, constructed with the relative velocities c_2 and v_2.

The *energy restored, or stored,* by the body in describing the path $A\ M\ B$ on the top, which has been gained or lost by the top, is

$$A = \pm \left(\frac{w_2^2 - w_1^2}{2g}\right) G.$$

If a body should transmit all its energy $\frac{w_1^2}{2g} G$ to the top, while describing the path $A\ M\ B$, the absolute velocity of exit must be $w_2 = 0$, and c_2 must be not only equal to v_2 but also exactly opposite to it; the path must therefore be tangent to the circumference at B.

Example.—If the interior radius of the top, represented in Fig. 499, is $CA = r_1 = 1$ foot and the exterior one $C\ B = r_2 = 1\frac{1}{2}$ feet and if it revolves 100 times per minute, the angular velocity is

$$\omega = \frac{\pi u}{30} = 3{,}1416 \,.\, \frac{10}{3} = 10{,}472 \text{ feet},$$

and consequently the velocity at the interior circumference is

$$v_1 = \omega\, r_1 = 10{,}472 \text{ feet, and at the exterior one}$$

$$v_2 = \omega\, r_2 = 10{,}472 \,.\, 1{,}5 = 15{,}708 \text{ feet.}$$

Now if we cause a body, whose velocity is $w_1 = 25$, to enter the top at A, in such a direction that the angle $w_1\ A\ v_1$ formed by its absolute motion with the direction of revolution is $\alpha = 30°$, we have for the relative velocity c_1, with which the body begins its motion on the top,

$$c_1^2 = v_1^2 + w_1^2 - 2\, v_1\, w_1 \cos.\, \alpha = 109{,}66 - 453{,}45 + 625{,}00 = 281{,}21,$$

and therefore

$$c_1 = 16{,}77 \text{ feet.}$$

If the body is to enter without impact, we must have for the angle $v_1\ A\ c_1 = \beta$ formed by the path with the inner circumference of the top

$$\frac{\sin.\, \beta}{\sin.\, \alpha} = \frac{w_1}{c_1}, \text{ or}$$

$$\sin.\beta = \frac{25 \sin.\, 30°}{16{,}77},$$

whence $\beta = 48° \ 12' \frac{1}{2}$.

For the relative velocity of exit c_2 we have

$$c_2^2 = c_1^2 + v_2^2 - v_1^2 = 281{,}21 + 109{,}66\, [(\tfrac{3}{2})^2 - 1^2] = 418{,}28,$$

and consequently

$$c_2 = 20{,}45 \text{ feet.}$$

And, on the contrary, for the absolute velocity of exit w_2, when the canal or groove $A\ M\ B$ forms with the exterior circumference an angle $\delta = 20°$ or $v_2\ B\ c_2 = 160°$, we have

$$w_2^2 = c_2^2 + v_2^2 - 2\, c_2\, v_2 \cos.\, \delta = 418{,}28 + 246{,}74 - 603{,}72 = 61{,}30,$$

and consequently

$$w_2 = 7{,}80 \text{ feet.}$$

Finally, the heights due to the velocities are

$$\frac{w_1^2}{2g} = 0{,}0155 \,.\, 625 = 9{,}69 \text{ feet, and } \frac{w_2^2}{2g} = 0{,}0155 \,.\, 61{,}31 = 0{,}95 \text{ feet,}$$

and the amount of mechanical effect imparted to the top by a body, whose weight is G, while passing over the top, is

$$A = \left(\frac{w_1^2 - w_2^2}{2g}\right) G = (9,69 - 0,95)\, G = 8,74\, G,$$

or, if its weight $G = 10$ pounds,

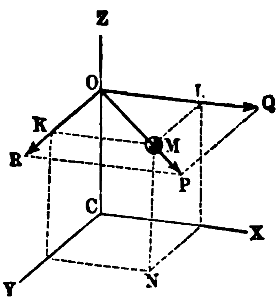
$$A = 8,74 \,.\, 10 = 87,4 \text{ foot-pounds.}$$

REMARK.—The foregoing theory of the motion of a body on a top is directly applicable to turbine wheels.

§ 305. Centrifugal Force of Masses of Finite Dimensions.

—The formulas for the centrifugal force found in the foregoing paragraphs are not directly applicable to an aggregate of masses or to a mass of finite extent; for we do not know what radius r of gyration must be substituted in the calculation. To determine this radius, the following method may be adopted. Let $C\,Z$, Fig. 500, be the axis of rotation and $C\,X$ and $C\,Y$ two rectangular co-ordinate axes and let M be an element of the mass and $M\,K = x$, $M\,L = y$ and $M\,N = z$ its distances from the co-ordinate planes $Y\,Z$, $X\,Z$ and $X\,Y$. Since the centrifugal force P acts in the direction of the radius, we can transfer its point of application to its point of intersection with the axis of rotation. If we decompose this force into two components in the directions of the axes $C\,X$ and $C\,Y$, we obtain $\overline{O\,Q} = Q$ and $\overline{O\,R} = R$, for which we have

FIG. 500.



$$O\,Q : O\,P = O\,L : O\,M \text{ and } O\,R : O\,P = O\,K : O\,M,$$

whence

$$Q = \frac{x}{r} P \text{ and } R = \frac{y}{r} P,$$

r designating the distance $O\,M$ of the element of the mass from the axis of rotation. If we proceed in the same way with all the elements of the mass, we obtain two systems of parallel forces, one in the plane $X\,Z$ and the other in the plane $Y\,Z$, and each of which acts at right angles to the axis $C\,Z$. Employing the indices 1, 2, 3, etc., to distinguish the various elements of the mass, I.E. putting them $= M_1, M_2, M_3$, etc., and their distances $= x_1, x_2, x_3$, etc., we have the resultant of one system of forces

$$Q = Q_1 + Q_2 + Q_3 + \ldots = \frac{P_1 x_1}{r_1} + \frac{P_2 x_2}{r_2} + \frac{P_3 x_3}{r_3} + \ldots$$
$$= \omega_2 \,.\, (M_1 x_1 + M_2 x_2 + \ldots),$$

and that of the other

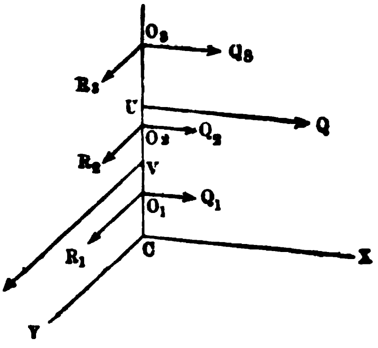
$$R = R_1 + R_2 + \ldots = \omega^2 . (M_1 y_1 + M_2 y_2 + \ldots).$$

If, finally, we put the distance $C\,O_1$, $C\,O_2$, etc., of the elements of the mass from the plane of $X\,Y_1 = z_1, z_2$, etc., we obtain for the points of application U and V of these resultants the ordinates $C\,U = u$ and $C\,V = v$ by means of the formulas

$$(Q_1 + Q_2 + \ldots)\,u = Q_1 z_1 + Q_2 z_2 + \ldots$$

and $(R_1 + R_2 + \ldots)\,v = R_1 z_1 + R_2 z_2 + \ldots$, whence

FIG. 501.



$$u = \frac{Q_1 z_1 + Q_2 z_2 + \ldots}{Q_1 + Q_2 + \ldots} = \frac{M_1 x_1 z_1 + M_2 x_2 z_2 + \ldots}{M_1 x_1 + M_2 x_2 + \ldots}$$

and

$$v = \frac{R_1 z_1 + R_2 z_2 + \ldots}{R_1 + R_2 + \ldots} = \frac{M_1 y_1 z_1 + M_2 y_2 z_2 + \ldots}{M_1 y_1 + M_2 y_2 + \ldots}.$$

Hence we see that generally the centrifugal forces of a system of masses or of finite bodies can be referred to two forces, which cannot be combined so as to give but a single resultant when u and v are unequal.

EXAMPLE.—Let the masses of a system be

$M_1 = 10$ pounds, $M_2 = 15$ pounds, $M_3 = 18$ pounds, $M_4 = 12$ pounds, and their distances

$x_1 = 0$ inches, $x_2 = 4$ inches, $x_3 = 2$ inches, $x_4 = 6$ inches,
$y_1 = 3$ " $y_2 = 1$ " $y_3 = 5$ " $y_4 = 3$ "
$z_1 = 2$ " $z_2 = 3$ " $z_3 = 3$ " $z_4 = 0$ "

then the resultants of the centrifugal forces are

$Q = \omega^2 . (10 . 0 + 15 . 4 + 18 . 2 + 12 . 6) = 168 . \omega^2$ and
$R = \omega^2 . (10 . 3 + 15 . 1 + 18 . 5 + 12 . 3) = 171 . \omega^2$,

and consequently their distances from the origin C are

$$u = \frac{10 . 0 . 2 + 15 . 4 . 3 + 18 . 2 . 3 + 12 . 6 . 0}{10 . 0 + 15 . 4 + 18 . 2 + 12 . 6} = \frac{288}{168} = \frac{12}{7} = 1{,}714 \text{ inches},$$

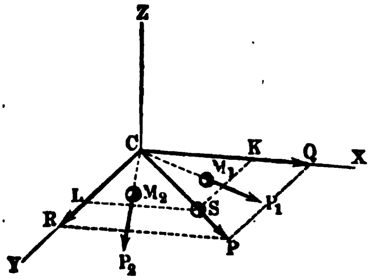
and

$$v = \frac{10 . 3 . 2 + 15 . 1 . 3 + 18 . 5 . 3 + 12 . 3 . 0}{10 . 3 + 15 . 1 + 18 . 5 + 12 . 3} = \frac{375}{171} = \frac{125}{57} = 2{,}193 \text{ inches}.$$

The difference of these values of u and v shows that the centrifugal forces cannot be replaced by a single force.

§ 306. If the elements of the mass lie in a plane of rotation, I.E. in a plane $X\ C\ Y$, Fig. 502, which is at right angles to the axis of rotation, as M_1, M_2..., do, their centrifugal forces will give a single resultant; for their directions cut each other at one point C of the axis $C\ Z$. If we retain the notations of the last paragraph, we obtain the resulting centrifugal force in this case

FIG. 502.



$$P = \sqrt{Q^2+R^2} = \omega^2 \sqrt{[(M_1 x_1+M_2 x_2+\ldots)^2+(M_1 y_1 + M_2 y_2+\ldots)^2]}.$$

Now if $C\ K = x$ and $C\ L = y$ are the co-ordinates of the centre of gravity of the system of masses $M = M_1 + M_2 + \ldots$, we have

$$M_1 x_1 + M_2 x_2 + \ldots = M x$$
$$M_1 y_1 + M_2 y_2 + \ldots = M y,$$

whence it follows that the centrifugal force is

$$P = \omega^2 \sqrt{M^2 x^2 + M^2 y^2} = \omega^2 M \sqrt{x^2 + y^2} = \omega^2 M r,$$

in which $r = \sqrt{x^2 + y^2}$ designates the distance $C\ S$ of the centre of gravity from the axis of rotation $C\ Z$.

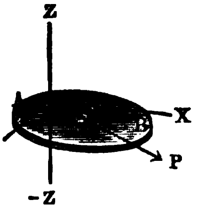
For the angle $P\ C\ X = a$, formed by this force with the axis $C\ X$, we have

$$tang.\ a = \frac{R}{Q} = \frac{M y}{M x} = \frac{y}{x};$$

consequently, *the direction of the centrifugal force passes through the centre of gravity of the system, and that force is precisely the same as it would be if all the elements of the mass were concentrated at the centre of gravity.*

For a disc $A\ B$ at right angles to the axis of rotation $Z\ \overline{Z}$, Fig. 503, the centrifugal force is also $= \omega^2 M r$, if M denotes its mass and r the distance $C\ S$ of its centre of gravity from the axis. If the centres of gravity of the elements of the mass of a body lie in a plane of rotation, or if this plane is a plane of symmetry of the body $A\ D\ F\ F_1$, Fig. 504, the centrifugal forces of the elements of the mass of the body can be combined so as to give a single resultant acting at the centre of gravity of the body, and

FIG. 503.



this resultant corresponds to the distance of this point S from the axis of rotation and can therefore be determined by the formula $P = \omega^2 M r$.

FIG. 504.

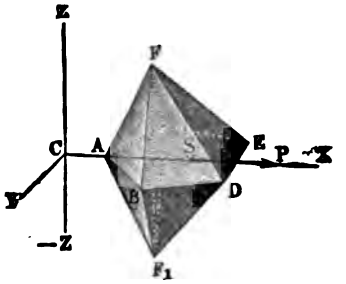
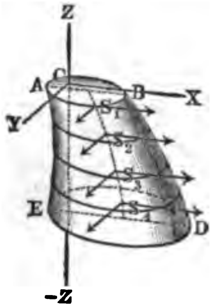


FIG. 505.



In order to find the centrifugal force of a body $A\ B\ D\ E$, Fig. 505, let us divide it into disc-shaped elements by planes perpendicular to the axis $\overline{Z\ Z}$, and then find their centres of gravity S_1, S_2, etc.; we can then determine by the aid of the latter the centrifugal forces, by decomposing these into their components in the directions of the axes $C\ X$ and $C\ Y$ and by combining the components in the plane $Z\ C\ X$, we obtain the resultant Q, and by combining those in the plane $Z\ C\ Y$, we obtain their resultant R.

If the centre of gravity of all the discs lie in a line parallel to the axis of rotation, we have $x = x_1 = x_2$, etc., and $y = y_1 = y_2$, etc., and therefore $r = r_1 = r_2$, etc., whence it follows that the centrifugal force of the whole body is

$$P = \omega^2 (M_1 r + M_2 r + \ldots) = \omega^2 M r,$$

and that the distance of the point of application from the plane $X\ Y$ is

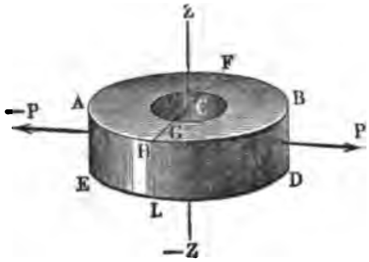
$$z = \frac{(M_1 z_1 + M_2 z_2 + \ldots) r}{(M_1 + M_2 + \ldots) r} = \frac{M_1 z_1 + M_2 z_2 + \ldots}{M_1 + M_2 + \ldots} = z.$$

From these equations we see that the centrifugal force of a body, which can be divided into discs, whose centres of gravity lie in a line parallel to the axis of rotation, is equal to the centrifugal force of the mass of the body concentrated at its centre of gravity, and the point of application of this force is at the centre of gravity.

Hence we can find in this manner the centrifugal forces of all *symmetrical bodies* (see § 106), whose axis of symmetry is parallel to their axis of rotation, and also that of all *solids of revolution*, whose geometrical axis is parallel to the axis of rotation. If the axis of rotation and the geometrical axis coincide the resulting centrifugal force is = 0.

EXAMPLE.—The dimensions, heaviness and strength of a mill-stone $A\ B\ D\ E$, Fig. 506, are given; required the angular velocity ω when the stone is torn apart by the centrifugal force. Putting the radius of the millstone $= r_1$, the radius of its eye $= r_2$, its height $A\ E = H\ L = l$, its heaviness $= \gamma$ and the modulus of ultimate strength $= K$, we have the force necessary to tear the stone apart in a diametral plane

FIG. 506.



$$P = 2\,(r_1 - r_2)\,l\,K,$$

the weight of the stone

$$G = \pi\,(r_1^2 - r_2^2)\,l\,\gamma,$$

and the radius of rotation for each half of the stone, I.E. the distance of its centre of gravity from the axis of revolution (see § 114),

$$r = \frac{4}{3\,\pi} \cdot \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}$$

At the moment of tearing apart the centrifugal force of one-half the stone is equal to the breaking load of the stone, and we have

$$\omega^2 \,.\, \tfrac{1}{2}\,\frac{G\,r}{g} = 2\,(r_1 - r_2)\,l\,K,$$

I.E.,

$$\omega^2 \,.\, \tfrac{2}{3}\,(r_1^3 - r_2^3)\,\frac{l\,\gamma}{g} = 2\,(r_1 - r_2)\,l\,K.$$

Cancelling $2\,l$ on both sides of the equation, we have

$$\omega = \sqrt{\frac{3\,g\,(r_1 - r_2)\,K}{(r_1^3 - r_2^3)\,\gamma}} = \sqrt{\frac{3\,g\,K}{(r_1^2 + r_1\,r_2 + r_2^2)\,\gamma}}$$

Now if $r_1 = 2$ feet $= 24$ inches, $r_2 = 4$ inches, $K = 750$ pounds and the specific gravity of the stone $= 2{,}5$, or the weight of a cubic inch of it $\gamma = \frac{62{,}4\ .\ 2{,}5}{1728} = 0{,}09028$ pounds, we have the angular velocity, when the tearing begins,

$$\omega = \sqrt{\frac{3\ .\ 12\ .\ 32{,}2\ .\ 750}{688\ .\ 0{,}09028}} = \sqrt{\frac{3375\ .\ 16{,}1}{43\ .\ 0{,}09028}} = 118{,}3 \text{ inches.}$$

If the number of revolutions in a minute $= u$, we have $\omega = \frac{2\,\pi\,u}{60}$ and inversely $u = \frac{30\,\omega}{\pi}$, or in this case, $= \frac{30\ .\ 118{,}3}{\pi} = 1129\tfrac{2}{3}$.

Generally the number of revolutions of such a stone is 120 or about nine times less. For a fly-wheel we can put $r_1^2 + r_1\,r_2 + r_2^2 = 3\,r^2$, r denoting the radius of the middle of the ring, and consequently we have

$$\omega = \sqrt{\frac{g\,K}{r^2\,\gamma}} \text{ or } v = \omega\,r = \sqrt{\frac{g\,K}{\gamma}}.$$

§ 307. If all the parts M_1, M_2 of a system of masses, Fig. 507, or *the centres of gravity of the elements* of a body are in *a plane* *passing through the axis of rotation,* the centrifugal forces form a system of parallel forces and can be referred to a *single force.* Let the distances of the elements of the mass from the axis of rotation $\overline{Z Z}$ be

$$O_1 M_1 = r_1, \; O_2 M_2 = r_2, \text{ etc.,}$$

then the centrifugal forces are

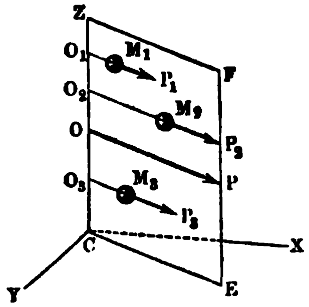
$$P_1 = \omega^2 M_1 r_1, \; P_2 = \omega^2 M_2 r_2, \text{ etc.,}$$

and their resultant is

$$P_1 = \omega^2 (M_1 r_1 + M_2 r_2 + \ldots)$$
$$= \omega^2 M r,$$

r denoting the distance of the centre of gravity of the whole mass M from the axis of rotation. The distance of the centre of gravity from the axis of rotation must be considered here as the radius of rotation. In order to find the point of application O of the resulting centrifugal force P, we substitute the distance of the elements of the mass from the normal plane, viz., $C O_1 = z_1$, $C O_2 = z_2$, etc., in the formula

FIG. 507.



$$C O = z = \frac{M_1 r_1 z_1 + M_2 r_2 z_2 + \ldots}{M_1 r_1 + M_2 r_2 + \ldots}.$$

By the aid of the formula $P = \omega^2 M r$ the centrifugal forces of *solids of revolution* and of other geometrical bodies can be determined, when the axis of these bodies is in the same plane as the axis of revolution.

For a *rod* $A \; C$, Fig. 508, whose length is $A \; C = l$ and whose angle of inclination $A \; C Z$ to the axis of rotation is $= a$, we have

$$r = \overline{K S} = \tfrac{1}{2} \, l \, sin \, a,$$

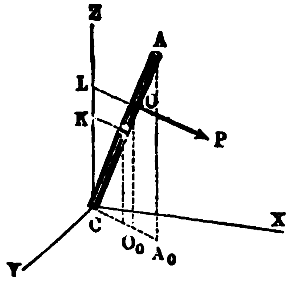
and consequently the centrifugal force

$$P = \omega^2 . \tfrac{1}{2} \, M l \, sin. \, a \, ;$$

but in order to find the point of application O of this force, we must substitute in the expression

$$\omega^2 . \frac{M}{n} \, x \, sin. \, a \, . \, x \, cos. \, a$$
$$= \omega^2 . \frac{M}{n} \, x^2 \, sin. \, a \, cos. \, a$$

FIG. 508.



for the moment $\frac{M}{n}$ of the rod successively, instead of x, the elements $\frac{l}{n}, \frac{2l}{n}, \frac{3l}{n}$, etc., and add the expressions thus obtained together. In this manner we find

$$P\,u = \omega^2 \frac{M}{n} \sin.\, a \cos.\, a \frac{l^2}{n^3} (1^2 + 2^2 + 3^2 + \ldots + n^2)$$
$$= \tfrac{1}{3}\, \omega^2 M\, l^2 \sin.\, a \cos.\, a,$$

hence the arm $C\,L = O_0\, O$ or

$$u = \tfrac{1}{3}\, \omega^2 M\, l^2 \sin.\, a \cos.\, a : \tfrac{1}{2}\, \omega^2 M\, l \sin.\, a = \tfrac{2}{3}\, l \cos.\, a,$$

and the distance of the point O from the end C of the rod, which lies on the axis, is

$$C\,O = \tfrac{2}{3}\, l.$$

If the rod $A\,B$, Fig. 509, does not reach the axis, we have

$$P = \tfrac{1}{2}\, \omega^2 F\, l_1^2 \sin.\, a - \tfrac{1}{2}\, \omega^2 F\, l_2^2 \sin.\, a$$
$$= \tfrac{1}{2}\, \omega^2 F \sin.\, a\, (l_1^2 - l_2^2),$$

and the moment

$$P\,u = \tfrac{1}{3}\, \omega^2 F \sin.\, a \cos.\, a\, (l_1^3 - l_2^3);$$

for the mass of $C\,A$ (= cross-section multiplied by the length) is $= F\,l_1$ and the mass of $C\,B, = F\,l_2$.

It follows, therefore, that the distance of the point of application O from the point of intersection C with the axis is

$$C\,O = \tfrac{2}{3}\, \frac{l_1^3 - l_2^3}{l_1^2 - l_2^2} \text{ or } C\,O = l + \frac{(l_1 - l_2)^2}{12\, l},$$

l denoting the distance $C\,S$ of the centre of gravity and $l_1 - l_2$ the length of the rod.

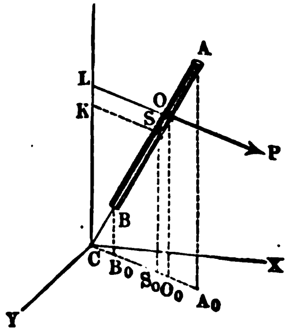


Fig. 509.

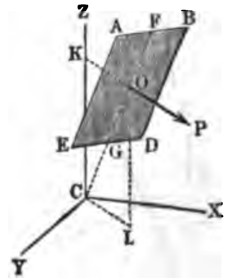
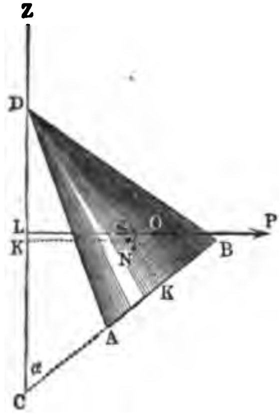


Fig. 510.

This formula holds good also for a *rectangular plate* $A\ B\ D\ E$, Fig. 510, which is divided into two similar rectangles by the axial plane $C\ O\ Z$, and whose plane is at right angles to this axial plane; for the points of application of the centrifugal forces of the slices, obtained by passing planes through it perpendicular to $C\ Z$, are in the medial line $F\ G$. Now if the distances $C\ F$ and $C\ G$ of the two bases $A\ B$ and $F\ E$ from the origin C are l_1 and l_2, we have here also

$$C\ O = \tfrac{2}{3} \cdot \frac{l_1^3 - l_2^3}{l_1^2 - l_2^2} = l + \frac{(l_1 - l_2)^2}{12\ l}.$$

FIG. 511.



In like manner the centrifugal force of a right cone $A\ B\ D$, with a circular base, Fig. 511, which turns about an axis $C\ D$ passing through its apex, is found by substituting in the formula $P = \omega^2\ M\ r$ for r the distance $K\ S$ of the centre of gravity S of this body from $C\ Z$. If h denote the altitude $K\ D$ of the cone, and a the angle $B\ C\ Z$ formed by the base of the cone with the axis of rotation, we will have

$$K\ S = \overline{D\ S}\ cos.\ D\ S\ K = \tfrac{3}{4}\ h\ cos.\ a,$$

and consequently the required centrifugal force is

$$P = \omega^2\ M\ \tfrac{3}{4}\ h\ cos.\ a.$$

The point of application O of this force is determined by the co-ordinates $D\ L = u$ and $L\ O = v$, for which we find with the aid of the Calculus, under the supposition that the axis of rotation $C\ Z$ does not pass through the cone, the following expression

$$u = \tfrac{4}{5}\ h\ sin.\ a \left[1 - \left(\frac{r}{2\ h}\right)^2\right]$$

and
$$v = \tfrac{4}{5}\ h\ cos.\ a \left[1 + \left(\frac{r\ tang.\ a}{2\ h}\right)^2\right],$$

r denoting the radius $K\ A = K\ B$ of the base.

§ **308.** If all the different parts of the body lie neither in a plane normal to the axis of revolution, nor in one containing that axis, the resulting centrifugal forces
$Q = \omega^2\ (M_1\ x_1 + M_2\ x_2 + \ldots)$ and $R = \omega^2\ (M_1\ y_1 + M_2\ y_2 + \ldots)$ will not give a single force, but it is possible to replace them by a force

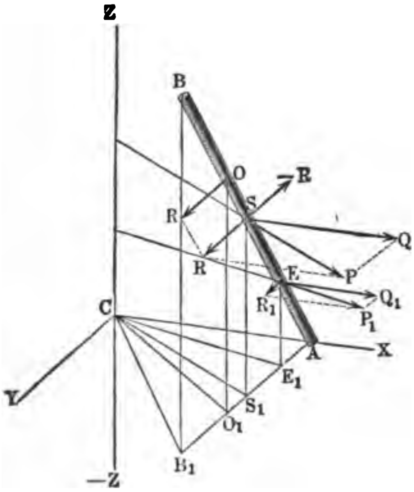
$$P = \sqrt{Q^2 + R^2} = \omega^2\ M\ r,$$

applied at the centre of gravity, and by a couple composed of Q and R. If we apply at the centre of gravity four forces $+ Q$ and $- Q$ as well as $+ R$ and $- R$, which balance each other, the positive forces will give the resultant

$$P = \sqrt{Q^2 + R^2},$$

while the negative ones $-Q$ and $-R$, together with the centrifugal forces applied at U and V (see Fig. 501) form the couples $(Q, -Q)$ and $(R, -R)$, which can be combined so as to form a single couple.

FIG. 512.



In order to understand better this referring of the centrifugal forces of a revolving body to one force and one couple, let us consider the following simple case. The rod $A\,B$, Fig. 512, which revolves about the axis $Z\,\bar{Z}$, is parallel to the plane $Y\,Z$ and its end A reposes upon the axis $C\,X$. Let us denote the length $A\,B$ of the rod by l, its weight by G, the angle $A\,B\,B_1$, formed by the rod with the axis of rotation, by α and its distance $C\,A$ from the plane $Y\,Z$, which is also its shortest distance from the axis $Z\,\bar{Z}$ by a. Now if E is an element $\frac{M}{n}$ of the rod, and $y = A\,E_1$ the horizontal projection of its distance $A\,E$ from the end A, we have the components of the centrifugal force P_1 of this element

$$Q_1 = \omega^2 \,.\, \frac{M}{n} \,.\, \overline{C\,A} = \omega^2 \,.\, \frac{M}{n}\, a \text{ and } R_1 = \omega^2 \,.\, \frac{M}{n} \,.\, \overline{A\,E_1} = \omega^2 \,.\, \frac{M}{n}\, y,$$

and their moments in reference to the plane $X\,C\,Y$ of the base, since the distance of the element from this plane $X\,Y$ is

$$E_1\,E = \overline{A\,E_1}\; cotg.\, \alpha = y\; cotg.\, \alpha, \text{ are}$$

$$Q_1\;z_1 = \omega^2 \,.\, \frac{M}{n} \,.\, \overline{C\,A} \,.\, \overline{E_1\,E} = \omega^2 \,.\, \frac{M}{n}\, a\, y\; cotg.\, \alpha \text{ and}$$

$$R_1\;z_1 = \omega^2 \,.\, \frac{M}{n}\, y^2 \,.\, cotg.\, \alpha.$$

The resultant of all the components parallel to $X\,Z$ is

$$Q = Q_1 + Q_2 + \ldots = n \,.\, \omega^2 \,.\, \frac{M}{n}\, a = \omega^2 \,.\, M\, a,$$

and its moment is

$$Q\,u = Q_1\,z_1 + Q_2\,z_2 + \ldots = \omega^2 . \frac{M}{n}\,a\,cotg.\,a\,(y_1 + y_2 + \ldots),$$

or, since $y_1 = \frac{l\,sin.\,a}{n}$, $y_2 = \frac{2\ l\,sin.\,a}{n}$, $y_3 = \frac{3\ l\,sin.\,a}{n}$, etc., and $cotg.\,a$. $sin.\,a = cos.\,a$, we have

$$Q\,u = \omega^2 . \frac{M}{n}\,a\,cos.\,a . . \frac{l}{n}\,(1 + 2 + 3 + \ldots + n) = \omega^2 . \frac{M}{n} a\,cos.\,a \frac{l}{n} . \frac{n^2}{2}$$

$$= \tfrac{1}{2}\,\omega^2 . M\,a\,l\,cos.\,a.$$

The distance of the point of application of this component from the plane $X\ Y$ of the base is

$$S_1\,S = u = \frac{\frac{1}{2}\,\omega^2\,M\,a\,l\,cos.\,a}{\omega^2\,M\,a} = \tfrac{1}{2}\,l\,cos.\,a,$$

I.E., this point coincides with *the centre of gravity of the rod.*

The resultant of the components parallel to $Y\ Z$ is

$$R = R_1 + R_2 + \ldots = \omega^2 . \frac{M}{n}\,(y_1 + y_2 + \ldots)$$

$$= \omega^2 . \frac{M}{n}\,\frac{l\,sin.\,a}{n} . \frac{n^2}{2} = \tfrac{1}{2}\,\omega^2\,M\,l\,sin.\,a,$$ and its moment is

$$R\,v = \omega^2\,\frac{M}{n} . cotg.\,a\,(y_1^2 + y_2^2 + \ldots)$$

$$= \omega^2 . \frac{M}{n} . cotg.\,a \left(\frac{(l\,sin.\,a)^2}{n^2} + \frac{(2\,l\,sin.\,a)^2}{n^2} + \ldots\right)$$

$$= \omega^2 . \frac{M}{n} . \frac{l^2}{n^2}\,(sin.\,a)^2\,cotg.\,a\,(1 + 4 + 9 + \ldots + n^2)$$

$$= \omega^2 . \frac{M}{n} . \frac{l^2}{n^2}\,sin.\,a\,cos.\,a . \frac{n^3}{3}$$

$$= \tfrac{1}{3}\,\omega^2\,M\,l^2\,sin.\,a\,cos.\,a.$$

Hence the distance of the point of application O of this force from the plane $X\ Y$ is

$$O_1\,O = v = \frac{\frac{1}{3}\,\omega^2\,M\,l^2\,sin.\,a\,cos.\,a}{\frac{1}{2}\,\omega^2\,M\,l\,sin.\,a} = \tfrac{2}{3}\,l\,cos.\,a,$$

I.E. this point lies at a distance $(\frac{2}{3} - \frac{1}{2})\,l\,cos.\,a = \frac{1}{6}\,l\,cos.\,a$ vertically above the centre of gravity, or, in general, $S\ O = \frac{1}{6}$ of the length of the rod $A\ B$.

From the two components $Q = \omega^2\ M\ a$ and $R = \frac{1}{2}\ \omega^2\,M\,l$ $sin.\,a$, it follows that the final resultant, which acts at the centre of gravity of the rod, is

$$P = \sqrt{Q^2 + R^2} = \omega^2\ M \sqrt{a^2 + \tfrac{1}{4}\,l^2\,sin.\,a^2},$$

that the couple is $(R, -\ R)$, and that its moment is

$$R \,.\, \overline{S\,O} = \tfrac{1}{2}\, \omega^2\, M\, l\, sin.\, a \,.\, \tfrac{1}{6}\, l\, cos.\, a$$
$$= \tfrac{1}{12}\, \omega^2\, M\, l^2\, sin.\, a\, cos.\, a = \tfrac{1}{24}\, \omega^2\, M\, l^2\, sin.\, 2\, a.$$

§ **309. Free Axes.**—The centrifugal forces of a body revolving uniformly upon its axis generally exert a pressure upon the axis, yet it is possible for these forces to balance each other, in which case the axis is subjected to no pressure from them. As examples of this case we may mention solids of revolution turning around their axis of symmetry, or geometrical axis, the wheel and axle, water wheels, etc. If a body in this condition is acted upon by no other forces, it will remain forever in revolution, although the axis is not fixed. This axis of rotation is therefore called a *free axis* (Fr. axe libre, Ger. freie Axe). From what precedes, we know the conditions, which are necessary when an axis of rotation becomes a free axis. It is necessary that not only the two resultants Q and R of the forces parallel to the co-ordinate planes $X\,Z$ and $Y\,Z$, but also that the sums of the moments of each of the two systems of forces shall be $= 0$, whence it follows that

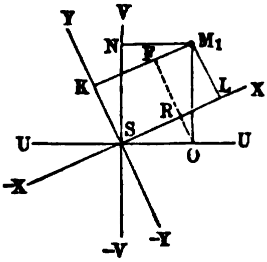
$$1)\ M_1\, x_1 + M_2\, x_2 + \ldots = 0,$$
$$2)\ M_1\, y_1 + M_2\, y_2 + \ldots = 0,$$
$$3)\ M_1\, x_1\, z_1 + M_2\, x_2\, z_2 + \ldots = 0 \text{ and}$$
$$4)\ M_1\, y_1\, z_1 + M_2\, y_2\, z_2 + \ldots = 0.$$

The first two conditions require the free axis to pass through the centre of gravity of the body or system of masses. The two latter, however, give the elements required for determining the position of this axis. It can also be proved that every body or system of masses has at least three free axes, and that these axes are at *right angles* to each other and intersect each other at the *centre of gravity* of the system.

The higher mechanics distinguishes from the free axes other axes, which may intersect each other at any point of the system and which are called *principal axes* (Fr. axes principaux, Ger. Hauptaxen). It is also proved that the moment of inertia of a body in reference to one of the principal axes is a maximum, and in relation to the second it is a minimum, and in relation to the third it has a mean value, and that for a point which lies in the free axes the principal axes are parallel to the free axes, I.E. to the principal axes passing through the centre of gravity.

§ **310. Free Axes of a System of Masses in a Plane.**—If the parts of a mass are in a plane, E.G., if they form a thin plate or plane figure, then the straight line, passing through the centre of gravity of the entire mass at right angle to that plane, is a free axis of the mass; for in this case the mass has no radius of rotation, and therefore the only possible centrifugal force is $= 0$. In order to find the other two free axes, we employ the following method. Let S, Fig. 513, be the centre of gravity of a mass and let $U\overline{U}$ and $V\overline{V}$ be two co-ordinate axes in the plane of the mass and let us determine the elements of the mass by means of co-ordinates parallel to these axes, E.G. the element M_1 by the co-ordinates $M_1 N = u_1$ and $M_1\, O = v_1$. Now if $X\overline{X}$ is one free axis and $Y\overline{Y}$ an axis at right-angles to the same and if the angle $U S X$, which the free axis makes with the axis of co-ordinates $S\, U$ and which is to be determined, $= \phi$, then putting for the co-ordinates of the elements of the mass in reference to $X\overline{X}$ and $Y\overline{Y}$, $x_1, x_2 \dots$ and $y_1, y_2 \dots$, E.G. for those of the mass M_1

Fig. 518.



$$M_1\, K = x_1 \text{ and } M_1\, L = y_1,$$

we obtain

$$x_1 = M_1\, K = S\, R + R\, L = S\, O \cos.\phi + O\, M_1 \sin.\phi = u_1 \cos.\phi + v_1 \sin.\phi,$$

$$y_1 = M_1\, L = -O\, R + O\, F = -S\, O \sin.\phi + O\, M_1 \cos.\phi$$
$$= -u_1 \sin.\phi + v_1 \cos.\phi,$$

and therefore the product

$$x_1\, y_1 = (u_1 \cos.\phi + v_1 \sin.\phi)\,.\,(-u_1 \sin.\phi + v_1 \cos.\phi)$$
$$= -(u_1^2 - v_1^2) \sin.\phi \cos.\phi + u_1\, v_1 (\cos.\phi^2 - \sin.\phi^2),$$

or, since $\sin.\phi \cos.\phi = \frac{1}{2} \sin. 2\,\phi$ and $\cos.\phi^2 - \sin.\phi^2 = \cos. 2\,\phi$,

$$x_1\, y_1 = -\tfrac{1}{2}(u_1^2 - v_1^2) \sin. 2\,\phi + u_1\, v_1 \cos. 2\,\phi,$$

and therefore the moment of the element M_1 is

$$M_1\, x_1\, y_1 = -\frac{M_1}{2}(u_1^2 - v_1^2) \sin. 2\,\phi + M_1\, u_1\, v_1 \cos. 2\,\phi,$$

and in like manner the moment of the element M_2 is

$$M_2\, x_2\, y_2 = -\frac{M_2}{2}(u_2^2 - v_2^2) \sin. 2\,\phi + M_2\, u_2\, v_2 \cos. 2\,\phi, \text{ etc.},$$

and the sum of the moments of all the elements or the moment of the entire mass itself is

$$M_1\, x_1\, y_1 + M_2\, x_2\, y_2 + \dots = -\tfrac{1}{2} \sin. 2\,\phi\, [(M_1\, u_1^2 + M_2\, u_2^2 + \dots)$$
$$-(M_1\, v_1^2 + M_2\, v_2^2 + \dots)] + \cos. 2\,\phi\, (M_1\, u_1\, v_1 + M_2\, u_2\, v_2 + \dots).$$

In order that $X\overline{X}$ shall be a free axis, this moment must be $= 0$; we must therefore put

$$\tfrac{1}{2}\,sin.\,2\,\phi\,[(M_1 u_1^2 + M_2 u_2^2 + \ldots) - (M_1 v_1^2 + M_2 v_2^2 + \ldots)]$$
$$- cos.\,2\,\phi\,(M_1\,u_1 v_1 + M_2 u_2 v_2 + \ldots) = 0,$$

from this we obtain the equation of condition

$$tang.\,2\,\phi = \frac{sin.\,2\,\phi}{cos.\,2\,\phi} = \frac{2\,(M_1 u_1 v_1 + M_2 u_2 v_2 + \ldots)}{(M_1 u_1^2 + M_2 u_2^2 + \ldots) - (M_1 v_1^2 + M_2 v_2^2 + \ldots)}$$
$$= \frac{\text{Double the moment of the centrifugal force}}{\text{Difference of the moments of inertia.}}.$$

This formula gives two values for $2\,\phi$, which differ from each other 180°, or two values of ϕ differing 90° from each other; this angle therefore determines not only the free axis $X\overline{X}$, but also the free axis $Y\overline{Y}$ perpendicular to it.

§ 311. The free axes of many surfaces and bodies can be given without any calculation. In a *symmetrical figure*, E.G., the axis of symmetry is a free axis, the perpendicular at the centre of gravity is the second, and the axis at right-angles to the surface of the figure the third free axis. For a *solid of revolution* $A\,B$, Fig. 514, the axis of rotation $Z\overline{Z}$ is one free axis and in like manner every normal $X\overline{X}$, $Y\overline{Y}\ldots$ to this line and passing through the centre of gravity is another. For a *sphere* every diameter is a free axis, and for a right *parallelopipedon* $A\,B\,D$, Fig. 515, bounded by 6 rectangles they are the three axes $X\overline{X}$, $Y\overline{Y}$ and $Z\overline{Z}$, passing through the centre of gravity perpendicular to the sides $B\,D$, $A\,B$ and $A\,D$, and parallel to the edges.

FIG. 514.

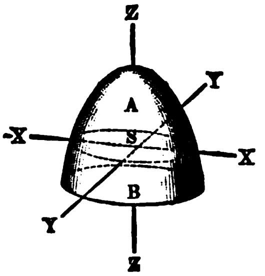
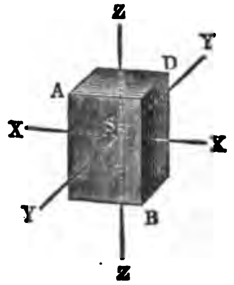
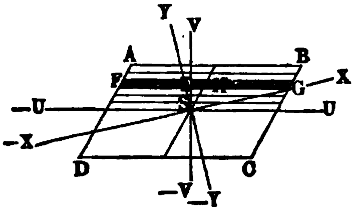


FIG. 515.



Let us now determine the three axes for a *rhomboid* $A\,B\,C\,D$, Fig. 516. We begin by passing two rectangular co-ordinate axes $U\overline{U}$ and $V\overline{V}$ through the centre of gravity, so that one is parallel to the side $A\ B$ of the rhomboid, and by decomposing the rhomboid by parallel lines in $2\ n$ equal strips, such as $F\ G$. Now if one side $A\ B = 2\ a$ and the other $A\ D = 2\ b$ and the acute angle $A\ D\ C$ between two sides $= a$, we have the length of the strip $E\ G$, situated at a distance $S\ E = x$ from $U\ \overline{U}$,

FIG. 516.



$= K\ G + E\ K = a + x\ cotg.\ a$, and that of the other part $E\ F$

$$= a - x\ cotg.\ a,$$

and since $\frac{b}{n}\ sin.\ a$ is the width of both, we have the areas of these strips

$$= \frac{b\ sin.\ a}{n}(a + x\ cotg.\ a) \text{ and } \frac{b\ sin.\ a}{n}(a - x\ cotg.\ a);$$

and consequently the measures of the centrifugal forces of the two portions in reference to the axis $V\ \overline{V}$ are

$$= \frac{b\ sin.\ a}{n}(a + x\ cotg.\ a) . \tfrac{1}{2}(a + x\ cotg.\ a) = \frac{b\ sin.\ a}{2\ n}(a + x\ cotg.\ a)^2$$

and

$$\frac{b\ sin.\ a}{2\ n}(a - x\ cotg.\ a)^2,$$

and their moments in reference to the axis $U\ \overline{U}$ are

$$\frac{b\ sin.\ a}{2\ n}(a + x\ cotg.\ a)^2\ x \text{ and } \frac{b\ sin.\ a}{2\ n}(a - x\ cotg.\ a)^2\ x.$$

Since the two forces act in opposition to each other in reference to $V\ \overline{V}$, by combining their moments we obtain the difference

$$\frac{b\ x\ sin.\ a}{2\ n}[(a + x\ cotg.\ a)^2 - (a - x\ cotg.\ a)^2] = \frac{2}{n}\ a\ b\ x^2\ cos.\ a.$$

If we substitute in this formula successively $\frac{b\ sin.\ a}{n}$, $\frac{2\ b\ sin.\ a}{n}$, $\frac{3\ b\ sin.\ a}{n}$, etc., and add the results, we obtain the measure of the moment of the centrifugal force of one-half the parallelogram

$$\frac{2\ a\ b}{n}\ cos.\ a . \frac{b^2\ sin.^2 a}{n^2}(1^2 + 2^2 + 3^2 + \ldots + n^2) = 2\ a\ b^3\ sin.^2 a\ cos.\ a . \frac{n^3}{3\ n^3}$$

$$= \tfrac{2}{3}\ a\ b^3\ sin.^2\ a\ cos.\ a,$$

and for the whole parallelogram we have

$$M_1\,u_1\,v_1 + M_2\,u_2\,v_2 + \ldots = \tfrac{4}{3}\,a\,b^3\,sin.^2\,a\,cos.\,a.$$

The moment of inertia of one strip $F\,G$ in reference to $V\,\overline{V}$ is

$$= \frac{b\,sin.\,a}{n}\left(\frac{(a + x\,cotg.\,a)^3}{3} + \frac{(a - x\,cotg.\,a)^3}{3}\right)$$

$$= \frac{2\,b\,sin.\,a}{3\,n}\,(a^3 + 3\,a\,x^2\,cotg.^2\,a\,) = \tfrac{2}{3}\,\frac{a\,b}{n}\,sin.\,a\,(a^2 + 3\,x^2\,cotg.^2\,a).$$

Substituting for x successively $\frac{b\,sin.\,a}{n}$, $\frac{2\,b\,sin.\,a}{n}$, $\frac{3\,b\,sin.\,a}{n}$, etc., and summing the resulting values, we obtain the moment of inertia of one-half the rhomboid, which is

$$= \tfrac{2}{3}\,a\,b\,sin.\,a\,(a^2 + b^2\,cos.^2\,a),$$

and for the whole rhomboid it is

$$= \tfrac{4}{3}\,a\,b\,sin.\,a\,(a^2 + b^2\,cos.^2\,a).$$

In reference to the axis of rotation $U\,\overline{U}$ the moment of inertia of the parallelogram is

$$= 4\,a\,b\,sin.\,a\,\frac{b^2\,sin.^2\,a}{3} = \tfrac{4}{3}\,a\,b^3\,sin.^3\,a \text{ (see § 287)},$$

and the required difference of the moments is given by the equation

$$\begin{aligned}
&(M_1\,u_1^2 + M_2\,u_2^2 + \ldots) - (M_1\,v_1^2 + M_2\,v_2^2 + \ldots)\\
&= \tfrac{4}{3}\,a\,b\,sin.\,a\,(a^2 + b^2\,cos.^2\,a) - \tfrac{4}{3}\,a\,b^3\,sin.^3\,a\\
&= \tfrac{4}{3}\,a\,b\,sin.\,a\,[a^2 + b^2\,(cos.^2\,a - sin.^2\,a)]\\
&= \tfrac{4}{3}\,a\,b\,sin.\,a\,(a^2 + b^2\,cos.\,2\,a).
\end{aligned}$$

Finally, we have for the angle $U\,S\,X = \phi$, which the free axis $X\,\overline{X}$ makes with the co-ordinate axis $U\,\overline{U}$ or with the side $A\,B$, according to § 310,

$$tang.\,2\,\phi = \frac{2\,(M_1\,u_1\,v_1 + M_2\,u_2\,v_2 + \ldots)}{(M_1\,u_1^2 + M_2\,u_2^2 + \ldots) - (M_1\,v_1^2 + M_2\,v_2^2 + \ldots)}$$

$$= \frac{2\,.\,\tfrac{4}{3}\,a\,b^3\,sin.^2\,a\,cos.\,a}{\tfrac{4}{3}\,a\,b\,sin.\,a\,(a^2 + b^2\,cos.\,2\,a)} = \frac{b^2\,sin.\,2\,a}{a^2 + b^2\,cos.\,2\,a}.$$

For the rhombus $a = b$, and

$$tang.\,2\,\phi = \frac{sin.\,2\,a}{1 + cos.\,2\,a} = \frac{2\,sin.\,a\,cos.\,a}{1 + cos.^2\,a - sin.^2\,a} = \frac{2\,sin.\,a\,cos.\,a}{2\,cos.^2\,a} = tang.\,a,$$

or $$2\,\phi = a \text{ and } \phi = \frac{a}{2}.$$

Since this angle gives the direction of the diagonal, it follows that the diagonals are free axes of the rhombus.

Example.—The sides of the rhomboid $A\ B\ C\ D$, Fig. 516, are $A\ B = 2\,a = 16$ inches, and $B\ C = 2\,b = 10$ inches, and the angle $A\ B\ C = \alpha = 60°$; what are the directions of the free axes?

Here we have

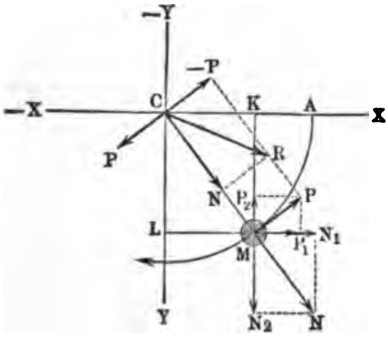
$$tang.\ 2\,\phi = \frac{5^2\,sin.\ 120°}{8^2 + 5^2\,cos.\ 120°} = \frac{25\ sin.\ 60°}{64 - 25\,cos.\ 60°} = \frac{25\ .\ 0{,}86603}{64 - 25\ .\ 0{,}5}$$

$$= 0{,}42040 = tang.\ 22°\ 48' \text{ or } tang.\ 202°\ 48';$$

hence it follows that the angles of inclination of the first two free axes to the side $A\ B$ are $\phi = 11°\ 24'$ and $101°\ 24'$. The third free axis is perpendicular to the plane of the parallelogram. These angles determine the free axes of a right parallelopipedon with a rhomboidal base.

§ **312. Action upon the Axis of Rotation.**—If a *material point M*, Fig. 517, revolves with a variable motion around a fixed axis C, the latter must counteract not only the centrifugal force, but also the force of inertia of this point. While the centrifugal force acts radially outwards, the force of inertia acts tangentially either in the opposite or in the same direction as the movement of rotation, according as the acceleration of this motion is positive or negative (Retardation). We can therefore assume that the centrifugal force $\overline{M\,N} = \overline{C\,N} = N$ acts directly upon the axis C, and that the force of inertia $\overline{M\,P} = -\,P$ is composed of a couple $(P, -\,P)$ and an axial force, $-\,P$, and consequently the entire force, acting upon the axis, $\overline{C\,R} = R$ is represented by the diagonal of a right-angled parallelogram formed of N and $-\,P$. If r is the distance $C\,M$ of the mass M from the axis of rotation C, ω the angular velocity and κ the angular acceleration, we have, according to § 302 and § 282,

FIG. 517.



$$N = \omega^2\,M\,r$$

and

$$P = \kappa\,M\,r,$$

and therefore the required resultant is

$$R = \sqrt{N^2 + P^2} = \sqrt{\omega^4 + \kappa^2}\,.\,M\,r,$$

and for the angle $R\,C\,N = \phi$, made by this force with the direction $C\,M$ of the centrifugal force, we have

$$tang.\,\phi = \frac{-P}{N} = -\frac{P}{N} = -\frac{\kappa}{\omega^2}.$$

Since in consequence of the acceleration κ, ω is variable, the centrifugal force N and the resultant R are variable.

In order to combine the centrifugal forces and the forces of inertia of the masses M_1, M_2, etc., we decompose each of these forces into two components parallel to the directions of two axes $X\,\overline{X}$ and $Y\,\overline{Y}$, then if we combine them by algebraical addition, so as to obtain two forces acting in the direction of each axis, we have only to determine the resultant of these two forces. If x and y are the co-ordinates $C\,K$ and $C\,L$ of the material point M in reference to the co-ordinate axes $\overline{X}\,X$ and $\overline{Y}\,Y$, we have the two components of the centrifugal force N

$$N_1 = \frac{x}{r}\,N = \omega^2\,M\,x \text{ and}$$

$$N_2 = \frac{y}{r}\,N = \omega^2\,M\,y,$$

and, on the contrary, those of the force of inertia

$$P_1 = \frac{y}{r}\,P = \kappa\,M\,y \text{ and}$$

$$P_2 = \frac{x}{r}\,P = \kappa\,M\,x,$$

and therefore the entire force in the axis $\overline{X}\,X$ is

$$Q = N_1 + P_1 = \omega^2\,M\,x + \kappa\,M\,y,$$

and that in the axis $\overline{Y}\,Y$ is

$$R = N_2 - P_2 = \omega^2\,M\,y - \kappa\,M\,x.$$

If we have a system of points or masses M_1, M_2, etc., which are revolving about a fixed axis C, Fig. 518, and if the co-ordinates of these points in reference to the axis $\overline{X}\,X$ are

$$C\,K_1 = x_1,\ C\,K_2 = x_2, \text{ etc.},$$

and those in reference to the axis $\overline{Y}\,Y$ are

$$C\,L_1 = y_1,\ C\,L_2 = y_2, \text{ etc.},$$

the entire force in the direction of the first axis is

$$Q = \omega^2\,M_1\,x_1 + \kappa\,M_1\,y_1 + \omega^2\,M_2\,x_2 + \kappa\,M_2\,y_2 + \ldots, \text{ i.e.}$$

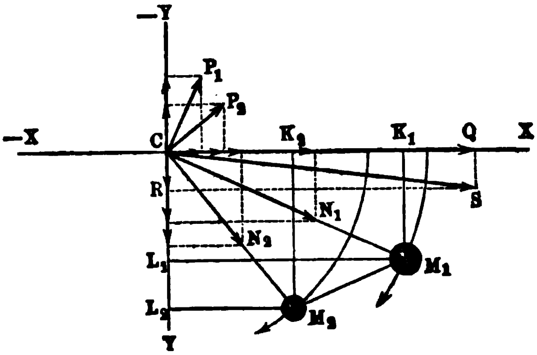
$$Q = \omega^2\,(M_1\,x_1 + M_2\,x_2 + \ldots) + \kappa\,(M_1\,y_1 + M_2\,y_2 + \ldots),$$

and that in the direction of the other axis is

$$R = \omega^2\,(M_1\,y_1 + M_2\,y_2 + \ldots) - \kappa\,(M_1\,x_1 + M_2\,x_2 + \ldots).$$

Now if we denote the entire mass $M_1 + M_2 + \ldots$ by M and the co-ordinates of its centre of gravity in reference to the axes $\overline{X}X$ and $\overline{Y}Y$ by x and y, we have (see § 305)

FIG. 518.



$$M_1 x_1 + M_2 x_2 + \ldots = M x$$
$$M_1 y_1 + M_2 y_2 + \ldots = M y,$$

and therefore, more simply,

$$Q = \omega^2 M x + \kappa M y \text{ and}$$
$$R = \omega^2 M y - \kappa M x.$$

From Q and R we obtain the resultant

$$S = \sqrt{Q^2 + R^2},$$

and for the angle $X C S = \phi$ of its direction

$$tang.\, \phi = \frac{R}{Q}$$

Since $M x$ and $M y$ are the statical moments of the centre of gravity, it follows that in determining the pressure S upon the axis of a system of masses, situated in one and the same plane of revolution, we can consider the whole mass to be concentrated at the centre of gravity; and since the distance of the centre of gravity of the system of masses from the axis of rotation is

$$r = \sqrt{x^2 + y^2},$$

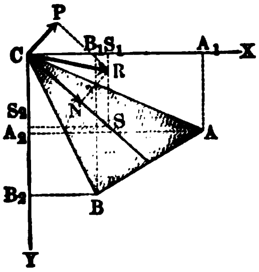
we have also

$$S = \sqrt{[(\omega^2 M x + \kappa M y)^2 + (\omega^2 M y - \kappa M x)^2]}$$
$$= M \sqrt{[\omega^4 (x^2 + y^2) + \kappa^2 (x^2 + y^2)]}$$
$$= M \sqrt{\omega^4 + \kappa^2} \sqrt{x^2 + y^2} = \sqrt{\omega^4 + \kappa^2} \,.\, M r$$

REMARK.—If a triangle $A B C$, Fig. 519, revolves about its corner C, and if the other corners A and B are determined by the co-ordinates

(x_1, y_1) and (x_2, y_2), we have, according to § 112, the co-ordinates of its centre of gravity S

FIG. 519.



$$C S_1 = x = \frac{x_1 + x_2}{3}$$

and
$$C S_2 = y = \frac{y_1 + y_2}{3},$$

and the mass, if we measure it by its superficial area, is

$$M = \frac{x_1 y_2 - x_2 y_1}{2}.$$

Its moment of inertia in reference to the axis of rotation C can be determined by the formula

$$W = \frac{M}{6} \left(\frac{x_1^3 - x_2^3}{x_1 - x_2} + \frac{y_1^3 - y_2^3}{y_1 - y_2} \right)$$

$$= \frac{M}{6} (x_1^2 + x_1 x_2 + x_2^2 + y_1^2 + y_1 y_2 + y_2^2).$$

This formula is also applicable to a *right prism*, whose base is the triangle $A\ B\ C$.

EXAMPLE.—A right prism with the triangular base $A\ B\ C$ is caused to revolve around its edge C by a force which acts uninterruptedly, so that at the end of the time $t = 1$ it has made $u = \frac{5}{2}$ revolutions; required not only the moment of this couple, but also the action of this motion upon the axis C. Let the base of this body be determined by the co-ordinates

$$x_1 = 1{,}5,\ y_1 = 0{,}5;\ x_2 = 0{,}4,\ y_2 = 1{,}0 \text{ feet},$$

and let its length or height be $l = 2$ feet, and its heaviness $\gamma = 30$ pounds. From these data we calculate, first, the area of the base

$$F = \frac{x_1 y_2 - x_2 y_1}{2} = \frac{1{,}5 \,.\, 1{,}0 - 0{,}4 \,.\, 0{,}5}{2} = \frac{1{,}3}{2} = 0{,}65 \text{ square feet},$$

and the mass of the whole body

$$M = \frac{F l \gamma}{g} = 0{,}031 \,.\, 0{,}65 \,.\, 2 \,.\, 30 = 1{,}209 \text{ pounds}.$$

Now

$$x_1^2 + x_1 x_2 + x_2^2 = 2{,}25 + 0{,}60 + 0{,}16 = 3{,}01 \text{ and}$$
$$y_1^2 + y_1 y_2 + y_2^2 = 0{,}25 + 0{,}50 + 1{,}00 = 1{,}75,$$

hence the moment of inertia of the body is

$$W = (3{,}01 + 1{,}75) \frac{M}{6} = 4{,}76 \,.\, \frac{1{,}209}{6} = 0{,}95914.$$

In consequence of the constant action of the couple, the movement of rotation is uniformly accelerated, and consequently the *angular velocity* of the body at the end of the time $t = 1$ second is (see § 10)

$$\omega = \frac{2\,s}{t} = \frac{2 \,.\, 2\,\pi\,u}{t} = \frac{2 \,.\, 2 \,.\, 5\,\pi}{2} = 31{,}416 \text{ feet},$$

and the *mechanical effect* required is

$$A = \tfrac{1}{2}\,\omega^2\,W = \tfrac{1}{2}\,(31{,}416)^2 \,.\, 0{,}95914 = 473{,}3 \text{ foot-pounds.}$$

The *angular acceleration* is

$$\kappa = \frac{\omega}{t} = \frac{31{,}416}{1} = 31{,}416 \text{ feet,}$$

and therefore the *moment of the couple*

$$P\,a = \kappa\,W = 31{,}416 \,.\, 0{,}95914 = 30{,}13 \text{ foot-pounds.}$$

The distances of the centre of gravity S of the base from the co-ordinate axes $\overline{X}\,X$ and $\overline{Y}\,Y$ are

$$x = \frac{x_1 + x_2}{3} = \frac{1{,}5 + 0{,}4}{3} = 0{,}6333 \text{ and}$$

$$y = \frac{y_1 + y_2}{3} = \frac{0{,}5 + 1{,}0}{3} = 0{,}5000,$$

consequently the distance of the centre of gravity from the *axis* is

$$C\,S = r = \sqrt{x^2 + y^2} = 0{,}6511.$$

Besides we have

$$\omega^4 = 31{,}416^4 = 974090 \text{ and}$$

$$\kappa^2 = 31{,}416^2 = 987,$$

whence

$$\sqrt{\omega^4 + \kappa^2} = \sqrt{975077} = 987{,}46,$$

and the *pressure upon the axis* increases during the accelerated rotation from

$$P = \kappa\,M\,r = 31{,}416 \,.\, 1{,}209 \,.\, 0{,}6511 = 24{,}73 \text{ pounds}$$

to

$$R = \sqrt{\omega^4 + \kappa^2} \,.\, M\,r = 987{,}46 \,.\, 1{,}209 \,.\, 0{,}6511 = 777{,}33 \text{ pounds.}$$

If after one second of time the couple ceases to act, the motion of rotation of the body becomes uniform, and the pressure upon the axis from that moment consists only of the centrifugal force, which is

$$N = \omega^2\,M\,r = 986{,}96 \,.\, 0{,}7872 = 776{,}94 \text{ pounds.}$$

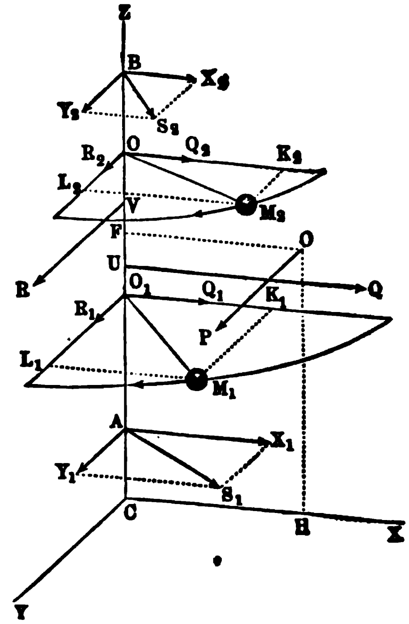
The pressure upon the axis, which increases gradually from 24,73 to 777,33 pounds, is in the beginning at right-angles to the central line of gravity $C\,S$, but approaches more and more this line as the velocity increases, so that at the end of the time $t = 1$ second, it makes but an angle ϕ with that line, and this angle is determined by the expression

$$tang.\ \phi = \frac{P}{N} = \frac{24{,}73}{776{,}94} = 0{,}03183,$$

for which $\phi = 1^\circ\,49'$. If the couple ceases to act, the direction of the axial force $N = 776{,}94$ pounds, coincides of course with the central line of gravity $C\,S$ and revolves with this line in a circle. If instead of the couple a single force P acts with the arm a upon the body, another pressure equal to this force P must be added to the pressure on the axis.

§ 313. Centre of Percussion.—If the different portions M_1, M_2, etc., Fig. 520, of a system of revolving masses are not in one and the same plane, the directions of the forces

FIG. 520.



$Q_1 = \omega^2 M_1 x_1 + \kappa M_1 y_1$,
$Q_2 = \omega^2 M_2 x_2 + \kappa M_2 y_2$, etc.,

no longer coincide with the co-ordinate axis $\overline{X} X$, but lie in the co-ordinate plane $X Z$, and those of the forces

$R_1 = \omega^2 M_1 y_1 - \kappa M_1 x_1$,
$R_2 = \omega^2 M_2 y_2 - \kappa M_2 x_2$, etc.,

no longer lie in the axis $\overline{Y} Y$, but in the co-ordinate plane $Y Z$. The system of forces Q_1, Q_2, etc., and R_1, R_2, etc., give, according to § 305, the resultants

$Q = Q_1 + Q_2 + \dots$ and
$R = R_1 + R_2 + \dots$

Now since the lines of application $U Q$ and $V R$ do not generally lie in the same plane, but cut the axis $C Z$ of rotation at different points U and V, it is impossible to obtain a single resultant by combining them, but we can refer them to a single force and a couple. The components are, of course, as above,

$$Q = \omega^2 (M_1 x_1 + M_2 x_2 + \dots) + \kappa (M_1 y_1 + M_2 y_2 + \dots)$$
$$= \omega^2 M x + \kappa M y$$

and

$$R = \omega^2 (M_1 y_1 + M_2 y_2 + \dots) - \kappa (M_1 x_1 + M_2 x_2 + \dots)$$
$$= \omega^2 M y + \kappa M x,$$

M denoting the entire mass $M_1 + M_2 + \dots$ and x and y the distances of its centre of gravity S from the co-ordinate planes $Y Z$ and $X Z$.

Now if we put the distances of the masses M_1, M_2, etc., from the plane of rotation $X Y$, which is perpendicular to the axis of rotation $C Z$, equal to z_1, z_2, etc., we obtain, as in § 305, the distances of the points of application U and V of the forces Q and R from the origin C.

$$u = \frac{Q_1 z_1 + Q_2 z_2 + \ldots}{Q_1 + Q_2 + \ldots}$$

$$= \frac{\omega^2 (M_1 x_1 z_1 + M_2 x_2 z_2 + \ldots) + \kappa (M_1 y_1 z_1 + M_2 y_2 z_2 + \ldots)}{\omega^2 (M_1 x_1 + M_2 x_2 + \ldots) + \kappa (M_1 y_1 + M_2 y_2 + \ldots)}$$

and

$$v = \frac{R_1 z_1 + R_2 z_2 + \ldots}{R_1 + R_2 + \ldots}$$

$$= \frac{\omega^2 (M_1 y_1 z_1 + M_2 y_2 z_2 + \ldots) - \kappa (M_1 x_1 z_1 + M_2 x_2 z_2 + \ldots)}{\omega^2 (M_1 y_1 + M_2 y_2 + \ldots) - \kappa (M_1 x_1 + M_2 x_2 + \ldots)}.$$

If the axis $C\,Z$ is retained at two points A and B (the pillow blocks), which are at the distance $C\,A = l_1$ and $C\,B = l_2$ from the origin of co-ordinates, the force Q is decomposed into two components

$$X_1 = \left(\frac{l_2 - u}{l_2 - l_1}\right) Q \text{ and } X_2 = \left(\frac{u - l_1}{l_2 - l_1}\right) Q,$$

and the force R into the components

$$Y_1 = \left(\frac{l_2 - v}{l_2 - l_1}\right) R \text{ and } Y_2 = \left(\frac{v - l_1}{l_2 - l_1}\right) R.$$

Now the pressure upon the bearing A is

$$S_1 = \sqrt{X_1^2 + Y_1^2},$$

and that upon the bearing B is

$$S_2 = \sqrt{X_2^2 + Y_2^2}.$$

If the acceleration of the rotation is produced not by a couple, whose moment is $P\,a$, but by a force P, whose arm is a, a third pressure equal to the force P is added to the two axial forces Q and R. If we cause this force P to act, at the distance $F\,O = a$ from the axis of rotation, parallel to the axis $C\,Y$ and perpendicular to the plane $X\,Z$, and if we assume that its line of application is at a distance $C\,F = H\,O = b$ from the co-ordinate plane $X\,Y$, the force R only will be increased by an amount P, and the portion of it Y_1 at the bearing A will be increased by

$$Y_3 = \left(\frac{l_2 - b}{l_2 - l_1}\right) P$$

and the part Y_2 at the bearing B by

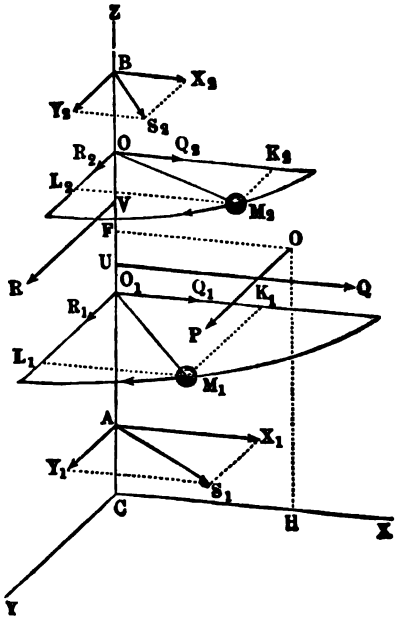
$$Y_4 = \left(\frac{b - l_1}{l_2 - l_1}\right) P.$$

If
$$M_1 x_1 + M_2 x_2 + \dots = 0,$$
$$M_1 y_1 + M_2 y_2 + \dots = 0,$$
$$M_1 x_1 z_1 + M_2 x_2 z_2 + \dots = 0 \text{ and}$$
$$M_1 y_1 z_1 + M_2 y_2 z_2 + \dots = 0,$$
$C Z$ is a *free axis*, and not only the forces Q and R, but also their moments $Q u$ and $R v$ become $= 0$; and we can, therefore, conclude that when a system of masses rotates about a free axis not only the centrifugal forces, but also the moments of inertia balance each other (compare § 309).

Let us assume that the system of masses is at rest, I.E., $\omega = 0$, or let us neglect the action of the centrifugal force upon the axis of rotation, then we have more simply for the pressures in the axes
$$Q = \kappa M y = \kappa (M_1 y_1 + M_2 y_2 + \dots) \text{ and}$$
$$R = - \kappa M x = - \kappa (M_1 x_1 + M_2 x_2 + \dots), \text{ and also}$$
$$Q u = \kappa (M_1 y_1 z_1 + M_2 y_2 z_2 + \dots) \text{ and}$$
$$R v = - \kappa (M_1 x_1 z_1 + M_2 x_2 z_2 + \dots).$$

When the plane of $X Z$ is plane of symmetry and consequently a plane of gravity,
$$M_1 y_1 + M_2 y_2 + \dots = 0$$
and
$$M_1 y_1 z_1 + M_2 y_2 z_2 + \dots = 0,$$
and, therefore,
$$Q = 0$$
and also
$$Q u = 0.$$

FIG. 521.



Now if we require that the force of rotation
$$P = \frac{\kappa W}{a}$$
shall be counteracted by the force of inertia R, so that there shall be no action upon the axis of rotation, we must have
$$P + R = 0$$
and
$$P b + R v = 0,$$
I.E.,

$$\frac{\kappa W}{a} - \kappa (M_1 x_1 + M_2 x_2 + \ldots) = 0$$

and

$$\frac{\kappa W b}{a} - \kappa (M_1 x_1 z_1 + M_2 x_2 z_2 + \ldots) = 0,$$

and consequently

$$a = \frac{W}{M x} = \frac{M_1 r_1^2 + M_2 r_2^2 + \ldots}{M_1 x_1 + M_2 x_2 + \ldots} = \frac{\text{Moment of inertia}}{\text{Statical moment}}$$

and

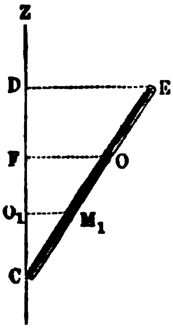
$$b = \left(\frac{M_1 x_1 z_1 + M_2 x_2 z_2 + \ldots}{W}\right) a = \frac{M_1 x_1 z_1 + M_2 x_2 z_2 + \ldots}{M_1 x_1 + M_2 x_2 + \ldots}$$

$$= \frac{\text{Moment of the centrifugal force}}{\text{Statical moment}}.$$

These co-ordinates determine a point O, which is called the *centre of percussion* (Fr. centre de percussion; Ger. Mittelpunkt des Stosses); for every force of impact P, whose direction passes through this point and is at right angles to the plane of symmetry $X Z$ of the body passing through the axis of rotation or fixed axis $C Z$, will be completely balanced, when the collision takes place, by the inertia of the mass, without producing any action upon the axis of the body.

EXAMPLE—1) The moment of inertia of a straight line or rod $C E$, Fig. 522, of uniform thickness throughout, which at one end C meets the axis $C Z$ at a given angle $Z C E$, when M is its mass and r the distance $D E$ of its other end from the axis of rotation, is

FIG. 522.



$$W = M k^2 = \tfrac{1}{3} M r^2 \text{ (see § 286)},$$

and, on the contrary, the statical moment is

$$M x = \tfrac{1}{2} M r,$$

and finally the moment of the centrifugal force, since, if h denotes the projection $C D$ of the length $C E$ of the rod on the axis of rotation $C Z$, we have

$$\frac{C O_1}{O_1 M_1} = \frac{z_1}{x_1} = \frac{h}{r},$$

or

$$M_1 x_1 z_1 = \frac{h}{r} M_1 x_1^2,\ M_2 x_2 z_2 = \frac{h}{r} M_2 x_2^2, \text{ etc.},$$

is

$$M_1 x_1 z_1 + M_2 x_2 z_2 + \ldots = \frac{h}{r} (M_1 x_1^2 + M_2 x_2^2 + \ldots) = \frac{h}{r} \cdot \tfrac{1}{3} M r^2 = \tfrac{1}{3} M h r.$$

Therefore, the co-ordinates of the centre of percussion of this rod are determined by the formulas

$$F\,O = a = \frac{\text{Moment of inertia}}{\text{Statical moment}} = \frac{\frac{1}{3}\,M\,r^2}{\frac{1}{2}\,M\,r} = \tfrac{2}{3}\,r$$

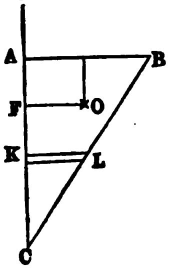
and

$$C\,F = b = \frac{\text{Moment of centrifugal force}}{\text{Statical moment}} = \frac{\frac{1}{3}\,M\,h\,r}{\frac{1}{2}\,M\,r} = \tfrac{2}{3}\,h,$$

and this centre is situated at $\frac{2}{3}$ of the length $C\,E$ of the rod from the end C and $\frac{1}{3}$ of the same from the end E.

2) The moment of inertia of a surface $A\ B\ C$, Fig. 523, whose form is a *right-angled triangle*, which turns around its base $C\,A$, is, when we denote the mass by M and its base and perpendicular $C\,A$ and $C\,B$ by h and r,

FIG. 523.



$$T = \frac{h\,r^3}{12} = \frac{h\,r}{2}\cdot\frac{r^2}{6} = \tfrac{1}{6}\,M\,r^2 \text{ (see § 229)},$$

and its statical moment, since the centre of gravity S is at a distance $\frac{r}{3}$ from the axis $C\,A$, is

$$M\,x = \frac{M\,r}{3},$$

consequently the distance of the centre of percussion O of this surface from this axis is

$$F\,O = a = \frac{\frac{1}{6}\,M\,r^2}{\frac{1}{3}\,M\,r} = \tfrac{1}{2}\,r.$$

For an element $K\,L$ of the triangle, whose shape is that of a strip, whose length is x and whose width is $\frac{h}{n}$, and which is situated at a distance $C\,K = z$ from the apex C, the moment of the centrifugal force is

$$M\,x\,z = \frac{h}{n}\,x\,.\,\tfrac{1}{2}\,x\,z,$$

or, since $\frac{x}{z} = \frac{r}{h}$, or $x = \frac{r}{h}\,z$,

$$M\,x\,z = \tfrac{1}{2}\,\frac{h}{n}\left(\frac{r}{h}\right)^2 z^3.$$

Substituting for z successively the values $1\left(\frac{h}{n}\right), 2\left(\frac{h}{n}\right), 3\left(\frac{h}{n}\right) \ldots n\left(\frac{h}{n}\right)$, and adding the values thus obtained for $M\,x\,z$, we have the total moment of the centrifugal forces

$$\begin{aligned} M_1\,x_1\,z_1 + M_2\,x_2\,z_2 + \ldots &= \tfrac{1}{2}\,\frac{h}{n}\left(\frac{r}{h}\right)^2 (1^3 + 2^3 + 3^3 + \ldots + n^3)\left(\frac{h}{n}\right)^3 \\ &= \tfrac{1}{2}\,\frac{h}{n}\left(\frac{r}{h}\right)^2 \cdot \frac{n^4}{4}\left(\frac{h}{n}\right)^3 = \tfrac{1}{8}\,r^2\,h^2 = \tfrac{1}{4}\,\frac{r\,h}{2}\,r\,h \\ &= \tfrac{1}{4}\,M\,r\,h, \end{aligned}$$

and, therefore, the distance of the centre of percussion O from the corner C is

$$C\,F = b = \frac{\frac{1}{4}\,M\,r\,h}{\frac{1}{3}\,M\,r} = \tfrac{3}{4}\,h.$$

CHAPTER III.

OF THE ACTION OF GRAVITY UPON THE MOTION OF BODIES IN PRESCRIBED PATHS.

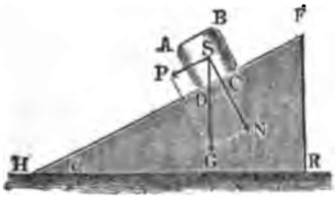
§ 314. Sliding upon an Inclined Plane.—A heavy body can be hindered in many ways from falling freely. We will, however, consider but two cases here, viz., the case of a body supported by an inclined plane and the case of a body movable around a horizontal axis. In both cases the paths of the bodies are contained in a vertical plane. If a body lies upon an *inclined plane,* its weight is decomposed into two components, one of which is normal to the plane and is counteracted by it, and the other is parallel to the plane and acts upon the body as a motive force. Let G be the weight of the body $A\ B\ C\ D$, Fig. 524, and a angle of inclination of the inclined plane $F\ H\ R$ to the horizon, according to § 146 the normal force is

$$N = G \cos. a,$$

and the motive force is

$$P = G \sin. a.$$

FIG. 524.



The motion of the body can be either a sliding or a rolling one. Let us consider the former case first. In this case all the parts of the body participate equally in its motion, and have therefore a common acceleration p, determined by the well-known formula

$$p = \frac{\text{force}}{\text{mass}}, = \frac{P}{M} = \frac{G \sin. a}{G} \cdot g = g \sin. a;$$

hence $$p : g = \sin. a : 1,$$

I.E., *the acceleration of a body upon an inclined plane is to the acceleration of gravity as the sine of the angle of inclination of the plane is to unity.* But on account of the friction this formula is seldom sufficiently accurate. It is, therefore, very often necessary in practice to take the friction into consideration.

If a body moves upon a *curved surface* the acceleration is variable, and is in every point equal to the acceleration corresponding to the plane, which is tangent to the curved surface at that point.

§ **315**. If a body slides down an inclined plane without friction and its initial velocity is = 0, then, according to § 11, the final velocity after t seconds is

$$v = g \sin. a \,.\, t = 32{,}2 \sin. a \,.\, t \text{ feet} = 9{,}81 \sin. a \,.\, t \text{ meters},$$

and the space described is

$$s = \tfrac{1}{2}\, g \sin. a \,.\, t^2 = 16{,}1 \sin. a \,.\, t^2 \text{ feet} = 4{,}905 \sin. a \,.\, t^2 \text{ meters}.$$

When a body falls freely $v_1 = g\, t$ and $s_1 = \frac{1}{2}\, g\, t^2$, and we can therefore put

$$v : v_1 = s : s_1 = \sin. a : 1,$$

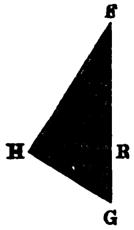
I.E., *the final velocity and the space described by a body sliding upon the inclined plane are to the velocity and the space described by a body falling freely as the sine of the angle of inclination of the plane is to unity.*

In the right-angled triangle $F\,G\,H$, Fig. 525, whose hypothenuse $F\,G$ is vertical, the base is $F\,H = F\,G \sin. F\,G\,H = F\,G \sin. F\,H\,R = \overline{F\,G} \sin. a$, when a denotes the inclination of the base to the horizon, and therefore

$$F\,H : F\,G = \sin. a : 1;$$

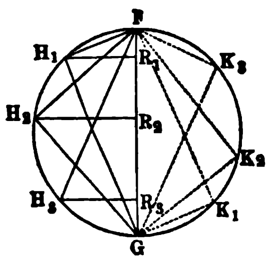
the body, therefore, describes the vertical hypothenuse $F\,G$ and the inclined base $F\,H$ in the same time. Hence the space described by a body upon an inclined plane in the time, in which, if falling freely, it would describe a given space, can be found by construction.

FIG. 525.



Since all the angles $F\,H_1\,G$, $F\,H_2\,G$, etc., inscribed in a semicircle $F\,H_2\,G$, Fig. 526, are right angles, the semicircle subtended by $F\,G$ will cut off from all inclined planes beginning at F the distances $F\,H_1$, $F\,H_2$, etc., described simultaneously with the diameter. For this reason we say that the *chords or diameter of a circle are described simultaneously or isochronously.* This is true not only when the chords, as, E.G., $F\,H_1$, $F\,H_2$, etc., begin at the highest point F, but also when the chords, as, E.G., $K_1\,G$, $K_2\,G$, etc., end at its lowest point G; for we

FIG. 526.



can draw through F the chords $F K_1$, $F K_2$, etc., which have the same length and position as the chords $G H_1$, $G H_2$, etc.

§ **316.** From the equation

$$s = \frac{v^2}{2p} = \frac{v^2}{2g \,.\, sin.\, a} \text{ for the space described,}$$

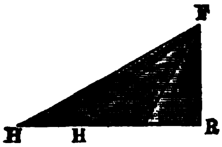
we obtain

$$s \, sin.\, a = \frac{v^2}{2g}, \text{ and inversely,}$$

$$v = \sqrt{2 g s \, sin.\, a}.$$

Now $s \, sin.\, a$ is the height $F R$ (Fig. 527) of the inclined plane or the vertical projection h of the space $F H = s$. If, therefore, several bodies, whose initial velocities are $= 0$, descend inclined planes $F H$, $F H_1$, etc., of different inclinations, but of the same height, their final velocity will be the same and equal to that acquired by a body falling freely through the distance $F R$ (compare § 43 and § 84).

FIG. 527.



From the equation $s = \frac{1}{2} g \, sin.\, a \,.\, t^2$ we obtain the formula for the time

$$t = \sqrt{\frac{2 s}{g \, sin.\, a}} = \frac{1}{sin.\, a} \sqrt{\frac{2 s \, sin.\, a}{g}} = \frac{1}{sin.\, a} \,.\, \sqrt{\frac{2 h}{g}}.$$

If a body falls freely through the height $F R = h$, the time is

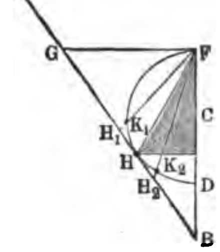
$$t_1 = \sqrt{\frac{2 h}{g}}, \text{ whence}$$

$$t : t_1 = 1 : sin.\, a = s : h = F H : F R.$$

The time required by a body to descend an inclined plane is to the time of falling freely through the height of this plane as the length of the plane is to its height.

EXAMPLE—1) The top F of an inclined plane $F H$, Fig. 528, is given, and we are required to determine the other extremity H, which is situated in such a position upon a line $A B$ that a body descending the plane will reach this line in the shortest time. If through F we draw the horizontal line $F G$ until it cuts $A B$, and make $G H = G F$, we obtain in H the point required, and in $F H$ the plane of the quickest descent; for if we pass through F and H a circle, to which the lines $F H$ and $G H$ are tangents, the chords $F K_1$, $F K_2$, etc., described simultaneously, are shorter than the lines $F H_1$, $F H_2$, etc., drawn from F to the line $A B$; consequently the time required to descend this chord is less than that required to descend these lines, and the inclined plane $F H$, which coincides with that chord, is the plane of quickest descent.

FIG. 528.



2) Required the inclination of the inclined plane $F H$, Fig. 527, which a body will descend in the same time as it will fall freely through the height $F R$ and move with the acquired velocity upon a horizontal plane to H. The time required to fall through the vertical distance $F R = h$ is

$$t_1 = \sqrt{\frac{2\,h}{g}},\text{ and the velocity acquired is}$$

$$v = \sqrt{2\,g\,h}.$$

If no velocity is lost in passing from the vertical to the horizontal motion, which is the case when the corner R is rounded off, the space $R H = h\ cotg.\ a$ will be described uniformly and in the time

$$t_2 = \frac{h\ cotg.\ a}{v} = \frac{h\ cotg.\ a}{\sqrt{2\,g\,h}} = \tfrac{1}{2}\ cotg.\ a\sqrt{\frac{2\,h}{g}}.$$

The time in which a body will descend the inclined plane is

$$t = \frac{1}{sin.\ a}\sqrt{\frac{2\,h}{g}}.$$

Now if we put $t = t_1 + t_2$, we obtain the equation of condition

$$\frac{1}{sin.\ a} = 1 + \tfrac{1}{2}\ cotg.\ a \text{ or } \frac{tang.\ a}{sin.\ a} = tang.\ a + \tfrac{1}{2}.$$

Resolving this equation, we obtain $tang.\ a = \tfrac{3}{4}$. In the corresponding inclined plane the height is to the base is to the length as 3 is to 4 is to 5, and the angle of inclination is $a = 36° 52' 11''$.

3) The time in which a body will slide down an inclined plane, whose base is a, is

$$t = \sqrt{\frac{2\,s}{g\ sin.\ a}} = \sqrt{\frac{2\,a}{g\ sin.\ a\ cos.\ a}} = \sqrt{\frac{4\,a}{g\ sin.\ 2\,a}};$$

this is a minimum when $sin.\ 2\,a$ is a maximum, I.E. $= 1$; then $2\,a° = 90$ or $a° = 45°$. Water flows quickest down roofs whose pitch is 45°.

§ 317. If the *initial velocity* of a body upon an inclined plane is c, we must employ the formula found in § 13 and § 14; hence, when a body ascends an inclined plane, we have the velocity

$$v = c - g\ sin.\ a\,.\,t,$$

and the space described

$$s = c\,t - \tfrac{1}{2}\,g\,\sin.\,a\,.\,t^2,$$

and for a body descending the inclined plane we must put

$$v = c + g\,\sin.\,a\,.\,t \text{ and } s = c\,t + \tfrac{1}{2}\,g\,\sin.\,a\,.\,t^2.$$

In both cases, however, the following formula

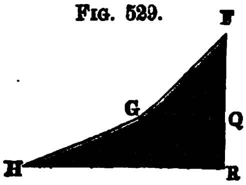
$$s = \frac{v^2 - c^2}{2\,g\,\sin.\,a}, \text{ or } s\,\sin.\,a = h = \frac{v^2 - c^2}{2\,g} = \frac{v^2}{2\,g} - \frac{c^2}{2\,g}$$

is applicable.

The vertical projection (h) *of the space* (s) *described upon the inclined plane is always equal to the difference of the heights due to the velocities.*

When two inclined planes $F\,G\,Q$ and $G\,H\,R$, Fig. 529, meet in a rounded edge, a body descending the plane will experience no impact in passing from one to the other; hence, if we have such a combination of planes, there will be no loss of velocity, and the following rule will be applicable to the case of a body descending these planes: *height of fall* equal to height due to velocity. We can easily understand that when a body ascends or descends a series of such planes or a curved line or surface, its motion will take place according to the same law.

FIG. 529.



EXAMPLE—1) A body ascends, with an initial velocity of 21 feet, an inclined plane, the inclination of which is 22°. What is its velocity and what is the space described after 1½ seconds?

The velocity is

$$v = 21 - 32{,}2\,\sin.\,22^\circ\,.\,1{,}5 = 21 - 32{,}2\,.\,0{,}3746\,.\,1{,}5 = 21 - 18{,}09$$
$$= 2{,}91 \text{ feet},$$

and the space is

$$s = \frac{c + v}{2}\,.\,t = \frac{21 + 2{,}91}{2}\,.\,\tfrac{3}{2} = \frac{23{,}91\,.\,3}{4} = 17{,}93 \text{ feet.}$$

2) How high will a body, whose initial velocity is 36 feet, rise upon a plane inclined at 48° to the horizon? The vertical height is

$$h = \frac{v^2}{2\,g} = 0{,}0155\,.\,v^2 = 0{,}0155\,.\,36^2 = 20{,}088 \text{ feet},$$

and therefore the entire space described upon the inclined plane is

$$s = \frac{h}{\sin.\,a} = \frac{20{,}088}{\sin.\,48^\circ} = 27{,}031 \text{ feet},$$

and the time required to describe it is

$$t = \frac{2\,.\,s}{v} = \frac{2\,.\,27{,}031}{36} = \frac{27{,}031}{18} = 1{,}5 \text{ seconds.}$$

§ 318. Sliding upon an Inclined Plane when the Friction is taken into Consideration.—The sliding friction has great influence upon the ascent or descent of a body upon an inclined plane. From the weight G of the body and from the angle of inclination a we obtain the normal pressure

$$N = G \cos. a,$$

and consequently the friction

$$F = \phi N = \phi G \cos. a.$$

If we subtract the latter from the force $P_1 = G \sin. a$, with which the gravity pulls it down the plane, there remains the motive force

$$P = G \sin. a - \phi G \cos. a,$$

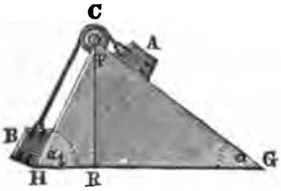
and we have for acceleration of a body moving down the inclined plane

$$p = \frac{\text{force}}{\text{mass}} = \left(\frac{G \sin. a - \phi G \cos. a}{G}\right) g = (\sin. a - \phi \cos. a)\, g.$$

For a body ascending an inclined plane the motive force is negative and $= G \sin. a + \phi \,.\, G \cos. a$, and the acceleration p is also negative and $= -(\sin. a + \phi \cos. a)\, g$.

If two bodies placed upon two *different inclined planes* $F G$ and $F H$, Fig. 530, are united by a perfectly flexible cord, which passes over a pulley C, it is possible that one of the bodies will descend and raise the other. Denoting the weight of these bodies by G and G_1, and the angles of inclination of the inclined planes, upon which they rest, by a and a_1, and assuming that G descends and draws up G_1, we obtain the motive force

FIG. 530.



$$P = G \sin. a - G_1 \sin. a_1 - \phi G \cos. a - \phi G_1 \cos. a_1$$
$$= G (\sin. a - \phi \cos. a) - G_1 (\sin. a_1 + \phi \cos. a_1),$$

and the mass moved

$$M = \frac{G + G_1}{g},$$

and therefore the acceleration with which G descends and G_1 ascends is

$$p = \frac{G (\sin. a - \phi \cos. a) - G_1 (\sin. a_1 + \phi \cos. a_1)}{G + G_1} \cdot g.$$

Since the friction, which is a resistance, cannot produce motion, we must have, if G descends and G_1 ascends,

$$G (\sin. a - \phi \cos. a) > G_1 (\sin. a_1 + \phi \cos. a_1), \text{ or}$$

$$\frac{G}{G_1} > \frac{\sin. a_1 + \phi \cos. a_1}{\sin. a - \phi \cos. a}, \text{ I.E. } \frac{G}{G_1} > \frac{\sin. (a_1 + \rho)}{\sin. (a - \rho)}.$$

If, on the contrary, G_1 descends and G ascends, we must have

$$\frac{G_1}{G} > \frac{sin.\, a + \phi\, cos.\, a}{sin.\, a_1 - \phi\, cos.\, a_1}, \text{ or}$$

$$\frac{G}{G_1} < \frac{sin.\, a_1 - \phi\, cos.\, a_1}{sin.\, a + \phi\, cos.\, a}, \text{ i.e. } \frac{G}{G_1} < \frac{sin.\, (a_1 - \rho)}{sin.\, (a + \rho)}.$$

As long as the ratio $\frac{G}{G_1}$ is within the limits

$$\frac{sin.\, a_1 + \phi\, cos.\, a_1}{sin.\, a - \phi\, cos.\, a} \text{ and } \frac{sin.\, a_1 - \phi\, cos.\, a_1}{sin.\, a + \phi\, cos.\, a}, \text{ or}$$

$$\frac{sin.\, (a_1 + \rho)}{sin.\, (a - \rho)} \text{ and } \frac{sin.\, (a_1 - \rho)}{sin.\, (a + \rho)},$$

the friction will prevent any motion.

EXAMPLE—1) A sled slides down an inclined plane covered with snow, 150 feet long and inclined at an angle of 20 degrees, and on arriving at the bottom it slides forward upon a horizontal plane until the friction brings it to rest. If the coefficient of friction between the snow and the sled is $= 0{,}03$, what space will the sled describe upon the horizontal plane (the resistance of the air being neglected)?

The acceleration of the sled is

$$p = (sin.\, a - \phi\, cos.\, a)\, g = (sin.\, 20^\circ - 0{,}03\, .\, cos.\, 20^\circ)\, .\, 32{,}2$$
$$= (0{,}3420 - 0{,}03\, .\, 0{,}9397)\, .\, 32{,}2 = 0{,}3138\, .\, 32{,}2 = 10{,}104 \text{ feet},$$

and therefore its velocity on arriving at the bottom of the inclined plane is

$$v = \sqrt{2\, p\, s} = \sqrt{2\, .\, 10{,}104\, .\, 150} = \sqrt{3031{,}2} = 55{,}06 \text{ feet.}$$

Upon the horizontal plane the acceleration is

$$p_1 = -\phi\, g = -0{,}03\, .\, 32{,}2 = -0{,}966 \text{ feet},$$

and therefore the space described is

$$s_1 = \frac{v^2}{2\, \phi\, g} = \frac{3031{,}2}{1{,}932} = 1569 \text{ feet.}$$

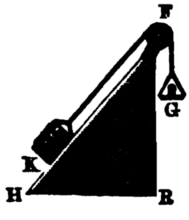
The time required to slide down the inclined plane is

$$t = \frac{2\, s}{v} = \frac{300}{55{,}06} = 5{,}45 \text{ seconds;}$$

that required to slide along on the horizontal plane is

FIG. 531.

$$t_1 = \frac{2\, s_1}{v} = \frac{3138}{55{,}06} = 57 \text{ seconds,}$$



and therefore the duration of the entire journey is

$$t + t_1 = 62{,}45 \text{ seconds} = 1 \text{ minute } 2{,}45 \text{ seconds.}$$

2) A bucket K, Fig. 531, which, when filled, weighs 250 pounds, is drawn up a plane, 70 feet long and inclined at an angle of 50°, by a weight $G = 260$; what time will be required when the coefficient of the friction of the bucket upon the floor is 0,36?

The motive force is

$$= G - (sin.\, a + \phi\, cos.\, a)\, K = 260 - (sin.\, 50^\circ + 0{,}36\, cos.\, 50^\circ) \,.\, 250$$

$$= 260 - 0{,}9974 \,.\, 250 = 10{,}6 \text{ pounds},$$

and therefore the acceleration is

$$p = \frac{10{,}6}{250 + 260} = \frac{10{,}6}{510} = 0{,}0208 \text{ feet};$$

the time of the motion is

$$t = \sqrt{\frac{2\, s}{p}} = \sqrt{\frac{140}{0{,}0208}} = \sqrt{6781} = 82{,}04 \text{ sec.} = 1 \text{ min. } 22 \text{ sec.},$$

and the final velocity

$$v = \frac{2\, s}{t} = \frac{140}{82} = 1{,}70 \text{ feet.}$$

§ **319. Rolling Motion upon an Inclined Plane.**—When a wagon runs down an inclined plane, it is the friction on the axle which offers the principal resistance to the acceleration. If G is the weight of the wagon, r the radius of the axle and a that of the wheel, we have

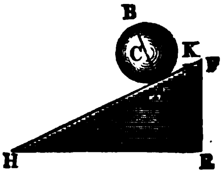
$$\frac{\phi\, r}{a} N = \frac{\phi\, r}{a}\, G\, cos.\, a,$$

and therefore the acceleration

$$p = \left(sin.\, a - \frac{\phi\, r}{a}\, cos.\, a\right) g.$$

If a *round body* $A\ B$, as, E.G., a cylinder or a sphere, etc., rolls down an inclined plane $F\,H$, Fig. 532, we have at the same time a motion of translation and of rotation. As the acceleration of translation p is generally equal to that of rotation (§ 169), if we put the moment of inertia of the rotating body $= G\, k^2$ and the radius $C\,A$ of rotation $= a$, we obtain for the force $\overline{A\,K} = K$, with which the roller (in consequence of the mutual penetration of its surface and that of the inclined plane) is set in rotation,

FIG. 532.



$$K = p \,.\, \frac{G\, k^2}{g\, a^2}.$$

But the force K opposes the force $G\ sin.\, a$, which tends to cause the body to slide down the plane, and therefore the motive force for the motion of translation is

$$P = G\ sin.\, a - K,$$

and its acceleration is

$$p = \frac{G\ sin.\, a - K}{G} \,.\, g$$

Eliminating K from the two equations, we obtain

$$G p = G g \sin. a - \frac{G k^2}{a^2} . p,$$

and consequently the required acceleration

$$p = \frac{g \sin. a}{1 + \frac{k^2}{a^2}}.$$

For a homogeneous cylinder $k^2 = \frac{1}{2} a^2$ (§ 288), and therefore

$$p = \frac{g \sin. a}{1 + \frac{1}{2}} = \frac{2}{3} g \sin. a,$$

but for a *sphere* $k^2 = \frac{2}{5} a^2$ (§ 290), and therefore

$$p = \frac{g \sin. a}{1 + \frac{2}{5}} = \frac{5}{7} g \sin. a;$$

the acceleration of a rolling *cylinder* is but $\frac{2}{3}$ and that of a rolling *sphere* is but $\frac{5}{7}$ as great as that of a body sliding without friction.

The force which produces the rotation is

$$K = \frac{g \sin. a}{1 + \frac{k^2}{a^2}} . \frac{G k^2}{g a^2} = \frac{G k^2 \sin. a}{a^2 + k^2}$$

As long as this force is less than the sliding friction $\phi\ G \cos. a$, so long will the body descend the plane with a perfect rolling motion. But if

$$K > \phi\ G \cos. a, \text{ I.E., if } tang.\ a > \phi \left(1 + \frac{a^2}{k^2}\right),$$

the friction is no longer sufficient to impart a velocity of rotation equal to that of translation; the acceleration of translation becomes, as in the case of sliding friction,

$$p = \frac{G \sin. a - \phi\ G \cos. a}{G} . g = (\sin. a - \phi \cos. a)\, g,$$

and that of rotation

$$p_1 = \frac{\phi\ G \cos. a}{G k^2 : a^2} . g = \phi \frac{a^2}{k^2} g \cos. a.$$

If the weight of a wagon is G, the radius of its wheels a and their moment of inertia $G k^2$, we will have

$$K = p \frac{G_1 k_1^2}{g a^2} \text{ and } p = \frac{G \sin. a - \phi \frac{r}{a} G \cos. a - K}{G} . g,$$

I.E.,

$$p = \frac{g\,(sin.\,a - \phi\,\frac{r}{a}\,cos.\,a)}{1 + \frac{G_1\,k_1^2}{G\,a^2}}.$$

EXAMPLE—1) A wagon, which, when loaded, weighs 3600 pounds and whose wheels are 4 feet high and have a moment of inertia of 2000 foot-pounds, rolls down a plane whose inclination is 12°; required the acceleration, when the coefficient of friction upon the axles is $\phi = 0,15$ and the thickness of the axles is $2\,r = 3$ inches.

Here we have

$$\frac{G_1\,k_1^2}{G\,a^2} = \frac{2000}{3600\,.\,2^2} = \frac{5}{36} = 0,139 \text{ and } \phi\,\frac{r}{a} = 0,15\,.\,\frac{1}{4\,.\,4} = 0,0094,$$

and therefore the required acceleration is

$$p = \frac{32,2\,(sin.\,12° - 0,0094\,.\,cos.\,12°)}{1 + 0,139} = \frac{32,2\,.\,(0,2079 - 0,0094\,.\,0,978)}{1,139}$$

$$= \frac{32,2\,.\,0,1987}{1,139} = 5,617 \text{ feet.}$$

2) With what acceleration will a massive roller roll down a plane whose angle of inclination is $a = 40°$?

If the coefficient of sliding friction of the roller upon the plane is $\phi = 0,24$, we have

$$\phi\left(1 + \frac{a^2}{k^2}\right) = 0,24\,(1 + 2) = 0,72.$$

Now $tang.\,40° = 0,839$, and $tang.\,a$ is therefore greater than $\phi\left(1 + \frac{a^2}{k^2}\right)$, and the acceleration of the rolling motion is smaller than that of the motion of translation.

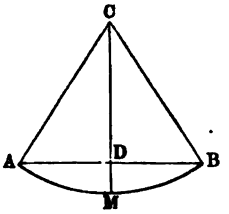
The latter is

$p = (sin.\,a - \phi\,cos.\,a)\,g = (0,643 - 0,24\,.\,0,7660)\,.\,32,2 = 0,459\,.\,32,2$ $= 14,78$ feet, and the former is

$$p_1 = 0,24\,.\,2\,.\,32,20\;cos.\,40° = 15,456\,.\,0,776 = 11,99 \text{ feet.}$$

§ 320. **The Circular Pendulum.**—A body suspended from a horizontal axis is in equilibrium as long as its centre of gravity is vertically under this axis; but if we move the centre of gravity out of the vertical plane containing the axis and abandon the body to itself, it assumes an *oscillating* or *vibrating motion* (Fr. oscillation, Ger. Schwingende Bewegung), I.E., a reciprocating motion in a circle. A body oscillating about a horizontal axis is called a *pendulum* (Fr. pendule, Ger. Pendel or Kreispendel). If the oscillating body is a material point, and if it is connected with the axis of rotation by a line without weight, we have a *simple* or *theoretical pendulum* (Fr. p. simple, Ger. einfaches or mathematisches P.); but if the pendulum consists of a body or of several bodies of finite dimensions, it is called a *compound pendulum* (Fr. pendule composé, Ger. zuzammengeseztes, physisches or materielles Pendel). Such a pendulum can be considered as a rigid combination of a number of simple pendulums, oscillating around a common axis. The simple pendulum has no real existence, but it is of great use in discussing the theory of the compound pendulum, which can be deduced from that of the simple one. If the pendulum, which is suspended in C, Fig. 533, is moved from its vertical position $C M$ to the position $C A$ and left to itself, by virtue of its weight it will return towards $C M$ with an accelerated motion, and it will arrive at the point M with a velocity, the height due to which is equal to $D M$. In consequence of this velocity it describes upon the other side the arc $M B = M A$, and rises to the height $D M$. It falls back again from B to M and A and continues to move backwards and forwards in the arc $A B$. If we could do away with the friction on the axis and the resistance of the air, this oscillating motion of the pendulum would continue forever; but since these resistances can never be entirely removed, the arc in which the oscillation takes place will gradually decrease until the pendulum comes to rest.

FIG. 533.



The motion of the pendulum from A to B is called *an oscillation* (Fr. oscillation, Ger. Schwung or Pendelschlag), the arc $A B$, *the amplitude* (Fr. amplitude, Ger. Swingungsbogen), and the angle measured by half the amplitude is called *the angle of displacement*. The time in which the pendulum makes an oscillation is called the *time, duration*, or *period of an oscillation* (Fr. durée d'une oscillation, Ger. Schwingungszeit or Schwingungsdauer).

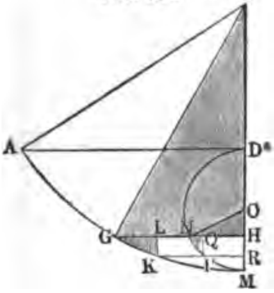
§ 321. Theory of the Simple Pendulum.—In consequence of the frequent use of the pendulum in common life, viz. for clocks, it is important to know the duration of an oscillation; its demonstration is therefore one of the most important problems in Mechanics. To solve this problem, let us put the length of the pendulum $A C = M C = r$, Fig. 534, and the height of rise and fall during an oscillation $M D = h$. Assuming that the pendulum has fallen from A to G, and making the vertical height $D\,H$ of fall corresponding to this motion $= x$, we have the velocity acquired at G

$$v = \sqrt{2\,g\,x},$$

and the element of time, during which the element of its path $G\,K$ is described,

$$\tau = \frac{G\,K}{v} = \frac{G\,K}{\sqrt{2\,g\,x}}.$$

FIG. 534.



If we describe from the middle O of $M\,D = h$ with the radius $O\,M = O\,D = \frac{1}{2}\,h$ the semicircle $M\,N\,D$, we can cut from the latter an elementary arc $N\,P$, which will have the same altitude $P\,Q = K\,L = R\,H$ as $G\,K$, and whose relation to the latter can be very simply expressed. In consequence of the similarity of the triangles $G\,K\,L$ and $C\,G\,H$ we have

$$\frac{G\,K}{K\,L} = \frac{C\,G}{G\,H},$$

and in consequence of the similarity of the triangles $N\,P\,Q$ and $O\,N\,H$

$$\frac{N\,P}{P\,Q} = \frac{O\,N}{N\,H},$$

dividing the first of these proportions by the second and remembering that $K\,L = P\,Q$, we obtain the ratio of the above elements of the arcs

$$\frac{G\,K}{N\,P} = \frac{C\,G\,.\,N\,H}{G\,H\,.\,O\,N}.$$

From a well-known property of the circle we have

$$\overline{G\,H}^2 = M\,H\,(2\,C\,M - M\,H) \text{ and } \overline{N\,H}^2 = M\,H\,.\,D\,H,$$

whence it follows that

$$\frac{G\,K}{N\,P} = \frac{C\,G\,.\,\sqrt{D\,H}}{O\,N\,.\,\sqrt{2\,C\,M - M\,H}} = \frac{r\,\sqrt{x}}{\frac{1}{2}\,h\,\sqrt{2\,r - (h - x)}},$$

and the time required to describe an element of the path is

$$\tau = \frac{r\,\sqrt{x}}{\frac{1}{2}\,h\,\sqrt{2\,r - (h - x)}}\,.\,\frac{N\,P}{\sqrt{2\,g\,x}} = \frac{2\,r}{h\,\sqrt{2\,g\,[2\,r - (h - x)]}}\,.\,N\,P$$

$$= \sqrt{\frac{r}{g}}\,\frac{N\,P}{h\,\sqrt{1 - \frac{h - x}{2\,r}}}.$$

Generally in practice the angle of displacement is small, and then $\frac{h}{2\,r}$, $\frac{x}{2\,r}$ and $\frac{h-x}{2\,r}$ are such small quantities, that we can neglect them and their higher powers and put

$$\tau = \sqrt{\frac{r}{g}} \cdot \frac{N\,P}{h}.$$

The duration of a semi-oscillation or the time within which the pendulum describes the arc $A\ M$ is equal to the sum of all the elements of the time corresponding to the elements $G\ K$ or $N\ P$. Now since $\frac{1}{h} \cdot \sqrt{\frac{r}{g}}$ is a constant factor, we can put the sum equal to $\frac{1}{h} \sqrt{\frac{r}{g}}$ times the sum of all the elements forming the semicircle $D\ N\ M$, I.E., $= \frac{1}{h} \sqrt{\frac{r}{g}}$ times the semicircle $\left(\frac{\pi\,h}{2}\right)$, or

$$t_1 = \frac{1}{h} \sqrt{\frac{r}{g}} \cdot \frac{\pi\,h}{2} = \frac{\pi}{2} \sqrt{\frac{r}{g}}.$$

The same time is required by the pendulum for its ascent; for the velocities are the same but opposite in direction, hence the duration of a complete oscillation is double the latter, or

$$t = 2\,t_1 = \pi \sqrt{\frac{r}{g}}.$$

(**§ 322.**) **More Exact Formula for the Duration of an Oscillation of the Circular Pendulum.**—In order to determine the duration of an oscillation with greater precision, as is sometimes necessary, when angles of displacement are large, we can transform the equation

$$\frac{1}{\sqrt{1 - \frac{h-x}{2\,r}}} = \left(1 - \frac{h-x}{2\,r}\right)^{-\frac{1}{2}}$$

into the series

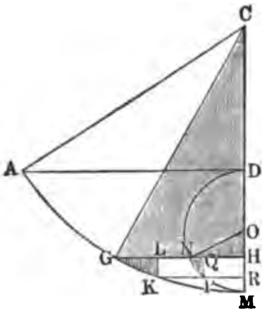
$$1 + \tfrac{1}{2} \cdot \frac{h-x}{2\,r} + \tfrac{3}{8} \cdot \left(\frac{h-x}{2\,r}\right)^2 + \ldots,$$

and then we have the time in which an element of the path is described

$$\tau = \left[1 + \tfrac{1}{2} \cdot \frac{h-x}{2\,r} + \tfrac{3}{8} \cdot \left(\frac{h-x}{2\,r}\right)^2 + \ldots\right] \sqrt{\frac{r}{g}} \cdot \frac{N\,P}{h}.$$

Putting the central angle $D\,O\,N = \phi^\circ$, or the arc

FIG. 535.



$$D\,N = D\,O\,.\,\phi = \frac{h\,\phi}{2},$$

we obtain the height

$$M\,H = h - x = M\,O - H\,O = \frac{h}{2} + \frac{h}{2}\,cos.\,\phi = (1 + cos.\,\phi)\,\frac{h}{2};$$

and therefore the element of time

$$\tau = \left[1 + \tfrac{1}{2}\,.\,(1 + cos.\,\phi)\,\frac{h}{4\,r} + \tfrac{3}{8}\,(1 + cos.\,\phi)^2\left(\frac{h}{4\,r}\right)^2 + \ldots\right]\sqrt{\frac{r}{g}}\,.\,\frac{N\,P}{h},$$

or, since

$$(1 + cos.\,\phi)^2 = 1 + 2\,cos.\,\phi + (cos.\,\phi)^2 = 1 + 2\,cos.\,\phi + \frac{1 + cos.\,2\,\phi}{2}$$
$$= \tfrac{3}{2} + 2\,cos.\,\phi + \tfrac{1}{2}\,cos.\,2\,\phi,$$

$$\tau = \left[1 + \tfrac{1}{2}\,(1 + cos.\,\phi)\,\frac{h}{4\,r} + \tfrac{3}{8}\,(\tfrac{3}{2} + 2\,cos.\,\phi\;.\, + \tfrac{1}{2}\,cos.\,2\,\phi)\left(\frac{h}{4\,r}\right)^2 + \ldots\right]\sqrt{\frac{r}{g}}\,.\,\frac{N\,P}{h}$$

$$= \left[1 + \tfrac{1}{2}\,\frac{h}{4\,r} + \tfrac{9}{16}\left(\frac{h}{4\,r}\right)^2 + \ldots + \left(\tfrac{1}{2}\,\frac{h}{4\,r} + \tfrac{3}{4}\left(\frac{h}{4\,r}\right)^2 + \ldots\right)cos.\,\phi + (\tfrac{3}{16} + \ldots)\left(\frac{h}{4\,r}\right)^2 cos.\,2\,\phi + \ldots\right]\sqrt{\frac{r}{g}}\,.\,\frac{N\,P}{h}$$

$$= \left(\left[1 + \tfrac{1}{2}\,\frac{h}{4\,r} + \tfrac{9}{16}\left(\frac{h}{4\,r}\right)^2 + \ldots\right]\frac{N\,P}{h} + \left[\tfrac{1}{2}\,\frac{h}{4\,r} + \tfrac{3}{4}\left(\frac{h}{4\,r}\right)^2 + \ldots\right]\frac{\overline{N\,P}\,cos.\,\phi}{h} + (\tfrac{3}{16} + \ldots)\left(\frac{h}{4\,r}\right)^2\frac{\overline{N\,P}\,cos.\,2\,\phi}{h}\right)\sqrt{\frac{r}{g}}.$$

Now the sum of all the elements $N\,P$ is $=$ the arc $D\,N\,P = \frac{\phi\,h}{2}$, $\overline{N\,P}\,cos.\,\phi$ is $= N\,Q$ and the sum of all the $N\,Q$ is $=$ the ordinate $N\,H = \frac{h}{2}\,sin.\,\phi$ and also the sum of all the $\frac{2\,\overline{N\,P}\,cos.\,2\,\phi}{h}$ is $= sin.\,2\,\phi$, therefore the time required to describe the arc $A\,G$ is

$$t_1 = \left(\left[1 + \tfrac{1}{2}\,.\,\frac{h}{4\,r} + \tfrac{9}{16}\left(\frac{h}{4\,r}\right)^2 + \ldots\right]\phi + \left[\tfrac{1}{2}\,\frac{h}{4\,r} + \tfrac{3}{4}\left(\frac{h}{4\,r}\right)^2 + \ldots\right]sin.\,\phi + (\tfrac{3}{16} + \ldots)\left(\frac{h}{4\,r}\right)^2\frac{sin.\,2\,\phi}{2}\right)\,.\,\tfrac{1}{2}\sqrt{\frac{r}{g}}.$$

The time required to describe the arc *A M* is, since we have here $\phi = \pi$, *sin.* ϕ = *sin.* π and *sin.* $2\,\phi$ = *sin.* $2\,\pi = 0$,

$$t_1 = \left[1 + \tfrac{1}{2}\,\frac{h}{4\,r} + \tfrac{3}{8}\cdot\tfrac{3}{2}\cdot\left(\frac{h}{4\,r}\right)^2 + \ldots\right]\pi\,.\,\tfrac{1}{2}\sqrt{\frac{r}{g}}$$

$$= \left[1 + (\tfrac{1}{2})^2\cdot\frac{h}{2\,r} + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2\cdot\left(\frac{h}{2\,r}\right)^2 + \ldots\right]\frac{\pi}{2}\sqrt{\frac{r}{g}}.$$

As the velocity decreases in the same manner, when the pendulum ascends on the other side, as it increased during the descent, the time required for describing the entire arc or the duration of the complete oscillation is

$$t = 2\,t_1 = \left[1 + (\tfrac{1}{2})^2\,\frac{h}{2\,r} + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2\left(\frac{h}{2\,r}\right)^2\right.$$

$$\left. + \left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2\left(\frac{h}{2\,r}\right)^3 + \ldots\right]\pi\sqrt{\frac{r}{g}}.$$

If the pendulum oscillates in a *semicircle*, we have $h = r$, and consequently the duration of an oscillation is

$$t = \left(1 + \tfrac{1}{8} + \frac{9}{256} + \frac{225}{18432} + \ldots\right)\pi\sqrt{\frac{r}{g}} = 1{,}180\,\pi\sqrt{\frac{r}{g}}.$$

In the most cases in practice the amplitude of the oscillations is much less than a semicircle, and the formula

$$t = \left(1 + \frac{h}{8\,r}\right)\pi\sqrt{\frac{r}{g}}$$

is sufficiently accurate.

If the *angle of displacement* be denoted by a, we have *cos.* $a = \dfrac{r-h}{r} = 1 - \dfrac{h}{r}$ or $\dfrac{h}{r} = 1 -$ *cos.* a, and therefore

$$\frac{h}{8\,r} = \tfrac{1}{4}\,.\,\frac{1 - cos.\ a}{2} = \tfrac{1}{4}\left(sin.\ \frac{a}{2}\right)^2;$$

from the latter formula we can determine the correction to be applied for any given amplitude. If, for example, this angle is $a = 15°$, we have

$$\frac{h}{8\,r} = \tfrac{1}{4}\left(sin.\ \frac{15°}{2}\right)^2 = 0{,}00426,$$

and, on the contrary, for $a = 5°$

$$\frac{h}{8\,r} = 0{,}00047;$$

for this last amplitude the duration of an oscillation is

$$t = 1{,}00047\,.\,\pi\sqrt{\frac{r}{g}}.$$

Consequently if the amplitude is less than 5°, we can put with sufficient accuracy the duration of an oscillation

$$t = \pi \sqrt{\frac{r}{g}} = \frac{\pi}{\sqrt{g}} \sqrt{r} = 0{,}554 \sqrt{r}.$$

§ **323. Length of the Pendulum.**—Since in the formula

$$t = \pi \sqrt{\frac{r}{g}}$$

the angle of displacement does not appear, it follows that the duration of small oscillations of a pendulum does not depend upon this angle, and that pendulums of the same lengths, when their amplitudes, although different, are small, oscillate *isochronally* or have the same duration of oscillation. A pendulum, when its amplitude is 4 degrees, make an oscillation in (almost) the same time as when it is 1 degree.

If we compare the duration t of an oscillation with the time t_1 of the free fall, we find the following relation. The time required by a body to fall freely a distance r is

$$t_1 = \sqrt{\frac{2r}{g}} = \sqrt{2} \, . \sqrt{\frac{r}{g}},$$

hence

$$t : t_1 = \pi : \sqrt{2};$$

the duration of an oscillation of a pendulum is to the time required by a body to fall freely a distance equal to the length of the pendulum as the number π is to the square root of 2. The time required to fall the distance $2\,r$ is

$$t_1 = \sqrt{\frac{2 \, . \, 2r}{g}} = 2 \sqrt{\frac{r}{g}};$$

therefore *the duration of an oscillation is to the time required to fall a height equal to twice the length of the pendulum as π is to 2.*

If we put the durations of the oscillations of two pendulums, whose lengths are r and r_1, equal to t and t_1, we obtain

$$t : t_1 = \sqrt{r} : \sqrt{r_1}.$$

When the acceleration of gravity is the same, *the durations of the oscillations are proportional to the square roots of the lengths of the pendulums.* Now if n is the number of oscillations made by one pendulum in a certain time, as, E.G., in a minute, and n_1 the number made in the same time by another pendulum, we have

$$\sqrt{r} : \sqrt{r_1} = \frac{1}{n} : \frac{1}{n_1},$$

and inversely $n : n_1 = \sqrt{r_1} : \sqrt{r}$,
I.E. *the number of oscillations is inversely proportional to the square root of the length of the pendulum.* A pendulum four times as long as another makes but one-half as many oscillations in the same time.

A pendulum is called a *second pendulum* (Fr. pendule à seconde, Ger. Secundenpendel), when the duration of its oscillation is a second. Substituting in the formula $t = \pi \sqrt{\frac{r}{g}}$, $t = 1$, we obtain the length of the second pendulum $r = \frac{g}{\pi^2}$; for English system of measures

$$r = 3{,}26255 \text{ feet} = 39{,}1506 \text{ inches},$$

and for the metrical system

$$r = 0{,}9938 \text{ metres}.$$

By inverting the formula $t = \pi \sqrt{\frac{r}{g}}$, we obtain $g = \left(\frac{\pi}{t}\right)^2 r$, by means of which we can deduce from the length r of the pendulum and the duration t of its oscillation the acceleration g of gravity. We can determine the value of g more simply and more accurately in this manner than with Atwood's machine.

REMARK.—By observations upon the pendulum, the decrease of the force of gravity, as we proceed from the equator to the poles, has been proved, and its intensity determined. This diminution is caused by the centrifugal force arising from the daily revolution of the earth upon its axis, and also by the increase of the radius of the earth from the poles to the equator. The centrifugal force diminishes the action of gravity at the equator $\frac{1}{289}$ of its value (§ 302), while at the poles the action of the centrifugal force is null. By observation upon the pendulum we can determine the acceleration of gravity at the place of observation. This acceleration, when β denotes the latitude of the place, is

$$g = 9{,}8056\,(1 - 0{,}00259 \cos. 2\,\beta) \text{ metres};$$

therefore at the equator, where $\beta = 0$ and $\cos. 2\,\beta = 1$, we have,

$$g = 9{,}8056\,(1 - 0{,}00259) = 9{,}780 \text{ metres},$$

and at the poles, where $\beta = 90°$, $\cos. 2\,\beta = \cos. 180° = -1$,

$$g = 9{,}8056 \,.\, 1{,}00259 = 9{,}831 \text{ metres}.$$

Upon mountains g is smaller than at the level of the sea.

§ **324. Cycloid.**—We can put a body in oscillation or cause it to assume a reciprocating motion in an infinite number of ways. Any body moving in such a manner is called a *pendulum.* We distinguish several kinds of pendulums, as, for example, the *circular pendulum,* which we have just discussed, the *cycloidal pendulum,* where the body, by virtue of its weight, swings backwards and for-

If we substitute $G\,K = \frac{2\sqrt{2\,r\,x}}{h} \,.\, \overline{N\,P}$ in the formula

$$\tau = \frac{G\,K}{\sqrt{2\,g\,x}},$$ we obtain

$$\tau = \frac{2\sqrt{2\,r\,x}}{\sqrt{2\,g\,x}.h} \,.\, N\,P = \frac{2}{h}\sqrt{\frac{r}{g}} \,.\, N\,P.$$

The time required to fall from A to M is the sum of all the values of τ, obtained by substituting for $N\,P$ all the divisions of the semicircle $D\,N\,M$, or

$$= \frac{2}{h}\sqrt{\frac{r}{g}} \text{ times the semicircle } D\,N\,M \left(\frac{\pi}{2}\,h\right).$$

Hence we have the time required to describe the arc $A\,M$

$$t_1 = \frac{\pi}{2}\,h \,.\, \frac{2}{h}\sqrt{\frac{r}{g}} = \pi\sqrt{\frac{r}{g}},$$

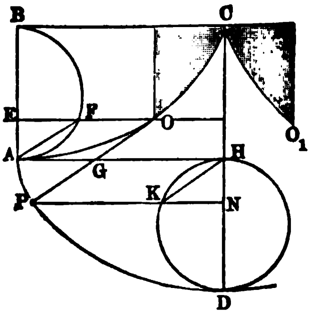
and since the time for ascending the arc $M\,B$ is equal to it, we have for the time required *to describe the whole arc A M B*

$$t = 2\,t_1 = 2\,\pi\sqrt{\frac{r}{g}} = \pi\sqrt{\frac{4\,r}{g}}.$$

Since this quantity is entirely independent of the length of the arc, it follows that the times of the oscillations for all arcs of the same cycloid are mathematically exactly equal, or that the *cycloidal pendulum* is perfectly *isochronal.* If we compare this formula with the formula for the duration of the oscillations of a circular pendulum, we find that the durations are the same for both pendulums, when the length of the circular pendulum is four times the radius of the generating circle of the cycloid.

REMARK.—In order to make a body suspended by a flexible cord oscillate in a cycloid and thereby to form a cycloidal pendulum, we must hang the same between two arcs $C\,O$ and $C\,O_1$, Fig. 538, of a cycloid, so that during each oscillation the cord will unwind from one and wind upon the other arc. It can easily be shown that, when the cord $C\,O\,P$ wraps and unwraps, the end P describes a cycloid equal to the given one, but in an inverted position. The length of the semi-cycloid is $C\,O\,A = C\,D = 2\,A\,B$ and the arc $O\,A$ is $=$ the straight line $O\,P$, which has been unwound; but the arc $O\,A =$ twice the chord $A\,F = 2\;G\;O$, and therefore

FIG. 538.



$P\,G = G\,O = A\,F$ and $H\,N = A\,E$. Describing upon $D\,H = A\,B$ a semicircle $D\,K\,H$ and drawing the ordinate $N\,P$, we have $K\,H = P\,G$ and, therefore, also

$P\,K = G\,H = A\,H - A\,G = A\,H - F\,O =$ arc $A\,F\,B -$ arc $A\,F =$ arc $B\,F =$ arc $D\,K$,

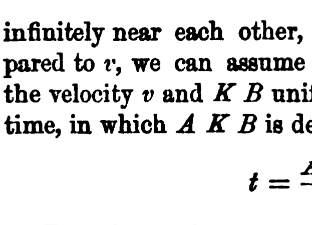
and, finally, $N\,P$ is $=$ the ordinate $N\,K$ of the circle plus the corresponding arc $D\,K$; $N\,P$ is therefore the ordinate of a cycloid $D\ P\ A$ corresponding to the generating circle $D\,K\,H$.

Upon the application of cycloidal pendulums to clocks, see "Jahrbücher des polytechn. Institutes in Wien," Vol. 20, Art. II. Also Prechtl's technologische Encyclopädie, Bd. 19.

(§ 326.) **The Curve of Quickest Descent.**—It can be proved by the Calculus that the cycloid, besides the property of *isochronism* or *tautochronism*, possesses also that of *brachystonism*, I.E. it is the line in which a body descends from one given point to another in the shortest time.

We can prove this (as Jacob Bernoulli did) in the following manner.

FIG. 539.



Let the relative position of two points A and B, Fig. 539, be given by the vertical distance $A\ C = a$ and the horizontal one $B\ C = b$, and that of a horizontal line $D\ E$ by the vertical distance $A\ D = h$; required the point K, in which a body falling from A to B must intersect the line $D\ E$ in order to reach B in the shortest time. If the body arrives at A with the velocity v, the velocity at K is

$$v_1 = \sqrt{v^2 + 2\,g\,h};$$

and supposing that A, K and B are infinitely near each other, or that a, b and h are very small compared to v, we can assume that $A\ K$ is described uniformly with the velocity v and $K\ B$ uniformly with the velocity v_1, or that the time, in which $A\ K\ B$ is described, is

$$t = \frac{A\,K}{v} + \frac{K\,B}{v_1}.$$

Denoting $D\ K$ by z, we have

$$A\ K = \sqrt{h^2 + z^2} \text{ and } K\,B = \sqrt{(a-h)^2 + (b-z)^2},$$

and therefore

$$t = \frac{\sqrt{h^2 + z^2}}{v} + \frac{\sqrt{(a-h)^2 + (b-z)^2}}{v_1}.$$

This quantity will be a minimum, when we make its first differential coefficient

$$\frac{d\,t}{d\,z} = \frac{z}{v\sqrt{h^2 + z^2}} - \frac{b - z}{v_1\sqrt{(a-h)^2 + (b-z)^2}} = 0.$$

But

$$\frac{z}{\sqrt{h^2 + z^2}} = \frac{K\,D}{K\,A} = \cos.\,A\,K\,D = \cos.\,\phi$$

and

$$\frac{b - z}{\sqrt{(a-h)^2 + (b-z)^2}} = \frac{B\,L}{B\,K} = \cos.\,K\,B\,L = \cos.\,\phi_1,$$

ϕ and ϕ_1 denoting the inclination of the paths $A\,K$ and $K\,B$ to the horizon; hence we have for the equation of condition

$$\frac{\cos.\,\phi}{v} = \frac{\cos.\,\phi_1}{v_1}.$$

Putting the heights due to the velocities v and v_1, $M\,A = y$ and $N\,K = y_1$, or

$$v = \sqrt{2\,g\,y} \text{ and } v_1 = \sqrt{2\,g\,y_1},$$

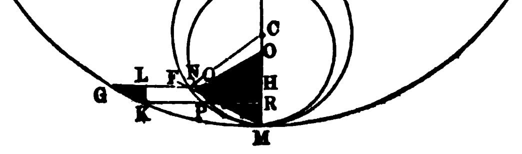
our equation becomes

$$\frac{\cos.\,\phi}{\sqrt{y}} = \frac{\cos.\,\phi_1}{\sqrt{y_1}},$$

and if we apply this formula to the case of a curved line $S\,A\,K\,B$, it follows that for every point of this curve the quotient $\frac{\cos.\,\phi}{\sqrt{y}}$ must be a constant quantity, such as $\frac{1}{\sqrt{2\,r}}$.

This property corresponds to a cycloid $S\,G\,M$, Fig. 540; for we have for an element $G\,K$ of this curve

FIG. 540.



$$\cos.\phi = \frac{G\,L}{G\,K} = \frac{F\,H}{F\,M} = \frac{\sqrt{M\,H\,.\,E\,H}}{\sqrt{M\,H\,.\,E\,M}} = \sqrt{\frac{E\,H}{E\,M}} = \sqrt{\frac{y}{2\,r}},$$

and therefore

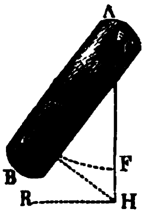
$$\frac{\cos.\phi}{\sqrt{y}} = \frac{1}{\sqrt{2\,r}}$$

r denoting the radius $C\,M = C\,E$ of the generating circle $E\,F\,M$.

An arc $S\,G$ of a cycloid is therefore the arc in which a body descends in the shortest time from one point S to another point G.

§ 327. **The Compound or Material Pendulum.**—In order to determine the duration of an oscillation of a compound pendulum or of any body $A\,B$, Fig. 541, oscillating about a horizontal axis C, we must first find the *centre of oscillation* (Fr. centre d'oscillation, Ger. Mittelpunkt des Schwunges or Schwingungspunkt), I.E., that point K of the body which, if it oscillates alone around C or forms a simple pendulum, has the same duration of oscillation as the entire body. We can easily perceive that there are several such points in a body, but we generally understand by it only that one, which lies in the same perpendicular to the horizontal axis as the centre of gravity does.

FIG. 541.



From the variable angle of displacement $K\,C\,F = \phi$ we obtain the acceleration of the isolated point K, which is

$$= g \sin.\phi;$$

for we can imagine that it slides down a plane, whose inclination is $K\,H\,R = K\,C\,F = \phi$. If $M\,k^2$ is the moment of inertia of the entire body or system of bodies $A\,B$, $M\,s$ its statical moment, I.E. the product of the mass and the distance $C\,S = s$ of its centre of gravity from the axis of oscillation C, and r the distance $C\,K$ of the centre of oscillation from the axis of rotation or the length of the simple pendulum, which vibrates isochronally with the material pendulum $A\,B$, we have the mass reduced to K

$$= \frac{M\,k^2}{r^2},$$

and therefore the rotary force reduced to this point is

$$= \frac{s}{r}\,M\,g \sin.\phi;$$

consequently the acceleration is

$$p = \frac{\text{force}}{\text{mass}} = \frac{s}{r} M g \sin. \phi : \frac{M k^2}{r^2} = \frac{M s r}{M k^2} \cdot g \sin. \phi.$$

In order that the duration of an oscillation of this pendulum shall be the same as that of the simple pendulum, it must have in every position the same acceleration as the other; hence

$$\frac{M s r}{M k^2} \cdot g \sin. \phi = g \sin. \phi.$$

This equation gives

$$r = \frac{M k^2}{M s} = \frac{\text{moment of inertia}}{\text{statical moment}}.$$

We find, then, *that the distance of the centre of oscillation from the point about which the rotation takes place, or the length of the simple pendulum having the same duration of oscillation as the compound pendulum, is equal to the moment of inertia of the compound pendulum divided by its statical moment or the moment of its weight.*

Substituting this value of r in the formula $t = \pi \sqrt{\frac{r}{g}}$, we obtain for the duration of an oscillation of a *compound pendulum*

$$t = \pi \sqrt{\frac{M k^2}{M g s}} = \pi \sqrt{\frac{k^2}{g s}},$$

or more accurately

$$t = \pi \left(1 + \frac{h}{8 r}\right) \sqrt{\frac{k^2}{g s}}.$$

By inversion we obtain from the duration of an oscillation of a suspended body its moment of inertia by putting

$$M k^2 = \left(\frac{t}{\pi}\right)^2 . M g s \text{ or } k^2 = \left(\frac{t}{\pi}\right)^2 g s.$$

REMARK—1) In order to determine the moment of inertia $M k^2$ of a body from the duration of one of its oscillations, it is necessary to know its statical moment $M g s = G s$. The latter is found by drawing the body $A C$, Fig. 542, out of its position of equilibrium by means of a rope $A B D$, which passes over a pulley and to which a weight P is suspended. The perpendicular $C N$, let fall from the axis C upon the direction of the rope $A B$, is the arm a of the weight P, and $P a$ is equal to the moment $G . \overline{C H}$ of the weight G, which acts vertically at the centre of gravity S. Denoting by α the angle $V C S = C S H$, which the body is raised by the weight P, we have

$$\overline{C H} = \overline{C S} \sin. \alpha = s \sin. \alpha,$$

and therefore

$$G s \sin. \alpha = P a,$$

from which we deduce the required statical moment

$$G s = \frac{P a}{\sin. \alpha}.$$

2) A very simple and useful pendulum *A D F*, Fig. 543, may be made of a ball of lead *A* about 1 inch in diameter, suspended by a silk thread, whose upper end is fastened into a ferrule *D* by a clamping screw. This ferrule has upon its end a screw, which passes through the arm *E F* and is made fast by a nut *G*, when the arm has been screwed into a door-frame or some other solid support. If the length is *C A* = 0,2485 or nearly ¼ meter, then this pendulum will beat half-seconds for almost an hour, although the arcs in which it oscillates will continually decrease.

FIG. 542.

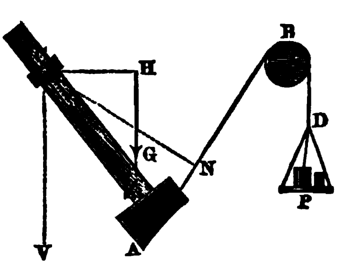
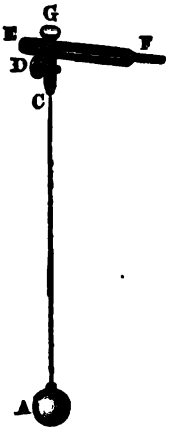


FIG. 543.



EXAMPLE—1) If the point of suspension of a prismatical rod *A B*, Fig. 544, is at a distance *C A* = l_1 from one end *A* and *C B* = l_2 from the other *B*, its moment of inertia, when *F* denotes its cross-section, is (§ 286)

$$M k^2 = \tfrac{1}{3} F (l_1^3 + l_2^3),$$

and its statical moment is

$$M s = \tfrac{1}{2} F (l_1^2 - l_2^2);$$

hence the length of the simple pendulum, which oscillates isochronally, is

$$r = \frac{M k^2}{M s} = \tfrac{2}{3} \cdot \frac{l_1^3 + l_2^3}{l_1^2 - l_2^2} = \frac{l^2 + 3 d^2}{6 d},$$

FIG. 544.

l denoting the sum $l_1 + l_2$ and d the difference $l_1 - l_2$. If this rod should beat half-seconds, we must make

$$r = \tfrac{1}{4} \cdot \frac{g}{\pi^2} = \tfrac{1}{4} \cdot 39{,}15 = 9{,}79 \text{ inches},$$

and if the rod is 12 inches long we must put

$$9{,}79 = \frac{144 + 3 d^2}{6 d} \text{ or } d^2 - 19{,}58\, d = -48,$$

hence
$$d = \frac{19{,}58 - \sqrt{191{,}3764}}{2} = \frac{19{,}58 - 13{,}83}{2} = 2\tfrac{7}{8} \text{ inches};$$

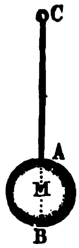
from which we obtain

$$l_1 = \frac{l+d}{2} = 6 + 1\tfrac{7}{16} = 7\tfrac{7}{16} \text{ and } l_2 = \frac{l-d}{2} = 6 - 1\tfrac{7}{16} = 4\tfrac{9}{16}.$$

2) If G is the weight and l the length of the rod of a pendulum with a spheroidal bob $A\,B$, Fig. 545, and if K is the weight and r_1 the diameter $M\,A = M\,B$ of the latter, we will have

$$r = \frac{\frac{1}{3}\,G\,l^2 + K\left[(l + r_1)^2 + \frac{2}{5}\,r_1^2\right]}{\frac{1}{2}\,G\,l + K\,(l + r_1)}$$

FIG. 545.



If the wire weighs 0,05 pounds and the ball 1,5 pounds, and if the length of the wire is 1 foot and the radius of the ball 1,15 inches, we have the distance of the centre of oscillation of this pendulum from the axis of rotation

$$r = \frac{\frac{1}{3}\,.\,0{,}05\,.\,12^2 + K\,(13{,}15^2 + \frac{2}{5}\,.\,1{,}15^2)}{\frac{1}{2}\,.\,0{,}05\,.\,12 + 1{,}5\,.\,13{,}15} = \frac{2{,}4 + 260{,}177}{0{,}3 + 19{,}725}$$

$$= \frac{262{,}577}{20{,}025} = 13{,}112 \text{ inches.}$$

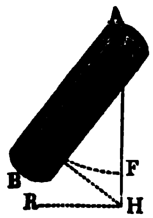
If we neglect the wire, $r = \frac{260{,}177}{19{,}725} = 13{,}190$ inches, and if we assume the mass of the ball to be concentrated at its centre $r = 13{,}15$ inches. The duration of an oscillation of this pendulum is

$$t = \pi\sqrt{\frac{r}{g}} = 0{,}554\sqrt{\frac{13{,}112}{12}} = 0{,}554\sqrt{1{,}0926} = 0{,}5791 \text{ seconds.}$$

§ 328. Reciprocity of the Point of Suspension and the Centre of Oscillation.

—The *point of suspension* and the *centre of oscillation* are reciprocal (Fr. réciproque; Ger. wechselseitig), i.e. one can be changed for the other, or the pendulum can be suspended at the centre of oscillation without changing the duration of the oscillation. This can be proved, by the aid of what was said in § 284, in the following manner. Let W be the moment of inertia of the compound pendulum $A\,B$, Fig. 546, referred to an axis of rotation passing through its centre of gravity S, for an axis of rotation passing through C, which is at a distance $C\,S = s$ from the centre of gravity S, we have

FIG. 546.



$$W_1 = W + M\,s^2,$$

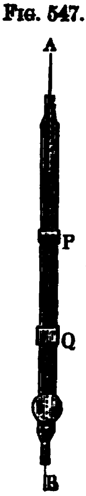
and therefore the distance of the centre of oscillation from the axis of rotation C is

$$r = \frac{W_1}{M\,s} = \frac{W + M\,s^2}{M\,s} = \frac{W}{M\,s} + s.$$

Denoting the distance $K\,S = r - s$ of the centre of oscillation K from the centre of gravity by s_1, we obtain the equation $s\,s_1 = \frac{W}{M}$, in which s and s_1 present themselves in the same manner, and therefore can be changed for one another. This formula is consequently applicable not only to the case, where s expresses the distance of centre of rotation and s_1 that of the centre of oscillation from the centre of gravity, but also to the case, where s expresses the distance of the centre of oscillation and s_1 that of the centre of rotation from the centre of gravity. Therefore C becomes the centre of oscillation, when K becomes the point of suspension. We employ this property in the *reversable pendulum A B*, Fig. 547, first suggested by Bohnenberger and afterwards employed by Kater. It is provided with two knife-edge axes C and K, which are so placed, that the duration of an oscillation remains the same, whether the pendulum is suspended from one axis or the other. In order to avoid changing the position of the axes in reference to each other, two sliding weights are applied to it, the smaller of which can be moved by a small screw. If by sliding the weights we have brought them to such a position, that the duration of an oscillation is the same, whether the pendulum be suspended in C or K, we obtain in the distance $C K$ the length r of the simple pendulum, which vibrates isochronally with the reversable pendulum, and the duration of the oscillation is given by the formula

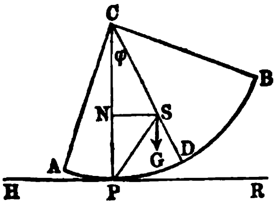
$$t = \pi \sqrt{\frac{r}{g}}.$$

Fig. 547.



§ **329. Rocking Pendulum.**—The *rocking* of a body with a cylindrical base can be compared to the oscillation of a pendulum. This rocking, like every other rolling motion, is composed of a motion of translation and one of rotation, but we can consider it as a rotation about a variable axis. This axis of rotation is the point of support, where the rocking body $A\ B\ C$, Fig. 548, rests upon the horizontal support $H\ R$. Let the radius $C\ D = C\ P$ of the cylindrical base $A\ D\ B$ be $= r$ and the distance $C\ S$ of the centre of gravity S of the whole body from the centre C of this base be $= s$, then we have for the distance $S\ P = y$ of the centre of gravity from the centre of rotation, corresponding to the angle $S\ C\ P = \phi$,

Fig. 548.



$$y^2 = r^2 + s^2 - 2\, r\, s \cos.\phi = (r - s)^2 + 4\, r\, s \left(\sin.\, \frac{\phi}{2}\right)^2.$$

If we denote the moment of inertia of the whole body in reference to the centre of gravity S by $M\, k^2$, we obtain the moment of inertia in reference to the point of support P

$$W = M\,(k^2 + y^2) = M\left[k^2 + (r - s)^2 + 4\, r\, s \left(\sin.\, \frac{\phi}{2}\right)^2\right],$$

for which for small angles we can put $M\,[k^2 + (r - s)^2 + r\, s\, \phi^2]$ or even $M\,[k^2 + (r - s)^2]$. Now since the moment of the force $=$ $G\,.\,\overline{S\,N} = M\,g\,.\,\overline{C\,S}\,\sin.\,\phi = M\,g\,s\,\sin.\,\phi$, we have the angular acceleration for a rotation around P

$$\kappa = \frac{\text{moment of force}}{\text{moment of inertia}} = \frac{M\,g\,s\,\sin.\,\phi}{M\,[k^2 + (r - s)^2]} = \frac{g\,s\,\sin.\,\phi}{k^2 + (r - s)^2}.$$

For the simple pendulum it is $= \frac{g\,\sin.\,\phi}{r_1}$, when r_1 denotes its length. If they should oscillate isochronally, we must have

$$\frac{g\,s\,\sin.\,\phi}{k^2 + (r - s)^2} = \frac{g\,\sin.\,\phi}{r_1}, \text{ i.e., } r_1 = \frac{k^2 + (r - s)^2}{s}.$$

The *duration of an oscillation of the rocking body* is, therefore,

$$t = \pi \sqrt{\frac{r_1}{g}} = \pi \sqrt{\frac{k^2 + (r - s)^2}{g\,s}}.$$

FIG. 549.

This theory is applicable to a pendulum $A\,B$, Fig. 549, with a rounded axis of rotation $C\,M$, when we substitute for r the radius of curvature $C\,M$ of this axis. If instead of the rounded axis a knife-edge axis D is used, the duration of an oscillation would be

$$t_1 = \pi \sqrt{\frac{k^2 + \overline{D\,S}^2}{g\,.\,\overline{D\,S}}} = \pi \sqrt{\frac{k^2 + (s - x)^2}{g\,(s - x)}}$$

when the distance $C\,D$ of the knife-edge D from the centre C of the rounded axis is denoted by x. The two pendulums will have the same duration of oscillation, when

$$\frac{k^2 + (s - x)^2}{s - x} = \frac{k^2 + (r - s)^2}{s}, \text{ or } \frac{k^2}{s - x} - x = \frac{k^2 + r^2}{s} - 2\,r;$$

putting approximatively $\frac{k^2}{s - x} = \frac{k^2}{s} + \frac{k^2\,x}{s^2}$ and neglecting r^2, we obtain

$$x = \frac{2\,r\,s^2}{s^2 - k^2}.$$

REMARK.—The conical pendulum will be discussed in the third part, in the article upon the "Governor."

In the appendix to this volume the subject of oscillation is treated at length.

CHAPTER IV.

THE THEORY OF IMPACT.

§ 330. Impact in General.—On account of the impenetrability of matter, two bodies cannot occupy the same space at the same time. If two bodies come together in such a way that one seeks to force itself into the space occupied by the other, a reciprocal action between them takes place, which causes a change in the conditions of motion of these bodies. This reciprocal action is what is called impact or collision (Fr. choc, Ger. Stoss).

The conditions of impact depend, in the first place, upon the *law of the equality of action and reaction* (§ 65); during the impact one body presses exactly as much upon the other as the other does upon it in the opposite direction. The straight line, normal to the surfaces, in which the two bodies touch each other, and passing through the point of tangency, is the direction of the force of impact. If the centre of gravity of the two bodies is upon this line, the impact is said to be *central;* if not, it is said to be *eccentric.* When the bodies A and B, Fig. 550, collide, the impact is central; for their centres of gravity S_1 and S_2 lie in the normal $N\overline{N}$ to the tangent plane. In the case represented in Fig. 551 the impact of A is central and that of B eccentric; for S_1 lies in and S_2 without the normal line or *line of impact* $N\overline{N}$.

FIG. 550.

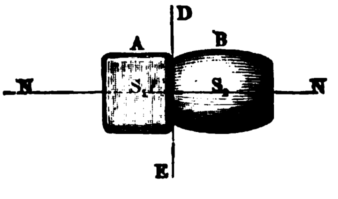
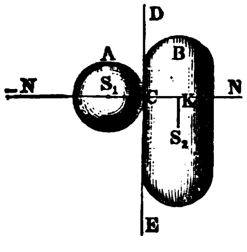
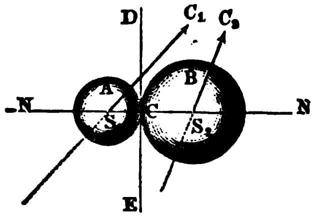


FIG. 551.



When we consider the direction of motion, we distinguish *direct impact* (Fr. choc direct, Ger. gerader Stoss) and *oblique impact* (Fr. choc oblique, Ger. shiefer Stoss). In direct impact the line of impact coincides with the direction of motion; in oblique impact the two directions diverge from each other. If the two bodies A and B, Fig. 552, move in the directions $S_1\,C_1$ and $S_2\,C_2$, which diverge from the line of impact $N\,N$, the impact which takes place is oblique, while, on the contrary, it would have been direct if the directions of motion had coincided with $N\,\overline{N}$.

FIG. 552.

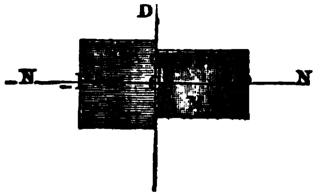


We distinguish, also, *the impact of free bodies from that of those partially or entirely retained.*

§ **331.** The time during which motion is imparted to a body or a change in its motion is produced is, it is true, very small, but by no means infinitely so; it depends not only upon the force of impact, but also upon the mass, velocity and elasticity of the colliding bodies. We can assume this time to consist of two parts. In the first period the bodies compress each other, and in the second they expand again, either totally or partially. The elasticity of the body, which is brought into action by the compression, puts itself into equilibrium with the inertia, and thus changes the condition of motion of the body. If during the compression the limit of elasticity is not surpassed, the body returns to exactly its former shape, and it is said to be *perfectly elastic;* but if the body, after the impact, only partially resumes its original form, we say it is *imperfectly elastic;* and if, finally, the body retains the shape it assumed under the maximum of compression or possesses no tendency to re-expand, we say that the body is *inelastic.* This classification of impact is correct within certain limits only; for it is possible that the same body will act as an elastic one when the impact is slight, and as an inelastic one when the impact is violent. Strictly speaking, perfectly elastic and perfectly inelastic bodies have no existence; but we will hereafter consider elastic bodies to be those which apparently resume their original form, and inelastic bodies to be those which undergo a considerable change of form in consequence of the impact.

In practical mechanics the bodies, such as wood, iron, etc., which are subjected to impact, are very often regarded as inelastic, because they either possess but little elasticity or lose the greater part of their elasticity in consequence of the repetition of the impact. It is very important in constructing machinery, etc., to avoid impacts as much as possible. If this cannot be done, we should diminish their intensity or change them into elastic ones; for they give rise to jars or concussions and cause the machinery to wear very fast, and in consequence a portion of the energy of the machine is consumed.

Fig. 553.



§ **332. Central Impact.**—Let us first investigate the laws of the direct central impact of bodies moving freely. Let us suppose the duration of the impact composed of the equal elements τ, and the pressure between the bodies during the first element of time to be $= P_1$, during the second to be $= P_2$, during the third to be $= P_3$, etc. Now if the mass of the body A, Fig. 553, $= M_1$, we have the corresponding accelerations

$$p_1 = \frac{P_1}{M_1}, p_2 = \frac{P_2}{M_1}$$

$$p_3 = \frac{P_3}{M_1}, \text{etc.}$$

But, according to § 19, the variation in velocity corresponding to p and to an element of the time τ is

$$\kappa = p\,t;$$

hence the elementary increments and diminutions of velocity in the foregoing case are

$$\kappa_1 = \frac{P_1\,\tau}{M_1}, \kappa_2 = \frac{P_2\,\tau}{M_1}, \kappa_3 = \frac{P_3\,\tau}{M_1}, \text{etc.},$$

and the increase or decrease in velocity of the mass M_1 after a certain time is

$$\kappa_1 + \kappa_2 + \kappa_3 + \ldots = (P_1 + P_2 + P_3 + \ldots)\frac{\tau}{M_1},$$

and the corresponding variation in velocity of the body B, whose mass is M_2, is

$$= (P_1 + P_2 + P_3 + \ldots)\frac{\tau}{M_2}.$$

The pressure acts in the following or impinging body in opposition to the velocity c, producing a diminution of velocity, and after a certain time the velocity, which the body still possesses, is

$$v_1 = c_1 - (P_1 + P_2 + \ldots)\frac{\tau}{M_1}.$$

The pressure acts upon the body B, which is in advance and which is impinged upon, in the direction of motion, its velocity c_2 is increased and becomes

$$v_2 = c_2 + (P_1 + P_2 + P_3 + \ldots)\frac{\tau}{M_2}.$$

Eliminating from the two equations $(P_1 + P_2 + P_3 + \ldots)\,\tau$, we have the general formula

$$\text{I. } M_1(c_1 - v_1) = M_2(v_2 - c_2), \text{ or}$$

$$M_1 v_1 + M_2 v_2 = M_1 c_1 + M_2 c_2.$$

The product of the mass of a body and its velocity is called its *momentum* (Fr. quantité de mouvement; Ger. Bewegungsmoment), and we can consequently assert *that at every instant of the impact the sum of the momentums* ($M_1 v_1 + M_2 v_2$) *of the two bodies is the same as before the impact took place.*

At the instant of greatest compression, the two bodies have the same velocity v, hence if we substitute this value v for v_1 and v_2 in the formula just found, we obtain

$$M_1 v + M_2 v = M_1 c_1 + M_2 c_2,$$

from which we deduce the velocity of the bodies at the moment of *greatest compression*

$$v = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2}.$$

If the bodies A and B are *inelastic*, I.E. if after compression they have no tendency to expand, all imparting or changing of motion ceases, when the bodies have been subjected to the maximum compression, and they then move on with the common velocity

$$v = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2}.$$

EXAMPLE—1) If an inelastic body B weighing 30 pounds is moving with a velocity of 3 feet and is impinged upon by another inelastic body A weighing 50 pounds and moving with a velocity of 7 feet, the two move on after the collision with a velocity

$$v = \frac{50\,.\,7 + 30\,.\,3}{50 + 30} = \frac{350 + 90}{80} = \frac{44}{8} = \frac{11}{2} = 5\tfrac{1}{2}\text{ feet.}$$

2) In order to cause a body weighing 120 pounds to change its velocity

from $c = 1\frac{1}{2}$ feet to $v = 2$ feet, we let a body weighing 50 pounds strike it; what velocity must the latter have? Here we have

$$c_1 = v + \frac{(v - c_2)\,M_2}{M_1} = 2 + \frac{(2 - 1{,}5)\,.\,120}{50} = 2 + \frac{6}{5} = 3{,}2 \text{ feet.}$$

§ 333. Elastic Impact.—If the colliding bodies are *perfectly elastic*, they expand gradually during the second period of the impact after having been compressed in the first one, and when they have finally assumed their original form, they continue their motion with different velocities. Since the work done in compressing an elastic body is equal to the energy restored by the body, when it expands again, no loss of vis viva is caused by the impact of elastic bodies. Hence we have for the vis viva the following equation

$$\text{II. } M_1\,v_1^2 + M_2\,v_2^2 = M_1\,c_1^2 + M_2\,c_2^2, \text{ or}$$

$$M_1\,(c_1^2 - v_1^2) = M_2\,(v_2^2 - c_2^2).$$

From equations I. and II. the velocities v_1 and v_2 of the bodies after the impact can be found. First by division we have

$$\frac{c_1^2 - v_1^2}{c_1 - v_1} = \frac{v_2^2 - c_2^2}{v_2 - c_2},$$

I.E.,

$$c_1 + v_1 = v_2 + c_2, \text{ or } v_2 - v_1 = c_1 - c_2;$$

substituting the value

$$v_2 = c_1 + v_1 - c_2,$$

deduced from the last equation, in equation I., we have

$$M_1\,v_1 + M_2\,v_1 + M_2\,(c_1 - c_2) = M_1\,c_1 + M_2\,c_2, \text{ or}$$

$$(M_1 + M_2)\,v_1 = (M_1 + M_2)\,c_1 - 2\,M_2\,(c_1 - c_2),$$

whence

$$v_1 = c_1 - \frac{2\,M_2\,(c_1 - c_2)}{M_1 + M_2} \text{ and}$$

$$v_2 = c_1 - c_2 + c_1 - \frac{2\,M_2\,(c_1 - c_2)}{M_1 + M_2} = c_2 + \frac{2\,M_1\,(c_1 - c_2)}{M_1 + M_2}.$$

Hence if the bodies are inelastic, the *loss of velocity of one body* is

$$c_1 - v = c_1 - \frac{M_1\,c_1 + M_2\,c_2}{M_1 + M_2} = \frac{M_2\,(c_1 - c_2)}{M_1 + M_2},$$

and when they are elastic, it is double that amount, or

$$c_1 - v_1 = \frac{2\,M_2\,(c_1 - c_2)}{M_1 + M_2},$$

and while for inelastic bodies we have the gain in velocity of the other body

$$v - c_2 = \frac{M_1\,c_1 + M_2\,c_2}{M_1 + M_2} - c_2 = \frac{M_1\,(c_1 - c_2)}{M_1 + M_2},$$

for elastic bodies it is

$$v_2 - c_2 = \frac{2\,M_1\,(c_1 - c_2)}{M_1 + M_2},$$

or *double as much.*

EXAMPLE.—Two perfectly elastic balls, one weighing 10 pounds and the other 16 pounds, collide with the velocities 12 and 6 feet. What are their velocities after the impact? Here $M_1 = 10$, $c_1 = 12$, $M_2 = 16$ and $c_2 = -6$ feet, and the loss of velocity of the first body is

$$c_1 - v_1 = \frac{2\,.\,16\,(12 + 6)}{10 + 16} = \frac{2\,.\,16\,.\,18}{26} = 22{,}154 \text{ feet.}$$

and the increase of the velocity of the other is

$$v_2 - c_2 = \frac{2\,.\,10\,.\,18}{26} = 13{,}846 \text{ feet.}$$

The first body, therefore, rebounds after the collision with the velocity $v_1 = 12 - 22{,}154 = -10{,}154$ feet, and the other with the velocity $v_2 = -6 + 13{,}846 = 7{,}846$ feet. The vis viva of these bodies after the impact is $= M_1\,v_1^2 + M_2\,v_2^2 = 10\,.\,10{,}154^2 + 16\,.\,7{,}846^2 = 1031 + 985 = 2016$ or the same as that before impact $M_1\,c_1 + M_2\,c_2 = 10\,.\,12^2 + 16\,.\,6^2 = 1440 + 576 = 2016$.

If the bodies were inelastic, the first body would lose but $\frac{c_1 - v_1}{2} = 11{,}077$ feet of its velocity and the other would gain $\frac{v_2 - c_2}{2} = 6{,}923$ feet; the velocity of the first body after the impact would be $12 - 11{,}077 = 0{,}923$ feet, and that of the second $-6 + 6{,}923 = 0{,}923$; a loss of mechanical effect
$[2016 - (10 + 16)\,0{,}923^2] : 2\,g = (2016 - 22{,}2)\,.\,0{,}0155 = 30{,}9$ foot-pounds, however, takes place.

§ 334. Particular Cases.—The formulas found in the foregoing paragraph for the final velocities of impact are of course applicable, when one of the bodies is at rest, or when the two bodies move in opposite directions and towards each other, or when the mass of one of the bodies is infinitely great compared to that of the other, etc. If the mass M_2 is *at rest*, we have $c_2 = 0$ and therefore for *inelastic* bodies

$$v = \frac{M_1 c_1}{M_1 + M_2},$$

and for *elastic ones*

$$v_1 = c_1 - \frac{2 M_2 c_1}{M_1 + M_2} = \frac{M_1 - M_2}{M_1 + M_2} c_1, \text{ and}$$

$$v_2 = 0 + \frac{2 M_1 c_1}{M_1 + M_2} = \frac{2 M_1}{M_1 + M_2} c_1.$$

If the bodies move *towards each other*, c_2 is negative, and therefore for inelastic bodies

$$v = \frac{M_1 c_1 - M_2 c_2}{M_1 + M_2}, \text{ and for elastic ones}$$

$$v_1 = c_1 - \frac{2 M_2 (c_1 + c_2)}{M_1 + M_2} \text{ and } v_2 = - c_2 + \frac{2 M_1 (c_1 + c_2)}{M_1 + M_2}.$$

If in this case the momenta of the bodies are equal, or $M_1 c_1 = M_2 c_2$, when the bodies are inelastic, $v = 0$, I.E., the bodies bring each other to rest, but if they are elastic,

$$v_1 = c_1 - \frac{2 (M_2 c_1 + M_1 c_1)}{M_1 + M_2} = c_1 - 2 c_1 = - c_1, \text{ and}$$

$$v_2 = - c_2 + \frac{2 (M_2 c_2 + M_1 c_2)}{M_1 + M_2} = - c_2 + 2 c_2 = + c_2;$$

the bodies after the impact proceed in the opposite direction with the same velocity they originally had. If, on the contrary, the masses are equal, we have for inelastic bodies

$$v = \frac{c_1 - c_2}{2},$$

and for elastic ones

$$v_1 = - c_2 \text{ and } v_2 = c_1,$$

I.E., each body returns with the same velocity that the other body had before the impact. If the bodies move in the same direction; and if the one in advance is *infinitely great*, we have for inelastic bodies

$$v = \frac{M_2 c_2}{M_2} = c_2,$$

and for elastic ones

$$v_1 = c_1 - 2 (c_1 - c_2) = 2 c_2 - c_1, \; v_2 = c_2 + 0 = c_2;$$

the velocity of the infinitely great body is not changed by the impact. If the infinitely great body is at rest, or if $c_2 = 0$, we have for inelastic bodies

$$v = 0,$$

and for elastic ones

$$v_1 = - c_1, \; v_2 = 0;$$

here the infinitely great body remains at rest; but in the first case the impinging body loses its velocity completely, and in the second case it is transformed into an equal opposite one.

EXAMPLE—1) With what velocity must a body weighing 8 pounds strike a body weighing 25 pounds in order to communicate to the latter a velocity of 2 feet? If the bodies are inelastic, we must put

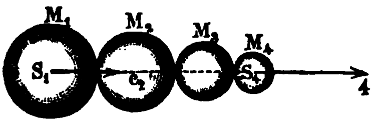
$$v = \frac{M_1 c_1}{M_1 + M_2},\ \text{I.E. } 2 = \frac{8 \,.\, c_1}{8 + 25},$$

whence we obtain $c_1 = \frac{33}{4} = 8\frac{1}{4}$ feet, which is the required velocity; if they were elastic, we would have

$$v_2 = \frac{2\, M_1\, c_1}{M_1 + M_2},\ \text{whence } c_1 = \frac{33}{8} = 4\frac{1}{8} \text{ feet.}$$

2) If a ball M_1, Fig. 554, strikes with the velocity c_1 the mass $M_2 = n\, M_1$, which is at rest, if the second mass strikes a third $M_3 = n\, M_2 = n^2\, M_1$, with the velocity imparted to it by the impact, and if this third mass strikes a fourth $M_4 = n\, M_3 = n^3\, M_1$, etc., we have, when these masses are perfectly elastic, the velocities

FIG. 554.



$$v_2 = \frac{2\, M_1}{M_1 + n\, M_1} c_1 = \frac{2}{1+n} \,.\, c_1,\ v_3 = \frac{2\, M_2}{M_2 + n\, M_2} v_2 = \frac{2}{1+n} \,.\, v_2 = \left(\frac{2}{1+n}\right)^2 c_1,\ v_4 = \left(\frac{2}{1+n}\right)^3 c_1.$$

If, for example, the weight of each mass is one-half that of the preceding one, we have the ratio of the geometrical series formed by the masses

$$n = \frac{1}{2},$$

hence

$$v_2 = \frac{4}{3} c_1,\ v_3 = \left(\frac{4}{3}\right)^2 c_1,\ v_4 = \left(\frac{4}{3}\right)^3 c_1 \ldots,\ v_{10} = \left(\frac{4}{3}\right)^9 c_1 = 13{,}32 \,.\, c_1.$$

§ 335. Loss of Energy.—When two inelastic bodies collide, *a loss of vis viva* always takes place, and therefore they do not possess so much energy after the impact as before. Before the impact the vis viva of the masses M_1 and M_2, which move with the velocities c_1 and c_2, is

$$M_1 c_1^2 + M_2 c_2^2,$$

but after the impact they move with the velocity

$$v = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2} \text{ and}$$

their vis viva is

$$M_1 v^2 + M_2 v^2;$$

by subtraction we obtain *the loss of vis viva* caused by the impact

$$K = M_1 (c_1^2 - v^2) + M_2 (c_2^2 - v^2)$$
$$= M_1 (c_1 + v)(c_1 - v) - M_2 (c_2 + v)(v - c_2), \text{ but}$$
$$M_1 (c_1 - v) = M_2 (v - c_2) = \frac{M_1 \, M_2 (c_1 - c_2)}{M_1 + M_2},$$

whence

$$K = (c_1 + v - c_2 - v) \frac{M_1 M_2 (c_1 - c_2)}{M_1 + M_2} = \frac{(c_1 - c_2)^2 M_1 M_2}{M_1 + M_2} = \frac{(c_1 - c_2)^2}{\frac{1}{M_1} + \frac{1}{M_2}}.$$

If the weights of the bodies are G_1 and G_2, or if

$$M_1 = \frac{G_1}{g} \text{ and } M_2 = \frac{G_2}{g},$$

we have the loss of energy or the work done

$$A = \frac{(c_1 - c_2)^2}{2g} \cdot \frac{G_1 \, G_2}{G_1 + G_2}.$$

We call $\frac{G_1 \, G_2}{G_1 + G_2}$ the *harmonic mean* between G_1 and G_2, and we can assert *that the loss of energy, caused by the impact of two inelastic bodies and expended in changing their form, is equal to the product of harmonic mean of the two masses and the height due to the difference of their velocities.*

If one of the masses M_2 is at rest, we have the loss of mechanical effect

$$A = \frac{c_1^2}{2g} \cdot \frac{G_1 \, G_2}{G_1 + G_2},$$

and if the moving mass M_1 is very great, compared to the mass at rest, G_2 disappears before G_1 and the formula becomes

$$A = \frac{c_1^2}{2g} \cdot G_2.$$

We can also put

$$K = M_1 (c_1^2 - v^2) + M_2 (c_2^2 - v^2)$$
$$= M_1 (c_1^2 - 2 c_1 v + v^2 + 2 c_1 v - 2 v^2) + M_2 (c_2^2 - 2 c_2 v + v^2 + 2 c_2 v - 2 v_2^2)$$
$$= M_1 (c_1 - v)^2 + 2 M_1 v (c_1 - v) + M_2 (c_2 - v)^2 + 2 M_2 v (c_2 - v)$$
$$= M_1 (c_1 - v)^2 + M_2 (c_2 - v)^2;$$

for $M_1 (c_1 - v) = M_2 (v - c_2)$.

From this we see *that the vis viva lost by the inelastic impact is* *equal to the sum of the products of the masses and the squares of their gain or loss of velocity.*

Example—1) If in a machine 16 impacts per minute take place between the masses

$$M_1 = \frac{1000}{g} \text{ lbs. and } M_2 = \frac{1200}{g} \text{ lbs.,}$$

whose velocities are $c_1 = 5$ feet and $c_2 = 2$ feet, the loss of energy, in consequence of these impacts, is

$$A = \tfrac{16}{60} \cdot \frac{(5-2)^2}{2\,g} \cdot \frac{1000\ .\ 1200}{2200} = \tfrac{4}{15} \cdot 9 \cdot 0{,}0155\ \tfrac{6000}{11} = 20{,}29 \text{ foot-lbs.}$$

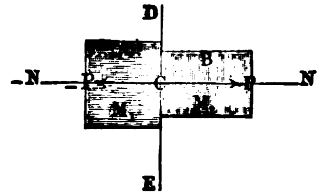
per second.

2) If two trains of cars, weighing 120000 and 160000 pounds, come into collision upon a railroad when their velocities are $c_1 = 20$ and $c_2 = 15$ feet, a loss of mechanical effect, which is expended in destroying the locomotives and cars, ensues; its value is

$$= \left(\frac{20+15}{2\,g}\right)^2 \cdot \frac{120000\ .\ 160000}{280000} = 35^2 \cdot 0{,}0155 \cdot \frac{1920000}{28} = 1302000 \text{ foot-lbs.}$$

§ 336. **Hardness.**—If we know the modulus of elasticity of the colliding bodies, we can find also the *compressive force* and the amount of compression. Let the cross-section of the bodies A and B, Fig. 555, be F_1 and F_2, their length l_1 and l_2, and their moduli of elasticity be E_1 and E_2. If they impinge upon one another, the compressions produced are, according to § 204,

FIG. 555.



$$\lambda_1 = \frac{P\,l_1}{F_1\,E_1} \text{ and } \lambda_2 = \frac{P\,l_2}{F_2\,E_2},$$

and their ratio is

$$\frac{\lambda_1}{\lambda_2} = \frac{F_2\,E_2}{F_1\,E_1} \cdot \frac{l_1}{l_2}.$$

If, for the sake of simplicity, we denote $\frac{F_1\,E_1}{l_1}$ by H_1 and $\frac{F_2\,E_2}{l_2}$ by H_2, we obtain

$$\lambda_1 = \frac{P}{H_1} \text{ and } \lambda_2 = \frac{P}{H_2},$$

and

$$\frac{\lambda_1}{\lambda_2} = \frac{H_2}{H_1}.$$

Calling, with Whewell (see the Mechanics of Engineering, § 207), the quantity $\frac{F\,E}{l}$ the hardness (Fr. dureté raideur, Ger.

Härte) of a body, it follows *that the depth of compression is inversely proportional to the hardness.*

If the mass $M = \frac{G}{g}$ impinges with the velocity c upon an immovable or infinitely great mass, all its vis viva is expended in compressing the latter body, whence, according to § 206,

$$\tfrac{1}{2} P \sigma = \frac{M c^2}{2} = \frac{c^2}{2 g} G.$$

But the space σ is equal to the sum of the compressions λ_1 and λ_2, and we have $\lambda_1 = \frac{P}{H_1}$ and $\lambda_2 = \frac{P}{H_2}$, whence

$$\sigma = \lambda_1 + \lambda_2 = P\left(\frac{1}{H_1} + \frac{1}{H_2}\right) = \frac{H_1 + H_2}{H_1 H_2} \cdot P,$$

or inversely

$$P = \frac{H_1 H_2}{H_1 + H_2} \sigma.$$

Substituting this value of P in the above equation, we obtain the equation of condition

$$\tfrac{1}{2} \cdot \frac{H_1 H_2}{H_1 + H_2} \cdot \sigma^2 = \frac{c^2}{2 g} G,$$

or

$$\sigma = c \sqrt{\frac{H_1 + H_2}{H_1 H_2} \cdot \frac{G}{g}},$$

by the aid of which the values P, λ_1 and λ_2 can be calculated.

Example.—If with a sledge, that weighs 50 pounds and is 6 inches long and the area of whose face is 4 square inches, we strike a lead plate one inch thick, and the area of whose cross-section is 2 square inches, with a velocity of 50 feet, the effect can be discussed as follows. Assuming $E_1 = 29000000$ as the modulus of elasticity of iron and $E_2 = 700000$ as that of lead, we find the hardness of the two bodies to be

$$H_1 = \frac{F_1 E_1}{l_1} = \frac{4 \cdot 29000000}{6} = 19333333 \text{ and}$$

$$H_2 = \frac{F_2 E_2}{l_2} = \frac{2 \cdot 700000}{1} = 1400000.$$

Substituting these values in the formula

$$\sigma = c \sqrt{\frac{H_1 + H_2}{H_1 H_2} \cdot \frac{G}{g}},$$

and putting the weight of the sledge = 4 . 6 . 0,29 = 7 pounds, or

$$\frac{G}{g} = 7 \cdot 0{,}031 = 0{,}217,$$

we have for the space described by the sledge in compressing the lead

$$\sigma = 50 \sqrt{\frac{20733333 \cdot 0{,}217}{19333333 \cdot 1400000}} = 50 \sqrt{\frac{0.44991}{2706666}} = 0{,}0204 \text{ inches} = 0{,}245 \text{ lines.}$$

Hence the pressure is

$$P = \frac{H_1 H_2}{H_1 + H_2} \cdot \sigma = \frac{19333333 \, . \, 1400000}{20733333} \, . \, 0{,}0204 = 26682 \text{ pounds};$$

the compression of the hammer is

$$\lambda_1 = \frac{P}{H_1} = \frac{26682}{19333333} = 0{,}0014 \text{ inches} = 0{,}016 \text{ lines},$$

and that of the lead

$$\lambda_2 = \frac{P}{H_2} = \frac{26682}{1400000} = 0{,}019 \text{ inches} = 0{,}228 \text{ lines}.$$

§ 337. **Elastic-inelastic Impact.**—If two masses M_1 and M_2 are moving with the velocities c_1 and c_2 in the same direction, their common velocity at the moment of maximum compression is, according to § 332,

$$v = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2},$$

and the work done during the compression, according to § 335, is

$$A = \frac{(c_1 - c_2)^2}{2} \cdot \frac{M_1 M_2}{M_1 + M_2} = \frac{(c_1 - c_2)^2}{2 g} \cdot \frac{G_1 G_2}{G_1 + G_2};$$

but this mechanical effect can be put

$$= \tfrac{1}{2} P \sigma = \tfrac{1}{2} P (\lambda_1 + \lambda_2) = \tfrac{1}{2} \cdot \frac{H_1 H_2}{H_1 + H_2} \sigma^2,$$

whence we obtain for the sum of the compressions of the two masses

$$\sigma = (c_1 - c_2) \sqrt{\frac{G_1 G_2}{g (G_1 + G_2)} \cdot \frac{H_1 + H_2}{H_1 H_2}},$$

from which the compressive force P and the compressions λ_1 and λ_2 of the two masses can be found.

If the bodies are inelastic, they remain compressed after the impact; but if one only is inelastic, the other resumes its original form in a second period, and the work done in expanding produces another change of velocity. If, for example, the mass $M_1 = \frac{G_1}{g}$ is elastic, the work done in the second period of the impact is

$$\tfrac{1}{2} P \lambda_1 = \tfrac{1}{2} \cdot \frac{P^2}{H_1} = \frac{1}{2 H_1} \left(\frac{H_1 H_2}{H_1 + H_2}\right)^2 \sigma^2$$

$$= \frac{(c_1 - c_2)^2}{2 g} \cdot \frac{G_1 G_2}{G_1 + G_2} \cdot \frac{H_2}{H_1 + H_2}.$$

We have, therefore, when the velocities after the impact are v_1 and v_2, the formulas

$$M_1 v_1 + M_2 v_2 = M_1 c_1 + M_2 c_2 \text{ and}$$

$$M_1 v_1^2 + M_2 v_2^2 = M_1 c_1^2 + M_2 c_2^2 + (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_2}{H_1 + H_2}$$

$$= M_1 c_1^2 + M_2 c_2^2 - (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} + (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_2}{H_1 + H_2},$$

i.e.

$$M_1 v_1^2 + M_2 v_2^2 = M_1 c_1^2 + M_2 c_2^2 - (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_1}{H_1 + H_2}.$$

If we put the loss of velocity $c_1 - v_1 = x$, we have the gain in velocity

$$v_2 - c_2 = \frac{M_1 x}{M_2},$$

and the last equation assumes the following form:

$$x(2 c_1 - x) - x\left(2 c_2 + \frac{M_1 x}{M_2}\right) - (c_1 - c_2)^2 \frac{M_2}{M_1 + M_2} \cdot \frac{H_1}{H_1 + H_2} = 0,$$

or

$$\frac{M_1 + M_2}{M_2} x^2 - 2(c_1 - c_2) x + (c_1 - c_2)^2 \cdot \frac{M_2}{M_1 + M_2} \cdot \frac{H_1}{H_1 + H_2} = 0.$$

Multiplying by $\frac{M_1}{M_1 + M_2}$ and remembering that

$$\frac{H_1}{H_1 + H_2} = 1 - \frac{H_2}{H_1 + H_2},$$

we obtain the quadratic equation

$$x^2 - 2(c_1 - c_2)\frac{M_2}{M_1 + M_2} x + (c_1 - c_2)^2 \left(\frac{M_2}{M_1 + M_2}\right)^2$$
$$= (c_1 - c_2)^2 \left(\frac{M_2}{M_1 + M_2}\right)^2 \frac{H_2}{H_1 + H_2}$$

or

$$\left(x - (c_1 - c_2)\frac{M_2}{M_1 + M_2}\right)^2 = (c_1 - c_2)^2 \left(\frac{M_2}{M_1 + M_2}\right)^2 \cdot \frac{H_2}{H_1 + H_2},$$

by resolving which we obtain the *loss of velocity* x of the first body

$$c_1 - v_1 = (c_1 - c_2)\frac{M_2}{M_1 + M_2}\left(1 + \sqrt{\frac{H_2}{H_1 + H_2}}\right),$$

and the *gain of velocity* of the other

$$v_2 - c_2 = (c_1 - c_2)\frac{M_1}{M_1 + M_2}\left(1 + \sqrt{\frac{H_2}{H_1 + H_2}}\right).$$

EXAMPLE.—If we assume that in the example of the foregoing paragraph the iron sledge is perfectly elastic and that the lead plate is perfectly inelastic, we obtain the loss of velocity of the hammer, which weighs 7 pounds and falls with the velocity of 50 feet, since we must put $c_2 = 0$ and $M_2 = \infty$,

$$c_1 - v_1 = c_1 \left(1 + \sqrt{\frac{H_2}{H_1 + H_2}}\right) = 50 \left(1 + \sqrt{\frac{1400000}{20783333}}\right)$$

$$= 50\,(1 + 0{,}26) = 63 \text{ feet};$$

hence the velocity of the sledge after the blow is

$$v_1 = c_1 - 63 = 50 - 63 = -13 \text{ feet.}$$

The velocity of the lead plate, which is retained, of course remains = 0.

§ 338. Imperfectly Elastic Impact.—If the colliding bodies are *imperfectly elastic,* they expand only partially in the second period of the impact and the mechanical effect expended in producing the compression in the first period is not entirely restored in the second period. If λ_1 and λ_2 again denote the amount of compression and P the pressure (called also *the force of distorsion*), we have the mechanical effects expended during the compression $= \frac{1}{2} P \lambda_1$ and $\frac{1}{2} P \lambda_2$, and if during the expansion but the μth part or more generally during the expansion of the first body but the μ_1th and during that of the other but the μ_2th part of the mechanical effect is restored, the entire loss of mechanical effect is

$$A = \tfrac{1}{2} P \left[(1 - \mu_1)\,\lambda_1 + (1 - \mu_2)\,\lambda_2\right],$$

or, putting $\lambda_1 = \frac{P}{H_1}$ and $\lambda_2 = \frac{P}{H_2}$,

$$A = \tfrac{1}{2} P^2 \left[\frac{1 - \mu_1}{H_1} + \frac{1 - \mu_2}{H_2}\right].$$

The force with which the bodies react in the second period is called the *force of restitution.*

But according to the foregoing paragraph we have

$$P = \frac{H_1 H_2 \sigma}{H_1 + H_2} \text{ and } \sigma = (c_1 - c_2) \sqrt{\frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_1 + H_2}{H_1 H_2}},$$

hence the required loss of mechanical effect is

$$A = \frac{(c_1 - c_2)^2}{2} \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_1 H_2}{H_1 + H_2} \left(\frac{1 - \mu_1}{H_1} + \frac{1 - \mu_2}{H_2}\right)$$

$$= \frac{(c_1 - c_2)^2}{2} \cdot \frac{M_1 M_2}{M_1 + M_2} \left(1 - \frac{\mu_1 H_2 + \mu_2 H_1}{H_1 + H_2}\right).$$

To find the velocities v_1 and v_2 after the impact, we employ the equations

$$M_1 v_1 + M_2 v_2 = M_1 c_1 + M_2 c_2 \text{ and}$$

$$M_1 v_1^2 + M_2 v_2^2 = M_1 c_1^2 + M_2 c_2^2$$

$$- (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{(1 - \mu_1) H_2 + (1 - \mu_2) H_1}{H_1 + H_2},$$

which we must combine and resolve. In exactly the same manner as in the last paragraph the *loss of velocity* of the first body is found to be

$$c_1 - v_1 = (c_1 - c_2)\frac{M_2}{M_1 + M_2}\left(1 + \sqrt{\frac{\mu_2 H_1 + \mu_1 H_2}{H_1 + H_2}}\right),$$

and *the gain in velocity of the body, which is in advance,*

$$v_2 - c_2 = (c_1 - c_2)\frac{M_1}{M_1 + M_2}\left(1 + \sqrt{\frac{\mu_2 H_1 + \mu_1 H_2}{H_1 + H_2}}\right).$$

These two formulas include also the laws of perfectly elastic and of inelastic impact. If we substitute in them $\mu_1 = \mu_2 = 1$, we obtain the formula already found for perfectly elastic bodies, and if we assume $\mu_1 = \mu_2 = 0$, we obtain the formulas for inelastic impact, etc. If both bodies are equally elastic, or $\mu_1 = \mu_2$, we have more simply

$$c_1 - v_1 = (c_1 - c_2)\frac{M_2}{M_1 + M_2}(1 + \sqrt{\mu})$$

and

$$v_2 - c_2 = (c_1 - c_2)\frac{M_1}{M_1 + M_2}(1 + \sqrt{\mu}).$$

If the mass M_2 is at rest and infinitely great, it follows that

$$c_1 - v_1 = c_1(1 + \sqrt{\mu}),\ \text{I.E.},$$

$$v_1 = -c_1\sqrt{\mu},\ \text{or inversely}$$

$$\mu = \left(\frac{v_1}{c_1}\right)^2$$

If we cause a mass M_1 to fall from a height h upon a rigidly supported mass M_2, and if it bounces back to a height h_1, we can determine the coefficient of imperfect elasticity of the body by the formula

$$\mu = \frac{h_1}{h}.$$

Newton found in this way for ivory,

$$\mu = (\tfrac{8}{9})^2 = \tfrac{64}{81} = 0{,}79,$$

for glass

$$\mu = (\tfrac{15}{16})^2 = 0{,}9375^2 = 0{,}879,$$

and for cork, steel and wool

$$\mu = (\tfrac{5}{9})^2 = 0{,}555^2 = 0{,}309.$$

We assume, in this case, that the falling body is a sphere and that the body upon which it falls is flat.

General Morin by causing cannon balls, weighing from 6 to 20 kilograms, to fall upon masses of *clay*, *wood* and *cast-iron*, which were suspended from a *spring balance* or *spring dynamometer* found that for clay and wood μ is nearly $= 0$, and that, on the contrary, for cast-iron it is nearly $= 1$, I.E. that the impact of bodies of former substances can be considered as inelastic and that of those of the latter substances as perfectly elastic (see A. Morin, Notions fondamentales de Mécanique, Art. 67–70).

EXAMPLE.—What will be the velocities of two steel plates after impact, if before the impact their velocities were $c_1 = 10$ and $c_1 = -6$ feet, and if one weighs 30 and the other 40 pounds? Here we have

$$c_1 - v_1 = (10 + 6) \,.\, \frac{40}{70} \left(1 + \frac{5}{9}\right) = 16 \,.\, \frac{4}{7} \,.\, \frac{14}{9} = \frac{16 \,.\, 8}{9} = 14{,}22 \text{ feet},$$

hence the required velocities are

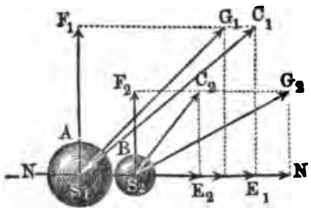
$$v_1 = c_1 - 14{,}22 = 10 - 14{,}22 = -4{,}22 \text{ feet}$$

and

$$v_2 = c_2 + 10{,}66 = -6 + 10{,}66 = 4{,}66 \text{ feet.}$$

§ 339. Oblique Impact.—If the directions of motion $S_1 C_1$ and $S_2 C_2$ of the two bodies A and B, Fig. 556, diverge from the normal $N N$ to the tangent plane, an *oblique* impact takes place. The theory of oblique impact can be referred to that of direct impact by decomposing the velocities $S_1 C_1 = c_1$ and $S_2 C_2 = c_2$ into their components in the direction of the normal and tangent; the components in the direction of the normal produce a direct impact, and are, therefore, changed exactly as in the case of direct impact, while the velocities parallel to the tangent plane cause no impact, and, therefore, remain unchanged. If we combine the normal velocity of any body, obtained according to the rules for direct impact, with the tangential velocity, which has remained unchanged, the resultant is the velocity of the body after the impact. Putting the angles formed by the directions of motion with the normal equal to a_1 and a_2, or $C_1 S_1 N = a_1$ and $C_2 S_2 N = a_2$, we obtain for the normal velocities $S_1 E_1$ and $S_2 E_2$ the values $c_1 \cos. a_1$ and $c_2 \cos. a_2$ and, on the contrary, for the tangential velocities $S_1 F_1$ and $S_2 F_2$ the values $c_1 \sin. a_1$ and $c_2 \sin. a_2$.

FIG. 556.



The normal velocities are changed by the collision, the first one becoming

$$v_1 = c_1 \cos. a_1 - (c_1 \cos. a_1 - c_2 \cos. a_2) \frac{M_2}{M_1 + M_2} (1 + \sqrt{\mu}),$$

and the second

$$v_2 = c_2 \cos. a_2 + (c_1 \cos. a_1 - c_2 \cos. a_2) \frac{M_1}{M_1 + M_2} (1 + \sqrt{\mu}),$$

in which M_1 and M_2 denote the masses of the two bodies.

From v_1 and c_1 *sin.* a_1 we obtain the velocity $S_1 G_1$ of the first body after the impact

$$w_1 = \sqrt{v_1^2 + c_1^2 \, sin.^2 \, a_1},$$

and from v_2 and c_2 *sin.* a_2 the velocity $S_2 G_2$ of the second body

$$w_2 = \sqrt{v_2^2 + c_2^2 \, sin.^2 \, a_2};$$

the angles formed by the directions of the velocities with the normal are given by the formulas

$$tang. \, \beta_1 = \frac{c_1 \, sin. \, a_1}{v_1} \text{ and } tang. \, \beta_2 = \frac{c_2 \, sin. \, a_2}{v_2},$$

β_1 denoting the angle $G_1 S_1 N$ and β_2 the angle $G_2 S_2 N$.

EXAMPLE—1) Two balls, weighing 30 and 50 pounds, strike each other with the velocities $c_1 = 20$ and $c_2 = 25$ feet, whose directions form the angles $a_1 = 21° \, 35'$ and $a_2 = 65° \, 20'$ with the direction of the normal to the tangent plane; in what direction and with what velocity will these bodies move after the impact? The constant components are

$$c_1 \, sin. \, a_1 = 20 \, . \, sin. \, 21° \, 35' = 7{,}357 \text{ feet and}$$
$$c_2 \, sin. \, a_2 = 25 \, . \, sin. \, 65° \, 20' = 22{,}719 \text{ feet,}$$

and the variable ones are

$$c_1 \, cos. \, a_1 = 20 \, . \, cos. \, 21° \, 35' = 18{,}598 \text{ feet and}$$
$$c_2 \, cos. \, a_2 = 25 \, . \, cos. \, 65° \, 20' = 10{,}433 \text{ feet.}$$

If the bodies are inelastic, we have $\mu = 0$, and therefore the normal velocities after the impact are

$$v_1 = 18{,}598 - (18{,}598 - 10{,}433) \, . \, \tfrac{50}{80} = 18{,}598 - 5{,}103 = 13{,}495 \text{ feet and}$$
$$v_2 = 10{,}433 + 8{,}165 \, . \, \tfrac{3}{8} = 10{,}433 + 3{,}062 = 13{,}495 \text{ feet.}$$

Hence the resulting velocities are

$$w_1 = \sqrt{13{,}495^2 + 7{,}357^2} = \sqrt{236{,}24} = 15{,}37 \text{ feet and}$$
$$w_2 = \sqrt{13{,}495^2 + 22{,}719^2} = \sqrt{698{,}27} = 26.42 \text{ feet;}$$

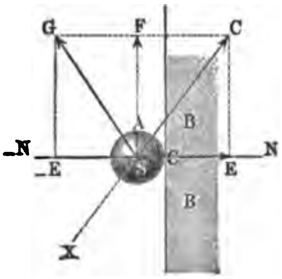
and their directions are determined by the formulas

$$tang. \, \beta_1 = \frac{7{,}357}{13{,}495}, \; log. \, tang. \, \beta_1 = 0{,}73658 - 1, \; \beta_1 = 28° \, 36' \text{ and}$$
$$tang. \, \beta_2 = \frac{22{,}719}{13{,}495}, \; log. \, tang. \, \beta_2 = 0{,}22622, \; \beta_2 = 59° \, 17'.$$

§ 340. Impact against an Infinitely Great Mass.—If the mass A, Fig. 557, strikes against another mass, which is infinitely great, or against an immovable object $B \, B$, or if $c_2 = 0$ and $M_2 = \infty$, we have

$$v_1 = c_1 \, cos. \, a_1 - c_1 \, cos. \, a_1 \, (1 + \sqrt{\mu}) = - \, c_1 \, cos. \, a_1 \sqrt{\mu} \text{ and}$$
$$v_2 = 0 + c_1 \, cos. \, a_1 \, \frac{M_1 \, (1 + \sqrt{\mu})}{\infty} = 0 + 0 = 0,$$

FIG. 557.



if in addition $\mu = 0$, we have $v_1 = 0$, but if $\mu = 1$, $v_1 = - c_1 \cos. a_1$, I.E., when *the impact is inelastic, the normal force is completely annihilated, but, on the contrary, when it is perfectly elastic, the normal force is changed into an equal opposite one.* The angle formed by the direction of motion after the impact with the normal is determined by the equation

$$\tan g. \beta_1 = \frac{c_1 \sin. a_1}{v_1} = - \frac{c_1 \sin. a_1}{c_1 \cos. a_1 \sqrt{\mu}}$$

$$= - \tan g. a_1 \sqrt{\frac{1}{\mu}}.$$

for inelastic bodies

$$\tan g. \beta_1 = \frac{\tan g. a_1}{0} = \infty; \text{ I.E. } \beta_1 = 90°;$$

and for elastic ones

$$\tan g. \beta_1 = - \tan g. a_1, \text{ I.E. } \beta_1 = - a_1.$$

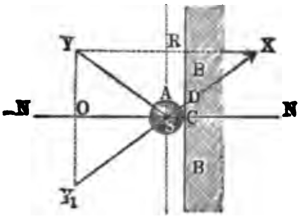
After an inelastic body has impinged upon an inelastic obstacle, it moves on with the velocity $c_1 \sin. a_1$ in the direction of the tangent plane. When an elastic body has impinged upon an elastic obstacle, it moves on with its velocity unchanged in the direction $S\,G$, which lies in the same plane as the normal $N\,\overline{N}$ and the original direction $X\,S$, and makes with the normal the same angle $G\,S\,\overline{N}$ that the direction of motion before the impact made with it. The angle $X\;S\;\overline{N}$, formed by the direction of motion before the impact with the normal or perpendicular, is called the *angle of incidence* (Fr. angle d'incidence; Ger. Einfalls-winkel), and the angle $G\,S\,\overline{N}$, formed by the direction of motion after the impact with the same, is called the *angle of reflexion* (Fr. angle de reflexion; Ger. Austritts- or Reflexionswinkel); *we can therefore assert that when the impact is perfectly elastic, the angles of incidence and of reflexion lie in the same plane as the normal and are equal to each other.*

When the impact is imperfectly elastic, the ratio $\sqrt{\mu}$ of the tangents of these angles is equal to the ratio of the velocity produced by the expansion to the velocity lost by the compression.

By the aid of this law we can easily find the direction in which a body A, Fig. 558, must strike against an immovable obstacle $B\,B$, when we wish it to take a given direction $S\;Y$ after the impact. If the impact is elastic, we let fall from a point Y of the given direction a perpendicular $Y\,O$ upon the normal $N\,\overline{N}$ and prolong it until the prolongation $O\,Y_1$ is equal to the perpendicular itself; $S\,Y_1$ is then the direction in question; for, according to the construction, the angle $\overline{N}\,S\,Y_1 = \overline{N}\,S\,Y$. If the impact is imperfectly elastic, we must make $O\,Y_1 = \sqrt{\mu}\,.\,O\,Y$; then $Y_1\,S$ is the required direction, for

$$\frac{tang.\,\alpha_1}{tang.\,\beta_1} = \frac{O\,Y_1}{O\,Y} = \sqrt{\mu},$$

FIG. 558.



If we let fall the perpendicular $Y\,R$ upon the line $S\,R$ parallel to the tangent plane and make the prolongation $R\,X = \sqrt{\frac{1}{\mu}}\;\overline{R\;Y}$, $S\,X$ will be, as we can easily see, the required direction of incidence.

REMARK.—The principal application of the theory of oblique impact is to the game of billiards. See "Théorie Mathematique des effets du jeu de billard, par Coriolis." According to Coriolis, when a billiard ball strikes the cushion the ratio of the velocity of recoil to the velocity of impact is = 0,5 to 0,6 or μ is $= 0{,}5^2 = 0{,}25$ to $0{,}6^2 = 0{,}36$. By the aid of these values the direction, in which a ball A must strike the cushion $B\,B$ when it is to be thrown back towards a point Y, can be determined. We let fall from Y the perpendicular $Y\,R$ to the line of gravity parallel to the cushion, prolong the same a distance $R\,X = \sqrt{\frac{1}{\mu}} = \frac{10}{5}$ to $\frac{10}{6}$ of its length, and draw the line $Y_1\;X$; the point of intersection D is the point towards which the ball must be driven, when we wish it to rebound towards Y. The twist of the ball causes this relation to vary somewhat.

§ **341. Friction of Impact.**—When oblique impact occurs, the pressure between the colliding bodies gives rise to friction, in consequence of which the components in the direction of the tangent plane are caused to vary. The friction F of impact is determined in the same way as that of pressure. If P denote the pressure of impact and ϕ the coefficient of friction, then $F = \phi\,P$. It differs from the friction of pressure in this only, that, like the impact itself, it acts but for an instant. The changes in velocity produced by it are not, however, immeasurably small; for the pressure P during impact (and therefore the portion ϕP of it) is generally very great. Denoting the impinging mass by M and the normal acceleration produced by the force of impact P by p, we have

$$P = M p \text{ and } F = \phi M p,$$

and also the retardation or negative acceleration of the friction during the impact

$$\frac{F}{M} = \phi p,$$

I.E. ϕ times that of the normal force. Now the duration of the action is the same for both forces; therefore the *change of velocity produced by the friction is ϕ times the change of the normal velocity produced by the impact.*

If a mass M falls vertically upon a horizontal sled, and if the velocity c of this mass is entirely lost by the collision, the retardation of the motion of the sled, whose mass is M_1, is

$$\frac{F}{M + M_1} = \frac{\phi M p}{M + M_1},$$

and consequently the loss of velocity is

$$v = \frac{\phi M}{M + M_1} c.$$

Morin has proved the correctness of this formula by experiment (see his Notions fondamentales de Mécanique).

If a body strikes an immovable mass $B B$ at an angle a, Fig. 559, the change in the normal velocity is, according to the last paragraph,

$$w = c \cos. a \, (1 + \sqrt{\mu});$$

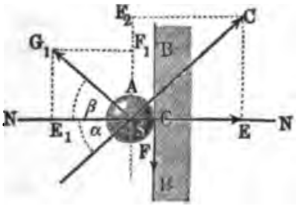
hence the variation produced in the tangential velocity is

$$= \phi w = \phi c \, (1 + \sqrt{\mu}) \cos. a.$$

After the impact the component $c \sin. a$ becomes

$$c \sin. a - \phi c \, (1 + \sqrt{\mu}) \cos. a$$
$$= [\sin. a - \phi \cos. a \, (1 + \sqrt{\mu})] \, c;$$

FIG. 559.



for perfectly elastic bodies it is

$$= (\sin. a - 2 \phi \cos. a) \, c,$$

and, on the contrary, for inelastic bodies it is

$$= (\sin. a - \phi \cos. a) \, c.$$

The friction very often causes the bodies to *turn around their centres of gravity*, or if, before the impact, a motion of rotation exists, it is changed. If the moment of inertia of a round body A in reference to its centre of gravity S is $= M k^2$, and if its radius $S C = a$, we have the mass of the body reduced to the point of tangency C

$$= \frac{M k^2}{a^2},$$

and therefore the acceleration of the rotation produced by the friction F is

$$p_1 = \frac{F}{M k^2 : a^2} = \frac{\phi M p}{M k^2 : a^2} = \phi p \cdot \frac{a^2}{k^2},$$

and the corresponding change of velocity is

$$w_1 = \phi \frac{a^2}{k^2} \cdot w = \phi \frac{a^2}{k^2} (1 + \sqrt{\mu}) \, c \cos. a.$$

For a cylinder $\frac{a^2}{k^2} = 2$, and for a sphere $\frac{a^2}{k^2} = \frac{5}{2}$, therefore, it follows that the changes of velocity of rotation of these round bodies, produced by impact against a plane, are

$$w_1 = 2 \phi (1 + \sqrt{\mu}) \cos. a \text{ and } w = \tfrac{5}{2} \phi (1 + \sqrt{\mu}) \cos. a.$$

EXAMPLE.—If a billiard ball strikes the cushion with a velocity of 15 feet, in such a manner that the angle of incidence $a = 45°$, what will be the conditions of motion after the impact? Putting for $\sqrt{\mu}$ its mean value 0,55, we have the normal component of the velocity after the impact

$$= - \sqrt{\mu} \, . \, c \cos. a = - 0{,}55 \, . \, 15 \, . \cos. 45° = - 8{,}25 \, . \, \sqrt{\tfrac{1}{2}} = - 5{,}833 \text{ feet},$$

and assuming, with Coriolis, $\phi = 0{,}20$, we obtain the component of the velocity parallel to the cushion, which is

$$= c \sin. a - \varphi (1 + \sqrt{\mu}) \, c \cos. a = (1 - 0{,}20 \, . \, 1{,}55) \, . \, 10{,}607 = 0{,}69 \, . \, 10{,}607$$
$$= 7{,}319 \text{ feet},$$

and consequently for the angle of reflection we have

$$tang. \beta = \frac{7{,}319}{5{,}833} = 1{,}2548 \text{ or } \beta = 51° \, 27';$$

hence the velocity after impact is

$$= \frac{5{,}833}{\cos. 51° \, 27'} = 9{,}360 \text{ feet}.$$

The ball also acquires the velocity of rotation

$$\tfrac{5}{7}\,\phi\,.\,1{,}55\,.\,10{,}607 = 8{,}220 \text{ feet}$$

about its vertical line of gravity.

Since the ball does not slide, but rolls upon the billiard table, we must assume that, besides its velocity $c = 15$ feet of translation, it has an equal velocity of rotation, and that this can also be resolved into the components

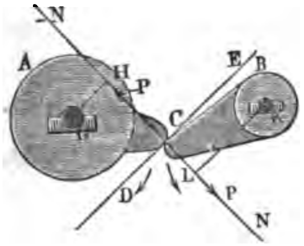
$$c \cos.\, a = 10{,}607 \text{ and } c \sin.\, a = 10{,}607.$$

The first component corresponds to a rotation about an axis parallel to the axis of the cushion, and becomes

$$c \cos.\, a - \tfrac{5}{7}\,\phi\,(1 + \sqrt{\mu})\, c \cos.\, a = 10{,}607 - 8{,}220 = 2{,}387 \text{ feet};$$

the other component $c \sin.\, a = 10{,}607$ feet corresponds to a rotation about an axis normal to the cushion and remains unchanged.

FIG. 560.



§ 342. Impact cf Revolving Bodies.—If *two bodies A* and *B*, Fig. 560, revolving around *the fixed axes G* and *K*, impinge upon one another, changes of velocity take place, which can be determined from the moments of inertia $M_1\, k_1^2$ and $M_2\, k_2^2$ of these bodies in reference to their fixed axes by the aid of the formulas found in the preceding paragraphs. If the perpendiculars $G\,H$ and $K\,L$, let fall from the axes of rotation upon the line of impact, be denoted by a_1 and a_2, we will have the masses reduced to the extremities H and L of these perpendicular to the line of impact $= \frac{M_1\, k_1^2}{a_1^2}$ and $\frac{M_2\, k_2^2}{a_2^2}$, substituting these values for M_1 and M_2 in the formula for central impact, we obtain the variations of velocity of the points H and L (§ 338).

$$c_1 - v_1 = (c_1 - c_2)\,\frac{M_2\, k_2^2 : a_2^2}{M_1\, k_1^2 : a_1^2 + M_2\, k_2^2 : a_2^2}\,(1 + \sqrt{\mu})$$

$$= (c_1 - c_2)\,\frac{M_2\, k_2^2\, a_1^2}{M_1\, k_1^2\, a_2^2 + M_2\, k_2^2\, a_1^2}\,(1 + \sqrt{\mu}) \text{ and}$$

$$v_2 - c_2 = (c_1 - c_2)\,\frac{M_1\, k_1^2 : a_1^2}{M_1\, k_1^2 : a_1^2 + M_2\, k_2^2 : a_2^2}\,(1 + \sqrt{\mu})$$

$$= (c_1 - c_2)\,\frac{M_1\, k_1^2\, a_2^2}{M_1\, k_1^2\, a_2^2 + M_2\, k_2^2\, a_1^2}\,(1 + \sqrt{\mu}),$$

in which c_1 and c_2 denote the velocities of these points before the impact.

To introduce the angular velocities, let us denote the angular velocities before the impact by ε_1 and ε_2 and those after the impact by ω_1 and ω_2, thus we obtain $c_1 = a_1 \varepsilon_1$, $c_2 = a_2 \varepsilon_2$, $v_1 = a_1 \omega_1$ and $v_2 = a_2 \omega_2$, and the loss of velocity of the impinging body is

$$\varepsilon_1 - \omega_1 = a_1 (a_1 \varepsilon_1 - a_2 \varepsilon_2) \frac{M_2 k_2^2}{M_1 k_1^2 a_2^2 + M_2 k_2^2 a_1^2} (1 + \sqrt{\mu}),$$

and the gain in velocity of the impinged body is

$$\omega_2 - \varepsilon_2 = a_2 (a_1 \varepsilon_1 - a_2 \varepsilon_2) \frac{M_1 k_1^2}{M_1 k_1^2 a_2^2 + M_2 k_2^2 a_1^2} (1 + \sqrt{\mu}).$$

The angular velocities after impact are

$$\omega_1 = \varepsilon_1 - a_1 (a_1 \varepsilon_1 - a_2 \varepsilon_2) (1 + \sqrt{\mu}) \frac{M_2 k_2^2}{M_1 k_1^2 a_2^2 + M_2 k_2^2 a_1^2}$$

and

$$\omega_2 = \varepsilon_2 + a_2 (a_1 \varepsilon_1 - a_2 \varepsilon_2) (1 + \sqrt{\mu}) \frac{M_1 k_1^2}{M_1 k_1^2 a_2^2 + M_2 k_2^2 a_1^2}.$$

If both bodies are perfectly elastic, we have $\mu = 1$, or

$$1 + \sqrt{\mu} = 2,$$

and if they are inelastic, $\mu = 0$, or

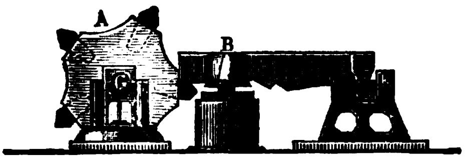
$$1 + \sqrt{\mu} = 1.$$

In the latter case the loss of vis viva occasioned by the impact is

$$= (a_1 \varepsilon_1 - a_2 \varepsilon_2)^2 \cdot \frac{M_1 k_1^2 \,.\, M_2 k_2^2}{M_1 k_1^2 a_2^2 + M_2 k_2^2 a_1^2}.$$

EXAMPLE.—The moment of inertia of the shaft $A\,G$, Fig. 561, in reference to its axis of rotation, G is

FIG. 561.



$= M_1 k_1^2 = 40000 : g$, and that of the tilt hammer $B\,K$ in reference to its axis K is $= 150000 : g$, the arm $G\,C$ of the shaft is two feet and that $K\,C$ of the hammer is 6 feet, and the angular velocity of the shaft at the moment it impinges upon the hammer is = 1,05 feet; how great is the velocity after the impact and how much mechanical effect is lost by each blow, supposing both bodies to be completely inelastic? The required angular velocity of the shaft is

$$\omega_1 = 1{,}05 - \frac{4 \,.\, 1{,}05 \,.\, 150000}{40000 \,.\, 36 + 150000 \,.\, 4} = 105 \left(1 - \frac{60}{204}\right) = 1{,}05 \,.\, 0{,}706$$
$$= 0{,}741 \text{ feet},$$

and that of the hammer is

$$\frac{2 \,.\, 6 \,.\, 1{,}05 \,.\, 4}{204} \text{ also } = \omega_1 \cdot \frac{G\,C}{K\,C} = 0{,}741 \,.\, \tfrac{1}{3} = 0{,}247 \text{ feet}$$

I.E., three times as small as that of the shaft. The loss of mechanical effect for each impact is

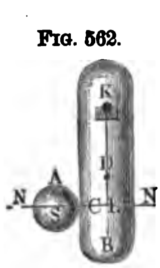
$$A = \frac{(2\,.\,1{,}05^2)}{2\,g} \cdot \frac{40000\,.\,150000}{40000\,.\,36 + 150000\,.\,4} = 0{,}0155\,(2{,}1)^2 \cdot \frac{600000}{144 + 60}$$

$$= 0{,}0155\,.\,4{,}41\,\frac{150000}{51} = \frac{10253{,}25}{51} = 201{,}05 \text{ foot-pounds.}$$

§ 343. Impact of an Oscillating Body.

—If a body A, Fig. 562, which has a *motion of translation* and is *unretained*, impinges upon a body $B\ C\ K$, *movable around an axis K*, we can find the velocities after impact by substituting in the formulas of the preceding paragraph instead of $a_1\,\varepsilon_1$ and $a_1\,\omega_1$ the velocities of translation c_1 and v_1 and instead of $\frac{M_1\,k_1^2}{a_1^2}$ the mass M_1 of the first body; the other notations remain unchanged. The velocity of the first mass after the impact is therefore

FIG. 562.



$$v_1 = c_1 - (c_1 - a_2\,\varepsilon_2)\,(1 + \sqrt{\mu}) \cdot \frac{M_2\,k_2^2}{M_1\,a_2^2 + M_2\,k_2^2},$$

and the angular velocity of the second is

$$\omega_2 = \varepsilon_2 + a_2\,(c_1 - a_2\,\varepsilon_2)\,(1 + \sqrt{\mu}) \cdot \frac{M_1}{M_1\,a_2^2 + M_2\,k_2^2}.$$

If the mass M_2 is at rest, or if $\varepsilon_2 = 0$, we have

$$v_1 = c_1 - c_1\,(1 + \sqrt{\mu}) \cdot \frac{M_2\,k_2^2}{M_1\,a_2^2 + M_2\,k_2^2}$$

and

$$\omega_2 = c_1\,(1 + \sqrt{\mu}) \cdot \frac{M_1\,a_2}{M_1\,a_2^2 + M_2\,k_2^2}.$$

If M_1 is at rest, I.E., if the oscillating mass impinges upon it, we have $c_1 = 0$, and hence

$$v_1 = a_2\,\varepsilon_2\,(1 + \sqrt{\mu}) \cdot \frac{M_2\,k_2^2}{M_1\,a_2^2 + M_2\,k_2^2}$$

and

$$\omega_2 = \varepsilon_2\left(1 - (1 + \sqrt{\mu})\,\frac{M_1\,a_2^2}{M_1\,a_2^2 + M_2\,k_2^2}\right).$$

The velocity, which is imparted to a mass at rest by another by a blow, depends not only upon the velocity of the blow and the masses of the bodies, but also upon the distance $K\ L = a_2$ at which the direction of the impact is situated from the axis K of the body which is capable of rotation. If the free body impinges upon the oscillating body, the angular velocity of the other becomes

$$\omega_2 = c_1 (1 + \sqrt{\mu}) \frac{M_1 a_2}{M_1 a_2^2 + M_2 k_2^2},$$

and if the oscillating body strikes against the free one, the latter acquires the velocity

$$v_1 = \varepsilon_2 (1 + \sqrt{\mu}) \frac{M_2 k_2^2 . a_2}{M_1 a_2^2 + M_2 k_2^2};$$

both velocities increase, therefore, when

$$\frac{a_2}{M_1 a_2^2 + M_2 k_2^2} \text{ or } \frac{1}{M_1 a_2 + \frac{M_2 k_2^2}{a^2}}$$

increases, or $M_1 a_2 + M_2 \frac{k_2^2}{a_2}$ decreases.

Substituting for a_2, $a \pm x$, x being very small, we obtain for the value of last expression

$$M_1 (a \pm x) + \frac{M_2 k_2^2}{a \pm x} = M_1 a \pm M_1 x + \frac{M_2 k_2^2}{a} \left(1 \mp \frac{x}{a} + \frac{x^2}{a^2} \mp \ldots\right),$$

or, since the powers of x are very small,

$$= M_1 a + \frac{M_2 k_2^2}{a} \pm \left(M_1 - \frac{M_2 k_2^2}{a^2}\right) x + \ldots$$

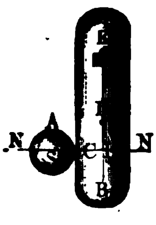
Now if a is to correspond to the minimum value of $M_1 a_2 + \frac{M_2 k_2^2}{a_2^2}$, the member $\pm \left(M_1 - \frac{M_2 k_2^2}{a^2}\right) x$ must disappear; for its sign is different, when a is increased a quantity (x) from what it becomes, when a is decreased by a quantity ($- x$); hence we must have

$$\left(M_1 - \frac{M_2 k_2^2}{a^2}\right) x = 0, \text{ I.E.,}$$

$$\frac{M_2 k_2^2}{a^2} = M_1, \text{ and consequently}$$

$$a = \sqrt{\frac{M_2 k_2^2}{M_1}} = k_2 \sqrt{\frac{M_2}{M_1}}.$$

FIG. 563.



Now if one body strike against the other at *this distance* (a), the latter assumes its maximum velocity, which is

$$1)\ \omega_2 = (1 + \sqrt{\mu}) \frac{c_1}{2 k_2} \sqrt{\frac{M_1}{M_2}} = (1 + \sqrt{\mu}) \frac{c_1}{2 a},$$

when the oscillating body is impinged upon; and

$$2)\ v_1 = \tfrac{1}{2} k_2 \varepsilon_2 (1 + \sqrt{\mu}) \sqrt{\frac{M_2}{M_1}} = (1 + \sqrt{\mu}) \frac{\varepsilon_2 a}{2},$$

when the free body receives the blow.

The extremity L of the distance or lever arm $a = k_2 \sqrt{\frac{M_2}{M_1}}$, which corresponds to the maximum velocity, or the point, where the latter line intersects the line of impact, is sometimes, though incorrectly, called the *centre of percussion;* a more correct term would be the *point of percussion.*

We should be careful not to confound it with the *centre of percussion* (§ 313), whose distance from the axis of rotation is expressed by the equation

$$a = \frac{M_2 k_2^2}{M_2 s} = \frac{k_2^2}{s},$$

in which s denotes the distance of the centre of gravity of the mass M_2 from the axis of rotation. If the direction $\overline{N} N$ of the impact of the masses M_1 and M_2 passes through the centre of percussion, the reaction upon the axis of rotation becomes $= 0$.

In order, for example, to prevent a hammer from jarring, I.E. reacting upon the hand, which holds it, or upon the axis, about which it turns, it is necessary that the direction of the blow shall pass through the centre of percussion.

If a suspended body $K B$ is struck by a mass M_1 with force P at the point of percussion, or at a distance $a = k_2 \sqrt{\frac{M_2}{M_1}}$ from the axis K, the reaction upon the axis is

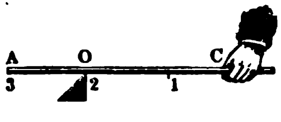
$$P_1 = P + R = P - \kappa M_2 s \text{ (see § 313).}$$

Since $P = \frac{\kappa M_2 k_2^2}{a}$, we have the angular acceleration $\kappa = \frac{P a}{M_2 k_2^2}$, and $\kappa M_2 s = \frac{M_2 s a}{M_2 k^2} P$; hence the required reaction is

$$P_1 = P\left(1 - \frac{M_2 s a}{M_2 k_2^2}\right) = P\left(1 - \frac{s a}{k_2^2}\right) = P\left(1 - \frac{s}{k_2}\sqrt{\frac{M_2}{M_1}}\right).$$

EXAMPLE—1) The centre of percussion of a prismatical rod $C A$, Fig. 564, which revolves about one of its ends, is at a distance

FIG. 564.



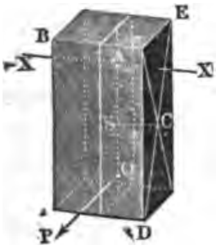
$$C O = a = \frac{\frac{1}{3} r^3}{\frac{1}{2} r} = \tfrac{2}{3} r = \tfrac{2}{3} C A$$

from the axis. Now if we grasp the rod at one end and strike with the point O, which is at the distance $C O = \frac{2}{3} C A$, upon an obstacle, we will feel no recoil. The point of percussion, on the contrary, is at a distance $r \sqrt{\frac{M_2}{3 M_1}}$ from C, and if the mass of the body struck $M_1 = M_2$, we have this distance $= \frac{r}{\sqrt{3}} = 0{,}5774\, r$. The rod $C A$ must therefore strike a mass at rest at this distance from C, when we wish to communicate the greatest possible velocity to the latter.

2) The distance of the centre of percussion O of a parallelopipedon $B\,D\,E$, Fig. 565, from an axis $X\,\overline{X}$, which is parallel to four of its sides and is at a distance $S\,A = s$ from the centre of gravity, and about which the body rotates, is

$$a = \frac{s^2 + \frac{1}{3}\,d^2}{s},$$

FIG. 565.



d denoting the semi-diagonal $C\,D$ of the sides, through which the axis $X\,\overline{X}$ passes (§ 287). If the force of impact passed through the point of percussion, we would have

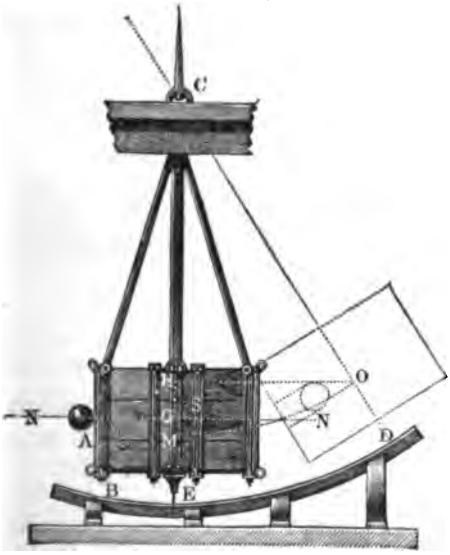
$$a = k_2 \sqrt{\frac{M_2}{M_1}} = \sqrt{(s^2 + \tfrac{1}{3}\,d^2)\,\frac{M_2}{M_1}},$$

and the reaction upon the axis would be

$$P_1 = P\left(1 - \frac{s\,a}{k_2{}^2}\right)$$
$$= P\left(1 - \frac{s}{\sqrt{s^2 + \frac{1}{3}\,d^2}}\sqrt{\frac{M_2}{M_1}}\right).$$

§ **344. Ballistic Pendulum.**—The principles discussed in the preceding paragraphs are applicable to the theory of the ballistic pendulum of Robins (Fr. pendule ballistique; Ger. ballistische Pendel). It consists of a large mass M, Fig. 566, which is capable of turning around a horizontal axis C. It is set in oscillation by means of a cannon-ball, which is shot against it, and serves to determine its velocity. In order to render the impact as inelastic as possible, upon the side where the ball strikes, a large cavity is made, which from time to time is filled with fresh wood or clay, etc. The ball remains, therefore, after every shot, sticking in this mass, and oscillates together with the whole body. In order to determine the velocity

FIG. 566.



of the ball, it is necessary to know the angle of displacement; to determine this angle, a graduated arc $B\,D$ is placed under it, along which a pointer, placed directly below the centre of gravity of the pendulum, moves.

According to the foregoing paragraph, the angular velocity of the ballistic pendulum, after the impact of the ball, is

$$\omega = \frac{M_1\, a_2\, c_1}{M_1\, a_2^2 + M_2\, k_2^2},$$

M_1 denoting the mass of the ball, $M_2\, k_2^2$ the moment of inertia of the pendulum, c_1 the velocity of the ball and a_2 the arm $C\,G$ of the impact or the distance of the line of impact $N\,\overline{N}$ from the axis of rotation. If the distance $C\,M$ of the centre of oscillation M of the entire mass, including the ball, from the axis of rotation C, I.E. if the length of the simple pendulum, oscillating isochronously with the ballistic one, $= r$, and the angle of displacement $E\,C\,D = a$, the height ascended by a pendulum oscillating isochronously will be

$$h = C\,M - C\,H = r - r\, cos.\, a = r\,(1 - cos.\, a) = 2\,r\left(sin.\,\frac{a}{2}\right)^2;$$

hence the velocity at the lowest point of its path is

$$v = \sqrt{2\,g\,h} = 2\,\sqrt{g\,r}\; sin.\,\frac{a}{2},$$

and the corresponding angular velocity

$$\omega = \frac{v}{r} = 2\,\sqrt{\frac{g}{r}}\,.\, sin.\,\frac{a}{2};$$

equating these values of the angular velocity, we have

$$c_1 = \frac{M_1\, a_2^2 + M_2\, k_2^2}{M_1\, a_2}\,.\, 2\,\sqrt{\frac{g}{r}}\,.\, sin.\,\frac{a}{2}.$$

Now, according to the theory of the simple pendulum,

$$r = \frac{\text{moment of inertia}}{\text{statical moment}} = \frac{M_1\, a_2^2 + M_2\, k_2^2}{(M_1 + M_2)\, s},$$

s denoting the distance $C\,S$ of the centre of gravity from the axis of rotation; hence it follows that

$$M_1\, a_2^2 + M_2\, k_2^2 = (M_1 + M_2)\, s\, r \text{ and}$$

$$c_1 = 2\left(\frac{M_1 + M_2}{M_1}\right).\,\frac{s}{a_2}\,\sqrt{g\,r}\,.\, sin.\,\frac{a}{2}.$$

If the pendulum makes n oscillations per minute, the duration of an oscillation is

$$\pi\sqrt{\frac{r}{g}} = \frac{60''}{n}, \text{ and therefore } \sqrt{g\,r} = \frac{60''\,.\,g}{n\,\pi};$$

hence the required velocity of the ball is

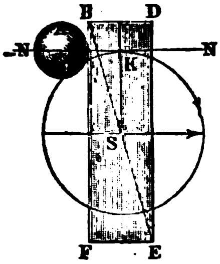
$$c_1 = \frac{M_1 + M_2}{M_1} \cdot \frac{120\, g\, s}{n\, \pi\, a_2} \cdot sin. \frac{a}{2}.$$

Example.—If a ballistic pendulum weighing 3000 pounds is set in oscillation by a 6-pound ball shot at it, and the angle of displacement is 15°, if the distance s of the centre of gravity from the axis = 5 feet and the distance of the direction of the shot from this axis is = 5,5 feet, and, finally, if the number of oscillations per minute is $n = 40$, the velocity of the ball, according to the above formula, is

$$c = \frac{3006}{6} \cdot \frac{120\ .\ 32{,}2\ .\ 5}{40\ .\ 3{,}1416\ .\ 5{,}5}\, sin.\, 7\tfrac{1}{2}^\circ = \frac{501\ .\ 3864\ .\ sin.\ 7^\circ\, 30'}{44\ .\ 3{,}1416} = 1828 \text{ feet.}$$

§ 345. Eccentric Impact.—Let us now examine a simple case of *eccentric impact*, where the *two masses are perfectly free.* If two bodies A and $B\ E$, Fig. 567, strike each other in such a manner that the direction $N\ \overline{N}$ of the impact passes through the centre of gravity S_1 of one body, and beyond the centre of gravity S of the other, the impact will be central for one body and eccentric for the other. The action of this eccentric impact can be found according to the theorem of § 281, if we assume, in the first place, that the second body is free and that the direction of impact passes through its centre of gravity S, and, in the second place, that this body is held fast at the centre of gravity, and that the force of impact acts as a rotating force. Now if c_1 is the initial velocity of A, c that of the centre of gravity of $B\ E$, and if the two velocities become v_1 and v, we have, as in § 332, $M_1 v_1 + M v = M_1 c_1 + M c$. If, further, ε is the initial angular velocity of the body $B\ E$, in turning about the axis passing through its centre of gravity perpendicular to the plane $N\ \overline{N}\ S$, and if, in consequence of the impact, this becomes ω, denoting the moment of inertia of this body in reference to S by $M k^2$, and the eccentricity or the distance $S\ K$ of the centre of gravity S from the line of impact by s, we have

FIG. 567.



$$M_1 v_1 + \frac{M k^2}{s^2} \cdot s\, \omega = M_1 c_1 + \frac{M k^2}{s^2}\, s\, \varepsilon.$$

If the bodies are inelastic, both points of tangency have the same velocity after impact, then $v_1 = v + s\, \omega$. Determining from the foregoing equations v and ω in terms of v_1, and substituting the values thus obtained in the last equation, we obtain

$$v_1 = \frac{M_1 (c_1 - v_1)}{M} + c + \frac{M_1 s^2 (c_1 - v_1)}{M k^2} + s\, \varepsilon,$$

from which we determine the loss of velocity of the first body

$$c_1 - v_1 = \frac{M k^2 (c_1 - c - s\,\varepsilon)}{(M_1 + M) k^2 + M_1 s^2},$$

the gain in velocity of translation of the second

$$v - c = \frac{M_1 k^2 (c_1 - c - s\,\varepsilon)}{(M_1 + M) k^2 + M_1 s^2},$$

and its gain in angular velocity

$$\omega - \varepsilon = \frac{M_1 s (c_1 - c - s\,\varepsilon)}{(M_1 + M) k^2 + M_1 s^2}.$$

When the impact is a perfectly elastic one, these values are doubled, and when it is imperfectly elastic, they are $(1 + \sqrt{\mu})$ times as great.

EXAMPLE.—If an iron ball A, weighing 65 pounds, strikes with a velocity of 36 feet the parallelopipedon $B\,E$, Fig. 567, which is at rest and is made of spruce, if this body is 5 feet long, 3 feet wide and 2 feet thick, and if the direction of impact $N\overline{N}$ is at a distance $S\,K = s = 1\frac{3}{4}$ feet from the centre of gravity S, we obtain the following values for the velocities after the impact. If the specific gravity of spruce is $= 0{,}45$, the weight of the parallelopipedon is $= 5 \,.\, 3 \,.\, 2 \,.\, 62{,}4 \,.\, 0{,}45$ pounds $= 842{,}4$ pounds. The square of the semi-diagonal of side $B\,D\,F$ parallel to the direction of the impact is

$$\left(\tfrac{5}{2}\right)^2 + \left(\tfrac{2}{2}\right)^2 = 7{,}25,$$

whence (according to § 287),

$$k^2 = \tfrac{1}{3} \,.\, 7{,}25 = 2{,}416,$$

$$g\,M\,k^2 = 842{,}4 \,.\, 2{,}416 = 2035{,}2,$$

and

$$g\,(M_1 + M)\,k^2 = 907{,}4 \,.\, 2{,}416 = 2192{,}3;$$

hence the velocity of the ball after the impact is

$$v_1 = c_1 - \frac{M k^2 c_1}{(M_1 + M) k^2 + M_1 s^2} = 36\left(1 - \frac{2035{,}2}{2192{,}3 + 65 \,.\, 1{,}75^2}\right)$$

$$= 36\left(1 - \frac{2035{,}2}{2391{,}4}\right) = 36 \,.\, 0{,}149 = 5{,}364 \text{ feet},$$

and that of the centre of gravity of the body struck is

$$v = \frac{M_1 k^2 c_1}{(M_1 + M) k^2 + M_1 s^2} = \frac{157{,}08 \,.\, 36}{2391{,}4} = 2{,}364 \text{ feet};$$

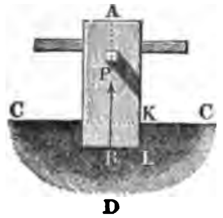
and finally the angular velocity is

$$\omega = \frac{M_1 s\, c_1}{(M_1 + M) k^2 + M_1 s^2} = \frac{113{,}75 \,.\, 36}{2391{,}4} = 1{,}712 \text{ feet}.$$

§ 346. Uses of the Force of Impact.—The weight of a body is a force which depends upon its mass alone and increases uniformly with it; the force of impact, on the contrary, increases not only with the mass, but also with the velocity and with the hardness of the colliding bodies (see § 336 and § 338), and it can therefore be increased at will. Impact is consequently an excellent method of obtaining great forces with small masses or weights, and it is very often made use of for breaking or stamping rock, cutting or compressing metals, driving nails, piles, etc. On the other hand, impact occasions not only a loss of mechanical effect, but also causes the different portions of the machine to wear rapidly or even to break, and the durability of the structure or machine is seriously affected by it. For this reason it is necessary to make the dimensions of those parts of the machine larger than when the latter are subjected to extension, compression, weight, etc., without impact.

If a *rigid body* $A\ B$, Fig. 568, strikes upon an unlimited mass $C\ D\ C$ of soft matter, it compresses the latter with a certain force, whose mean value P is determined by means of the depth of the impression $K\ L = s$, when we put the work done $P\ s$ during the compression equal to the energy of the mass of the striking body. If M be the mass, or $G = g\ M$ the weight, of this body $A\ B$ and v the velocity with which it strikes upon $C\ D\ C$, we will have

FIG. 568.



$$\tfrac{1}{2}\, M v^2 = \frac{v^2}{2\, g}\, G,$$

and the required force with which the soft matter will be compressed is

$$P = \tfrac{1}{2}\, \frac{M v^2}{s} = \frac{v^2}{2\, g\, s}\, G.$$

Dividing this force by the cross-section of the body F, we obtain the force with which each unit of surface of the soft material is compressed and which such a unit can bear without giving way,

$$p = \frac{P}{F} = \frac{v^2}{2\, g} \cdot \frac{G}{F\, s}.$$

For safety we only load such a mass with a small portion of p, for example with one-tenth part $\left(\frac{p}{10}\right)$.

The body M acquires its velocity v by being allowed to fall freely from a height $h = \frac{v^2}{2\, g}$. If we substitute this height, instead of $\frac{v^2}{2\, g}$, in the foregoing formula, we have

$$P = \frac{G\, h}{3}, \text{ or for the unit of surface } p = \frac{G\, h}{F\, s}.$$

The force or resistance P, with which soft or loose granular masses oppose the penetration of a rigid body $A\ B$, is generally variable and increases with the depth s of the penetration. In many cases we can assume it to increase directly with s, I.E., that it is null at the beginning and double at the end what it is in the middle. Now since the value of P, deduced from the above formula, is the mean value, the resistance or proof load P_1 of soft materials is twice as great as the value P obtained by the formula,

I.E., $$P_1 = 2\,P = \frac{2\,G\,h}{s}.$$

EXAMPLE.—If a commander $A\ B$, Fig. 568, whose weight $G = 120$ lbs. falls upon a mass of earth from a height $h = 4$ feet, and if the latter is compressed $\frac{1}{4}$ an inch by the last blow, a surface of this material equal to the cross-section of the stamper will support a weight

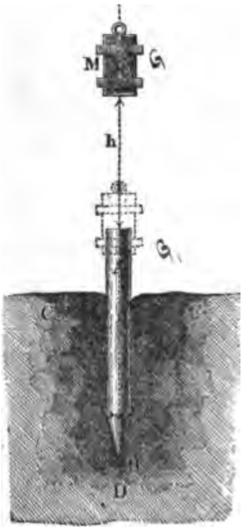
$$P = \frac{G\,h}{s} = \frac{120\,.\,4}{\frac{1}{48}} = 23040 \text{ pounds.}$$

Now if the cross-section F of the commander is $\frac{5}{4}$ square feet, the force per square foot supported by this mass of earth would be

$$p = \frac{P}{F} = \frac{23040}{1{,}25} = 18432 \text{ pounds};$$

instead of which, for the sake of safety, we should take but $\frac{1}{10}\ p = 1843{,}2$ pounds.

FIG. 569.



§ 347. Pile-driving.—If we drive piles such as $A\ B$, Fig. 569, into earth or any other soft material $C\ D\ C$, we increase its resistance much more than we would by simply stamping it. Such piles (Fr. pieux; Ger. Pfähle) are from 10 to 30 feet long, 8 to 20 inches thick, and are provided with an iron shoe B. The body M, the so-called ram (Fr. mouton; Ger. Rammklotz, Rammbär or Hoyer), which is allowed to fall from 3 to 30 feet upon the top of the pile, is generally made of cast iron, more rarely of oak, and weighs from 5 to 20 hundred weights. If the ram falls the vertical distance h, the velocity with which it strikes the pile is

$$c = \sqrt{2\,g\,h},$$

and if its weight $= G$ and that of the pile

$= G_1$, we have, when we suppose that both bodies are inelastic, the velocity of the same at the end of the impact (see § 332)

$$v = \frac{G c}{G + G_1},$$

hence the corresponding height due to the velocity is

$$\frac{v^2}{2g} = \left(\frac{G}{G + G_1}\right)^2 \cdot \frac{c^2}{2g} = \left(\frac{G}{G + G_1}\right)^2 h.$$

Now if the pile sink during the last blow a distance s, the resistance of the earth and the load which the pile can support is

$$P = \frac{v^2}{2gs}(G + G_1) = \frac{h}{s} \cdot \frac{G^2}{G + G_1},$$

or more correctly, since the weight $G + G_1$ of the pile and ram act in opposition to the resistance of the earth,

$$P = \frac{h}{s} \cdot \frac{G^2}{G + G_1} + (G + G_1).$$

In most cases $G + G_1$ is so small, compared to P, that we can neglect the latter part of the formula.

If the weight G_1 of the pile is much smaller than the weight G of the ram, we can write

$$v = \frac{G c}{G + G_1} = c$$

and simply $$P = \frac{h}{s} G.$$

The foregoing theory suffices in practice, when the resistance P is moderate and, consequently, the depth s of the impression is not very small; for in that case the compression of the pile, etc., can be neglected. If, on the contrary, the resistance P is very great and, consequently, the depth s of the impression very small, the compression σ of the pile can no longer be regarded as null, and must therefore be introduced into the calculation.

The pile of course does not begin to sink until the force of impact has become equal to the resistance P of the earth. Now if $H = \frac{F E}{l}$ and $H_1 = \frac{F_1 E_1}{l_1}$ denote the hardness of the ram and that of the pile (in the sense of § 336), the sum of the compressions of the two bodies, when the force of impact is P, is

$$\sigma = \frac{P}{H} + \frac{P}{H_1} = \left(\frac{1}{H} + \frac{1}{H_1}\right) P,$$

and the mechanical effect expended in producing this compression is

$$L = \frac{P \sigma}{2} = \left(\frac{1}{H} + \frac{1}{H_1}\right) \frac{P^2}{2}.$$

Now if this first impact of the two bodies causes the velocity c of the ram to become v, its mass $M = \frac{G}{g}$ performs the work

$$L = \frac{1}{2} M c^2 - \frac{1}{2} M v^2 = (c^2 - v^2) \frac{M}{2} = \left(\frac{c^2 - v^2}{2g}\right) G;$$

hence we can put

$$\left(\frac{c^2 - v^2}{2g}\right) G = \left(\frac{1}{H} + \frac{1}{H_1}\right) \frac{P^2}{2},$$

from which we obtain

$$\frac{v^2}{2g} = \frac{c^2}{2g} - \left(\frac{1}{H} + \frac{1}{H_1}\right) \frac{P^2}{2G};$$

consequently the velocity of the ram, when the pile begins to penetrate the earth, is

$$v = \sqrt{c^2 - 2g\left(\frac{1}{H} + \frac{1}{H_1}\right) \frac{P^2}{2G}}.$$

We infer from this that a pile (and also a bolt or nail in a wall) will begin to enter the resisting obstacle when

$$\frac{c^2}{2g} G > \left(\frac{1}{H} + \frac{1}{H_1}\right) \frac{P^2}{2},$$

or when the weight of the ram and its velocity have the proper relation to the resistance of the earth. During the penetration of the pile the force of impact and, consequently, the compression of the pile, etc., diminish as long as the velocity of the ram exceeds that of the pile; when both attain a common velocity v_1 and the force of impact becomes a maximum, the bodies begin to expand again. During this expansion not only the velocity of the ram, but also that of the pile becomes gradually $= 0$; the pressure between the two bodies becomes again P, and consequently at the moment, when the pile ceases to penetrate, the whole energy $\frac{c^2}{2g} G$ of the ram is consumed by the work

$$\left(\frac{1}{H} + \frac{1}{H_1}\right) \frac{P^2}{2}$$

expended in compressing, and by the work

$$P s$$

done in driving the pile to the depth s.

Hence we have

$$\frac{c^2}{2g} G = G h = \left(\frac{1}{H} + \frac{1}{H_1}\right) \frac{P^2}{2} + P s,$$

and therefore the load which corresponds to the depth of penetration s is

$$P = \left(\frac{H H_1}{H + H_1}\right)\left(\sqrt{2\left(\frac{H + H_1}{H H_1}\right)\frac{c^2}{2g}\,G + s^2} - s\right).$$

If the compression $\left(\frac{1}{H} + \frac{1}{H_1}\right)\frac{P^2}{2}$ is considerably smaller than the space s described by the pile, we can write simply

$$P = \frac{c^2}{2g}\,\frac{G}{s} = \frac{G\,h}{s}, \text{ or, more accurately,}$$

$$P = \frac{G\,h}{s + \left(\frac{1}{H} + \frac{1}{H_1}\right)\frac{G\,h^2}{2\,s}}.$$

Comparing the work done in driving in the pile

$$P\,s = \frac{G\,h}{1 + \left(\frac{1}{H} + \frac{1}{H_1}\right)\frac{P}{2\,s}}$$

with the work done $G\,h$ in raising the ram, we see that the former approaches the latter more and more as $\left(\frac{1}{H} + \frac{1}{H_1}\right)\frac{P}{2\,s}$ becomes smaller or as the hardness $H = \frac{F\,E}{l}$ of the ram and that $H_1 = \frac{F_1\,E_1}{l_1}$ of the pile become greater, I.E. the greater the cross-sections F and F_1 and the moduli of elasticity E and E_1 of these bodies are and the smaller the lengths are.

The action of the weights of the two bodies can be entirely neglected, since they generally form but a small portion of the resistance P. We can also neglect the energy, which the bodies possess in consequence of their elasticity (although the latter is imperfect) after the pile has come to rest; for the body, which is thrown back by their expansion, is generally, upon falling again, incapable of overcoming P and setting the pile in motion. For safety's sake, the pile, which has been driven in, is loaded with only $\frac{1}{10}$ part of the resistance P, just found, or perhaps with even less. According to some late experiments made by Major John Sanders, U. S. A., at Fort Delaware (communicated by letter) we can put, approximatively, the resistance

$$P = \frac{G\,h}{3\,s}.$$

EXAMPLE.—A pile, whose cross-section is 1 foot = 144 square inches, whose length is 25 feet = 25 . 12 = 300 inches, and whose weight is 1200 pounds, is driven by the last tally of ten blows of a ram, weighing 2000 pounds and falling 6 feet = 72 inches, 2 inches deeper, what is the resistance of the earth? If we neglect the inconsiderable compression of the cast iron and put (according to § 212) the modulus of elasticity of wood $E_1 = 1{,}560000$, we obtain

$$\frac{1}{2}\left(\frac{1}{H}+\frac{1}{H_1}\right) = 0 + \frac{l_1}{2\,F_1\,E_1} = \frac{300}{2\,.\,144\,.\,1{,}560000} = \frac{1}{1497600}.$$

Now since $G\,h = 2000\,.\,72 = 144000$ inch-pounds and the depth of the penetration after one blow is $s = \frac{2}{10} = 0{,}2$ inches, we obtain for the determination of P the following equation:

$$\frac{P^2}{1497600} + 0{,}2\,P = 144000 \text{ or } P^2 + 299520\,P = 215654400000.$$

Resolving this equation, we obtain

$$P = -\,149760 + \sqrt{238082457600} = 338177 \text{ pounds.}$$

According to Sanders' formula

$$P = \frac{G\,h}{8\,s} = \frac{144000}{0{,}6} = 240000,$$

while the old formula, on the contrary, gives

$$P = \frac{G^2\,h}{(G+G_2)\,s} = \frac{G}{G+G_1}\cdot\frac{G\,h}{s} = \frac{2000}{3200}\cdot\frac{144000}{0{,}2} = \frac{5}{8}\,.\,720000$$
$$= 450000 \text{ pounds.}$$

From $P = 338177$ pounds we obtain

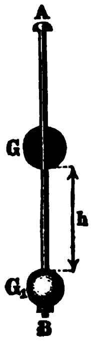
$$\left(\frac{1}{H}+\frac{1}{H_1}\right)\frac{P^2}{2} = 76365 \text{ inch-pounds,}$$

and therefore the height from which a ram weighing 2000 pounds must fall in order to move the pile is

$$h = \left(\frac{1}{H}+\frac{1}{H_1}\right)\frac{P^2}{2\,G} = \frac{76365}{2000} = 38{,}2 \text{ inches.}$$

§ 348. Absolute Strength of Impact.—By the aid of the *moduli of resilience and fragility* (see § 206) we can easily calculate the conditions under which a prismatical body $A\,B$, Fig. 570, will be stretched to the limit of elasticity or broken by a blow in the direction of its axis. If G be the weight and c the velocity of the impinging body, the work done, when the prismatical body, whose weight we will denote by G_1, is struck, is

FIG. 570.



$$L = \frac{c^2}{2g}\cdot\frac{G^2}{G+G_1},$$

or denoting the height due to the velocity $\frac{c^2}{2g}$ by h, we have more simply

$$L = \frac{G^2\,h}{G+G_1}.$$

This mechanical effect is chiefly expended in stretching the rod $A\,B$, upon which the second body hangs; if, therefore, H is the hardness, l the length, F the cross-section, E the modulus of elasticity, P the force of impact and λ the extension of the rod produced by it, we have

$$L = \frac{P\lambda}{2} = \frac{P^2}{2\,H} = \tfrac{1}{2}\,H\lambda^2 = \frac{F\,E}{2\,l}\,\lambda^2.$$

and consequently

$$\frac{F\,E}{2\,l}\,\lambda^2 = \frac{G^2\,h}{G + G_1},$$

from which the extension λ of the rod, caused by this impact, can be easily calculated.

If the rod is to be extended only to the limit of elasticity, we have, when A denotes the modulus of resilience (§ 206),

$$L = A\;V = A\,F\,l,$$

and therefore

$$A\,F\,l = \frac{G^2\,h}{G + G_1};$$

the velocity of impact $c = \sqrt{2\,g\,h}$, which is necessary to stretch it to the limit of elasticity, is determined by the height

$$h = \frac{G + G_1}{G^2}\,.\,A\,F\,l.$$

If we are required to find the conditions of rupture of the rod, we must substitute, instead of the modulus of resilience A, the modulus of fragility B.

We see from this that the greater the mass of the rod is, the greater is the blow it can bear. Hence we have the following important rule, that the mass of bodies subjected to impacts should be made as great as possible.

Since G and G_1 fall the distance λ during the impact, it is more correct to put

$$L = \frac{G^2\,h}{G + G_1} + (G + G_1)\,\lambda,$$

or for the case, when the limit of elasticity is reached,

$$A\;F = \frac{G^2}{G + G_1}\cdot\frac{h}{l} + (G + G_1)\,\frac{\lambda}{l},$$

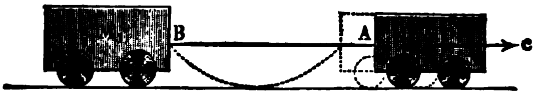
in which $\frac{\lambda}{l} = \sigma$ expresses the extension corresponding to the limit of elasticity.

If, finally, we wish to take into consideration the mass and weight G_2 of the rod, we have, since its centre of gravity sinks but $\frac{1}{2}\,\lambda$,

$$A\ F = \frac{G^2}{G + G_1 + G_2} \cdot \frac{h}{l} + (G + G_1 + \tfrac{1}{2}\ G_2)\ \sigma.$$

We have a similar instance of the action of impact, when a moving mass $M = \frac{G}{g}$, Fig. 571, puts another mass $M_1 = \frac{G_1}{g}$ in motion by means of a chain or rope. If c is the velocity of M at the moment, when the chain is stretched, v the velocity with which both bodies move after the impact, we have again

FIG. 571



$$v = \frac{M\,c}{M + M_1} = \frac{G\,G_1}{G + G_1},$$

while, on the contrary, the work expended in stretching the chain is

$$L = \tfrac{1}{2}\,M\,c^2 - \tfrac{1}{2}\,(M + M_1)\,v^2 = \left(M - \frac{M_2}{M + M_1}\right)\frac{c^2}{2}$$

$$= \frac{M\,M_1}{M + M_1} \cdot \frac{c^2}{2} = \frac{G\,G_1}{G + G_1} \cdot h.$$

If, therefore, this chain, etc., is to be stretched only to the limit of elasticity, we must put

$$A\,F\,l = \frac{G\,G_1}{G + G_1} \cdot h,$$

F denoting the cross-section and l the length of the chain.

EXAMPLE—1) If two opposite suspension-rods of a chain bridge support a constant weight of 5000 pounds, which is increased 6000 pounds by a passing wagon, if the modulus of resilience A of wrought iron is 7 inch-pounds and if the length of the suspension-rods is 200 inches and their cross-section 1,5 square inches, we have the dangerous height of fall

$$h = \frac{A\,F\,l\,(G + G_1)}{G^2} = \frac{7\,.\,2\,.\,1{,}5\,.\,200\,.\,11000}{36000000} = \frac{7\,.\,11}{60} = \frac{77}{60} = 1{,}28 \text{ inches.}$$

If the wagon passes over an obstacle 1,3 inches high, the suspension-rods would already be in danger of being stretched beyond the limit of elasticity.

2) If a full bucket or loaded cage in a shaft is not gradually set in motion, but if by means of the rope, which has been hanging loosely, it is suddenly brought to a certain velocity by the revolving drum, the rope will often be stretched beyond the limit of elasticity, and sometimes even

broken. If the mass of the drum and shaft, reduced to the circumference of the former, is $M = \frac{G}{g} = \frac{100000}{g}$, the weight of the full bucket or cage is $G_1 = 2000$ pounds, and the weight of the rope = 400 pounds, then if the weight of a cubic inch of rope is = 0,8 pounds, its volume will be

$$F l = \frac{G_1}{\lambda} = \frac{400}{0{,}8} = \frac{4000}{8} \text{ cubic inches,}$$

and, finally, if the modulus of fragility of this rope is = 850 pounds, we have the height due to the velocity, which will break the rope,

$$h = B F l \,.\, \frac{G + G_1}{G\, G_1} = 850 \,.\, \frac{4000}{8} \cdot \frac{100000 + 2000}{100000 \,.\, 2000} = \frac{1400000}{8} \cdot \frac{102}{200000}$$

$$= 238 \text{ inches} = 19{,}83 \text{ feet,}$$

and, therefore, the velocity of the rope at the beginning of the strain is

$$c = \sqrt{2 g h} = \sqrt{64{,}4 \,.\, 19{,}83} = 35{,}74 \text{ feet.}$$

§ 349. Relative Strength of Impact.—The foregoing theory is also applicable to the case of a prismatic body $B\ B$, Fig. 572, supported at both ends and exposed to the blows of a body A, which falls from the height $A\ C = h$ upon its middle C. Let $\frac{G}{g} = M$ be the mass of the falling body and M_1 that of the body $B\ B$, reduced to its middle C, then the energy of the bodies after the impact is

$$L = \frac{c^2}{2} \cdot \frac{M^2}{M + M_1} = \frac{c^2}{2 g} \cdot \frac{M}{M + M_1} \cdot M g = \frac{M}{M + M_1} G\, h.$$

FIG. 572.

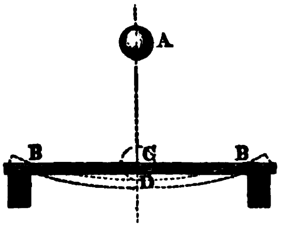
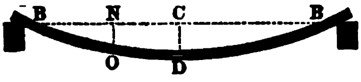


FIG. 573.



The mass M_1 of the beam $B\ B$ can be determined in the following manner. Let G_1 be the weight, l half the length $B\ D$, Fig. 573, of this beam, x the abscissa $B\ N$ and y the corresponding ordinate $N\ O$ of the curve, formed by $B\ B$ at the moment of greatest flexure, and, finally, let a denote the maximum deflection $C D$ of this curve. If we imagine $B\ C$ to be divided into n (an infinite number of) parts, the weight of an element O of the rod will be $\frac{G_1}{n}$, and therefore the mass of an element of the rod, reduced from N to D, is

$$= \frac{G_1}{n\,g} \cdot \left(\frac{N\,O}{C\,D}\right)^2 = \frac{G_1\,y^2}{n\,g\,a^2}.$$

But, according to § 217,

$$y = \frac{P\,x}{2\,W\,E}\left(l^2 - \frac{x^2}{3}\right), \text{ or}$$

$$y^2 = \frac{P^2\,x^2}{4\,W^2\,E^2}\left(l^4 - \tfrac{2}{3}\,l^2\,x^2 + \frac{x^4}{9}\right) \text{ and } a^2 = \frac{P^2\,l^6}{9\,W^2\,E^2},$$

whence it follows that the element of the mass of the rod is

$$= \frac{9\,G_1\,x^2\left(l^4 - \tfrac{2}{3}\,l^2\,x^2 + \frac{x^4}{9}\right)}{4\,n\,g\,l^6}.$$

Now if instead of x we substitute successively $\frac{l}{n}, 2\,\frac{l}{n}, 3\,\frac{l}{n} \ldots \frac{n\,l}{n}$ and add, etc., the values thus obtained, we obtain the mass of the rod $B\,B$, reduced to its middle C,

$$M_1 = \frac{9\,G_1}{4\,g\,l^6}\left(l^4 \cdot \frac{l^3}{3} - \tfrac{2}{3}\,l^2 \cdot \frac{l^5}{5} + \tfrac{1}{9}\,\frac{l^7}{7}\right) = \tfrac{17}{35} \cdot \frac{G_1}{g}.$$

If we substitute this value, we can put the work done by the impact

$$L = \frac{M}{M + M_1} \cdot G\,h = \frac{G^2\,h}{G + \frac{17}{35}\,G_1},$$

and obtain the condition of bending to the limit of elasticity (see § 235),

$$A \cdot \frac{W\,l}{3\,e^2} = \frac{G^2\,h}{G + \frac{17}{35}\,G_1}.$$

If the beam is a parallelopipedon, we have

$$\tfrac{1}{9}\,A\ V_1 = \frac{G^2\,h}{G + \frac{17}{35}\,G_1},$$

and therefore

$$h = \frac{A\,V_1\,(G + \frac{17}{35}\,G_1)}{9\,G^2}, \text{ or putting } V_1 = \frac{G_1}{\gamma},$$

$$h = \frac{A\,G_1\,(G + \frac{17}{35}\,G)}{9\,\gamma\,G^2}.$$

If we substitute B instead of A, the expression becomes

$$h = \frac{B\,G_1\,(G + \frac{17}{35}\,G_1)}{9\,\gamma\,G^2}$$

and gives the height, from which the weight G must fall in order to break the parallelopipedical rod.

EXAMPLE.—From what a height must an iron weight $G = 200$ pounds fall, in order to break by striking it in the middle a cast iron plate 36 inches long, 12 inches wide and 3 inches thick, which is supported at both ends?

The modulus of fragility

$$B = 14{,}8 \text{ inch-pounds}$$

(see § 211), and the volume of the plate is

$$V_1 = b\,h\,l = 12 \,.\, 3 \,.\, 36 = 1296 \text{ cubic inches,}$$

and, since a cubic inch of cast iron weighs $\gamma = 0{,}259$ pounds, its weight is

$$G_1 = 1296 \,.\, 0{,}259 = 335{,}7 \text{ pounds;}$$

the required height is

$$h = \frac{14{,}8 \,.\, 335{,}7\,(200 + \frac{17}{35} \,.\, 356{,}4)}{9 \,.\, 0{,}275 \,.\, 40000} = 18\tfrac{3}{4} \text{ inches.}$$

§ 350. Mechanical Effect of the Strength of Torsion.—

We can also investigate *the action of impact in twisting shafts.* According to § 262 the mechanical effect which is required to produce a torsion a in a shaft, whose length is l and the measure of whose moment of flexure is W, is

$$L = \frac{P\,a\,a}{2} = \frac{a^2 \,.\, W\,C}{2\,l} = \frac{P^2\,a^2\,l}{2\,W\,C},$$

we can also put

$$L = \frac{S^2}{2\,C}\,\frac{W\,l}{e^2} \text{ (see § 264),}$$

e denoting the distance of the most remote fibre from the neutral axis and S the strain in that fibre.

If we substitute for S the modulus of proof strength T, and for $\frac{T}{2\,C} = \frac{\sigma\,T}{2}$ the modulus of resilience A, we obtain the work to be performed in stretching the remotest fibre to the limit of elasticity

$$L = A \,.\, \frac{W\,l}{e^2},$$

and the mechanical effect necessary to rupture the shaft by wrenching, when we substitute for the modulus of resilience A the modulus of fragility B; its value is

$$L_1 = B \,.\, \frac{W\,l}{e^2}.$$

For a *cylindrical shaft* $W = \frac{\pi\,r^4}{2}$ and $e = r$, hence

$$L = \frac{A}{2} \,.\, \pi\,r^2\,l = \frac{A}{2}\,V \text{ and } L_1 = \frac{B}{2} \,.\, \pi\,r^2\,l = \frac{B}{2}\,V$$

when $V = \pi\,r^2\,l$ denotes the volume of this shaft.

For a shaft with a *square cross-section*, the length of whose side is b, we have

$$W = \frac{b^4}{6} \text{ and } e = b\sqrt{\tfrac{1}{2}},$$

and consequently

$$L = \frac{A}{3}\, b^2 l = \frac{A}{3}\, V \text{ and } L_1 = \frac{B}{3}\, V.$$

If a revolving wheel and axle, whose mass reduced to the point of impact is $M = \frac{G}{g}$, impinges upon a mass $M_1 = \frac{G_1}{g}$, which is at rest, with the velocity c, both will move on after the impact with the velocity

$$v = \frac{M c}{M + M_1} = \frac{G c}{G + G_1} \text{ (see § 334), and}$$

consequently the mechanical effect

$$L = \frac{G\, G_1}{G + G_1} \cdot \frac{c^2}{2 g},$$

which is expended in twisting the axle and bending the arms of the wheel, is lost (see § 335).

But L is also the sum of the mechanical effects expended in producing the torsion of the axle and in bending the arms of the wheel, etc., I.E.,

$$L = A \,.\, \frac{W l}{e^2} + A_1 \frac{W_1 l}{e_1^2},$$

when A_1 denotes the modulus of resilience, W_1 the measure of moment of flexure and e_1 the distance of the exterior fibre from the neutral axis (see § 235); we can therefore put

$$\frac{A\, W l}{e^2} + \frac{A_1\, W_1\, l_1}{3\, e_1^2} = \frac{G\, G_1}{G + G_1}\, \frac{c^2}{2 g}.$$

If the shaft is cylindrical, we have $\frac{W l}{e^2} = \frac{V}{2}$, and if it is four-sided, we have $\frac{W l}{e^2} = \frac{V}{3}$, when V denotes its volume; and for the four-sided arm we have $\frac{W_1 l_1}{3\, e_1^2} = \frac{V_1}{9}$, where V_1 denotes the volume of the arm.

Hence for a *cylindrical shaft* we have

$$\frac{A}{2}\, V + \frac{A_1}{9}\, V_1 = \frac{G\, G_1}{G + G_1}\, \frac{c^2}{2 g},$$

and, on the contrary, for a *four-sided shaft*

$$\frac{A}{3}\, V + \frac{A_1}{9}\, V_1 = \frac{G\, G_1}{G + G_1}\, \frac{c^2}{2 g}.$$

The volumes V and V_1 have a certain relation to each other, which can be expressed as follows. The moment of flexure of the arms is equal to the moment of torsion of the shaft. Hence

$$\frac{W\,T}{e} = \frac{W_1\,T_1}{e_1}, \text{ or}$$

$$1)\quad \frac{\pi\,d^3}{16}\,T = \frac{b^3\,T}{3\,\sqrt{2}} = \frac{b_1\,h_1^2\,T_1}{6},$$

T and T_1 denoting the moduli of proof strength for torsion and bending and d the diameter of a round, and b the length of the sides of a four-sided shaft, while h_1 is the thickness and b_1 the sum of the widths of all the arms of the wheel.

But we have also $V = \frac{\pi\,d^2}{4}\,l = b^2\,l$ and $V_1 = b_1\,h_1\,l_1$, and therefore

$$2)\quad \begin{cases} \dfrac{\pi\,d^2\,l\,A}{8} + \dfrac{b_1\,h_1\,l_1\,A_1}{9} = \dfrac{G\,G_1}{G+G_1}\,\dfrac{c^2}{2\,g} \text{ and} \\ \frac{1}{3}\,b^2\,l\,A + \dfrac{b_1\,h_1\,l_1\,A_1}{9} = \dfrac{G\,G_1}{G+G_1}\cdot\dfrac{c^2}{2\,g}. \end{cases}$$

Now if the ratio $\nu = \frac{b_1}{h_1}$ of the dimensions is given, we can calculate the thickness d or h of the shaft or the thickness h_1 and the width b_1 of the arms by means of equations (1) and (2). We must introduce into this calculation

1) for cast iron

$$A_1 = 3{,}16 \text{ and } A = \frac{T^2}{2\,C} = \frac{1906^2}{2\,.\,2840000} = 0{,}640 \text{ inch-lbs.},$$

2) for wrought iron

$$A_1 = 6{,}23 \text{ and } A = \frac{T^2}{2C} = \frac{5974^2}{2\,.\,9000000} = 1{,}983 \text{ inch-lbs.},$$

3) and for wood, the mean value

$$A_1 = 2{,}17 \text{ and } A = \frac{T^2}{2\,C} = \frac{395^2}{2\,.\,590000} = 0{,}132 \text{ inch-lbs.}$$

EXAMPLE.—Let the mass of the wheel, etc., of a tilt-hammer, reduced to the point of application of the cam, be $M = \frac{200000}{g}$ pounds, and the mass of the hammer reduced to the same point be $M = \frac{25000}{g}$ pounds, and let the distance from the wheel to the ring, in which the cams are set, be $l = 15$ feet $= 225$ inches, and the length of the arms of the wheel be $l_1 = 10$ feet $=$ 120 inches. Now if the hammer, every time it is lifted, is struck with a velocity of 2 feet, how thick must the shaft and the arms of the wheel be made in order to sustain this impact without being *damaged?* If the shaft and arms are of wood, we have

$$395 \frac{\pi d^3}{16} = 1000 \frac{b_1 h_1^2}{6},$$

and if the number of arms is $n = 16$, we can put

$$b_1 = \nu \,.\, n\, h_1 = 0{,}707 \,.\, 16\, h_1 = 11{,}3 \,.\, h_1,$$

whence we obtain

$$d = h_1 \sqrt[3]{\frac{16000 \,.\, 11{,}3}{6 \,.\, 395 \,.\, \pi}} = 2{,}9 \,.\, h_1.$$

But

$$\frac{\pi}{8} A\, l = 0{,}132 \,.\, 225 \frac{\pi}{8} = 11{,}66,$$

$$\tfrac{1}{9} A_1\, l_1 = \tfrac{1}{9} \,.\, 2{,}17 \,.\, 120 = 28{,}9,$$

and also

$$\frac{G\, G_1}{G + G_1} \,.\, \frac{c^2}{2g} = 12 \,.\, 0{,}0155 \,.\, 4 \,.\, \frac{200000 \,.\, 25000}{200000 + 25000} = 0{,}744 \,.\, \frac{5000000}{225}$$
$$= 16533 \text{ inch-pounds};$$

hence we have the equation of condition

$$(2{,}9)^2 \,.\, 11{,}66\, h_1^2 + 11{,}3 \,.\, 28{,}9\, h_1^2 = 16533, \text{ i.e.,}$$
$$98{,}1\, h_1^2 + 326{,}6\, h_1^2 = 16533,$$

hence the required thickness of the arm

$$h_1 = \sqrt{\frac{16533}{424{,}7}} = 6{,}24 \text{ inches}$$

the width of the arm

$$b_1 = 0{,}707\, h_1 = 4{,}41 \text{ inches},$$

and the thickness of the shaft

$$d = 2{,}9\, h_1 = 18{,}1 \text{ inches}.$$

For the sake of security we make the dimensions considerably larger.

REMARK.—It is only of late years that much attention has been paid to the strength of impact. We find something in regard to it in Tredgold's work on the strength of cast iron, in Poncelet's "Introduction à la Mécanique Industrielle," and in Rühlmann's "Grundzüge der Mechanik und Geostatik." The discussion in the latter work is based principally upon Hodgkinson's experiments on the resistance of prismatic bodies to impact, upon which subject an article by Bornemann is to be found in the "Zeitschrift für das gesammte Ingenieurwesen" (the Ingenieur).

The experiments of Hodgkinson agree essentially with the foregoing theory of the strength of impact; they apply particularly to relative strength, and were made in the following manner: large weights swinging like pendulums were caused to strike against rods supported at both ends. The formula $L = \frac{G^2 h}{G + \frac{1}{2} G_1}$, which we found by assuming that the impact was perfectly inelastic, was verified completely; the mechanical effect L was found not to depend upon the nature of the colliding bodies. Equally heavy bodies of different materials (cast iron, cast steel, bell metal, lead) produced, when they fell from the same height, equal deflections of the same rod (of cast iron or cast steel); the deflections were almost exactly the same as those given by the theory for a perfectly elastic rod.

FINAL REMARK.—For the study of the Mechanics of rigid bodies, besides the older works of Euler, Poisson, Poinsot, Poncelet, Navier and Coriolis, and those of Whewell, Mosely, Eytelwein and Gerstner, the following are recommended:

Duhamel, Cours de Mécanique, Paris, chez Mallet-Bachelier, 1854; Sohnke, Analytische Theorie der Statik und Dynamik, Halle, 1854; Broch's Lehrbuch der Mechanik, Berlin, 1854; Morin, Leçons de Mécanique pratique, Delaunay, Traité de Mécanique rationelle, Paris, 1856; Rankine, A Manual of Applied Mechanics, second edition, London, 1861—a valuable work, too little prized in England. A translation of a new Monograph upon impact, by Poinsot, has lately appeared in the third year of Schlömich's Zeitschrift für Mathematik und Physik.

SIXTH SECTION.

STATICS OF FLUIDS.

CHAPTER I.

OF THE EQUILIBRIUM AND PRESSURE OF WATER IN VESSELS.

§ 351. Fluids.—We consider *fluids* to be bodies composed of material points, whose coherence is so slight that the smallest force suffices to separate them from each other (§ 62). Many bodies which are met with in nature, such as air, water, etc., possess this distinguishing property of fluids in an eminent degree, while others, on the contrary, such as oil, tallow, softened clay, etc., possess a less degree of fluidity. The former are called *perfectly,* and the latter *imperfectly fluid,* or viscous bodies. Certain bodies, as, E.G., *dough,* lie midway between the solids and the fluids.

Perfectly fluid bodies, of which only we will treat in the discussion which is to follow, are at the same time perfectly elastic, I.E. they can be compressed by extraneous forces, and when these forces are removed, they reassume the primitive volume. But the amount of change of volume corresponding to a certain pressure is very different for different fluids; while in *liquids* this change is quite unimportant, in *gaseous* or *aeriform fluids* it is very great, and they are therefore called *elastic* or *compressible fluids.* On account of the slight degree of compressibility of liquids, they are treated in most of the researches in hydrostatics (§ 66) as incompressible or inelastic fluids. As water is the most generally diffused of all liquids and is the most generally employed in practical life, we regard it as the representative of all these fluids, and in the researches in the mechanics of liquids we speak only of water, with the tacit understanding that the mechanical relations of other liquids are the same.

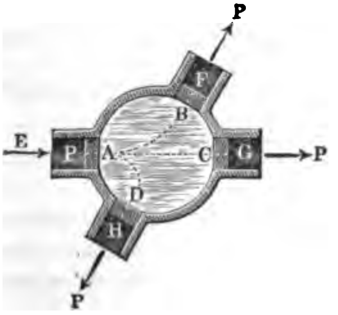
For the same reason, in the mechanics of elastic fluids we speak only of common atmospheric air.

REMARK.—A column of water, whose cross-section is one square inch, is compressed by a weight of 14,7 pounds, corresponding to the weight of the atmosphere, about 0,00005 or one fifty millionth of its volume, while a column of air under the same pressure occupies but one-half of its primitive volume. See Aimé "Ueber die Zusammendrückung der Flüssigkeiten" in Poggendorff's Annalen, Ergänzungsband (to Vol. 72), 1848. According to the formula $P = \frac{\lambda}{l} F E$ (§ 204), we have, when $P = 14,7$ pounds, $F =$ 1 square inch and $\frac{\lambda}{l} = \frac{5}{100000} = \frac{1}{20000}$, the modulus of elasticity of water

$$E = \frac{P l}{F \lambda} = 14,7 \,.\, 20000 = 294000 \text{ pounds.}$$

§ **352. Principle of Equal Pressure.**—The characteristic property of fluids, by which they are principally distinguished from solid bodies and which forms the basis of the theory of the equilibrium of fluids, is *the capacity of transmitting the pressure exerted upon a portion of their surface unchanged in all directions.* In solid bodies the pressure is transmitted only in its own direction (§ 86); if, on the contrary, water is subjected to pressure on one side, the same pressure is exerted throughout all the mass of fluid and can consequently be observed at all parts of the surface. In order to convince ourselves of the correctness of this law, we can employ an apparatus filled with water, like the one whose horizontal cross-section is represented in Fig. 574. The tubes $A\ E$, $B\ F$, etc., which are of the same size and at the same distance above the base, are closed by pistons, which are easily movable and which fit the tubes perfectly; the water will then press upon each of them, by virtue of its weight, exactly as much as upon the others. Let us for the present disregard this pressure and regard the water as imponderable. If we exert against one of the pistons A a certain pressure P, the water will

FIG. 574.



transmit the same pressure to the other pistons *B*, *C*, *D*, and to preserve the equilibrium or to prevent these pistons from moving backwards, an equal opposite pressure P (Fig. 575) must be exerted against each of the other pistons. We are therefore authorized to assume that the pressure P exerted upon a portion *A* of the surface produces a strain which is propagated not only in the straight line *A C*, but also in every other direction *B F*, *D H*, etc., upon any equally large portions *C*, *B*, *D* of the surface.

Fig. 575.

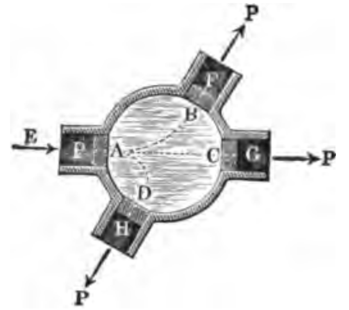
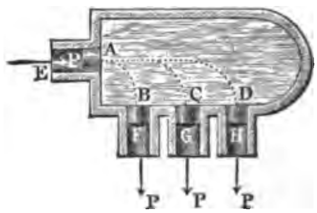


Fig. 576.



If the axes of the pipes *B F*, *C G*, etc., Fig. 576, are parallel to each other, the forces acting on the pistons can be combined so as to give a single resultant; if n is the number of the equally large pistons, the total pressure upon them will be

$$P_1 = n\,P;$$

in the case represented in the figure

$$P_1 = 3\,P.$$

Now the aggregate area F_1 of the surfaces *B*, *C*, *D*, upon which the pressures are exerted, is also $= n$ times the area F of one of the pistons; n is therefore not only $= \frac{P_1}{P}$, but also $\frac{F_1}{F}$, or in general

$$\frac{P_1}{P} = \frac{F_1}{F} \text{ and } P_1 = \frac{F_1}{F}\,P.$$

Now if we cause the tubes *B*, *C*, *D* to approach each other, until they form, as in Fig. 577, a single one, and if we close the latter by a single piston, F_1 becomes a single surface and P_1 is the pressure exerted upon it; hence we have the general law: ***the pressures exerted by a fluid upon the different parts of the walls of the vessel are proportional to the areas of those parts.***

This law corresponds also to the *principle of virtual velocities.* If the piston $A\,D = F$, Fig. 578, moves a distance $A\,A_1 = s$ inwards, it presses the prism of water $F\,s$ out of its tube, and the piston $B\,E = F_1$ moves outwards the distance $B\,B_1 = s_1$ and leaves behind it the prismatical space $F_1\,s_1$. Now as we have assumed that water can be neither expanded nor compressed, its volume must remain the same after the pistons have been moved, or the increase $F\,s$ must be equal to the decrease $F_1\,s_1$. But the equation $F_1\,s_1 = F\,s$ gives

FIG. 577.

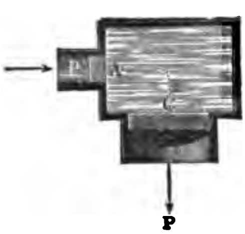
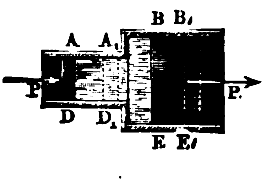


FIG. 578.



$$\frac{F_1}{F} = \frac{s}{s_1},$$

and by combining this proportion with the proportion $\frac{P_1}{P} = \frac{F_1}{F}$ we obtain

$$\frac{P_1}{P} = \frac{s}{s_1};$$

hence the mechanical effect $P_1\,s_1 = P\,s$ (see § 83).

EXAMPLE.—If the diameter of the piston $A\,D$ is 1½ inches and that of the piston $B\,E$ is 10 inches, and if the pressure exerted by the former upon the water is 36 pounds, that exerted upon the latter piston is

$$P_1 = \frac{F_1}{F}\,P = \frac{10^2}{1,5^2}\,.\,36 = \frac{400}{9}\,.\,36 = 1600 \text{ pounds.}$$

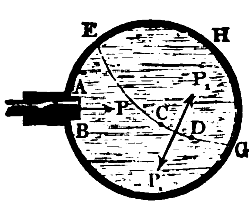
If the first piston moves 6 inches, the second moves but

$$s_1 = \frac{F}{F_1}\,s = \frac{9\,.\,6}{400} = \frac{27}{200} = 0,135 \text{ inches.}$$

REMARK.—In the following pages we will meet with many applications of this law, E.G., to the hydraulic press, water pressure engines, pumps, etc.

§ 353. Pressure in the Water.—The pressure exerted by the particles of water against each other must be estimated in exactly the same manner as the pressure of the water against the wall of the vessel. The pressures upon both sides of any surface $E\,C\,G$, which divides the water in a vessel $B\,G\,H$, Fig. 579, into two parts, when equilibrium exists, are equal. Now as a rigid body counter-

FIG. 579.



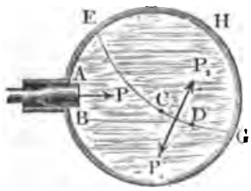
acts all forces whose directions are at right angles to its surface, the conditions of equilibrium will not be disturbed, when one-half $E\ G\ H$ of the liquid becomes rigid, or if its limiting surface becomes a wall of the vessel. If the fluid half $E\ B\ G$ in one portion $C\ D = F_1$ of the imaginary surface of separation $E\ C\ G$ exerts a pressure P_1 upon the rigid half $E\ G\ H$, the latter counteracts this pressure completely and will react with an equal opposite pressure $(-P_1)$ upon $C\ D = F_1$. Since the conditions of equilibrium will not be changed, when this mass of water $E\ G\ H$ becomes fluid again, the latter will react with an equal pressure $(-P)$ upon the mass of water $E\ B\ G$; hence the pressure of the water upon both sides of a surface $\overline{C\ D} = F$ is also determined by the proportion

$$\frac{P_1}{P} = \frac{F_1}{F},$$

when all the water is pressed in a surface $\overline{A\ B} = F$ by a force P. Hence the pressure upon any given surface F_1 in any arbitrary position is

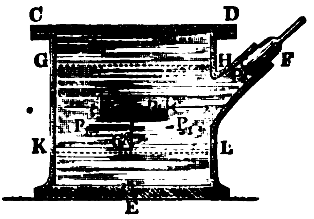
$$P_1 = \frac{F_1}{F} P.$$

FIG. 580.



The law of the transmission of pressure in water, expressed by the last proportion, is only applicable when we consider water as an imponderable fluid, and it must therefore be modified, when it is required to determine in addition the pressure arising from *the weight of the water.* If we imagine a part of the water in a vessel $C\ D\ E$, Fig. 581, to become rigid and to have the form of an infinitely thin horizontal prism $A\ B$, it is easy to see, that the pressures of the water, that remains fluid, upon the sides of the rigid part balances the weight G of the prism and that the horizontal pressures upon the vertical bases A and B of this part counteract each other. These pressures (P and $-P$) must therefore be equal and opposite to each other. Since the state of equilibrium is not changed, when $A\ B$ again becomes fluid, it follows that the pressures of the

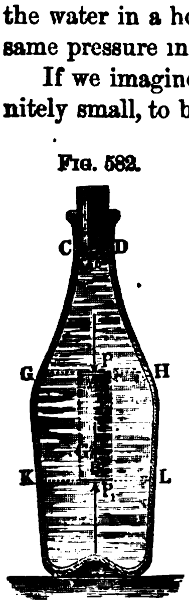
FIG. 581.



water against the vertical elements A and B of the surface, which are situated in one and the same horizontal plane, must be equal to each other, and since the pressure upon an element does not change, when its inclination or direction changes, it follows that the water in a horizontal layer, as, E.G., $G\,H$, $K\,L$, etc., exerts the same pressure in all directions and in all positions.

If we imagine a vertical prism $A\,B$, whose cross-section is infinitely small, to become rigid in the mass of water $C\,H\,K$, Fig. 582, we can conclude from the conditions of its equilibrium with the remaining liquid that the pressures exerted by the latter upon the vertical sides of the prism balance each other and that the weight G of the latter body is in equilibrium with the excess $P_1 - P$ of the pressure P_1 upon lower base B above the pressure P upon the upper base A. Hence $P_1 - P = G$, I.E. the pressure P_1 of the water upon any elementary surface B is equal to its pressure P upon an element A, of equal size and situated above it, plus the weight G of a column of water $A\,B$, whose base is one or other elementary surface and whose height is the vertical distance between the two elements. According to what precedes this rule is not only applicable to two elements, situated vertically above one another, but can also be employed for determining the pressure upon the walls of the vessel; for the two pressures P and P_1 are transmitted unchanged in the horizontal planes $G\,H$ and $K\,L$. Hence the pressure P_1 upon an elementary surface B, K or L of the horizontal plane $K\,L$ is equal to the pressure P upon an equally great element A, G or H in a higher horizontal plane plus the weight of the column of water, whose base is this element F and whose height is the distance $A\,B = h$ of the horizontal planes $G\,H$ and $K\,L$ from one another. If γ is the heaviness of water, this weight is

FIG. 582.



$$G = F\,h\,\gamma, \text{ and therefore } P_1 = P + G = P + F\,h\,\gamma.$$

If the areas of the elements of surface are unequal; if, E.G., the area of the upper one (in $G\,H$) is F and that of the lower one (in $K\,L$) is F_1, the pressure upon the latter is

$$P_1 = \frac{F_1}{F}\,(P + F\,h\,\gamma) = \frac{F_1}{F}\,P + F_1\,h\,\gamma.$$

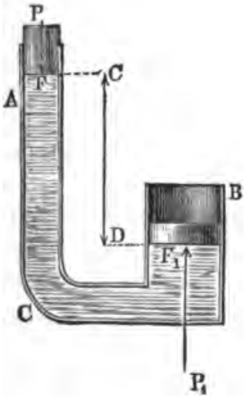
By means of the same formula the pressure P upon an element

F in the horizontal plane $G\,H$ can be determined, when the exterior pressure P_0 upon an element of the surface $C\,D = F_0$, which is at a distance h above or below $G\,H$, is known. It is

$$P = \frac{F}{F_0} P_0 \pm F\,h\,\gamma.$$

Since the pressures upon equal elements in a horizontal plane are equal to each other, it follows that the foregoing formula is applicable to horizontal surfaces of finite dimension, as, E.G., where the water serves to transmit the force P, which acts upon a horizontal piston F, Fig. 583, to another horizontal piston F_1. This formula

FIG. 583.



$$P_1 = \frac{F_1}{F} P + F_1\,h\,\gamma = F_1 \left(\frac{P}{F} + h\,\gamma\right)$$

gives directly the pressure P_1 upon this surface, when h denotes the vertical height $C\,D$ between the surfaces of the two pistons.

If we denote the pressures $\frac{P}{F}$ and $\frac{P_1}{F_1}$ upon the units of surface by p and p_1, we have more simply

$$p_1 = p + h\,\gamma.$$

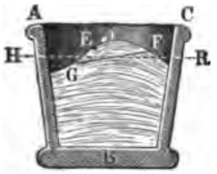
EXAMPLE.—If the diameters of the two pistons F and F_1 of a hydrostatic press $A\ C\ B$, Fig. 583, are $d = 2\frac{1}{2}$ inches and $d_1 = 9$ inches, and if they are situated at the distance $C\,D = h = 60$ inches above one another, and if the larger piston is to exert a pressure $P_1 = 1600$ pounds, we have the force which must be applied to the smaller piston

$$P = \frac{F}{F_1} P_1 - F\,h\,\gamma = \left(\frac{d}{d_1}\right)^2 P_1 - \frac{\pi\,d^2}{4}\,h\,\gamma$$

$$= \left(\frac{5}{18}\right)^2 . 1600 - \frac{\pi}{4} . \frac{25}{4} . \frac{60 . 62,5}{1728} = 123,46 - 10,66 = 112,8 \text{ pounds.}$$

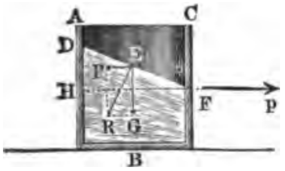
§ **354. Surface of Water.**—In consequence of the action of gravity upon water, all the elements of it tend to descend, and really do so when they are not prevented. In order to keep a quantity of water together, it is necessary to confine it in a vessel. The water in a vessel $A\ B\ C$, Fig. 584, can only be in equilibrium when the free surface $H\,R$ is at right angles to the direction of gravity, or horizontal; for so long as this surface is curved or inclined to the horizon there will be elements of the water, such as E, which, being situated above the others, will, in consequence of their great mobility and their weight, slide down those below them as upon an inclined plane. Since, when the distances are very great, the directions of gravity cannot be considered as parallel lines, the free surface or the surface of the water in a very large vessel, E.G. in a large sea, will not, under these circumstances, form a plane surface, but a portion of the surface of a sphere. If another force acts, in addition to gravity, upon the elements of the water, then, when equilibrium exists, the free surface of the water is at right angles to the resultant of this force and that of gravity.

FIG. 584.



If a vessel $A\ B\ C$, Fig. 585, is moved forward with the *constant acceleration* p, the free surface of the water forms an *inclined plane* $D\ F$; for in this case every element E of this surface is drawn vertically downwards by its weight G and in a horizontal direction by its inertia $P = \frac{p}{g}\ G$, the two forces giving rise to a resultant R, whose direction forms, with that of gravity, a constant angle $R\ E\ G = a$. This angle is at the same time the angle $D\ F\ H$ formed by the surface of the water (which is at right angles to the resultant) with the horizon. It is determined by the equation

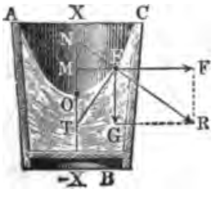
FIG. 585.



$$tang.\ a = \frac{P}{G} = \frac{p}{g}.$$

If, on the contrary, a vessel $A\ B\ C$, Fig. 586, is caused *to revolve uniformly* about its vertical axis $X\ \overline{X}$, the surface of the revolving water forms a hollow $A\ O\ C$, whose cross-section through the axis is a *parabola*. If ω is the angular velocity of the vessel and of the water in it, G the weight of an element E of the water, and y its distance $M\ E$ from the vertical axis, we have the centrifugal force of this element

FIG. 586.



$$F = \omega^2 \frac{G y}{g} \text{ (§ 302)},$$

and therefore for the angle $R E G = T E M = \phi$, formed by the resultant with the vertical or by the tangent $E T$ to the profile of the water with the horizontal line $M E$,

$$tang. \phi = \frac{F}{G} = \frac{\omega^2 y}{g}$$

From this formula we see that the tangent of the angle, formed by the tangent line with the ordinate, is proportional to the ordinate. Since this is one of the properties of the common parabola (see § 157), the vertical cross-section $A\ O\ C$ of the surface of the water is a parabola, whose axis coincides with the axis of rotation $X \overline{X}$.

If the *velocity of rotation* of the water in the vessel $A\ B\ D$, Fig. 588, were constant and $= c$, we would have $F = \frac{c^2\ G}{g\ y}$, and therefore $tang. \phi = \frac{c^2}{g y}$; hence the subtangent of the curve, formed by the cross-section $A\ E\ B$ of the water, $M\ T = m = \frac{c^2}{g}$ or constant. According to Article 20 of the Introduction to the Calculus, the equation of such a curve is

FIG. 587.

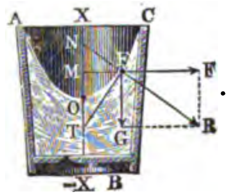


FIG. 588.

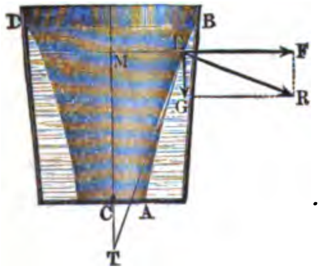
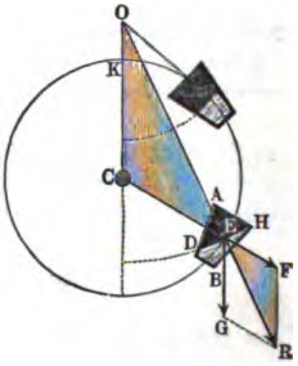


FIG. 589.



$$y = r\,e^{\frac{x}{m}} = r\,e^{g x c^{-2}},$$

r denoting the ordinate of the beginning A.

If we cause a vessel $A\ B\ H$, Fig. 589, to move uniformly in a vertical circle around a horizontal axis C, the surface of the water will assume a *cylindrical form*, with a circular cross-section $D\ E\ H$. If we prolong the direction of the resultant R of the weight G and of the centrifugal force F of an element E until it cuts the vertical line $C\ K$, passing through the centre of rotation, we obtain the two similar triangles $E\ C\ O$ and $E\ F\ R$, for which we have

$$\frac{C\ O}{E\ C} = \frac{F\ R}{E\ F} = \frac{G}{F};$$

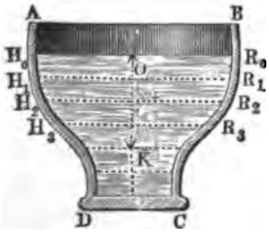
but if we put the radius of gyration $E\ C = y$ and retain the last notations, we have $F = \frac{\omega^2\, G y}{g}$, whence it follows that the line

$$C\ O = \frac{g}{\omega^2} = g\left(\frac{30}{\pi\ u}\right)^2 = \frac{2936}{u^2}\ \text{feet} = \frac{894{,}6}{u^2}\ \text{meters},$$

u denoting the number of revolutions per minute. Since this value of $C\ O$ is the same for all the elements of the water, it follows that the resultants of all the elements of the water forming the cross-section $D\ E\ H$ are directed towards O, and that the cross-section, which is at right-angles to all these directions, is the arc of a circle described from O. Hence the surfaces of the water in the buckets of an overshot water-wheel are always *cylindrical* ones, described from the same horizontal axis.

§ 355. Pressure upon the Bottom.—The pressure in a vessel $A\ B\ C\ D$, Fig. 590, is a minimum immediately below the surface, increases with the depth, and is a maximum at the bottom. This, although a consequence of § 353, can also be proved as follows. Let us suppose that the area of the surface $H_0\ R_0$ of the water is F_0 and that a pressure P_0 is exerted uniformly upon it, E.G. by the atmosphere lying above it or by a piston, and let us imagine the entire mass of water to be divided by very many horizontal planes, such as $H_1\ R_1$, $H_2\ R_2$, etc., into equally thick layers. If F_1 is the area of the first layer $H_1\ R_1$, λ its thickness, and γ the heaviness of water, we have the weight of the first layer $G_1 = F_1\,\lambda\,\gamma$, and that portion of the pressure in $H_1\ R_1$ produced by the pressure

FIG. 590.



P_0 upon the surface of the water $H_0\, R_0$, according to the principles enunciated in § 352, is

$$= \frac{P_0\, F_1}{F_0}.$$

Adding both these pressures, we obtain the pressure in the horizontal section $H_1\, R_1$

$$P_1 = \frac{P_0\, F_1}{F_0} + F_1\, \lambda\, \gamma.$$

Dividing by F_1, we obtain the equation

$$\frac{P_1}{F_1} = \frac{P_0}{F_0} + \lambda\, \gamma,$$

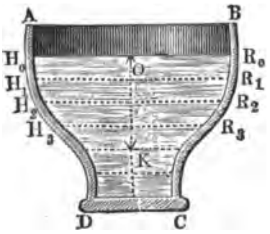
or, since $\frac{P_0}{F_0}$ and $\frac{P_1}{F_1}$ denote the pressures p_0 and p_1 in $H_0\, R_0$ and $H_1\, R_1$ referred to the unit of surface, we have

$$p_1 = p_0 + \lambda\, \gamma.$$

The pressure in the following horizontal layer $H_2\, R_2$ is determined exactly in the same manner as the pressure in the layer $H_1\, R_1$, but we must not forget that the initial pressure upon an element of the surface is in this case $p_1 = p_0 + \lambda\, \gamma$, while in the first case it was p_0. Hence the pressure in the horizontal layer $H_2\, R_2$ is

$$p_2 = p_1 + \lambda\, \gamma = p_0 + \lambda\, \gamma + \lambda\, \gamma = p_0 + 2\, \lambda\, \gamma,$$

FIG. 591.



in like manner the pressure in the third layer $H_3\, R_3$ is

$$= p_0 + 3\, \lambda\, \gamma,$$

in the fourth

$$= p_0 + 4\, \lambda\, \gamma,$$

and in the nth

$$= p_0 + n\, \lambda\, \gamma.$$

But $n\, \lambda$ is the depth $O\, K = h$ of this nth layer below the surface of the water; we can therefore put the pressure upon each unit of surface in the nth horizontal layer

$$p = p_0 + h\, \gamma \text{ (compare § 353).}$$

We call the depth h of one element of surface below the level of the water its *head or height of water* (Fr. charge d'eau; Ger. Druckhöhe), and we find the pressure of the water upon any unit of surface by adding to the pressure applied from without the weight of a column of water, whose base is unity and whose height is the head of water. When a surface is horizontal, as E.G. the bottom $C\, D$ (Fig. 591), the head of water h is the same for all positions, and if its area is $= F$, the pressure of the water upon it is

$$P = (p_0 + h\gamma)\,F = F p_0 + F h \gamma = P_0 + F h \gamma,$$

or, if we neglect the external pressure, $P = F h \gamma$. *The pressure of the water upon a horizontal surface is therefore equal to the weight of the column* of water $F h$ above it.

This pressure of the water upon a horizontal surface, E.G. upon the horizontal bottom or upon a horizontal portion of the wall of a vessel, is entirely independent of the form of the vessel; whether the vessel $A\ C$, Fig. 592, is prismatic as in a, or wider above than below as in b, or wider below than above as in c, or inclined as in d, or with spherical walls as in e, etc., the pressure upon the bottom is always equal to the weight of a column of water, whose base is the bottom of the vessel and whose height is its depth below the level of the water. Since the pressure of water is transmitted in all directions, this law is also applicable when the surface, as E.G. $B\ C$, in Fig. 593, is pressed from below upwards. Each unit of surface of the layer of water $B\ K$, touching $B\ C$, is subjected to the pressure of a column of water, whose height is $H B = R K = h$, and the pressure against the surface $C B$ is $= F h \gamma$, F denoting the area of that surface.

FIG. 592.

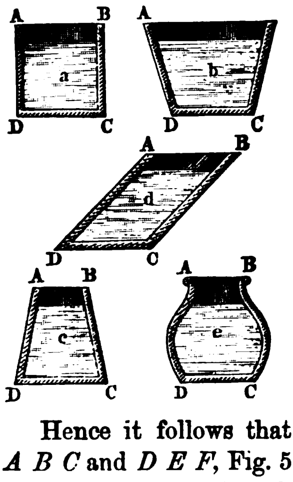


FIG. 593.

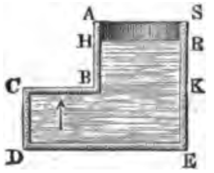
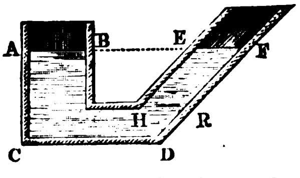


FIG. 594.



Hence it follows that the water in the communicating tubes $A\ B\ C$ and $D\ E\ F$, Fig. 594, will stand at the same height, when in equilibrium, or that the surfaces $A\ B$ and $E\ F$ will be in the same horizontal plane. In order to preserve the equilibrium it is necessary that the layer of water $H\ R$ shall be pressed downwards by the column of water $E\ R$ above it as much as it is pressed upwards by the mass of water below it. Since in both cases the surface pressed upon is the same, the head of water must be the same, and the level of the water at $A\ B$ must be at the same height above $H\ R$ as that at $E\ F$.

§ **356. Lateral Pressure.**—The formula just found for the pressure of water against a horizontal surface, is not directly applicable to *a plane surface inclined to the horizon;* for in this case the head of water is different at different points.

The pressure $p = h\,\gamma$ upon every unit of surface within the horizontal layer at the depth h below the surface of the water acts in all directions (§ 352), and, consequently, at right angles to the walls of the vessel, by which (§ 138) it is entirely counteracted. Now if F_1 is the area of an element of the side $A\ B\ C$, Fig. 595, and h_1 its head of water $F_1\ H_1$, we have the pressure perpendicular to it

$$P_1 = F_1 \,.\, h_1\,\gamma;$$

if F_2 is the surface of a second element and h_2 its head of water, we have the normal pressure upon it

$$P_2 = F_2\,h_2\,\gamma;$$

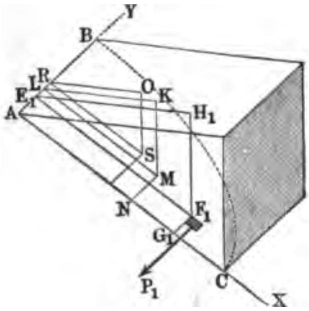
and in like manner for a third element

$$P_3 = F_3\,h_3\,\gamma, \text{ etc.}$$

These normal pressures form a system of parallel forces, whose resultant P is the sum of these pressures, I.E.,

$$P = (F_1\,h_1 + F_2\,h_2 + \ldots)\,\gamma.$$

FIG. 595.



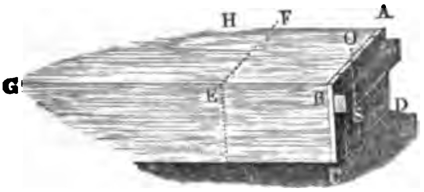
But $F_1\,h_1 + F_2\,h_2 + \ldots$ is the sum of the statical moments of F_1, F_2, etc., in reference to the surface $A\ O\ B$ of the water and $= F\,h$, when F denotes the area of the whole surface and h the depth $S\ O$ of its centre of gravity S below the surface of the water; hence the entire normal pressure against the plane surface is

$$P = F\,h\,\gamma.$$

If we understand by the *head of water* of a surface the depth of its centre of gravity below the surface of the water, the following rule will be generally applicable, viz.: *the pressure of water against a plane surface is equal to the weight of a column of water, whose base is the surface and whose height is its head of water.*

We must here observe that this pressure does not depend upon the quantity of water above or in front of the surface pressed, thus, E.G., if the other circumstances are the same, a wall $A\ B\ C\ D$, Fig. 596, has to resist the same pressure whether it dams up the water of a small trough $A\ C\ E\ F$ or that of a large dam $A\ C\ G\ H$ or that of a lake. From the width $A\ B = C\ D = b$ and the height $A\ D = B\ C = a$ of the rectangular wall we obtain the surface of the same $F = a\ b$ and the head of water $S\ O = \frac{a}{2}$, and, therefore, the pressure of the water against it is

FIG. 596.



$$P = a\ b \,.\, \frac{a}{2}\, \gamma = \tfrac{1}{2}\, a^2\, b\, \gamma.$$

The pressure increases therefore with the width and with the square of the height of the surface pressed upon.

EXAMPLE.—If the water in front of a sluice gate, made of oak, 4 feet wide, 5 feet high and 2½ inches thick, stands 3½ feet high, how great a force is required to lift it?

The volume of this gate

$$4 \,.\, 5 \,.\, \tfrac{5}{24} = \tfrac{25}{6} \text{ cubic feet.}$$

Assuming the heaviness of oak, saturated with water, to be according to § 61, 62,5 . 1,11 = 69,375 pounds, the weight of this gate is

$$G = \tfrac{25}{6} \,.\, 69{,}375 = 25 \,.\, 11{,}5625 = 289{,}06 \text{ pounds.}$$

The pressure of the water against the gate and the pressure of the latter against its guides is

$$P = \tfrac{1}{2}\, (\tfrac{7}{2})^2 \,.\, 4 \,.\, 62{,}5 = 49 \,.\, 31{,}25 = 1531{,}25 \text{ pounds;}$$

putting the coefficient of friction for wet wood (§ 174) $\phi = 0{,}68$, we have the friction of the gate upon its guides

$$F = \phi\, P = 0{,}68 \,.\, 1531{,}25 = 1041{,}25.$$

Adding to the latter the weight of the gate, we have the force necessary to draw it up

$$= 1041{,}25 + 289{,}06 = 1330{,}31 \text{ pounds.}$$

357. Centre of Pressure of Water.—The resultant $P = F h \gamma$ of all the elementary pressures $F_1 h_1 \gamma$, $F_2 h_2 \gamma$, etc., has, like the resultant of any other system of parallel forces, a definite point of application, which is called the *centre of pressure.* By retaining or supporting this point the whole pressure of the water upon a surface will be held in equilibrium. The statical moments of the elementary pressures $F_1 h_1 \gamma$, $F_2 h_2 \gamma$, etc., in reference to the plane of the surface of the water $A\ B\ O$, Fig. 595, are

$$F_1 h_1 \gamma \,.\, h_1 = F_1 h_1^2 \gamma,\ F_2 h_2 \gamma \,.\, h_2 = F_2 h_2^2 \gamma, \text{ etc.,}$$

and the statical moment of the entire pressure of the water in reference to this plane is

$$(F_1 h_1^2 + F_2 h_2^2 + \ldots) \gamma.$$

Denoting the distance $K M$ of *the centre M of this pressure* from the surface of the water by z, we have the moment of the pressure of the water

$$P z = (F_1 h_1 + F_2 h_2 + \ldots) z \gamma,$$

and by putting these moments equal to each other we obtain the distance of *this centre M* below the surface of the water

$$1) \quad z = \frac{F_1 h_1^2 + F_2 h_2^2 + \ldots}{F_1 h_1 + F_2 h_2 + \ldots} \text{ or } = \frac{F_1 h_1^2 + F_2 h_2^2 + \ldots}{F h},$$

when, as above, F denotes the area of the entire surface and h the depth of the centre of gravity below the surface of the water.

In order to determine completely this point of pressure, we must find its distance from another line or plane. If we put the distances $F_1 G_1$, $F_2 G_2$, etc., of the elements F_1, F_2, etc., of the surface from the line $A C$, which determines the angle of inclination of the plane, $= y_1, y_2$, etc., we have the moments of the elementary pressures in reference to this line

$$= F_1 h_1 y_1 \gamma, F_2 h_2 y_2 \gamma, \text{etc.},$$

and the moment of the entire surface

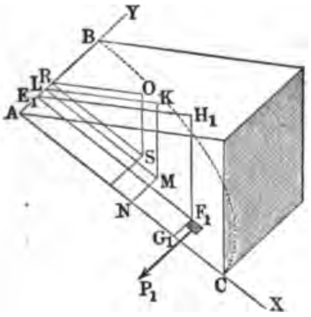
$$= (F_1 h_1 y_1 + F_2 h_2 y_2 + \ldots) \gamma;$$

denoting the distance $M N$ of this centre M from that line by v, we have also this moment

$$= (F_1 h_1 + F_2 h_2 + \ldots) v \gamma.$$

Equating these two moments, we obtain the second ordinate

FIG. 597.



$$2) \quad v = \frac{F_1 h_1 y_1 + F_2 h_2 y_2 + \ldots}{F_1 h_1 + F_2 h_2 + \ldots} \text{ or } = \frac{F_1 h_1 y_1 + F_2 h_2 y_2 + \ldots}{F h}.$$

If a denote the angle of inclination of the plane $A B C$ to the horizon, x_1, x_2, etc., the distances $E_1 F_1$, $E_2 F_2$, etc., of the elements F_1, F_2, etc., and u is the distance $L M$ of the centre of pressure M from the line of intersection $A B$ of the plane with the surface of the water, we have $h_1 = x_1 \sin. a$, $h_2 = x_2 \sin. a$, etc., and also $z = u \sin. a$; substituting these values of z and v in the expression, we obtain

$$u = \frac{F_1 x_1^2 + F_2 x_2^2 + \ldots}{F_1 x_1 + F_2 x_2 + \ldots} = \frac{\text{Moment of inertia}}{\text{Statical moment}} \text{ and}$$

$$v = \frac{F_1 x_1 y_1 + F_2 x_2 y_2 + \ldots}{F_1 x_1 + F_2 x_2 + \ldots} = \frac{\text{Moment of the centrifugal force}}{\text{Statical moment}}$$

We find then the distances u and v of the centre of pressure from the horizontal axis $A\ Y$ and from the axis $A\ X$, formed by the line of dip, when we divide by the statical moment of the surface with reference to the first axis, in the first place, the moment of inertia in reference to the same axis, and, in the second, the moment of the centrifugal force of the same in reference to both axes. The first distance is also that of the *centre of oscillation* from the line of intersection with the surface of the water. Besides it is easy to perceive that the *centre of pressure* of water coincides perfectly with the *centre of percussion*, determined in § 313, when the line of intersection $A\ Y$ of the surface with the surface of the water is regarded as the axis of rotation.

§ 358. Pressure of Water against Rectangles and Triangles.—If the surface pressed upon is a *rectangle* $A\ C$, Fig. 598, with a horizontal base line $C\ D$, the centre M of pressure is found in the line of dip $K\ L$, which bisects the base line, and it is at a distance equal to two-thirds of this line from the side $A\ B$, which lies in the surface of water. If the rectangle, as in Fig. 599, does not reach the surface of the water, then, if the distance $K\ L$ of the lower line $C\ D$ from the surface of the water $= l_1$ and that $K\ O$ of the upper one $A\ B$, $= l_2$, we have the distance $K\ M$ of the centre of pressure from the surface of the water

FIG. 598.

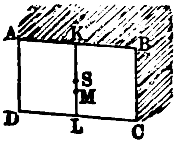


FIG. 599.

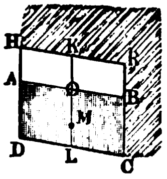
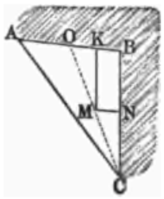


FIG. 600.



$$u = \tfrac{2}{3} \cdot \frac{l_1^3 - l_2^3}{l_1^2 - l_2^2}.$$

The distance $K\ M$ of the centre of pressure M of a *right-angle triangle* $A\ B\ C$, Fig. 600, whose base $A\ B$ lies in the surface of the water, from $A\ B$ (Example § 313) is

$$u = \frac{\frac{1}{6} F . l^2}{\frac{1}{3} F . l} = \tfrac{1}{2}\, l,$$

when l denotes the altitude $B\ C$ of the triangle.

The distance of this point M from the other side $B\ C$ is, since this point lies in the line $C\ O$, which bisects the triangle and runs from the apex C to the middle of the base, $N M = v = \frac{1}{4}\, b$, b denoting the base $A\ B$.

If the apex C is situated at the surface of the water, as in Fig. 601, and if the base $A\ B$ is below the apex, we have

$$K M = u = \frac{\frac{1}{2} F l^2}{\frac{2}{3} F l} = \tfrac{3}{4}\, l \text{ and}$$

$$N M = v = \tfrac{3}{4} \cdot \frac{b}{2} = \tfrac{3}{8}\, b.$$

If the whole triangle $A\ B\ C$, Fig. 602, is immersed in the water, and the base $A\ B$ is at a distance $A\ H = l_2$ and the apex C at the distance $C\ H = l_1$ from surface $H\ R$, we determine the distance $M\ K$ of the centre of pressure M below the surface of the water $H\ R$ by means of the formula

FIG. 601.

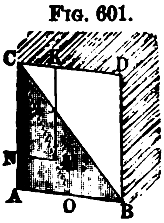
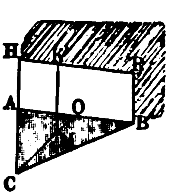


FIG. 602.



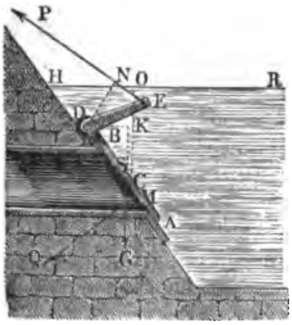
$$u = \frac{\frac{1}{18} F (l_1 - l_2)^2 + F\left(l_2 + \frac{l_1 - l_2}{3}\right)^2}{F\left(l_2 + \frac{l_1 - l_2}{3}\right)}.$$

$$= \frac{\frac{1}{18}(l_1 - l_2)^2 + \frac{1}{9}(2\, l_2 + l_1)^2}{\frac{1}{3}(2\, l_2 + l_1)} = \frac{l_1^2 + 2\, l_1\, l_2 + 3\, l_2^2}{2\,(l_1 + 2\, l_2)}.$$

The centre of pressure of other plane figures can be determined in the same manner.

EXAMPLE.—What force P must we employ to raise a circular clack-valve $A\ B$, Fig. 603, which is movable about a horizontal axis D? Let the length of this valve be $= 1\frac{1}{4}$ feet, its diameter $A\ B$ be $= 1\frac{1}{4}$ feet, and the distance of its centre of gravity S from the axis D be $D\ S = 0{,}75$ feet, and its weight be $G = 35$ pounds; further, let the distance $D\ H$ of the axis of rotation D from the surface of the water, measured in the plane of the valve, be $= 1$ foot and the angle of inclination of this plane to the horizon be $a = 68°$.

FIG. 603.



The surface upon which the pressure is exerted is

$$F = \pi\, r^2 = \frac{\pi\, d^2}{4} = 0{,}7854 \cdot \frac{25}{16} = 1{,}2272 \text{ square feet,}$$

and the head of water or depth of its centre C below the water level is

$$O\,C = h = H\,C.\,sin.\,a = (HD + D\,C)\,sin.\,a = (H\,D + DB + B\,C)\,sin.\,a$$
$$= (1 + 0{,}25 + 0{,}625)\ sin.\ 68^\circ = 1{,}875\,.\,0{,}9272 = 1{,}7385 \text{ feet},$$

and, therefore, the pressure of the water upon the surface $A\,B = F$ is

$$Q = F\,h\,\gamma = 1{,}2272\,.\,1{,}7385\,.\,62{,}5 = 133{,}34;$$

the arm b of this force with reference to the axis of rotation D is the distance $D\,M$ of the centre of pressure M from it, hence

$$b = H\,M - H\,D.$$

But we have

$$H\,M = H\,C + \frac{r^2}{4\,H\,C} = 1{,}875 + \frac{1}{4\,.\,1{,}875}\,.\left(\frac{5}{8}\right)^2 = 1{,}9271 \text{ feet},$$

whence $b = 1{,}9271 - 1{,}0000 = 0{,}9271$ feet,

and the required statical moment of the pressure is

$$Q\,b = 133{,}34\,.\,0{,}9271 = 123{,}62 \text{ foot-pounds.}$$

The arm of the weight of the valve is

$$D\,K = D\,\overline{S}\,cos.\,a = 0{,}75\,.\,cos.\ 68^\circ = 0{,}75\,.\,0{,}3746 = 0{,}2810 \text{ feet},$$

and therefore its statical moment is

$$= 35\,.\,0{,}2810 = 9{,}84 \text{ foot-pounds.}$$

By adding these moments, we obtain the entire moment necessary to open the valve

$$P\,a = 123{,}62 + 9{,}84 = 133{,}46 \text{ foot-pounds.}$$

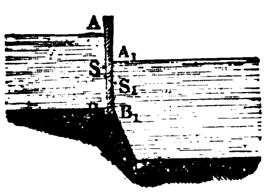
Now if the arm of the force, which opens the valve, is $D\,N = a = 0{,}75$ feet, the intensity of that force must be

$$P = \frac{133{,}46}{0{,}75} = 177{,}95 \text{ pounds.}$$

§ 359. Pressure upon Both Sides of a Surface.

—If a *plane surface* $A\,B$, Fig. 604, is subjected upon *both sides* to the pressure of water, the two resultants of the pressures on the two sides give rise to a new resultant, which, as they act in opposite directions, is obtained by subtracting one from the other.

FIG. 604.



If F is the area of the portion $A\,B$ subjected to pressure on one side of the surface, and h the depth $A\,S$ of its centre of gravity below the surface of the water, and if F_1 is the area of the portion $A_1\,B_1$ on the other side, which is subjected to the pressure of the water, and h_1 the depth $A_1\,S_1$ of its centre of gravity below the corresponding surface of the water, the required resultant will be

$$P = F\,h\,\gamma - F_1\,h_1\,\gamma = (F\,h - F_1\,h_1)\,\gamma.$$

If the moment of inertia of the first portion of the surface with reference to the line, in which the plane of the surface cuts the first surface of the water, $= F k^2$, we have the statical moment of the pressure of the water upon one side

$$= F k^2 \gamma,$$

and if the moment of inertia of the second portion of the surface, with reference to its line of intersection with the other surface of the water, $= F_1 k_1^2$, we will have in like manner the statical moment of the pressure of the water on the other side, with reference to the axis in the second surface of the water,

$$= F_1 k_1^2 \gamma.$$

Putting the difference of level $A A_1$ of the two surfaces of the water $= a$, we have the increase of the latter moment, when we pass from the axis A_1 to the axis A,

$$= F_1 h_1 a \gamma,$$

and consequently the statical moment of the pressure $F_1 h_1 \gamma$, in reference to the axis A in the first surface of water, is

$$= F_1 k_1^2 \gamma + F_1 h_1 . a . \gamma = (F_1 k_1^2 + F_1 a h_1) \gamma.$$

Hence it follows that the statical moment of the difference of the two resultants is

$$= (F k^2 - F_1 k_1^2 - a F_1 h_1) \gamma,$$

and the arm of this difference or the distance of the centre of pressure from the axis in the first surface of water is

$$u = \frac{F k^2 - F_1 k_1^2 - a F_1 h_1}{F h - F_1 h_1}.$$

If the portions of surface which are subjected to pressure are equal, as is represented in Fig. 605, where the whole surface $A B = F$ is submerged, we have more simply

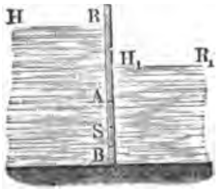
$$P = F (h - h_1) \gamma,$$

and since $k^2 = k_1^2 + 2 a h_1 + a^2$ (see § 224) and $h - h_1 = a$, we have

$$u = \frac{k^2 - k_1^2 - a h_1}{h - h_1} = \frac{a h_1 + a^2}{a}$$

$$= h_1 + a = h.$$

FIG. 605.



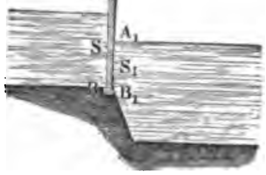
In the latter case the pressure is equal to the weight of a column of water, whose base is the surface pressed upon and whose height is the difference of level $R H_1$ of the water on the two sides of the surfaces, and the centre of pressure coincides with the centre of gravity S of the surface. This law is also correct when the two surfaces of water are subjected to equal pressure, E.G. by means of pistons or by the atmosphere; for if the pressure upon each unit of surface $= p$ and the height of the corresponding column of water is $l = \frac{p}{\gamma}$ (§ 355), we must substitute, instead of h, $h + l$, and instead of h_1, $h_1 + l$; by subtraction we obtain the pressure

$$P = (h + l - [h_1 + l])\, F\gamma = (h - h_1)\, F\gamma.$$

For this reason we generally neglect the pressure of the air in hydrostatical experiments.

Example.—The depth $A\,B$ of the water in the head-bay, Fig. 606, is 7 feet, the water in the chamber of the lock rises 4 feet upon the gate, and the width of the canal and lock-chamber is 7,5 feet; what is the resulting pressure upon the gate of the lock?

Fig. 606.



Here

$$F = 7\,.\,7{,}5 = 52{,}5 \text{ square feet},$$
$$F_1 = 4\,.\,7{,}5 = 30{,}0 \text{ square feet},$$
$$h = \frac{1}{2}\,.\,7 = \frac{7}{2},\ h_1 = \frac{4}{2} = 2 \text{ feet},$$
$$a = 7 - 4 = 3 \text{ feet},$$
$$k^2 = \frac{1}{3}\,.\,7^2 = \frac{49}{3} \text{ and } k_1^2 = \frac{1}{3}\,.\,4^2 = \frac{16}{3};$$

hence the required resultant is

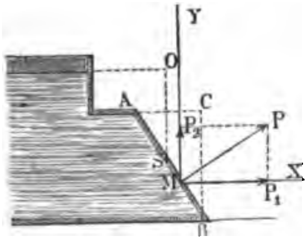
$$P = (F h - F_1\, h_1)\,\gamma = \left(52{,}5\,.\,\frac{7}{2} - 30\,.\,2\right).\,62{,}5$$
$$= 123{,}75\,.\,62{,}5 = 7734{,}4 \text{ pounds},$$

and the depth of the point of application below the surface of the water is

$$u = \frac{52{,}5\,.\,\frac{49}{3} - 30\,.\,\frac{16}{3} - 3\,.\,60}{52{,}5\,.\,\frac{7}{2} - 60} = \frac{517{,}5}{123{,}75} = 4{,}182 \text{ feet.}$$

§ 360. Pressure in a Given Direction.—In many cases we wish to know but one part of the pressure, viz.: that exerted in a certain direction. In order to find such a component, we decompose the normal pressure $\overline{M\,P} = P$ on the surface $\overline{A\,B} = F$, Fig. 607, into two components, one in the given direction $M\,X$ and one at right angles to it, viz.:

Fig. 607.



$$M\,P_1 = P_1 \text{ and } M\,P_2 = P_2.$$

Now if a is the angle $P\,M\,X$ formed by the direction of the normal pressure with the given direction $M\,X$ of the component, the components will be

$$P_1 = P \cos.\, a \text{ and } P_2 = P \sin.\, a.$$

If we project the surface $A\,B$ upon a plane perpendicular to the given direction $M\,X$, we have the area of the projection $B\,C$

$$F_1 = F \,.\, cos.\ A\ B\ C,$$

or, since the angle of inclination $A\ B\ C$ of the surface to its projection is equal to the angle $P\ M\ X = a$, formed by the direction of the normal pressure and that of its component P_1,

$$F_1 = F\ cos.\ a, \text{ and inversely}$$

$$cos.\ a = \frac{F_1}{F};$$

the required component is therefore

$$P_1 = P \,.\, \frac{F_1}{F}.$$

Now, since the value of the normal pressure is $P = F\,h\,\gamma$, we have

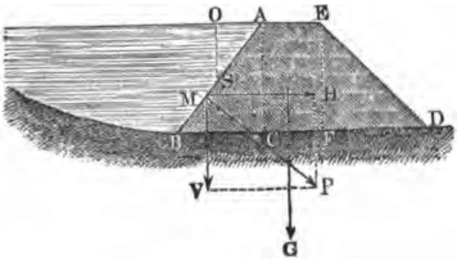
$$P_1 = F_1\,h\,\gamma,$$

I.E., *the pressure exerted by water in any direction upon a surface is equal to the weight of a column of water, whose base is the projection of the surface at right angles to the given direction and whose height is the depth of the centre of gravity of the surface below the surface of the water.*

In most cases in practice we are only required to determine the vertical or a horizontal component of the pressure of the water against the surface. Since the projection at right angles to the vertical direction is the horizontal projection and that at right angles to a horizontal direction is a vertical one, we find the *vertical pressure* of the water against a surface by treating its *horizontal projection* as the surface subjected to pressure, and, on the contrary, the *horizontal pressure* of the water in any direction by treating the *vertical projection*, or elevation, of the surface at right angles to the given direction as the surface pressed upon, and in both cases we must regard the depth $O\ S$ of the centre of gravity S of the surface below the surface of the water as the head of water.

Hence, if we wish to determine in the case of a *prismatical embankment* or *dam $A\ B\ D\ E$,* Fig. 608, the horizontal pressure of the water, we must consider the longitudinal elevation $A\ C$, and if the vertical pressure is to be determined, the horizontal projection $B\ C$ of the surface $A\ B$ must be considered as the surface pressed upon. Putting the length of the

FIG. 608.



dam $= l$, its height $A\,C = h$ and horizontal projection of the slope $B\,C = a$, we have the horizontal pressure of the water

$$H = l\,h\,.\,\frac{h}{2}\,\gamma = \tfrac{1}{2}\,h^2\,l\,\gamma$$

and its vertical pressure

$$V = a\,l\,.\,\frac{h}{2}\,\gamma = \tfrac{1}{2}\,a\,l\,h\,\gamma.$$

Now if the width of the top of this dam is $A\,E = b$, the horizontal projection of the other slope $D\,F = a_1$ and the heaviness of the material of the dam $= \gamma_1$, the weight of the dam is

$$G = \left(b + \frac{a + a_1}{2}\right) h\,l\,\gamma_1,$$

and the entire vertical pressure of the dam upon its horizontal base is

$$V + G = \tfrac{1}{2}\,a\,l\,h\,\gamma + \left(b + \frac{a + a_1}{2}\right) h\,l\,\gamma_1 = \left[\tfrac{1}{2}\,a\,\gamma + \left(b + \frac{a + a_1}{2}\right)\gamma_1\right] h\,l.$$

Putting the coefficient of friction $= \phi$, we have the friction or force necessary to *push the dam forward*

$$F = \phi\,(V + G) = \left[\tfrac{1}{2}\,a\,\gamma + \left(b + \frac{a + a_1}{2}\right)\gamma_1\right]\phi\,h\,l.$$

When the horizontal pressure pushes the embankment forward, we must have

$$\tfrac{1}{2}\,h^2\,l\,\gamma = \left[\tfrac{1}{2}\,\gamma\,a + \left(b + \frac{a + a_1}{2}\right)\gamma_1\right] . \,\phi\,h\,l,$$

or more simply

$$h = \phi\left(a + (2\,b + a + a_1)\,\frac{\gamma_1}{\gamma}\right).$$

If we wish to prevent the dam from being moved, we must make

$$h < \phi\left(a + (2\,b + a + a_1)\,\frac{\gamma_1}{\gamma}\right), \text{ or}$$

$$b > \tfrac{1}{2}\left[\left(\frac{h}{\phi} - a\right)\frac{\gamma}{\gamma_1} - (a + a_1)\right].$$

For the sake of greater security we assume that the water has penetrated below the base of the dam to a great extent, and for this reason, in the worst case, we must consider that an opposite pressure $= (b + a + a_1)\;l\;h\;\gamma$ is acting from below upwards; hence we must put

$$h < \phi\left[(2\,b + a + a_1)\left(\frac{\gamma_1}{\gamma} - 1\right) - a_1\right].$$

Example.—If the density of the clay composing the dam is nearly double that of water, or

$$\frac{\gamma_1}{\gamma} = 2 \text{ and } \frac{\gamma_1}{\gamma} - 1 = 1,$$

we can write simply

$$h < \phi\,(2\,b + a).$$

It has been found by experiment that a dam resists sufficiently, when its height, top and the horizontal projections of its slopes are equal to each other. Hence, if we substitute in the last formula

$h = b = a$, we obtain $\phi = \frac{1}{3}$,

for which reason in other cases we must put

$$h = \tfrac{1}{3}\left[(2\,b + a + a_1)\left(\frac{\gamma_1}{\gamma} - 1\right) - a_1\right],$$

and for clay dams in particular

$h = \frac{1}{3}\,(2\,b + a)$, or inversely

$$b = \frac{3\,h - a}{2}.$$

If the height of the dam is 20 feet and the angle of inclination of the slope is $\alpha = 36^\circ$, the horizontal projection is

$$a = h\,cotg.\,\alpha = 20\,.\,cotg.\,36^\circ = 20\,.\,1{,}3764 = 27{,}53 \text{ feet},$$

and therefore the width of the top of the dam must be

$$b = \frac{60 - 27{,}53}{2} = 16{,}24 \text{ feet.}$$

§ 361. Pressure upon Curved Surfaces.—The law of the pressure of water in a given direction, deduced in the foregoing paragraph, is applicable only to plane surfaces or to a single element of a curved surface, but not to curved surfaces in general. The normal pressures upon the different elements of curved surfaces can be decomposed into components parallel to a given direction and into others perpendicular to the first. The first set of parallel components forms a system of parallel forces, whose resultant gives the pressure in the given direction, and the other set of components can also be combined so as to form a single resultant, but the two resultants are not capable of further combination, unless their directions intersect each other (§ 97). Hence we are generally unable to combine all the pressures upon the elements of curved surfaces so as to form a single resultant; there are, however, cases where it is possible.

If G_1, G_2, G_3, etc., are the projections and h_1, h_2, h_3, etc., the heads of water of the elements F_1, F_2, F_3, etc., of a curved surface, the pressure of the water in the direction perpendicular to the plane of projection is

$$P = (G_1\,h_1 + G_2\,h_2 + G_3\,h_3 + \ldots)\,\gamma,$$

and its moment in reference to the plane of the surface of the water is

$$P\,u = (G_1\,h_1^2 + G_2\,h_2^2 + G_3\,h_3^2 + \ldots)\,\gamma.$$

If we can decompose the curved surface subjected to the pressure into elements, which have a constant ratio to their projections, I.E., if we can put

$$\frac{G_1}{F_1} = \frac{G_2}{F_2} = \frac{G_3}{F_3} \text{ etc.} = n, \text{ we will have}$$

$$G_1 = \frac{F_1}{n},\ G_2 = \frac{F_2}{n}, \text{ etc., and therefore}$$

$$P = \left(\frac{F_1 h_1}{n} + \frac{F_2 h_2}{n} + \ldots\right)\gamma = \left(\frac{F_1 h_1 + F_2 h_2 + \ldots}{n}\right)\gamma = \frac{F h}{n}\gamma,$$

F denoting the area and h the depth of the centre of gravity of the entire surface below the level of the water. But we have

$$F = F_1 + F_2 + \ldots = n\,G_1 + n\,G_2 + \ldots = n\,(G_1 + G_2 + \ldots) = n\,G,$$

G denoting the area of the projection of the entire surface; hence

$$P = \frac{F h}{n}\gamma = G\,h\,\gamma,$$

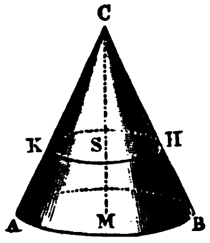
as in the case of a plane surface, or *the pressure of water in one direction is equal to the weight of a prism of water, whose base is the projection of the curved surface upon a plane perpendicular to the given direction and whose height is the depth of the centre of gravity of the curved surface below the surface of the water.*

Thus, E.G., the vertical pressure against the side of a conical vessel $A\,C\,B$, Fig. 609, which is filled with water, is equal to the weight of a column of water, whose base is the base of the cone and whose height is two-thirds the length of the axis $C\,M$; for the horizontal projections of the surface of a right cone, as well as the surface itself, can be decomposed into elementary triangles, and the centre of gravity S of the surface of the cone is at a distance from the apex of the cone equal to two-thirds of its height h (§ 116). If r is the radius of the base and h the height of the cone, we have the pressure upon the base $= \pi\,r^2\,h\,\gamma$ and the vertical pressure upon the sides $= \frac{2}{3}\,\pi\,r^2\,h\,\gamma$; now as the base and the side are united together and the pressures are in opposite directions, it follows that the force with which the entire vessel is pressed downwards is

$$= (1 - \tfrac{2}{3})\,\pi\,r^2\,h\,\gamma = \tfrac{1}{3}\,\pi\,r^2\,h\,\gamma$$

= the weight of the entire mass of water. If we cut the base loose from the conical portion of the vessel it will exert a pressure upon its support $= \pi\,r^2\,h\,\gamma$, and to prevent the side of the vessel from being raised by the water we would have to exert a pressure upon it $= \frac{2}{3}\,\pi\,r^2\,h\,\gamma$.

FIG. 609.

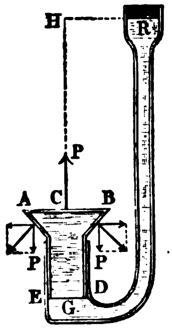


REMARK.—The pressure exerted by the steam of a steam-engine or the water of a water-pressure engine is perfectly independent of the shape of the piston. No matter how much we may increase the surface pressed upon by hollowing out or rounding the piston, the force, with which the water or steam moves the piston, remains the same and is equal to the product of the cross-section or horizontal projection of the piston and the pressure upon the unit of surface. If the piston $A\ B$, Fig. 610, is funnel-shaped and if its greater radius is $C\ A = C\ B = r$ and its smaller $G\ D = G\ E = r_1$, the pressure upon the base is $= \pi r^2 p$ and the reaction upon the conical surface is $= \pi (r^2 - r_1^2) p$; hence the resulting pressure is

$$P = \pi r^2 p - \pi (r^2 - r_1^2) p = \pi r_1^2 p$$

= the cross-section of the cylinder multiplied by the pressure upon the unit of surface.

FIG. 610.



§ 362. **Horizontal and Vertical Pressure.**—Whatever may be the form of a curved surface $A\ B$, Fig. 611, the ***horizontal pressure*** of the water against it is always equal to the weight of a column of water, whose base is the vertical projection $A_0\ B_0$ of the surface at right angles to the given direction and whose height is the depth $O\ S$ of the centre of gravity S of this projection below the surface of the water. The correctness of this assumption is shown directly by the formula

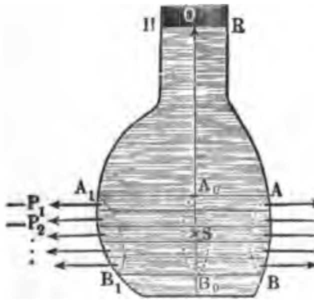
$$P = (G_1 h_1 + G_2 h_2 + \ldots) \gamma,$$

when we remember that the heads of water h_1, h_2, etc., of the elements of the surface are also the heads of water of their projections or that $G_1 h_1 + G_2 h_2 + \ldots$ is the statical moment of the entire projection, I.E., the product $G\ h$ of the vertical projection G multiplied by the depth h of its centre of gravity below the surface of the water. Hence we must again put

$$P = G\ h\ \gamma$$

and remember that h is the head of water of the vertical projection.

FIG. 611.

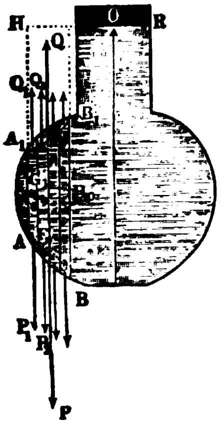


The vertical section, by which we divide a vessel and the water contained in it into two equal or unequal parts, is at the same time the vertical projection of both parts, the horizontal pressure upon one part of the vessel is proportional to its vertical projection multiplied by the depth of its centre of gravity below the surface of the water; consequently the horizontal pressure upon one portion $A\,B$ of the wall of the vessel is exactly as great as the horizontal pressure upon the opposite portion $A_1\,B_1$, which acts in the opposite direction, and the two pressures balance each other. The vessel will therefore be subjected to equal pressure in all directions by the water contained in it.

The vertical pressure $P_1 = G_1\,h_1\,\gamma$ of the water against an element F_1, Fig. 612, of the wall of the vessel is, since the horizontal projection G_1 of the element can be regarded as the cross-section, and the head of water h_1 as the height, or $G_1\,h_1$ as the volume, of a prism, equal to the weight of a column of water $H\,F_1$, extending above the element to the plane $H\,R$ of the surface of the water. Hence the elementary surfaces, which form a finite portion $A\;B$ of the bottom or wall of the vessel, support a pressure which is equal to the weight of the columns of water above them, I.E. to the weight of the column of water above the entire portion. Putting its volume equal to V_1, we obtain the vertical pressure of the water

$$P = V_1\,\gamma.$$

FIG. 612.



The vertical pressure upon another portion $A_1\;B_1$ of the wall of the vessel, which lies vertically above the former and which limits the volume $A_1\,B_1\,H = V_2$, is

$$Q = V_2\,\gamma;$$

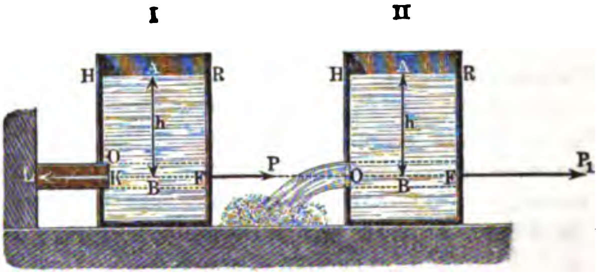
but if the two portions are rigidly connected together, the resultant of the two forces, which acts vertically downwards, will be

$$R = (P - Q) = (V_1 - V_2)\,\gamma = V\,\gamma$$

= *to the weight of the column of water contained between the two surfaces.* If we apply this rule to the entire vessel, it follows *that the entire vertical pressure of the water against the vessel is equal to the weight of the water contained in it.*

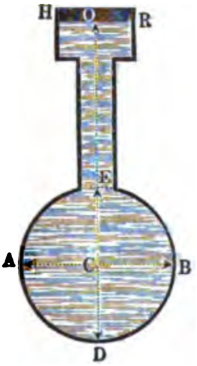
If we make an opening O in the side of the vessel $H\;B\;R$, Fig. 613, I and II, that portion of the pressure, which corresponds to the cross-section of this opening, is wanting and the pressure upon the surface F opposite to it remains unbalanced. If the opening (as in I) is closed by a stopper K, which is prevented from yielding by a resisting object L on the outside, an equal distribution of the horizontal pressure upon the walls of the vessel no longer takes place, but, on the contrary, the vessel is moved forward with a force $P = F h \gamma$, which is counteracted on the opposite side by the stopper. If the stopper is removed and the water allowed to flow through the opening O, as in II, the reaction of the discharging water increases this pressure P from $F h \gamma$ to $P_1 = 2 F h \gamma$, as will be shown hereafter.

FIG. 613.



EXAMPLE.—The vertical pressure P_1 upon the lower hemispherical surface $A\ D\ B$, Fig. 614, is equal to the weight of a column of water bounded above by the surface of the water $H\ R$ and below by the hemispherical surface. If r is the radius $C\ A = C\ D$ of this surface and h the height $C\ O$ of the surface of the water above the horizontal plane $A\ B$, which limits it, the volume of the hemisphere $A\ B\ D$ will be $V_1 = \frac{2}{3} \pi r^3$, and that of the cylinder above $A\ B$, $V_2 = \pi r^2 h$; hence

FIG. 614.



$$P_1 = (V_1 + V_2) \gamma = (\tfrac{2}{3} \pi r^3 + \pi r^2 h) \gamma = (h + \tfrac{2}{3} r) \pi r^2 \gamma.$$

The pressure, which is directed vertically upwards upon the upper hemisphere $A\ E\ B$, is, on the contrary,

$$P_2 = (V_2 - V_1) \gamma = (h - \tfrac{2}{3} r) \pi r^2 \gamma,$$

and therefore the entire vertical pressure

$$P = P_1 - P_2 = 2\ V_2 = \tfrac{4}{3} \pi r^3 \gamma$$

is equal to the weight of *water* in the entire sphere.

The horizontal pressure upon one of the hemispheres $D\ A\ E$ and $D\ B\ E$, which join each other in the vertical plane $D\ C\ E$, is measured by the weight of a prism, whose base is $D\ C\ E = \pi r^2$ and whose height is $C\ O = h$; this pressure is

$$R = \pi r^2 h \gamma.$$

§ 363. Thickness of Pipes.

—The application of the theory of the pressure of water to the determination of the thickness of *pipes, boilers*, etc., is of great importance. In order that these vessels shall sufficiently resist the pressure of the water and not be broken, their *walls* must be made of a *certain thickness*, which depends upon the head of water and the internal diameter of the vessel. The rupture of the pipe may be caused either by a transverse or by a longitudinal tearing. The latter form of rupture is most likely to occur, as will appear from the following discussion.

If the head of the water in a pipe $= h$ or the pressure upon the unit of surface of the pipe is $p = h\,\gamma$, the width of the pipe $M\,N = 2\,C\,M = 2\,r$, Fig. 615, and the cross-section of the body of water in it $F = \pi\,r^2$, the pressure, which is exerted upon the *surface of the end* of the pipe and which must be sustained by the cross-section of the tube, is

$$P = F\,p = \pi\,r^2\,h\,\gamma = \pi\,r^2\,p.$$

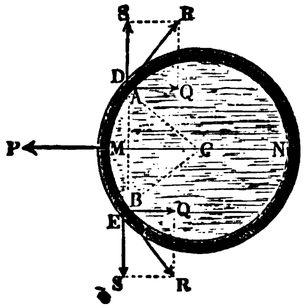
Now if the thickness of the pipe is $A\,D = B\,E = e$, its cross-section is

$$= \pi\,(r + e)^2 - \pi\,r^2 = 2\,\pi\,r\,e + e^2 = 2\,\pi\,r\,e\left(1 + \frac{e}{2\,r}\right),$$

and if we denote the *modulus of proof strength* of the material, of which the pipe is composed, by T, the proof strength of the entire tube in the direction of the axis is

$$P = \left(1 + \frac{e}{2\,r}\right) 2\,\pi\,r\,e\,T.$$

FIG. 615.



Hence we can put

$$\left(1 + \frac{e}{2\,r}\right) 2\,\pi\,r\,e\,T = \pi\,r^2\,p, \text{ or}$$

$$\left(1 + \frac{e}{2\,r}\right) 2\,e\,T = r\,p \text{ (see § 205);}$$

the resolution of this equation gives the thickness

$$e = \frac{r\,p}{2\left(1 + \frac{e}{2\,r}\right) T}$$

of the pipe, for which we can generally write with sufficient accuracy

$$e = \frac{r\,p}{2\,T} = \frac{r\,h\,\gamma}{2\,T}.$$

The mean pressure, which the water exerts upon a portion of the wall $A\,M\,B$, whose length is l and whose central angle is $A\,C\,B = 2\,a^{\circ}$, is, since the projection of this portion at right angles to the line $C\,M$ passing through the centre is a rectangle, whose area is $\overline{A\,B}\,.\,l = 2\,r\,l\,sin.\,a$,

$$P = 2\,r\,l\,sin.\,a\,.\,p = 2\,r\,l\,h\,sin.\,a\,.\,\gamma.$$

This force is held in equilibrium by the forces of cohesion R, R in the cross-sections $\overline{AD}\,.\,l$ and $\overline{BE}\,.\,l = e\,l$ of the wall of the pipe; it is therefore equal to the sum $2\,Q$ of the components $\overline{DQ} = Q$ and $\overline{EQ} = Q$ of the latter forces, which are parallel to the line $C\,M$. Now if we put $R = e\,l\,T$, we obtain

$$Q = R \sin. A\,R\,Q = R \sin. A\,C\,M = e\,l\,T \sin. a,$$

and therefore

$$2\,e\,l\,T \sin. a = 2\,r\,l\,p \sin. a, \text{ I.E. } e\,T = r\,p;$$

hence the required thickness of the pipe is

$$e = \frac{r\,p}{T} = \frac{r\,h\,\gamma}{T},$$

which is entirely independent of its length.

Since the first calculation gave e only $= \frac{r\,p}{2\,T}$, it follows that to prevent a longitudinal tear we must make the wall twice as thick as would be necessary to prevent a transverse one.

From the formula

$$e = \frac{r\,p}{T} = \frac{r\,h\,\gamma}{T}$$

just found, it follows *that the thickness of similar pipes must be proportional to the width and to the head of water or pressure upon the unit of surface.* A pipe, which is three times as wide as another and which has to bear a pressure five times as great as the first, must be fifteen times as thick.

We must give to *hollow spheres* which sustain a pressure p upon each unit of surface the thickness

$$e = \frac{r\,p}{2\,T};$$

for here the projection of the surface pressed upon is the great circle $\pi\,r^2$, and the surface of separation of the ring is $2\,\pi\,r\,e \left(1 + \frac{e}{2\,r}\right)$, or approximatively, when the thickness is small, $= 2\,\pi\,r\,e$.

The formulas just found, give for $p = 0$ also $e = 0$; hence pipes, which have no internal pressure to resist, can be made infinitely thin; but since every pipe in consequence of its own weight must sustain a certain pressure and also must be made of a certain thickness to be water-tight, we must add to the value found a certain thickness e in order to have a pipe, which under all circumstances will be strong enough. Hence for a cylindrical tube or boiler we have

$$e = e_1 + \frac{r h \gamma}{T},$$

or more simply, if d is the interior width of the tube, p the pressure in atmospheres, each corresponding to a column of water 34 feet high, and μ a coefficient determined by experiment,

$$e = e_1 + \mu\, p\, d.$$

It has been experimentally determined that for tubes made of

Sheet iron . . . . $e = 0,00086\, p\, d + 0,12$ inches,
Cast iron . . . . . $e = 0,00238\, p\, d + 0,34$ "
Copper . . . . . . $e = 0,00148\, p\, d + 0,16$ "
Lead . . . . . . $e = 0,00507\, p\, d + 0,21$ "
Zinc . . . . $e = 0,00242\, p\, d + 0,16$, "
Wood . . . . . . $e = 0,0323\, p\, d + 1,07$ "
Natural stone . . . $e = 0,0369\, p\, d + 1,18$ "
Artificial stone . . . $e = 0,0538\, p\, d + 1,58$ "

Example.—If a vertical water-pressure engine has an inlet cast-iron pipe 10 inches wide inside, how thick must its walls be for a depth of 100, 200 and 300 feet? For a depth of 100 feet this thickness is $0,00238 \cdot \frac{100}{34} \cdot 10 + 0.34 = 0,07 + 0,34 = 0,41$ inches; for a depth of 200 feet $= 0,14 + 0,34 = 0,48$ inches; and for a depth of 300 feet $= 0,21 + 0,34 = 0,55$ inches. Cast-iron conduit pipes are generally tested to 10 atmospheres, in which case we have

$$e = 0,0238\, d + 0,34 \text{ inches},$$

and for pipes of 10 inches internal diameter we must make the thickness

$$e = 0,24 + 0,34 = 0,58 \text{ inches}.$$

Remark—1) In the second part of this work the thickness of tubes exposed not only to hydrostatic pressure, but also to hydraulic impact, will be calculated.

2) In the second part the thickness of the walls of steam-boilers will be treated. Upon the theory of the thickness of pipes, we can consult the treatise of Geh. Regierungsrath Brix in the proceedings of the "Vereins zur Beförderung des Gewerbefleisses, in Preussen," year 1834, and Wiebes "Lehre von den einfachen Maschinentheilen," Vol. I, and also Rankine's "Manual of Applied Mechanics," page 289, and Scheffler's "Monographien über die Gitter- und Bogenträger, und über die Festigkeit der *Gefasswände.*" The technical relations and the testing of pipes are treated in Hagen's "Handbuch der Wasserbaukunst," Part 1st, and also in Geniey's "Essai sur les moyens de conduire, etc., les eaux," and in the "Traité théoretique et pratique de la conduite et de la distribution des eaux," par Dupuit, Paris. 1854.

CHAPTER II.

EQUILIBRIUM OF WATER WITH OTHER BODIES.

§ 364. Upward Pressure, Buoyant Effort.—A body *immersed in water* is subjected to pressure upon all sides, and the question arises, what is the magnitude, direction, and point of application of the resultant of all these pressures? Let us imagine this resultant composed of a vertical and two horizontal components, and let us determine them according to the rules of § 362. The *horizontal pressure* of the water against a body is equal to the horizontal pressure against its vertical projection; but every elevation *A C*, Fig. 616, of a body is at the same time the projection of the rear part *A D C* and of the fore part *A B C* of its surface, and consequently the pressure *P* upon the hind part of the surface of a body is equal to the pressure $-P$ upon the fore part; and as the directions of these pressures are opposite, their resultant is $= 0$. Since this relation exists for any given horizontal direction and its corresponding vertical projection, it follows that the resultant of all the horizontal pressures is equal to zero, and that the body *A C*, which is under water, is *subjected to equal pressure in all horizontal directions*, and therefore has no tendency to move horizontally.

FIG. 616.

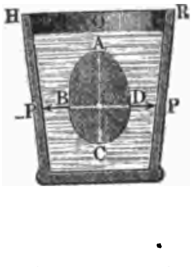
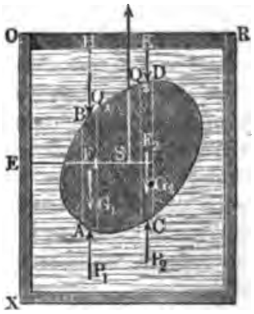


FIG. 617.



In order to find the *vertical pressure* of the water upon a body *A B D*, Fig. 617, immersed in it, let us imagine it to be decomposed into the vertical elementary prisms $A\ B$, $C\ D$, etc., and let us determine the vertical pressure upon their bases A and B, C and D, etc. Let the lengths of these columns be l_1, l_2, etc., the depths $H\ B$, $K\ D$ of their upper ends B, D below the surface of the water $O\ R$ be h_1, h_2, etc., and their horizontal cross-sections be F_1, F_2, etc., then we have the vertical pressures which act from above downwards upon their ends B, D, etc.,

$$Q_1,\ Q_2,\ \text{etc.},\ = F_1\, h_1\, \gamma,\ F_2\, h_2\, \gamma,\ \text{etc.},$$

and, on the contrary, the vertical pressures which act from below upwards against the ends A, C, etc., are

$$P_1,\ P_2,\ \text{etc.},\ = F_1\,(h_1 + l_1)\,\gamma,\ F_2\,(h_2 + l_2)\,\gamma,\ \text{etc.}$$

By combining these parallel forces we obtain the resultant

$$\begin{aligned} P &= P_1 + P_2 + \ldots - (Q_1 + Q_2 + \ldots) \\ &= F_1\,(h_1 + l_1)\,\gamma + F_2\,(h_2 + l_2)\,\gamma + \ldots - F_1\,h_1\,\gamma - F_2\,h_2\,\gamma - \ldots \\ &= (F_1\,l_1 + F_2\,l_2 + \ldots)\,\gamma = V\gamma, \end{aligned}$$

in which V denotes the volume of the immersed body or of the water displaced by it. *Hence the upward pressure or buoyant effort, with which water tends to raise up a body immersed in it, is equal to the weight of the water displaced or of a quantity of water which has the same volume as the submerged body.*

Finally, in order to determine the point of application of this resultant, let us put the distances $E\ F_1$, $E\ F_2$, etc., of the elementary columns $A\ B$, $C\ D$, etc., from a vertical plane $O\ X$ equal to a_1, a_2, etc., and let us determine their moments in reference to this plane. If S is the point of application of the upward thrust, which is called the *centre of buoyancy*, and $E\ S = x$ its distance from that plane, we have

$$V\gamma x = F_1\, l_1\, \gamma \,.\, a_1 + F_2\, l_2\, \gamma \,.\, a_2 + \ldots,$$

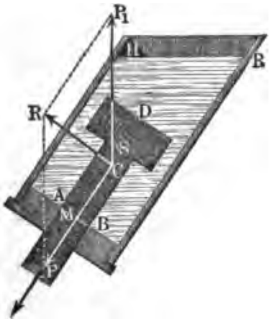
and therefore

$$x = \frac{F_1\, l_1\, a_1 + F_2\, l_2\, a_2 + \ldots}{F_1\, l_1 + F_2\, l_2 + \ldots} = \frac{V_1\, a_1 + V_2\, a_2 + \ldots}{V_1 + V_2 + \ldots},$$

V_1, V_2, etc., denoting the contents of the elementary columns. Since (according to § 105) the centre of gravity of a body is determined by exactly the same formula, *it follows that the point of application S of the upward thrust coincides with the centre of gravity of the water displaced.* The direction of the buoyant effort is called the *line of support;* when it passes through the centre of gravity of the body, it is called the *line of rest.*

§ 365. Upward Pressure, or Buoyant Effort, when the Body is Partially Surrounded by Water.—If a body, such as $A\ B\ D$, Fig. 618, is not entirely surrounded by the water $A\ H\ R$, and the surface $\overline{A\ B}$, whose area is F, is united to the wall of the vessel, or if the body, where its cross-section is $\overline{A\ B} = F$, passes through the wall of the vessel, the pressure which the water would have exerted upon this surface $A\ B$, if the body was free or in contact with the water alone, is absent.

FIG. 618.



If we denote the head of water upon $A\ B$, I.E. the depth of its centre of gravity below the surface of water $H\ R$, by h, the pressure of the water upon $A\ B$ will be $P = F\ h\ \gamma$; and if V_1 denotes the volume of water displaced by $A\ B\ D$, the buoyant effort of the water, or the force, with which the body would tend to rise if it were free, is $P_1 = V_1\ \gamma$.

However, since the pressure upon $A\ B$ is wanting, the entire action of the water upon the body is the resultant R of $P_1 = V_1\ \gamma$ and $- P = - F\ h\ \gamma$. In order to determine this resultant, we prolong the vertical line of gravity of the water displaced and the right line passing through the centre M of the pressure perpendicular to $A\ B$ until they meet at the point C; then, assuming the forces P_1 and $- P$ to be applied at this point, we combine them by means of the parallelogram of forces and obtain the resultant $C\ R = R$.

If the inclination of the surface $A\ B$ to the horizon as well as the deviation of the force P from the vertical $= a$, the angle formed by the directions of the forces P and $- P_1$ with each other will be $= M\ C\ P_1 = 180 - a$, and therefore the resultant, which measures the whole effect of the pressure of the water upon the body $A\ B\ D$, will be

$$R = \sqrt{P_1^2 + P^2 - 2\ P\ P_1 \cos. a}$$
$$= \gamma \sqrt{V_1^2 + (F\ h)^2 - 2\ V_1\ F\ h \cos. a.}$$

According to the principle of action and reaction, the body will react with a pressure $- R$ upon the water. If V_0 is the volume of the water in the vessel or $V_0\ \gamma$ its weight G, the pressure, which acts vertically downwards upon the vessel, is

$$Q = V_0\ \gamma + P_1 = (V_0 + V_1)\ \gamma, \text{ I.E. } Q = V\ \gamma,$$

when $V = V_0 + V_1$ denotes the volume of the space occupied by the water and the body $A\ B\ D$.

Combining this with the pressure $P = F\ h\ \gamma$, we have the entire pressure sustained by the vessel

$$R_1 = \sqrt{Q^2 + P^2 - 2\,Q\,P\,cos.\,a}$$
$$= \gamma\sqrt{V^2 + (F h)^2 - 2\,V\,F\,h\,cos.\,a.}$$

If the surface $A\ B$ were horizontal or $a = 0°$, we would have

$$R = (V_1 - F\,h)\,\gamma \text{ and } R_1 = (V - F\,h)\,\gamma.$$

If also $V_1 = 0$, R would be $= -F\,h\,\gamma$ (see § 355).

§ 366. Equilibrium of Floating Bodies.—The *buoyant effort* P upon a body floating or immersed in water is accompanied by the weight G of the body, which acts in the opposite direction, and the resultant of the two forces is

$$R = G - P \text{ or } = (\varepsilon - 1)\,V\gamma,$$

in which ε denotes the specific gravity of the body.

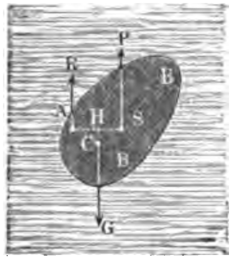
If the body is *homogeneous*, its centre of gravity and that of the water displaced coincide, and this point is consequently the point of application of the resultant $R = G - P$; but if the body is heterogeneous, the two centres of gravity do not coincide and the point of application of the resultant does not coincide with either of them. Putting the horizontal distance $S\ H$, Fig. 619, of the two centres of gravity from each other $= b$ and the horizontal distance $S\ A$ of the required point of application A from the centre of gravity S of the water displaced, $= a$, we have the equation

$$G\,b = R\,a,$$

whence we obtain

$$a = \frac{G\,b}{R} = \frac{G\,b}{G - P}.$$

FIG. 619.



If the immersed body is abandoned to the action of gravity, one of three cases may occur. Either the specific gravity ε of the body is equal to that of the water, or it is greater, or it is less. In the first case the buoyant effort is equal to the weight, in the second it is smaller, and in the third it is greater. While in the first case the buoyant effort and the weight are in equilibrium, in the second case the body will sink with the force

$$G - V\gamma = (\varepsilon - 1)\,V\gamma,$$

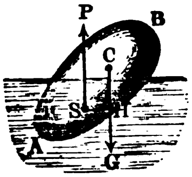
and in the third case it will rise with the force

$$V\gamma - G = (1 - \varepsilon)\,V\gamma.$$

The body will continue to rise until the volume V_1 of the water displaced by the body and limited by the plane of the surface of the water has the same weight as the entire body. The weight $G = V \varepsilon \gamma$ of the body $A\ B$, Fig. 620, and the buoyant effort $P = V_1 \gamma$ form a couple, by which the body is turned until the directions of these forces coincide or until the centre of gravity of the body and the centre of buoyancy come into the same vertical line, or until the line of support becomes a line of rest. From the equality of the forces P and G we have the expression

$$V_1 = \varepsilon\, V, \text{ or } \frac{V_1}{V} = \frac{\varepsilon}{1}.$$

FIG. 620.



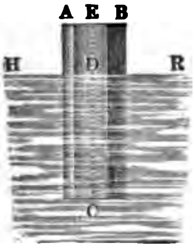
The line passing through the centre of gravity of the floating body and the centre of buoyancy is called the axis of floatation (Fr. axe de flottaison; Ger. Schwimmaxe), and the section of the floating body formed by the plane of the surface of the water is called the plane of floatation (Fr. plan de flottaison; Ger. Schwimmebene). From what precedes we see that any plane, which divides the body in such a manner that the centres of gravity of the two portions will be in a line perpendicular to it, and that one portion of the body will be to the whole as the specific gravity of the body is to that of the liquid, will be a plane of floatation of the body.

§ 367. **Depth of Floatation.**—If we know the form and weight of a floating body, we can calculate beforehand by the aid of the foregoing rule the *depth of immersion.* If G is the weight of the body, we can put the volume of the water displaced

$$V_1 = \frac{G}{\gamma};$$

if we combine this with the stereometric formula for this volume V_1, we obtain the required equation of condition.

FIG. 621.



For a *prism* $A\ B\ C$, Fig. 621, whose axis is vertical, we have $V_1 = F\,y$, when F denotes the cross-section and y the depth $C\ D$ of immersion; hence it follows that

$$F\,y = \frac{G}{\gamma} \text{ and } y = \frac{G}{F\gamma} = \frac{G\,h}{V\gamma},$$

in which V denotes the volume and h the length of the floating prism.

For a pyramid $A\ B\ C$, Fig. 622, floating with its apex below the surface of the water, we have, since the contents of similar pyramids are proportional to the cubes of their heights,

$\frac{V_1}{V} = \frac{y^3}{h^3}$, and consequently the *depth of immersion*, is

$$C D = y = h \sqrt[3]{\frac{V_1}{V}} = h \sqrt[3]{\frac{G}{V \gamma}},$$

in which V denotes the volume and h the height of the pyramid.

FIG. 622.

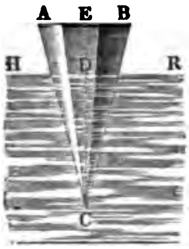
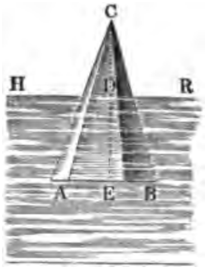


FIG. 623.



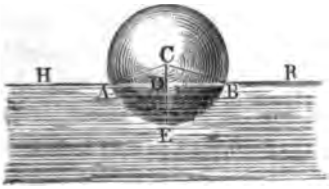
For a *pyramid, $A\ B\ C$,* Fig. 623, floating with its *base under water*, we obtain, on the contrary, the distance $C D = y_1$ from the apex to the surface of the water by putting

$$\frac{V_1}{V} = \frac{h^3 - y_1^3}{h^3}, \text{ whence } y_1 = h \sqrt[3]{1 - \frac{V_1}{V}} = h \sqrt[3]{1 - \frac{G}{V \gamma}}.$$

For a sphere $A\ B$, Fig. 624, whose radius is $C A = r$,

$$V_1 = \pi y^2 \left(r - \frac{y}{3}\right),$$

FIG. 624.



we have therefore, in this case, to solve the cubic equation

$$y^3 - 3 r y^2 + \frac{3 G}{\pi \gamma} = 0$$

in order to find the depth of the immersion $D\ E = y$ of the sphere.

If a *cylinder* $A\ K$, Fig. 625, floats with its axis horizontal and its radius is $A\ C = B\ C = r$, we have, when a° denotes the central angle $A\ C\ B$ of the immersed arc, for the depth of immersion $D\ E$

$$y = r\,(1 - \cos.\ \tfrac{1}{2}\, a);$$

now in order to find the arc a we must put the volume of the water displaced = sector $\left(\frac{r^2 a}{2}\right)$ minus the triangle $\left(\frac{r^2 \sin. a}{2}\right)$, multiplied by the length $B\,K = l$ of the cylinder, or

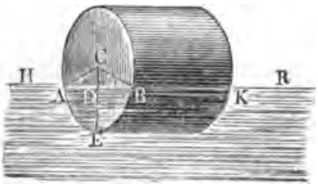
$$(a - \sin. a)\,\frac{l\,r^2}{2} = \frac{G}{\gamma},$$

and resolve the equation

$$a - \sin. a = \frac{2\,G}{l\,r^2\,\gamma}$$

by approximation with reference to a.

FIG. 625.



EXAMPLE—1) If a wooden sphere 10 inches in diameter, which is floating, is immersed 4½ inches in the water, the volume of the water displaced is

$$V_1 = \pi\left(\tfrac{9}{2}\right)^2\left(5 - \tfrac{3}{2}\right) = \frac{\pi \,.\, 81 \,.\, 7}{8} = \frac{567 \,.\, \pi}{8} = 222{,}66 \text{ cubic inches,}$$

while the volume of the sphere itself is

$$\frac{\pi\,d^3}{6} = \frac{\pi \,.\, 10^3}{6} = 523{,}6 \text{ cubic inches.}$$

Therefore 523,6 cubic inches of the material of the sphere weigh as much as 222,66 cubic inches of water, and the specific gravity of the former is

$$\varepsilon = \frac{222{,}66}{523{,}6} = 0{,}425.$$

2) How deep will a wooden cylinder 10 inches in diameter sink, when floating, if its specific gravity is $\varepsilon = 0{,}425$? Here

$$\frac{a - \sin. a}{2} = \frac{\pi\,r^2\,l \,.\, \varepsilon\,\gamma}{l\,r^2\,\gamma} = \pi\,\varepsilon = 0{,}425 \,.\, \pi = .1{,}3352.$$

Now the table of segments in the "Ingenieur," page 154, gives for the area $\frac{a - \sin. a}{2} = 1{,}32766$ a segment of a circle, whose central angle is $a^0 = 166^0$, and for $\frac{a - \sin. a}{2} = 1{,}34487$ an angle $a^0 = 167^0$; we can, therefore, put the angle at the centre, corresponding to the sector 1,3352

$$a^0 = 166^0 + \frac{1{,}33520 - 1{,}32766}{1{,}34487 - 1{,}32766} \,.\, 1^0 = 166^0 + \frac{754^0}{1721} = 166^0\,26'.$$

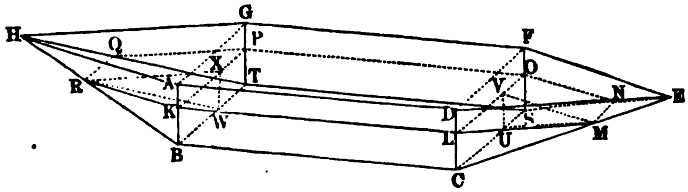
The depth of immersion is, therefore,

$$y = r\,(1 - \cos. \tfrac{1}{2}\,a) = 5\,(1 - \cos. 83^0\,13') = 5 \,.\, 0{,}8819 = 4{,}41 \text{ inches.}$$

§ 368 The most important application of the above principle is to the determination of the depth of immersion of *boats* and *ships*. If the boats have a regular form this depth can be calculated by geometrical formulas; but if the form is irregular, or if its equation is unknown, or if it is composed of very many forms, the depth of immersion must be determined by experiment.

An example of the first case is furnished by the boat *A C E G H*, represented in Fig. 626, whose sides are plane surfaces. It consists of a parallelopipedon *A C F* and two four-sided pyramids *C E F* and *B G H*, which form the bow and stern, and its plane of floatation is composed of a parallelogram *K L O P* and of two trapezoids *L M N O* and *K P Q R* which limit the space, from which the water is displaced and which can be decomposed into a parallelopipedon *K C O T*, into two triangular prisms *U V M N* and *W X R Q*, and into two four-sided pyramids *C V M* and *B X R*. Let us put the length $A D = B C$ of the central portion $= l$, its width $A G = b$ and its height $A B = h$, the length of each of the two beaks $= c$ and the depth of immersion under water, i.e. $B K = C L = y$. It follows that the immersed portion *K C O T* of the middle piece is

FIG. 626.



$$= \overline{B C} . \overline{C S} . \overline{C L} = l b y.$$

Putting the width *C U* of the base of the pyramid *C V M*, $= x$ and the height of this pyramid $= z$, we have

$$\frac{x}{b} = \frac{z}{c} = \frac{y}{h}, \text{ whence}$$

$$x = \frac{b}{h} y \text{ and } z = \frac{c}{h} y;$$

hence the volume of this pyramid is

$$= \tfrac{1}{3} x y z = \frac{b c y^3}{3 h^2},$$

and those of the two pyramids (*C V M* and *B X R*) together are

$$= \tfrac{2}{3} \frac{b c y^3}{h^2}$$

The cross-section of the triangular pyramid *U V N* is

$$= \tfrac{1}{2} y z = \frac{c y^2}{2 h} \text{ and the side } M N = V O$$

$$= b - \frac{b y}{h} = b\left(1 - \frac{y}{h}\right);$$

hence the contents of the two prisms $V\,U\,N$ and $X\,W\,Q$ together are

$$= 2 \,.\, \frac{c\,y^2}{2\,h} \,.\, b\left(1 - \frac{y}{h}\right) = \frac{b\,c\,y^2}{h}\left(1 - \frac{y}{h}\right).$$

Finally, by adding the volumes first found, we obtain that of the water displaced

$$V = b\,l\,y + \tfrac{2}{3}\,\frac{b\,c\,y^3}{h^2} + \frac{b\,c\,y^2}{h} - \frac{b\,c\,y^3}{h^2} = \left(l + \frac{c\,y}{h} - \tfrac{1}{3} \,.\, \frac{c\,y^2}{h^2}\right) b\,y.$$

Now if the gross weight of the boat $= G$, we must put

$$\left(l + \frac{c\,y}{h} - \tfrac{1}{3} \,.\, \frac{c\,y^2}{h^2}\right) b\,h\,\gamma = G, \text{ or}$$

$$y^3 - 3\,h\,y^2 - \frac{3\,l\,h^2}{c} \,.\, y + \frac{3\,h^2\,G}{b\,c\,\gamma} = 0.$$

By resolving this cubic equation we obtain from the gross weight G of the boat its *depth y of floatation.*

Example—1) If the length of the middle portion of a boat is $l = 50$ feet, the length of each terminal pyramid is $c = 15$ feet, the width $b = 12$ feet and the depth $h = 4$ feet, the total load for an immersion of 2 feet is

$$G = [50 + 15 \,.\, \tfrac{2}{4} - \tfrac{1}{3} \,.\, 15 \,.\, (\tfrac{2}{4})^2] \,.\, 12 \,.\, 2 \,.\, 62{,}5$$
$$= [50 + 7{,}5 - 1{,}25]\; 24 \,.\, 62{,}5 = 84375 \text{ pounds.}$$

2) If the gross weight of the above boat was 50000 pounds, we would have for the *depth of immersion*

$$y^3 - 12\,y^2 - 160\,y + 213{,}33 = 0.$$

From this we obtain

$$y = \frac{213{,}33 + y^3 - 12\,y^2}{160} = 1{,}333 + 0{,}00625\,y^3 - 0{,}075\,y^2,$$

approximatively, $y = 1{,}333 + 0{,}00625\,(1{,}333)^3 - 0{,}075\,(1{,}333)^2$
$= 1{,}333 + 0{,}0148 - 0{,}1333 = 1{,}215$, and more exactly
$y = 1{,}333 + 0{,}00625\,(1{,}215)^3 - 0{,}075\,(1{,}215)^2 = 1{,}2338$ feet.

Remark.—In order to find the weight of the cargo, vessels are provided on both sides with a scale. The divisions of the scale are generally determined empirically by finding the immersion for given loads. This subject will be treated more at length in the third volume.

§ 369. Stability of Floating Bodies.—A body floats either in an *upright* or *inclined* position, and *with* or *without stability.* A body, E.G. a ship, floats in an upright position, when at least one of the planes passing through the axis of floatation is a plane of symmetry of the body, and in an inclined position, when the body cannot be divided into two symmetrical parts by any plane passing through the axis of floatation. A floating body is in stable equilibrium, when it tends to maintain its position of equilibrium (compare § 141), I.E. if work must be done to move it out of this position, or if it returns to its original position of equilibrium after having been moved from it. A body floats in unstable equilibrium, when it passes into a new position of equilibrium as soon as it has been moved from its original one by being shaken, by a blow, etc.

If a body *A B C*, Fig. 627, which was floating in an upright position, is brought into an inclined one, the centre of buoyancy S moves from the plane of symmetry and assumes a position S_1 in the half of the body most immersed. The buoyant effort $P = V \gamma$, which is applied at S_1, and the weight of the ship $G = - P$, which is applied at C, form a couple which will always turn the body (see § 93). No matter around what point this rotation takes place, the point C, yielding to the weight G, will always sink, and the point S_1 or another M, situated in the vertical line $S_1\ P$, yielding to the action P, will rise, and the axis or plane of symmetry $E\ F$ will be drawn downwards at C and upwards at M, and therefore the body will right itself when M, as in Fig. 627, is above C, and, on the contrary, it will incline itself more and more when, as is represented in Fig. 628, M is situated below C. Hence the stability of a floating body, such as a ship, depends upon the point M, where the vertical line, which passes through the centre of buoyancy S_1, cuts the plane of symmetry. This point is called the *metacentre* (Fr. métacentre; Ger. Metacentrum). A ship or any other body floats with stability when its metacentre lies above its centre of gravity, and without stability when it lies below it; it is in indifferent equilibrium when these two points coincide.

FIG. 627

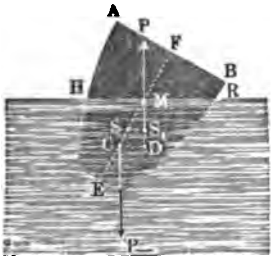
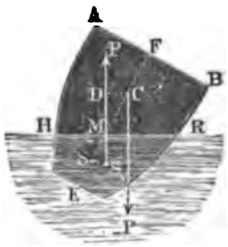


FIG. 628.



The horizontal distance $C\ D$ of the metacentre M from the centre of gravity C of the ship is the arm of the couple formed by P and $G = - P$, and its moment, which is the measure of the stability, is $= P \, . \, \overline{CD}$. If we denote the distance CM by c, and the angle $S\,M\,S_1$, through which the ship rolls or through which its axis is turned, by ϕ, we have for the measure of the stability of the ship

$$S = P\,c\,sin.\,\phi\,;$$

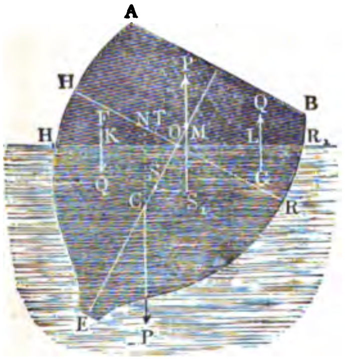
it increases, therefore, with the weight, with the distance of the metacentre from the centre of gravity of the ship and with the angle of inclination.

§ 370. **Determination of the Moment of Stability.**—In the last formula

$$S = P\,c\,sin.\,\phi,$$

the *stability* of the ship depends principally upon the distance of the metacentre from the centre of gravity of the ship, and it is, therefore, important to obtain a formula for the determination of this distance. While the ship $A\,B\,E$, Fig. 629, passes from its upright to its inclined position, the centre of buoyancy S moves to S_1, and the wedge-shaped space $H\,O\,H_1$ passes out of the water drawing the wedge-shaped piece $R\,O\,R_1$ into it, and the buoyant effort on one side is diminished by the force Q, acting at the centre of gravity F of the space $H\,O\,H_1$ and upon the other side it is increased by an equal force Q, acting at the centre of gravity G of the space $R\,O\,R_1$. Therefore the force P applied at S_1 replaces the force originally applied at S and the couple $(Q, -Q)$, or, what amounts to the same thing, an opposite force $-P$, acting in S_1, balances the force P applied at S together with the couple $(Q, -Q)$, or more simply a couple $(P, -P)$, whose points of application are S and S_1, balances the couple $(Q, -Q)$. Now if the cross-section $H\,E\,R = H_1\,E\,R_1$ of the immersed portion of the ship $= F$ and the cross-section $H\,O\,H_1 = R\,O\,R_1$ of the space, which is drawn out the water on one side and immersed on the other, $= F_1$, if the horizontal distance $K\,L$ of the centres of gravity of these spaces from each other $= a$ and the horizontal distance $M\,T$ of the centres of gravity S and S_1 from each other, or the horizontal projection

FIG. 629



$S\,S_1$ of the space described by S, during the rolling, $= s$, we have, since the couples balance each other,

$$F\,s = F_1\,a, \text{ whence } s = \frac{F_1}{F}\,a \text{ and}$$

$$\overline{S\,M} = \frac{M\,T}{sin.\,\phi} = \frac{s}{sin.\,\phi} = \frac{F_1\,a}{F\,sin.\,\phi}.$$

The line $C\,M = c$, which enters as a factor into the measure of the stability, is $= C\,S + S\,M$; denoting, therefore, the distance $C\,S$ of the centre of gravity C of the ship from the centre of buoyancy S by e, we obtain the *measure of the stability*

$$S = P\,c\,sin.\,\phi = P\left(\frac{F_1\,a}{F} + e\,sin.\,\phi\right).$$

If the angle through which the ship rolls is small, the cross-sections $H\,O\,H_1$ and $R\,O\,R_1$ can be treated as isosceles triangles. If we denote the width $H\,R = H_1\,R_1$ of the ship at the surface of the water by b, we can put

$$F_1 = \tfrac{1}{2}\,.\,\tfrac{1}{2}\,b\,.\,\tfrac{1}{2}\,b\,\phi = \tfrac{1}{8}\,b^2\,\phi \text{ and } K\,L = a = 2\,.\,\tfrac{2}{3}\,\frac{b}{2} = \tfrac{2}{3}\,b,$$

as well as $sin.\,\phi = \phi$; hence the measure of the stability of the ship is

$$S = P\left(\tfrac{1}{12}\,\frac{b^2\,\phi}{F} + e\,\phi\right) = \left(\frac{b^2}{12\,F} + e\right)P\,\phi.$$

If the centre of gravity C of the ship coincides with the centre of buoyancy S, we have $e = 0$, whence

$$S = \frac{b^2}{12\,F}\,.\,P\,\phi,$$

and if the centre of gravity of the ship lies above the centre of buoyancy, e, on the contrary, is negative and

$$S = \left(\frac{b^2}{12\,F} - e\right)P\,\phi.$$

It also follows that the stability of a ship becomes null, when e is negative and at the same time $= \dfrac{b^2}{12\,F}$.

We see from the results obtained that a ship's stability is greater the wider the ship is and the lower the centre of gravity is.

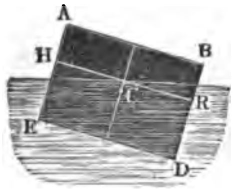
EXAMPLE.—The measure of the stability of a parallelopipedon $A\,D$, Fig. 680, whose width $A\,B = b$, whose height $A\,E = h$ and whose depth of immersion $E\,H = y$ is, since $F = b\,y$ and $e = -\dfrac{h-y}{2}$,

$$S = P\phi\left(\frac{b^2}{12\,b\,y} - \frac{h}{2} + \frac{y}{2}\right),$$

or if the specific gravity of the material of which the parallelopipedon is composed be put $= \varepsilon$

$$S = P\phi\left(\frac{b^2}{12\,\varepsilon\,h} - \frac{h}{2}(1 - \varepsilon)\right)$$

FIG. 630.



From this we see that the stability ceases when

$$b^2 = 6\,h^2\,\varepsilon\,(1 - \varepsilon),\text{ I.E., when}$$

$$\frac{b}{h} = \sqrt{6\,\varepsilon\,(1 - \varepsilon)}.$$

For $\varepsilon = \frac{1}{2}$ we have

$$\frac{b}{h} = \sqrt{\frac{3}{2}\cdot\frac{1}{2}} = \sqrt{\frac{3}{2}} = 1{,}225.$$

If in this case the width is not at least 1,225 times the height, the parallelopipedon floats in unstable equilibrium.

371. Inclined Floating.

—The formula

$$S = P\left(\frac{F_1\,a}{F} \pm e\,sin.\,\phi\right)$$

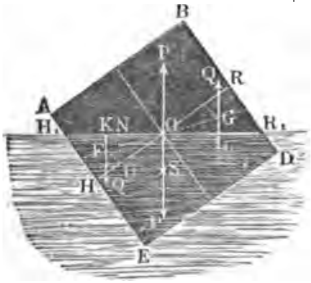
for the stability of a floating body can also be employed to determine the various positions of floating bodies; for if we put $S = 0$, we obtain the equation of condition of the position of equilibrium, and by resolving it we obtain the corresponding angle of inclination. We have, therefore, to resolve the equation

$$\frac{F_1\,a}{F} \pm e\,sin.\,\phi = 0$$

in reference to ϕ.

In the case of a parallelopipedon $A\,B\,D\,E$, Fig. 631, the cross-section F is $= H\,R\,D\,E = H_1\,R_1\,D\,E = b\,y$, b denoting the width $A\,B = H\,R$ and y the depth of immersion $E\,H = D\,R$, and the cross-section

FIG. 631.



$$F_1 \doteq H\,O\,H_1 = R\,O\,R_1$$

is a right-angled triangle, whose base is $O\,H = O\,R = \frac{1}{2}\,b$,

and whose altitude is

$$H\,H_1 = R\,R_1 = \tfrac{1}{2}\,b\,tang.\,\phi,$$

whence

$$F_1 = \tfrac{1}{8}\,b^2\,tang.\,\phi.$$

Now since the centre of gravity F is at a distance

$$F\,U = \tfrac{1}{3}\,H\,H_1 = \tfrac{1}{6}\,b\,tang.\,\phi$$

from the base $H\,R$ and at a distance $O\,U = \frac{2}{3}\,O\,H = \frac{1}{3}\,b$ from the centre O, it follows that the horizontal distance of the centre of gravity F from the centre O

$$= O\,K = O\,N + N\,K = O\,U \cos.\phi + F\,U \sin.\phi$$
$$= \tfrac{1}{3}\,b \cos.\phi + \tfrac{1}{6}\,b \operatorname{tang.}\phi \sin.\phi,$$

and the arm of the lever is

$$a = \overline{K\,L} = 2\,\overline{O\,K} = \tfrac{2}{3}\,b \cos.\phi + \tfrac{1}{3}\,b\,\frac{\sin.^2\phi}{\cos.\phi}.$$

Hence the equation of condition of the inclined position of equilibrium is

$$\frac{\frac{1}{8}\,b^2 \operatorname{tang.}\phi\,(\frac{2}{3}\,b \cos.^2\phi + \frac{1}{3}\,b \sin.^2\phi)}{b\,y \cos.\phi} - e \sin.\phi = 0,$$

or, substituting $\frac{\sin.\phi}{\cos.\phi} = \operatorname{tang.}\phi$,

$$\sin.\phi\,[(\tfrac{1}{12} + \tfrac{1}{24} \operatorname{tang.}^2\phi)\,b^2 - e\,y] = 0,$$

which equation is satisfied by

$$\sin.\phi = 0 \text{ and by}$$
$$\operatorname{tang.}\phi = \sqrt{2}\,\sqrt{\frac{12\,e\,y}{b^2} - 1}.$$

The angle $\phi = 0$, determined by the first equation, is applicable to the body when in an upright position, and that, given by the second equation, to the body when floating in an inclined position. If the latter case is possible, $\frac{e\,y}{b^2}$ must be $> \frac{1}{12}$. Now if h is the height and ε the specific gravity of the parallelopipedon, we have

$$y = \varepsilon\,h \text{ and } e = \frac{h - y}{2} = (1 - \varepsilon)\,\frac{h}{2},$$

whence it follows that

$$\operatorname{tang.}\phi = \sqrt{2}\,\sqrt{\frac{6\,\varepsilon\,(1 - \varepsilon)\,h^2}{b^2} - 1};$$

and the equation of condition for inclined floating is

$$\frac{h}{b} > \sqrt{\frac{1}{6\,\varepsilon\,(1 - \varepsilon)}}.$$

Example 1) If the floating parallelopipedon is as high as wide, and if its specific gravity is $\varepsilon = \frac{1}{2}$, we have

$$\operatorname{tang.}\phi = \sqrt{2}\,\sqrt{3\,.\,\tfrac{1}{2} - 1} = \sqrt{3 - 2} = 1, \text{ whence } \phi = 45°.$$

2) If the height $h = 0{,}9$ of the width b and the specific gravity is again $\frac{1}{2}$, we have

$$\operatorname{tang.}\phi = \sqrt{3\,.\,0{,}81 - 2} = \sqrt{0{,}43} = 0{,}6557, \text{ whence } \phi = 33°\ 15'.$$

§ **372. Specific Gravity.**—The law of the buoyant effort or upward thrust of water can be made use of to determine the heaviness or *specific gravity* of bodies. According to § 364 the buoyant effort of the water is equal to the weight of liquid displaced; hence, if we denote by V the volume of the body and by γ_1 the heaviness of the liquid, we have the buoyant effort $P = V\gamma_1$. Now if γ_2 is the heaviness of the material of the body, we have its weight $G = V\gamma_2$, whence the ratio of the heavinesses is

$$\frac{\gamma_2}{\gamma_1} = \frac{G}{P},$$

I.E., *the heaviness of the immersed body is to the heaviness of the fluid as the absolute weight of the body is to the buoyant effort or loss of weight during immersion.*

Hence $\gamma_2 = \frac{G}{P}\gamma_1$ and $\gamma_1 = \frac{P}{G}\gamma_2$, or if γ denotes the heaviness of water, ε_1 the specific gravity of the fluid, and ε_2 that of the body, we have, putting $\gamma_1 = \varepsilon_1\gamma$ and $\gamma_2 = \varepsilon_2\gamma$,

$$\varepsilon_2 = \frac{G}{P}\varepsilon_1 \text{ and } \varepsilon_1 = \frac{P}{G}\varepsilon_2.$$

If we know the weight of a body and its loss of weight when immersed in a liquid, we can find from the heaviness or specific gravity of the liquid the heaviness and specific gravity of the material of which the body is composed, and, inversely, from the heaviness or specific gravity of the latter, the heaviness and specific gravity of the former.

If the liquid in which we weigh solid bodies is water, we have $\varepsilon_1 = 1$ and $\gamma_1 = \gamma = 1000$ kilograms $= 62{,}425$ pounds; the former when we employ the cubic meter and the latter when we employ the cubic foot as unit of volume; therefore in this case the heaviness of the body is

$$\gamma_2 = \frac{G}{P}\gamma = \frac{\text{absolute weight}}{\text{loss of weight}} \text{ multiplied by the heaviness of water,}$$

and its specific gravity is

$$\varepsilon_2 = \frac{G}{P} = \frac{\text{absolute weight}}{\text{loss of weight}}.$$

In order to find the buoyant effort or loss of weight, we employ, as we do when determining the weight G, an ordinary balance. To the under side of one of its scale-pans is attached a small hook, from which the body is suspended by means of a hair, wire or fine thread before it is immersed in the water, which is contained in a vessel placed under the dish of the scale. A scale thus arranged for weighing under water is generally called *a hydrostatic balance* (Fr. balance hydrostatique; Ger. hydrostatische Wage).

If the body whose specific gravity is to be determined is less dense than water, we can combine it mechanically with some other heavy body, so that the compound mass will tend to sink in the water. If the heavy body loses in the water a weight P_2 and the compound mass P_1, the loss of weight of the lighter body is

$$P = P_1 - P_2.$$

Now if G denotes the weight of the lighter body, we have its specific gravity

$$\varepsilon_2 = \frac{G}{P} = \frac{G}{P_1 - P_2}.$$

If we know the specific gravity ε of a mechanical combination of two bodies, and also the specific gravities ε_1 and ε_2 of the components, we can calculate from the weight G of the whole mass, by means of the well-known *principle of Archimedes*, the weights G_1 and G_2 of the components.

We have $G_1 + G_2 = G$, and also

$$\text{volume } \frac{G}{\varepsilon_1 \gamma} + \text{volume } \frac{G_2}{\varepsilon_2 \gamma} = \text{volume } \frac{G}{\varepsilon \gamma}, \text{ or}$$

$$\frac{G_1}{\varepsilon_1} + \frac{G_2}{\varepsilon_2} = \frac{G}{\varepsilon}.$$

Combining the two equations, we obtain

$$G_1 = G\left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon_2}\right) : \left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}\right), \text{ or}$$

$$G_2 = G\left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon_1}\right) : \left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right).$$

EXAMPLE—1) If a piece of limestone weighing 310 grams becomes 121,5 grams lighter in water, the specific gravity of this body is

$$\varepsilon = \frac{310}{121{,}5} = 2{,}55.$$

2) In order to find the specific gravity of a piece of oak, a piece of lead wire, which lost 10,5 grams in weight when weighed in water, was wrapped around the piece of wood, which weighed 426,5 grams. The compound mass was 484,5 grains lighter in the water than in the air; hence the specific gravity of the wood is

$$\varepsilon = \frac{426{,}5}{484{,}5 - 10{,}5} = \frac{426{,}5}{474} = 0{,}9.$$

3) An iron vessel completely filled with mercury weighed 500 pounds, and lost, when weighed in water, 40 pounds. If the specific gravity of the cast iron is $= 7{,}2$ and that of the mercury is $= 13{,}6$, the weight of the empty vessel is

$$G_1 = 500\left(\frac{40}{500} - \frac{1}{13,6}\right) : \left(\frac{1}{7,2} - \frac{1}{13,6}\right) = 500\,(0,08 - 0,07353) :$$

$$(0,1388 - 0,0735) = \frac{500\,.\,0,00647}{0,0653} = \frac{3235}{65,3} = 49,5 \text{ pounds},$$

and the weight of the mercury contained in it is

$$G_2 = 500\,.\,(0,08 - 0,1388) : (0,07353 - 0,1388) = \frac{500\,.\,0,0588}{0,0653} = \frac{2940}{6,53}$$

$$= 450,2 \text{ pounds}.$$

REMARK—1) We can determine the specific gravity of fluids, loose granular masses, etc., by simply weighing them in the air; for by enclosing them in vessels, we can obtain any desired volume of them. If the weight of an empty bottle is $= G$, and when filled with water it is $= G_1$, and if, when filled with some other liquid, its weight is G_2, the specific gravity of the latter liquid is

$$\varepsilon = \frac{G_2 - G}{G_1 - G}.$$

In order, E.G., to obtain the specific gravity of rye (in bulk), we filled a bottle with grains of rye, and, after shaking it well, weighed it. After subtracting the weight of the bottle, that of the rye was found to be $= 120,75$ grams, and the weight of an equal quantity of water was 155,65; hence the specific gravity of the rye in bulk is

$$= \frac{120,75}{155,65} = 0,776,$$

and a cubic foot of this grain weighs

$$0,776\,.\,62,5 = 48,5 \text{ pounds}.$$

2) The problem, first solved by Archimedes, of determining from the specific gravity of a composition, and those of its components, the proportion of each of the components, is of but very limited application to chemical combinations, alloys of metals, etc.; for in such cases a contraction generally, and an expansion sometimes, takes place, so that the volume of the composition is no longer equal to the sum of the volumes of the components.

§ 373. **Hydrometers, Areometers.**—We generally employ for the determination of the density of fluids *areometers* or *hydrometers* (Fr. aréomètres; Ger. Aräometer, Senkwagen). These instruments are hollow, symmetrical about an axis, have their centre of gravity very low down, and give, when we float them in any liquid, its specific gravity. They are made of glass, sheet brass, etc., and, according to the uses they are applied to, are called hydrometers, lactometers, salinometers, alcoholmeters, etc. There are two kinds of areometers, viz.: those *with weights* (Fr. à volume constant; Ger. Gewichtsaräometer), and the *graduated areometers* (Fr. à poids constant; Ger. Scalenaräometer). The first are often used to determine the weight or specific gravity of solid bodies.

1) If V is the volume of the part of an areometer $A\ B\ C$, Fig. 632, which is under water, when the latter floats vertically immersed to a point O, G the weight of the whole apparatus, P that of the weights placed upon the dish A, when the apparatus floats in water, whose heaviness $= \gamma$, and P_1 their weight when the apparatus floats in another liquid whose density is γ_1, we will have

$$V\gamma = P + G,$$

$$V\gamma_1 = P_1 + G.$$

FIG. 632.

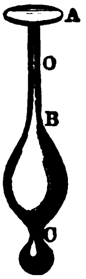


FIG. 634.

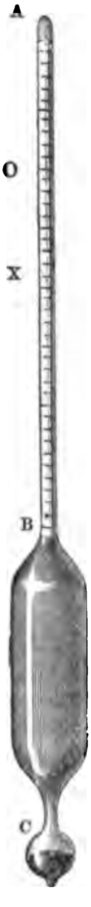
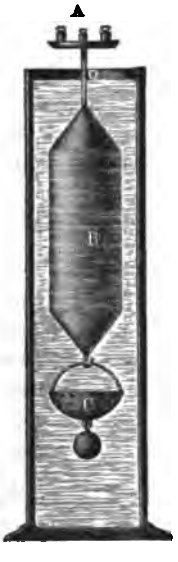


FIG. 633.



Hence the ratio of the heavinesses or specific gravities of these liquids is

$$\frac{\gamma_1}{\gamma} = \frac{P_1 + G}{P + G}.$$

2) Let P be the weight, which must be placed upon the dish in order to immerse the areometer $A\ B\ C$, Fig. 633, to a point O, and let P_1 be the weight, which must be placed upon the dish A with the body to be weighed in order to produce the same immersion, then we have simply

$$G_1 = P - P_1.$$

But if we must increase P_1 by P_2 when the body to be weighed is placed in the lower dish C, which is under water, in order to preserve the same depth of immersion, the upward thrust is $= P_2$ and the specific gravity of the body is

$$\varepsilon = \frac{G_1}{P_2} = \frac{P - P_1}{P_2}.$$

The hydrometer with the dish suspended below is employed for the determination of the specific gravity of solid bodies, such as minerals, etc., and is called *Nicholson's hydrometer.*

3) If we put the weight of an areometer $B\ C$ with a graduated *scale* $A\ B$, Fig. 634, $= G$, and the immersed volume, when it floats on water, $= V$, we have $G = V\gamma$. If the areometer rises a distance $O\ X = x$, when immersed in another

liquid, we have, when the cross-section of the shaft is denoted by F, the volume immersed

$$= V - F\,x, \text{ and therefore } G = (V - F\,x)\,\gamma_1.$$

Dividing the two formulas by each other, we obtain the heaviness of the liquid

$$\gamma_1 = \frac{V}{V - F\,x} \cdot \gamma = \gamma : \left(1 - \frac{F}{V}\,x\right) = \frac{\gamma}{1 - \mu\,x},$$

in which μ denotes the constant quotient $\frac{F}{V}$.

If the liquid in which the areometer floats is lighter than water, it will sink in it a distance x, and we will have

$$G = (V + F\,x)\,\gamma, \text{ and therefore}$$

$$\gamma_1 = \frac{\gamma}{1 + \mu\,x}.$$

In order to find the coefficient $\mu = \frac{F}{V}$, we increase its weight by an amount P, E.G. by pouring mercury in the areometer at the upper end, so that it passes to the bottom of it, rendering the apparatus so much heavier that, when floating in water, a considerable portion of the length of the stem, to which the scale is applied, is immersed. Putting $P = F\,l\,\gamma$, l denoting the immersion produced by P, we obtain

$$\mu = \frac{F}{V} = \frac{P}{V\,l\,\gamma} = \frac{P}{G\,l}.$$

EXAMPLE—1) If an areometer, weighing 65 grams, must have 13,5 grams removed from the dish in order to float at the same depth in alcohol as it had done in water, the specific gravity of alcohol is

$$= \frac{65 - 13{,}5}{65} = 1 - 0{,}208 = 0{,}792.$$

2) The normal weight of a Nicholson hydrometer is 100 grams; that is, we must place 100 grams upon the dish in order to sink the instrument to 0, but we must take away 66,5 grams after having laid a piece of brass which we wish to weigh upon the upper dish, and 7,85 grams had to be added when the brass was removed to the lower dish. The absolute weight of the brass is then 66,5 grams and its specific gravity is

$$\frac{66{,}5}{7{,}85} = 8{,}47.$$

3) A graduated areometer, weighing 75 grams, rises, after 31 grams of the substance, with which it was filled, has been removed, a distance l = 6 inches = 72 lines; the coefficient μ is therefore

$$= \frac{31}{75\,.\,72} = 0{,}00574.$$

It was then refilled until its weight became again 75 grams, when it was placed in a solution of salt; it then rose a distance of 29 lines; the specific gravity of the liquid is therefore

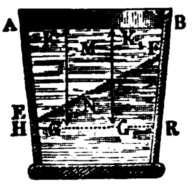
$$= 1 : (1 - 0{,}00574 \,.\, 29) = 1 : 0{,}833 = 1{,}2.$$

REMARK.—A more extended treatment of this subject belongs to the province of chemistry, physics and technology.

§ 374. Liquids of Different Densities.

§ 374. Liquids of Different Densities.—If several *liquids of different densities* exist in a vessel at the same time without exerting any chemical action upon one another, they will place themselves, in consequence of the ease with which their particles are displaced, above each other in the order of their specific gravities, viz: the most dense at the bottom, the less dense above it and the least dense on top. When in equilibrium the limiting surfaces are horizontal; for so long as the limiting surface $E\ F$ between the masses M and N, Fig. 635, is inclined so long will there be columns of liquid, such as $G\ K$, $G_1\ K_1$, etc., of different weights above the horizontal layer $H\ R$; hence the pressure upon this layer cannot be the same everywhere and consequently equilibrium cannot exist.

FIG. 635.



The liquids arrange themselves also in the *communicating tubes* $A\ B$ and $C\ D$, Fig. 636, according to their specific gravities above one another, but their surfaces $A\ O$ and $D\ G$ do not lie in one and the same horizontal plane.

FIG. 636

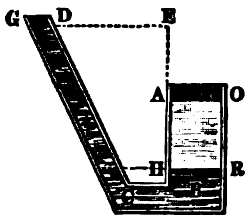
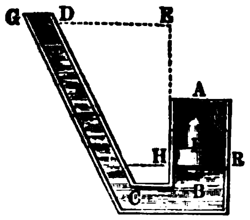


FIG. 637.



If F is the area of the cross-section $H\ R$ of a piston, Fig. 637, in one leg $A\ B$ of two communicating tubes and h the head of water or the height $E\ H$ of the surface of the water in the second tube $C\ D$ above $H\ R$, we have the pressure upon the surface of the piston

$$P = F h \gamma.$$

If we replace the force, exerted by the piston, by a column of liquid $H\ A\ O\ R$, Fig. 636, whose height is h_1 and whose heaviness is γ_1, we have

$$P = F h_1 \gamma_1;$$

equating the two expressions we obtain

$$h_1 \gamma_1 = h \gamma,$$

or the proportion

$$\frac{h_1}{h} = \frac{\gamma}{\gamma_1}.$$

Therefore the heads or the heights of the columns of liquid, measured from the common plane of contact of two different liquids, which are in equilibrium in two communicating tubes, are to each other inversely as the heavinesses or specific gravities of these liquids.

Since mercury is about 13,6 times as heavy as water, a column of mercury in communicating tubes will hold in equilibrium a column of water 13,6 times as long.

CHAPTER III.

OF THE MOLECULAR ACTION OF WATER.

§ **375. Molecular Forces.**—Although the cohesion of water is very slight, it is not null. The *molecules* (Fr. molécules; Ger. Theile or Moleküle) not only cohere together, but also adhere to other bodies, E.G., to the sides of a vessel, so that a certain force is necessary to destroy this union, which we call the *adhesion* (Fr. adhérence; Ger. Adhäsion) of the water. A drop of water, which hangs from a solid body, demonstrates the existence of the cohesion and of the adhesion of the water. Without cohesion the water could not form a drop and without adhesion it could not remain hanging from the solid body; gravity is here overcome not only by the cohesion, but also by the adhesion. The actions, arising from the combination of the forces of cohesion and adhesion, are called, to distinguish them from the actions of inertia, of gravity, etc., the *molecular actions. Capillarity* or the raising or depressing of the surface of water or mercury in narrow tubes or between plates, placed close together, is an important instance of molecular action.

§ **376. Adhesion Plates.**—The cohesion and adhesion of water have been determined by means of adhesion plates. To accomplish this object, such a plate is suspended (instead of the scale pan) at one end of the beam of a balance, which is brought into equilibrium by means of weights; the vessel containing the liquid to be examined is then caused to approach gradually, until the surface of the liquid comes in contact with the plate. Weights are now gradually placed upon the dish at the other end of the beam, until the plate is torn away from the surface of the water. The results of such experiments depend particularly upon the fact whether the plate is *moistened* by the water or *not*. In the first case after the contact a thin sheet of water remains hanging to the plate; hence in tearing the latter from the water, we overcome not the adhesion, but the cohesion of the water. Hence the force necessary to tear different plates from the surface of the water does not depend upon the nature of the material, of which the plates are composed. Other liquids, on the contrary, require different forces to be applied to the adhesion plates. *Du Buat* found that the adhesion between water and tin plate was from 65 to 70 grains per square inch (old Prussian measure). This gives a force of about 5 kilograms for a square meter, or 1,024 pounds per square foot. *Achard* found values differing but little from the above for lead, iron, copper, brass, tin and zinc. *Gay Lussac* obtained the same results with a glass disc, and *Huth* with different kinds of wooden plates.

If, on the contrary, the surface of the disc is *not moistened* by the surface of the water, the results obtained are totally different; for in this case it is not the cohesion, but the adhesion of the water which is overcome. It appears that the duration of contact has a great influence upon the force necessary to tear the disc loose, E.G., Gay Lussac found that, with a glass plate 120 millimeters in diameter, a force varying from 150 to 300 grams, according as the duration of contact was long or short, was necessary to tear it loose from a surface of mercury.

REMARK.—In Frankenheim's "Lehre der Cohäsion" the phenomena of cohesion, as, E.G., those presented when moistened plates are torn from the surface of water, are called "*Synaphy*," and, on the contrary, the phenomena of adhesion, as, E.G., those occurring during the separation of unmoistened plates from the surface of a liquid, "*Prosaphy*."

§ 377. Adhesion to the Sides of a Vessel.

—If a drop of water spreads itself out upon the surface of another body and moistens it, the adhesion is in this case predominant; but if, on the contrary, the drop retains its spherical form upon the surface of a solid or fluid body, the cohesion is the strongest. The combined action of these two forces upon the surface of a liquid near the walls of the vessel is particularly remarkable; the water rises up and forms a concave surface when the cohesion is less powerful than the adhesion, and the wall becomes moistened in consequence; the surface of the water, on the contrary, is curved downwards in the neighborhood of the walls of the vessel and forms a convex surface when the side of the vessel is not moistened or when the cohesion is predominant.

These phenomena can be easily explained as follows.

A molecule E in the surface $H\,R$ of the water (Fig. 638) is drawn downwards in all directions by the surrounding water, and the resultant of all these attractions is a single force A acting vertically downwards; on the contrary, a molecule E at the vertical wall $B\,E$, Fig. 639, of the vessel is acted upon by the wall with a horizontal force P and by the water filling the quadrant $B\,E\,O$ with a force A, whose direction is inclined downwards to the horizon; the direction of the resultant R of these two forces is at right angles to the surface of the water (see § 354). According as the attractive force of the wall of the vessel is greater or less than the horizontal component A_1 of the mean force of cohesion A of the water, the resultant R will assume a direction either from without inward or from within outward. In the first case (Fig. 639) the surface of the water at E rises along the wall, and in the second case it descends along the wall $B\,E$, as is represented in Fig. 640.

FIG. 638.

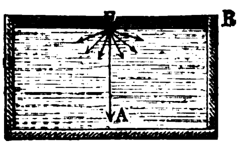


FIG. 639.

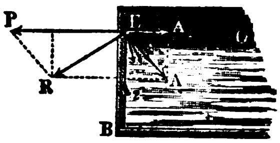
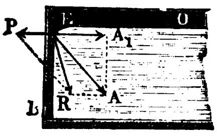
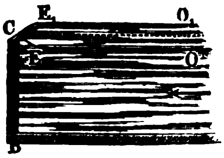


FIG. 640.



These relations change, when the water reaches to the brim of the vessel; for the direction of the attractive force of the wall of the vessel is then different. If, E.G., the surface of the water $E\,O$, Fig. 641, which in the beginning reached to the brim C of the vessel $B\,C\,O$, is caused to rise gradually by adding water, the inclination of the force of adhesion to the horizon will gradually increase, and its horizontal component will, in consequence, gradually decrease, until it becomes less than the horizontal component A_1 of the force of cohesion A. Consequently the form of the surface of the water at E changes continually, until its concavity becomes a convexity and the depression below the brim of the vessel becomes an elevation, which must attain a certain height before the water will flow over the side of the vessel.

FIG. 641.



§ 378. Tension of the Surface of the Water.—Since each molecule in the surface $H R$, Fig. 638, of a liquid is attracted downwards by the mass of liquid below it with a force A, we can assume that a condensation and a coherence among the molecules of the liquid upon the surface will be the result and that a certain force will therefore be necessary to overcome this coherence or to tear the surface of the liquid. This coherence of the surface of a liquid shows itself not only whenever a foreign body is dipped into it, but also whenever the surface of the liquid becomes curved, as, E.G., in the neighborhood of the wall of the vessel. If we assume with *Young* that the tension or cohesion of the surface of a liquid is the same in all parts of it, we can deduce, as *Geheimer Oberbaurath Hagen* has proved, from that hypothesis all the laws of capillary attraction which coincide best with the results of experiment.

FIG. 642.

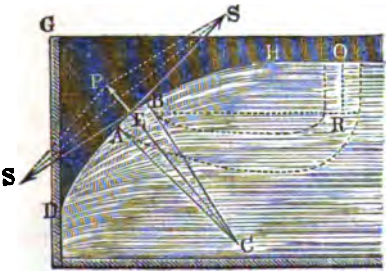
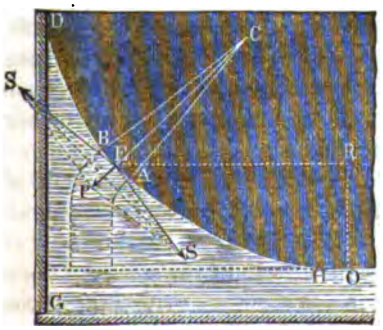


FIG. 643.



In the neighborhood of a plane wall $D\ G$, Figs. 642 and 643, the surface of the liquid forms a cylindrical surface $D\ A\ H$, which is convex either upwards or downwards. If P is the normal force upon an element $A\ E\ B = \sigma$ of this surface, S the tension of this element and r its radius of curvature $CA = CB$, we have, in consequence of the similarity of the triangles EPS and ABC,

$$\frac{P}{S} = \frac{AB}{CA} = \frac{\sigma}{r},$$

and, therefore, the normal or bending force is

$$P = \frac{\sigma}{r} S.$$

Now if the element AEB of the surface is at the vertical distance $OR = y$ above or below the surface of the water which is free from the influence of the wall DG, and if γ denotes the heaviness of the liquid, we have, according to (§ 356) the well-known law of hydrostatics, the pressure of the water upon the element $\overline{AB} = \sigma$

$$P = \sigma y \gamma;$$

we can therefore put

$$\sigma y \gamma = \frac{\sigma}{r} S \text{ and}$$

$$y = \frac{S}{r\gamma}.$$

Fig. 644.

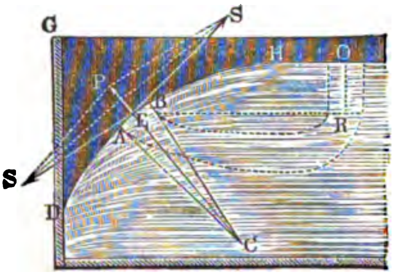
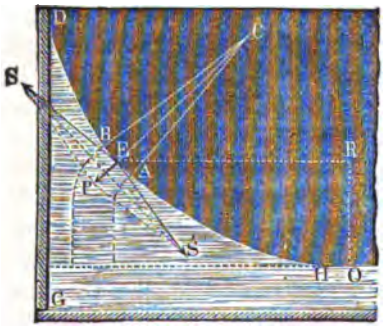


Fig. 645.



Hence the *depression* or *elevation* of an element of the surface of a liquid in reference to the free or unaffected part of this surface is *inversely proportional to the radius of curvature.*

§ **379.** In the vicinity of a *curved wall*, E.G., of a vertical cylindrical surface, the surface of the water forms a surface of double curvature and the column of water below the rectangular element $FGHK$, Fig. 646, of the surface is solicited by two forces P_1 and P_2, one of which is the resultant of the tensions S_1, S_1 in the normal plane ABE, parallel to the side $FG = HK$; the other is the resultant of the tensions S_2, S_2 in the normal plane CDE, parallel to the side $GH = FK$. The former plane corresponds to the greater and the latter to the least radius of curvature; putting the two radii $= r_1$ and r_2 and the length of the sides $FG = \sigma_1$ and $GH = \sigma_2$ and denoting the tension for a width $=$ unity by S, we have the tensions acting in the two planes

$$S_1 = \sigma_2\, S \text{ and } S_2 = \sigma_1\, S$$

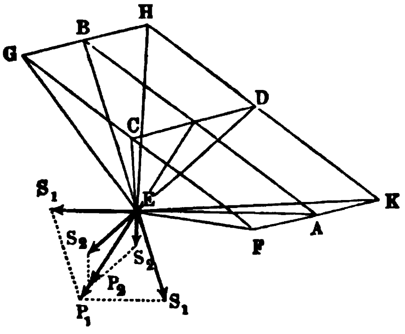
and the normal forces resulting from them

$$P_1 = \sigma_2\, S \frac{\sigma_1}{r_1} = \frac{S\,\sigma_1\,\sigma_2}{r_1} \text{ and}$$

$$P_2 = \sigma_1\, S \frac{\sigma_2}{r_2} = \frac{S\,\sigma_1\,\sigma_2}{r_2}, \text{ and their resultant is}$$

$$P = P_1 + P_2 = S\,\sigma_1\,\sigma_2\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

Fig. 646.



If here also y denote the height of the element $F\,G\,H\,K$ of the surface (which may be regarded as a rectangle, whose area is $\sigma_1\,\sigma_2$) above the lowest or general surface of the water, we have the force, with which this element is drawn normally upwards or downwards by the water above or below it,

$$P = y\,\sigma_1\,\sigma_2\,\gamma;$$

equating the two values for P, we obtain

$$y\,\sigma_1\,\sigma_2\,\gamma = S\,\sigma_1\,\sigma_2\left(\frac{1}{r_1} + \frac{1}{r_2}\right), \text{ whence}$$

$$y = \frac{S}{\gamma}\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

When the wall is *cylindrical* the elevation (depression) of the surface of the water above (below) the general water level is at every point proportional to the sum of the reciprocals of the maximum and minimum radii of curvature. This formula contains also that of the foregoing paragraph; for if the normal section $C\,E\,D$ is a right line, we have

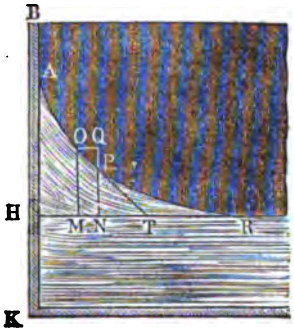
$$r_2 = \infty, \text{ whence}$$

$$\frac{1}{r_2} = 0 \text{ and}$$

$$y = \frac{S}{\gamma} \cdot \frac{1}{r_1}.$$

(§ 380.) **Curve of the Surface of Water.**—The curve formed by the vertical cross-section of the surface of the water near a plane wall, can be found, according to Hagen, in the following manner. Let $A\ R$, Fig. 647, be the surface of the water attracted by the vertical wall $B\ K$, $H\ R$ the general level of the water, and let the point of intersection H of the two surfaces be the origin of co-ordinates. Let us put the co-ordinates of a point O of the surface $A\ O\ R$, $H\ M = x$ and $M\ O = y$, the arc $A\ O = s$, the tangential angle $O\ T\ M = a$, and the elements $O\ Q$, $Q\ P$ and $O\ P$ respectively $= d\,x$, $d\,y$ and $d\,s$.

FIG. 647.



Since $y = \frac{S}{r\,\gamma}$, and, according to Article 33 of the Introduction to the Calculus,

$$r = -\frac{d\,s}{d\,a} \text{ and } d\,y = -\,d\,s\ sin.\ a, \text{ we have}$$

$$y = -\frac{S\,d\,a}{\gamma\,d\,s} = \frac{S\ sin.\ a\,.\,d\,a}{\gamma\,d\,y}, \text{ or}$$

$$y\,d\,y = \frac{S}{\gamma}\ sin.\ a\,.\,d\,a,$$

by integrating which we obtain

$$\tfrac{1}{2}\,y^2 = \frac{S}{\gamma}\int sin.\ a\,.\,d\,a = Con. - \frac{S}{\gamma}\ cos.\ a.$$

Since for the point R, a and y are both $= 0$, we have

$$0 = Con. - \frac{S}{\gamma}\ cos.\ 0, \text{ whence } Con. = \frac{S}{\gamma} \text{ and}$$

$$y^2 = \frac{2\,S}{\gamma}\,(1 - cos.\ a) = \frac{4\,S}{\gamma}\,\frac{(1 - cos.\ a)}{2} = \frac{4\,S}{\gamma}\,(sin.\ \tfrac{1}{2}\,a)^2,$$

hence

$$y = 2\sqrt{\frac{S}{\gamma}}\ sin.\ \tfrac{1}{2}\,a.$$

For $a = 90°$, we have $sin.\ \frac{1}{2}\,a = sin.\ 45° = \sqrt{\frac{1}{2}}$; hence the maximum elevation of the water immediately against the wall is

$$h = 2\sqrt{\frac{S}{\gamma}}\cdot\sqrt{\tfrac{1}{2}} = \sqrt{\frac{2\,S}{\gamma}}, \text{ or inversely}$$

$$\frac{S}{\gamma} = \tfrac{1}{2}\,h^2 \text{ and}$$

1) $y = h\sqrt{2}\,.\,sin.\ \frac{1}{2}\,a.$

Differentiating this expression, we obtain

$$d\,y = \tfrac{1}{2}\,h\sqrt{2}\,cos.\,\tfrac{1}{2}\,a\,.\,d\,a = h\sqrt{\tfrac{1}{2}}\,cos.\,\tfrac{1}{2}\,a\,.\,d\,a,$$

and since $d\,y = -\,d\,x\,.\,tang.\,a$, it follows that

$$d\,x = -\,h\sqrt{\tfrac{1}{2}}\,.\,\frac{cos.\,\frac{1}{2}\,a}{tang.\,a}\,.\,d\,a = -\,h\sqrt{\tfrac{1}{2}}\,.\,\frac{cos.\,\frac{1}{2}\,a\;cos.\,a}{sin.\,a}\,.\,d\,a$$

$$= -\,h\sqrt{\tfrac{1}{2}}\,.\,\frac{cos.\,\frac{1}{2}\,a\left[(cos.\,\frac{1}{2}\,a)^2 - (sin.\,\frac{1}{2}\,a)^2\right]}{2\,sin.\,\frac{1}{2}\,a\,.\,cos.\,\frac{1}{2}\,a}\,d\,a$$

$$= -\,h\sqrt{\tfrac{1}{2}}\,.\,\frac{1-2\,(sin.\,\frac{1}{2}\,a)^2}{2\,sin.\,\frac{1}{2}\,a}\,d\,a$$

$$= -\,h\sqrt{\tfrac{1}{2}}\,.\left(\frac{\frac{1}{2}}{sin.\,\frac{1}{2}\,a} - sin.\,\tfrac{1}{2}\,a\right)d\,a.$$

But now

$$\int sin.\,\tfrac{1}{2}\,a\,.\,d\,a = -\,2\,cos.\,\tfrac{1}{2}\,a \text{ and}$$

$$\int\frac{d\,a}{sin.\,\frac{1}{2}\,a} = 2\,l\,tang.\,\tfrac{1}{4}\,a$$

(see Introduction to the Calculus, Art. 29);

hence we have

$$x = -\,h\sqrt{\tfrac{1}{2}}\,(l\,tang.\,\tfrac{1}{4}\,a + 2\,cos.\,\tfrac{1}{2}\,a) + Con.$$

Now since for $x = 0$, $a^\circ = 90^\circ$, $tang.\,\frac{1}{4}\,a = tang.\,22\frac{1}{2}^\circ = \sqrt{2} - 1$ and $cos.\,\frac{1}{2}\,a = \sqrt{\frac{1}{2}}$, it follows that

$$Con. = h\sqrt{\tfrac{1}{2}}\,[l\,(\sqrt{2} - 1) + 2\sqrt{\tfrac{1}{2}}], \text{ and}$$

$$2)\quad x = h\sqrt{\tfrac{1}{2}}\left[l\left(\frac{\sqrt{2} - 1}{tang.\,\frac{1}{4}\,a}\right) + 2\,(\sqrt{\tfrac{1}{2}} - cos.\,\tfrac{1}{2}\,a)\right]$$

$$= h\,[1 - \sqrt{2}\,.\,cos.\,\tfrac{1}{2}\,a - \sqrt{\tfrac{1}{2}}\,l\,(\sqrt{2} + 1)\,tang.\,\tfrac{1}{4}\,a].$$

For $a = 0$ we have

$$cos.\,\tfrac{1}{2}\,a = 1 \text{ and } l\,tang.\,\tfrac{1}{4}\,a = -\,\infty,$$

and therefore

$$x = +\,\infty\,;$$

$H\,R$ is consequently the asymptote, which the section $A\;O\;R$ of the surface of the water continually approaches.

REMARK.—If we invert the formula (1) and put

$$sin.\,\tfrac{1}{2}\,a = \frac{y}{h}\sqrt{\tfrac{1}{2}}$$

we can calculate for every value of y, first a and then by means of (2) the corresponding value of x.

i.e. as a mean value

$$\frac{S}{\gamma} = 0,9243 \text{ and } S = 0,01059 \text{ grams.}$$

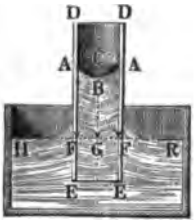
(Compare the foregoing paragraph.)

§ **382. Capillary Tubes.**—We can easily calculate the height to which the surface of water will rise in narrow vertical tubes, called *capillary tubes* (Fr. tubes capillaires; Ger. Haarröhrchen), by starting from the formula

$$y = \frac{S}{\gamma}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

of § 379 as a basis and assuming that the surface (the meniscus) forms a semi-spheroid $A\,B\,A$, Fig. 649, whose circular base $A\,A$ coincides with the cross-section of the tube. If we retain the notations of the foregoing paragraph, i.e. if we put the radius $C\,A$ of the tube $= a$ and minimum and maximum heights $B\,G$ and $A\,F$ of the water in the tube above the general level of the water $H\,R$, $= h_1$ and h_2, we must substitute in

FIG. 649.



$$h_2 = \frac{S}{\gamma}\left(\frac{1}{r_1} + \frac{1}{r_2}\right),\ r_1 = a \text{ and } r_2 = \frac{(h_2 - h_1)^2}{a}, \text{ and in}$$

$$h_1 = \frac{S}{\gamma}\left(\frac{1}{r_1} + \frac{1}{r_2}\right),\ r_1 = r_2 = \frac{a^2}{h_2 - h_1}; \text{ thus we obtain}$$

$$h_2 = \frac{S}{\gamma}\left(\frac{1}{a} + \frac{a}{(h_2 - h_1)^2}\right) \text{ and}$$

$$h_1 = \frac{2\,S}{\gamma} \cdot \frac{(h_2 - h_1)}{a^2}.$$

Subtracting the last equation from the one preceding it, we obtain

$$h_2 - h_1 = \frac{S}{\gamma}\left(\frac{1}{a} + \frac{a}{(h_2 - h_1)^2} - \frac{2\,(h_2 - h_1)}{a^2}\right),$$

or

$$1 = \frac{S}{\gamma}\left(\frac{1}{a\,(h_2 - h_1)} + \frac{a}{(h_2 - h_1)^3} - \frac{2}{a^2}\right),$$

and also

$$\left(\frac{\gamma}{S} + \frac{2}{a^2}\right)(h_2 - h_1)^3 - \frac{1}{a}\,(h_2 - h_1)^2 = a.$$

If a is small, we can put

$$\frac{2}{a^2}(h_2 - h_1)^3 - \frac{1}{a}(h_2 - h_1)^2 = a,$$

whence it follows that

$$h_2 - h_1 = a;$$

assuming $h_2 - h_1 = a + \delta$ and putting $(h_2 - h_1)^2 = a^2 + 2\,a\,\delta$, and also $(h_2 - h_1)^3 = a^3 + 3\,a^2\,\delta$, we obtain

$$\left(\frac{\gamma}{S} + \frac{2}{a^2}\right)(a^3 + 3\,a^2\,\delta) - \frac{1}{a}(a^2 + 2\,a\,\delta) = a,$$

or

$$\frac{\gamma}{S}\,a^3 + \left(\frac{\gamma}{S} + \frac{2}{a^2}\right).\,3\,a^2\,\delta - 2\,\delta = 0,$$

whence it follows that

$$\delta = -\frac{\gamma\,a^3}{3\,\gamma\,a^2 + 4\,S}, \text{ or approximatively, } \delta = -\frac{\gamma\,a^3}{4\,S}.$$

Hence we have

$$h_2 - h_1 = a - \frac{\gamma\,a^3}{4\,S},$$

whence

$$h_1 = \frac{2\,S}{\gamma}\cdot\frac{1}{a^2}\left(a - \frac{\gamma\,a^3}{4\,S}\right) = \frac{2}{a}\cdot\frac{S}{\gamma} - \frac{a}{2} \text{ and}$$

$$h_2 = \frac{S}{\gamma}\left(\frac{1}{a} + \frac{a}{\left(a - \frac{\gamma\,a^3}{4\,S}\right)^2}\right) = \frac{S}{\gamma}\left[\frac{1}{a} + \frac{a}{a^2}\left(1 + \frac{\gamma\,a^2}{4\,S}\right)^2\right]$$

$$= \frac{S}{\gamma}\left[\frac{1}{a} + \frac{1}{a}\left(1 + \frac{\gamma\,a^2}{2\,S}\right)\right] = \frac{2}{a}\cdot\frac{S}{\gamma} + \frac{a}{2}.$$

The mean elevation in capillary tubes is inversely proportional to the width of the tube.

We have also for the determination of S the formula

$$\frac{S}{\gamma} = \tfrac{1}{2}\,a\,h_1 + \frac{a^2}{4}.$$

Observations made by Hagen with capillary tubes in spring water gave the following results:

Width of tube a, lines	0,295	0,336	0,413	0,546	0,647	0,751	0,765
Elevation h_1, "	10,08	8,50	6,87	5,17	4,23	3,72	3,59
Measure of tension } $\frac{S}{\gamma}$, grams	1,508	1,455	1,453	1,478	1,473	1,512	1,494

According to these experiments the mean values are

$$\frac{S}{\gamma} = 1,482 \text{ and } S = 0,0170 \text{ grams.}$$

The variations in these values are due to the fact that the tension S of the surface of the water diminishes with the time, and is much smaller in water that has been boiled, than in fresh. We can now assume that the tension of the water in every strip 1 line wide is $S = 0,0106$ to $0,0170$ grams.

§ **383**. The foregoing theory is also applicable, ***when the wall is** not moistened by the liquid;* here, however, it is not an elevation but a *sinking* of the surface which takes place, and the latter is concave instead of convex. The vertical force P, which is due to the difference of level $B\ G$ and acts from below upwards, is balanced by the tensions S and S of the surface $A\ B\ A$, Fig. 650, of the liquid in the tube. The force of adhesion of the solid body does not, according to the foregoing theory, come into play in this case.

FIG. 650.

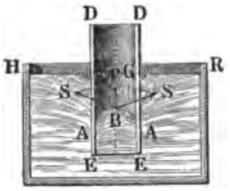
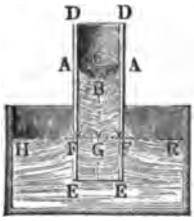


FIG. 651.



If we make the force, with which the wall of the tube attracts to itself the column of fluid $B\ G$, Fig. 651, proportional to the circumference of the tube, if, E.G., for a *cylindrical tube* we put this force $P = \mu\ 2\ \pi\ a$, in which μ denotes a coefficient, we have

$$\pi\ a^2\ h = 2\ \mu\ \pi\ a,$$

and, therefore, the mean elevation of the water in the tube is

$$h = \frac{2\ \mu}{a}.$$

For two *parallel plates*, on the contrary, we have $P = 2\ \mu\ l$ and $P = 2\ a\ h\ l\ \gamma$, l denoting the undetermined length of the column of water, and, therefore,

$$h = \frac{\mu}{a},$$

I.E., half as great as in a tube, when the distance $2\ a$ of the plates from each other is equal to the diameter of the tube. This agrees also with the results of the last paragraph.

According to *Hagen's* experiments the strength or tension of the surface of liquid does not depend upon its degree of fluidity, but it increases in intensity, the more the liquid adheres to other bodies. According to others, particularly *Brunner* and *Frankenheim* (see Poggendorf's Annalen, Vols. 70 and 72), the height h, to which water rises in capillary tubes, increases and S consequently diminishes, when the temperature of the liquid is augmented. For alcohol S is about one-half and for mercury about eight times the strength of the surface of water.

REMARK—1) Hagen found by measuring and weighing drops of liquid, which tore themselves loose from the base of small cylinders, about the same values as he did by his observations upon capillary plates. In like manner the experiments with adhesion plates have furnished results, which coincide very well with the former, when we assume that the force necessary to tear the plate loose is balanced by the weight of the cylinder of liquid raised and by the tension upon the surface of this cylinder.

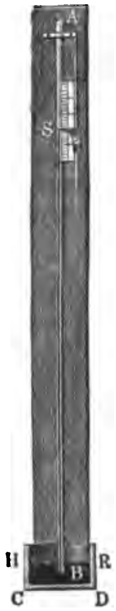
2) The number of treatises upon capillary attraction is so great that we cannot cite them all here. The greatest mathematicians, such as La Place, Poisson, Gauss, etc., have given their attention to it. A complete account of the older literature is to be found in Frankenheim's "Lehre von der Cohäsion." The treatise which was specially used in preparing this chapter is the following: "Ueber die Oberfläche der Flüssigkeiten," by Hagen, a memoir read in the Royal Academy of Science in Berlin, in 1845. A new physical theory of capillary attraction, by J. Mille, is contained in Vol. 45 of Poggendorff's Annalen (1838). Here also belong Boutigny's Studies of Bodies in a Spheroidal Condition.

CHAPTER IV.

OF THE EQUILIBRIUM AND PRESSURE OF THE AIR.

§ 384. **Tension of Gases.**—The *atmospheric air*, which surrounds us, as well as all other *gases* (Fr. gaz; Ger. gase) possess, in consequence of the repulsion between their molecules, a tendency to expand into a greater space. We can therefore obtain a limited quantity of air only by enclosing it in a perfectly tight vessel. The force with which the gases seek to expand is called their tension (Fr. tension; Ger. Spannkraft, Elasticität or Expansivkraft). It shows itself by the pressure exerted by the gas upon the walls of the vessel enclosing it, and differs from the elasticity of solids or liquids in this: it is in action, no matter what the density of the gas may be, while the expansive force of solids and liquids is null, when they are extended to a certain degree. The pressure or tension of the air and other gases is measured by *barometers*, *manometers* and valves. The *barometer* (Fr. baromètre; Ger. Barometer) is employed principally to measure the pressure of the atmosphere. The most common kind is the so-called *cistern barometer*, Fig. 652; it consists of a glass tube, closed at one end A and open at the other B, which, after being filled with mercury, is turned over and placed with its open end under the mercury contained in the vessel $C\,D$. After the instrument has been inverted, there remains in the tube a column $B\,S$ of mercury, which (see § 374) is balanced by the pressure of the air upon the surface $H\,R$. Since the space $A\,S$ above the column of mercury is free from air, the column has no pressure upon it from above, and the height of this column, or rather that of the mercury in the same, above the level $H\,R$ of the mercury in the vessel can be employed as a measure of the pressure of the air. In order to measure easily and correctly this height, an accurately graduated scale is added, which can be moved along the tube and which is sometimes provided with a movable pointer S.

FIG. 652.



REMARK.—It is the province of physics to give more detailed descriptions of different barometers, to explain their use, etc. (See Müller's Lehrbuch der Physik und Meteorologie, Vol. I.)

§ 385. Pressure of the Atmosphere.—By means of the barometer it has been found that in places situated near the level of the sea, when the atmosphere is in its average condition, the *pressure of the air* is balanced by a column of mercury at a temperature of 32° Fahr., 76 centimetres long or about 28 Paris inches = 29 Prussian inches = 29,92 English inches. Since the specific gravity of mercury at 32° temperature is 13,6, it follows that the pressure of the air is equal to the weight of a column of water 0,76 . 13,6 = 10,336 metres = 31,73 Paris feet = 32,84 Prussian feet = 33,91 English feet. We often measure the tension of the air by the pressure upon the unit of surface. Since a cubic centimetre of mercury weighs 0,0136 kilograms, the atmospheric pressure or the weight of a column of mercury 76 centimetres high, the base of which is 1 square centimetre, is

$$p = 0{,}0136 \,.\, 76 = 1{,}0336 \text{ kilograms.}$$

But a square inch is 6,451 square centimetres, and therefore the mean pressure of the air is also measured by 1,0336 . 6,451 = 6,678 kilograms = 14,701 pounds upon a square inch = 2116,9 pounds upon a square foot. Assuming the exact height of the barometer to be 28 Paris inches = 29 Prussian inches, we obtain for the pressure of the air upon one square inch 14,103 Prussian pounds and upon the square foot 2030 Prussian pounds.

The standard usually adopted, where the English system of measure is used, is 14,7 pounds upon the square inch, which corresponds to a column of mercury about 30 (exactly 29,922) inches and to a column of water about 34 (exactly 33,9) feet high. It is very common in mechanics to take the pressure of the atmosphere as the unit and to refer other tensions to it; they are then given in pressures of the atmosphere, or simply in atmospheres. Thus a column of mercury $30 \,.\, n$ inches high, or a weight of $14{,}7 \,.\, n$ English pounds, corresponds to the pressure of n atmospheres, and, inversely, a column of mercury h inches high to a tension $\frac{h}{30} = 0{,}03333\, h$ atmospheres and the tension $\frac{p}{14{,}7} = 0{,}06803\, p$ atmospheres to a pressure of p pounds upon a square inch. Besides the equation $\frac{h}{29{,}922} = \frac{p}{14{,}7}$ gives the formulas for reduction

$$h = 2{,}0355\, p \text{ inches and } p = 0{,}4913\, h \text{ pounds,}$$

For a tension of h inches $= p$ pounds the pressure upon a surface of F square inches is

$$P = Fp = 0{,}4913\ Fh \text{ pounds}$$
$$= Fh\gamma = 2{,}0355\ Fp \text{ inches.}$$

EXAMPLE—1) If the level of the water is 250 feet above the piston of a water-pressure engine, the pressure upon the piston is

$$= \frac{250}{34} = 7{,}4 \text{ atmospheres.}$$

2) If the air in a blowing-cylinder has a tension of 1,2 atmospheres, the pressure upon every square inch of the same is

$$= 1{,}2 \,.\, 14{,}7 = 17{,}64 \text{ pounds,}$$

and upon the piston, whose diameter is 50 inches,

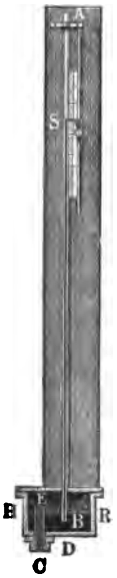
$$= \frac{\pi\ 50^2}{4} \,.\, 17{,}64 = 34636 \text{ pounds.}$$

Since the atmosphere exerts an opposite pressure $\frac{\pi \,.\, 50^2}{4} \,.\, 14{,}7 = 28863$ lbs., the force of the piston is

$$P = 34636 - 28863 = 5773 \text{ pounds.}$$

§ 386. Manometer.—In order to determine the tension of gases or vapors which are enclosed in vessels, we employ instruments, which resemble barometers and are called *manometers* (Fr. manomètres; Ger. Manometer). These instruments are filled with mercury or water and are either open or closed; in the latter case the upper part may be free from air or filled with it. The manometer *with* a *vacuum* above the column of mercury, as is represented in Fig. 653, is like the common barometer. In order to be able to measure with it the tension of the air in a gasholder, a tube $C\ E$ is added to it, one end of which C opens into the gasholder and the other end E enters above the level of the mercury $H\ R$ into the case $H\ D\ R$ of the instrument. The space $H\ E\ R$ above the mercury is thus put in communication with the gasholder; the air existing in this space assumes the tension of the air or gas in the gasholder and presses a column of mercury $B\ S$ into the tube, which balances the tension of the air that is to be measured.

FIG. 653.



The *syphon manometer* $A\ B\ C$, Fig. 654, which is open at the end A, gives the excess of the tension of the gas in a vessel above the pressure of the atmosphere; for that tension is balanced by the combination of the pressure of the atmosphere upon S and of the column of mercury $R\ S$. If b

is the height of the barometer and h that of the manometer, or the distance $R\,S$ between the surfaces H and S of the quicksilver in the two legs of the manometer, the pressure of the air which is in communication with the short leg will be expressed by the height of the column of mercury

$$b_1 = b + h,$$

or by the pressure upon a square inch

$$p = 0{,}4913\ (b + h) \text{ pounds},$$

or, if b is the mean height of the barometer,

$$p = 14{,}7 + 0{,}4913\ h \text{ pounds.}$$

The cistern manometer $A\,B\,C\,D$, Fig. 655, is more common than the syphon manometer. Since in the former the air acts upon the column of liquid through the medium of a large mass of mercury or water, the vibrations of the air are not so quickly communicated to the column of liquid, and consequently the measurement of the column, which is less agitated, can be made more easily and more accurately. In order to facilitate the reading of the instrument, a float, which communicates by means of a string, passing over a pulley, with a pointer, which is movable along the scale, is often placed on top of the mercury in the tube.

FIG. 654.

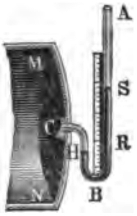


FIG. 655.

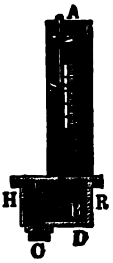
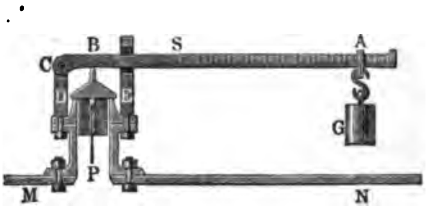


FIG. 656.



Manometers can also be used for the purpose of measuring the pressure of water and other liquids; in this case they are called *piezometers* (Fr. piézometres; Ger. Piezometer).

By the aid of a valve $D\,E$, Fig. 656, the tension of the gas or steam, contained in a vessel $M\,N$, can be determined, although not with the same accuracy, by placing the sliding weight G in such a position that it balances the pressure of the steam. If $C\,S = s$ is the distance of the centre of gravity of the lever from the axis of rotation C, $C\,A = a$ the arm of the lever of the sliding weight and Q the combined weight of the valve and lever, we have the statical moment, with which the valve is pressed downwards by the weights,

$$= G\,a + Q\,s;$$

now if the pressure of the gas or steam upwards $= P$, the pressure of the atmosphere downwards $= P_1$ and the arm of the lever $C\,B$ of the valve $= b$, we have the statical moment with which the valve tends to open

$$= (P - P_1)\,b,$$

equating the two moments, we obtain

$$P\,b - P_1\,b = G\,a + Q\,s, \text{ and consequently,}$$

$$P = P_1 + \frac{G\,a + Q\,s}{b}.$$

If r denote the radius of the valve $D\,E$, p the interior and p_1 the exterior tension, measured by the pressure upon a square inch, we have $P = \pi\,r^2\,p$ and $P_1 = \pi\,r^2\,p_1$, whence

$$p = p_1 + \frac{G\,a + Q\,s}{\pi\,r^2\,b}.$$

Example—1) If the height of the mercury in an open manometer is 3,5 inches and that of the barometer 30 inches, the corresponding tension is

$$h = b + h_1 = 30 + 3,5 = 33,5 \text{ inches, or}$$

$$p = 0,4913\,.\,h = 0,4913\,.\,33,5 = 16,46 \text{ pounds.}$$

2) If the height of a water manometer is 21 inches and that of the barometer is 29 inches, the corresponding tension is

$$h = 29 + \frac{21}{13,6} = 30,54 \text{ inches} = 15,0 \text{ pounds.}$$

3) If the statical moment of a safety valve, when not loaded, is 10 inch-pounds, if the arm of the lever of the valve, measured from the valve to the axis of rotation, is $b = 4$ inches and its radius is $r = 1,5$ inches, the difference of the pressures upon the valve is

$$p - p_1 = \frac{150 + 10}{\pi\,(1,5)^2\,.\,4} = \frac{160}{9\,\pi} = 5,66 \text{ pounds.}$$

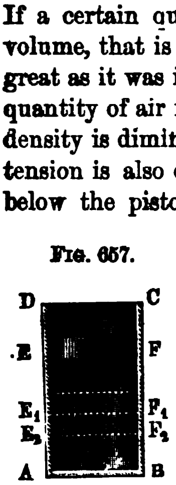
If the pressure of the atmosphere were $p_1 = 14,6$ pounds, the tension of the air under the valve would be

$$p = 20,26 \text{ pounds.}$$

§ 387. Mariotte's Law.—The tension of a gas increases with the condensation; the more we compress a certain quantity of air, the greater the tension becomes, and the more we expand or attenuate it, the less the tension becomes. The relation between the tension and the density or volume of gases is expressed by the law discovered by Mariotte (or Boyle) and named after him. It asserts, *that the density of one and the same quantity of air is proportional to its tension*, or, since the spaces occupied by one and the same mass are inversely proportional to their densities, *that the volumes of one and the same mass of air are inversely proportional to their tensions.*

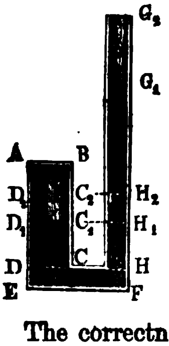
If a certain quantity of air is compressed into half its original volume, that is if its density doubled, its tension becomes twice as great as it was in the beginning, and if, on the contrary, a certain quantity of air is expanded to three times its original volume, its density is diminished to one-third of what it was, and its original tension is also diminished in the same proportion. If the space below the piston $E\ F$ of a cylinder $A\ C$, Fig. 657, is filled with ordinary atmospheric air, which in the beginning acts with a pressure of 14,7 pounds upon each square inch, it will act with a pressure of 29,4 pounds, when we move the piston to $E_1\ F_1$ and thus compress the inclosed air into one-half its initial volume; the pressure will become 3 . 14,7 ≐ 44,1 pounds, when the piston in passing to $E_2\ F_2$ describes two-thirds of the entire height. If the area of the surface of the piston is one square foot, the pressure of the atmosphere against it is = 144 . 14,7 = 2116,8 pounds; hence, if we wish to depress the piston one-half the height of the cylinder, we must place upon it a gradually increasing weight of 2116,8 pounds, and if we wish to depress it two-thirds of the height of the cylinder, 2 . 2116,8 = 4233,6 pounds must gradually be added, etc.

FIG. 657.



We can also prove *Mariotte's Law* by pouring mercury into the tube $G_2\ H$, which communicates with the cylindrical air vessel $A\ C$, Fig. 658. If we begin by cutting off a certain volume $A\ C$ of air, of the same tension as the exterior air, by means of a quantity $D\ E\ F\ H$ of mercury, and if we then compress it by pouring in quicksilver, until it occupies one-half, one-quarter, etc., of its original volume, we will find that heights $G_1\ H_1$, $G_2\ H_2$, etc., of the surface of the mercury in the tube are equal to the height of the barometer b multiplied by one, three, etc. Consequently, if we add the height corresponding to the pressure of the atmosphere, we find that the tension is double, quadruple, etc., that of the original volume.

FIG. 658.

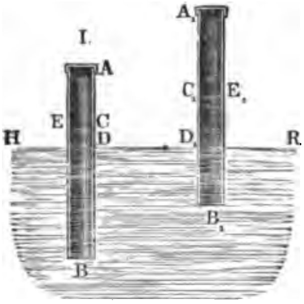


The correctness of the law of Mariotte in regard to expansion can easily be proved by dipping a cylindrical tube (of regular calibre) $A\ B$, Fig. 659, vertically into mercury (water) and, after properly closing the upper end A, expanding the enclosed volume of air $A\,E$ (I) by carefully drawing up the tube so that the air shall occupy a volume $A_1\,E_1$ (II). The densities of the air in the spaces $A\,E$ and $A_1\,E_1$ are inversely proportional to the heights $A\,C$ and $A_1\,C_1$, and its tensions are directly proportional to the differences between the height b of the barometer and the heights $C\,D$ and $C_1\,D_1$ of the columns $D\,E$ and $D_1\,E_1$ of mercury standing above the level $H\,R$ of the mercury; hence, according to Mariotte's law,

$$\frac{A\,C}{A_1\,C_1} = \frac{b - C_1\,D_1}{b - C\,D},$$

which can be verified by observing any given immersion of the tube $A\,B$.

FIG. 659.



If h and h_1 or p and p_1 are the tensions, γ and γ_1 the corresponding densities or heavinesses, and V and V_1 the corresponding volumes of the same quantity of air, we have, according to the above law,

$$\frac{\gamma}{\gamma_1} = \frac{V_1}{V} = \frac{h}{h_1} = \frac{p}{p_1}, \text{ or } V\gamma = V_1\,\gamma_1 \text{ and } V_1\,p_1 = V\,p, \text{ whence}$$

$$\gamma_1 = \frac{h_1}{h}\,\gamma = \frac{p_1}{p}\,\gamma \text{ and } V_1 = \frac{h}{h_1}\,V = \frac{p}{p_1}\,V.$$

By means of these formulas we can reduce the density and also the volume of the air of one tension to those of another.

REMARK.—It is only when the pressures are very great that variations from the law of Mariotte are observed. According to Regnault, when the volume V of atmospheric air at one meter pressure becomes the volume V_1, the pressure is

$$p = \frac{V_0}{V}\left[1 - 0{,}0011054\left(\frac{V_0}{V} - 1\right) + 0{,}000019881\left(\frac{V_0}{V} - 1\right)^2\right] \text{meters},$$

so that for .... $\frac{V_0}{V}$ = 5	10	15	20
we have ...... p = 4,97944	9,91622	14,82484	19,71988 meters.

EXAMPLE 1) If the manometer of a blowing machine marks 8 inches, and the barometer stands at 80 inches, the density of the blast is $\frac{80 + 8}{80} = \frac{88}{80} = 1{,}1$ times as great as that of the exterior air.

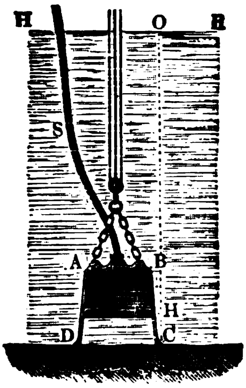
2) If a cubic foot of air, when the barometer stands at 30,05 inches,

weighs $\frac{62,425}{770}$ pounds, what is its weight when the barometer stands at 34 inches? Its weight is

$$\frac{62,425}{770} \cdot \frac{34}{30,05} = \frac{42,449}{462,77} = 0,09173 \text{ pounds.}$$

3) How deep can a diving-bell (Fr. cloche à plongeur; Ger. Taucherglocke) *A B C D*, Fig. 660, be immersed in water, when the water is not to rise in it above a certain height $CH = y$. In the beginning the bell with its opening *C D* stands above the level of the water *H R*, so that the whole space *V* is filled with air at a pressure equal to that of a column of water, whose height is $= b$. If afterwards the bell sinks to a depth $O\,C = x$ and a volume *W* of water is thus introduced into it, the volume of the inclosed air, when none is pressed back through the hose, becomes $V - W$ and the height of the water barometer becomes $b + x - y$; hence

FIG. 660.



$$\frac{b + x - y}{b} = \frac{V}{V - W},$$

whence we obtain

$$x = y - b + \frac{Vb}{V - W} = y + \frac{Wb}{V - W}.$$

If the mean cross-section of the lower part of the bell $= F$, we can put $W = Fy$ and therefore

$$x = y\left(1 + \frac{Fb}{V - Fy}\right).$$

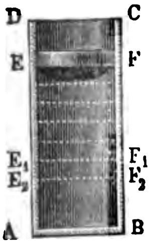
If the height of barometer $= 34$ feet of water, the volume of the bell $V = 100$ cubic feet, the mean cross-section of the lower half $F = 20$ square feet, and the height, to which the water is to be admitted, is $y = 3$ feet, the volume of this water is $W = Fy = 20 \,.\, 3 = 60$ cubic feet; hence that of the confined air is $V - W = 40$ cubic feet, and its density is $= \frac{100}{40} = 2\frac{1}{2}$ times that of the exterior air, and the corresponding depth of immersion is

$$x = 3 + \frac{60 \,.\, 34}{40} = 3 + 51 = 54 \text{ feet.}$$

§ 388. Work Done by Compressed Air.—*The energy stored by a given quantity of air when it is compressed to a certain degree*, as well as that restored by it when it expands again, can not be determined at once; for the tension varies at every moment of the expansion or compression. We must therefore seek out a particular formula for the calculation of this quantity. Let us imagine a certain quantity of air *A F* to be shut off in a cylinder *A C*, Fig. 661, by a piston *E F*, and let us calculate what mechanical effect is necessary to move the piston a certain distance $E E_1 = F F_1$. If the initial tension $= p$ and the initial height of the space in the cylinder $A\ E = s$, and if, on the contrary, the tension after the space $E E_1$ has been described $= p_1$ and the height $E_1\ A$ of the remaining volume of air $= s_1$, we have the proportion

$$p_1 : p = s : s_1, \text{ whence } p_1 = \frac{s}{s_1} p.$$

FIG. 661.



While the piston describes a very small portion $E_1 E_2 = \sigma$ of the space, the tension p_1 can be regarded as constant, and the work done is $= F p_1 \sigma = \frac{F p s \sigma}{s_1}$, F denoting the area of the piston.

According to the theory of logarithms,* a very small quantity

$$x = l\,(1 + x) = 2{,}3026 \ log.\ (1 + x),$$

l denoting the Naperian and *log.* the common logarithm; consequently we can put

$$F p s \frac{\sigma}{s_1} = F p s\, l \left(1 + \frac{\sigma}{s_1}\right)$$
$$= 2{,}3026\, F p s \, log. \left(1 + \frac{\sigma}{s_1}\right).$$

But now

$$l \left(1 + \frac{\sigma}{s_1}\right) = l \left(\frac{s_1 + \sigma}{s_1}\right) = l\,(s_1 + \sigma) - l\, s_1;$$

hence the elementary work done is

$$F p s \frac{\sigma}{s_1} = F p s\, [l\,(s_1 + \sigma) - l\, s_1].$$

Let us imagine the whole space $E E_1$ to be composed of n parts, such as σ, I.E., let us put $E E_1 = n\, \sigma$, we will then find the work corresponding to all these parts by substituting in the last formula successively, instead of s_1, the values $s_1 + \sigma$, $s_1 + 2\,\sigma$, $s_1 + 3\,\sigma, \ldots$ up to $s_1 + (n - 1)\,\sigma$, and instead of $s_1 + \sigma$, the values $s_1 + 2\,\sigma$, $s_1 + 3\,\sigma$, etc., up to $s_1 + n\,\sigma$ or s, and if we add the values deduced, we will obtain the whole work done while the space $s - s_1$ is described

---

* According to the series $e^x = 1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \ldots$ (see § 194 and also the Introduction to the Calculus, Art. 19) for a very small x, we have $e^x = 1 + x$, and therefore

$$l\,(1 + x) = x.$$

$$A = F p s \left\{ \begin{array}{l} l(s_1 + \sigma) - l\, s_1 \\ l(s_1 + 2\,\sigma) - l(s_1 + \sigma) \\ l(s_1 + 3\,\sigma) - l(s_1 + 2\,\sigma) \\ . \\ . \\ . \\ l(s_1 + n\,\sigma) - l[s_1 + (n-1)\,\sigma] \end{array} \right\}$$

$$= F p s\,[l(s_1 + n\,\sigma) - l\, s_1]$$

$$= F p s\,(l\, s - l\, s_1) = F p s\, l\left(\frac{s}{s_1}\right);$$

for the first term in each line is cancelled by the second term in the next.

Since $\frac{s}{s_1} = \frac{h_1}{h} = \frac{p_1}{p}$, we can put the work done

$$A = F p s\, l\left(\frac{h_1}{h}\right) = F p s\, l\left(\frac{p_1}{p}\right).$$

If we make the space described by the piston $s - s_1 = x$, we find for the mean value of the pressure on the piston, when the air is compressed in the ratio

$$\frac{h_1}{h} = \frac{p_1}{p},$$

$$P = \frac{A}{x} = F p \frac{s}{x} l\left(\frac{p_1}{p}\right).$$

Putting $F = 1$ (square foot) and $s = 1$ (foot), we obtain the following formula for the work done

$$A = p\, l\left(\frac{p_1}{p}\right) = 2{,}3026\, p\, log. \left(\frac{p_1}{p}\right).$$

This formula gives the mechanical effect necessary to transform a unit of volume (1 cubic foot) of air from a lower pressure or tension p to a higher one p_1, and in so doing to compress the air into a volume of $\left(\frac{p}{p_1}\right)$ cubic feet. On the contrary,

$$A = p_1\, l\left(\frac{p_1}{p}\right) = 2{,}3026 p_1\, log. \left(\frac{p_1}{p}\right)$$

expresses the work done by the unit of volume of a gas which passes from a greater tension p_1 to a lesser one p.

In order to compress a quantity of air, whose volume is V and whose tension is p, into a volume V_1 of the tension $p_1 = \frac{V}{V_1} p$, the work to be done is $V p\, l\left(\frac{V}{V_1}\right)$, and if, on the contrary, the volume

V_1 of the tension p_1 becomes a volume V, whose tension is $p = \frac{V_1}{V} p_1$, the energy restored is

$$V p \, l \left(\frac{V}{V_1}\right) = V_1 \, p_1 l \left(\frac{V}{V_1}\right).$$

Remark.—The mechanical effect necessary to produce moderate differences of tension $(p_1 - p)$, or small changes of volume $(V_1 - V)$ can be expressed more simply by the formula

$$A = F\left(\frac{p + p_1}{2}\right)(s - s_1) = F s\left(1 - \frac{p}{p_1}\right)\left(\frac{p + p_1}{2}\right)$$
$$= V\left(1 - \frac{p}{p_1}\right)\left(\frac{p + p_1}{2}\right),$$

or more accurately by the aid of Simpson's rule, when z denotes the pressure at the middle of the path $\frac{s + s_1}{2}$ of the piston, by the formula

$$A = V\left(1 - \frac{p}{p_1}\right)\left(\frac{p + 4z + p_1}{6}\right).$$

But now

$$\frac{z}{p} = \frac{s}{\frac{1}{2}(s + s_1)} = \frac{2s}{s + s_1} = \frac{2}{1 + \frac{p}{p_1}} = \frac{2p_1}{p + p_1},$$

whence it follows that

$$A = \tfrac{1}{6} V\left(1 - \frac{p}{p_1}\right)\left(p + \frac{8 p p_1}{p + p_1} + p_1\right)$$
$$= \tfrac{1}{6} V p \left(\frac{p_1}{p} + \frac{8(p_1 - p)}{p_1 + p} - \frac{p}{p_1}\right)$$

Example—1) If a blowing machine changes per second 10 cubic feet of air, at a pressure of 28 inches, into a blast at a pressure of 30 inches, the work to be done in every second is

$$A = 17280 \,.\, 0{,}4913 \,.\, 28 \,.\, l\left(\frac{30}{28}\right) = 237711 \,.\, (l\,15 - l\,16)$$
$$= 237711 \,.\, (2{,}708050 - 2{,}639057) = 237711 \,.\, 0{,}068993$$
$$= 16400{,}4 \text{ inch-pounds} = 1366{,}7 \text{ foot-pounds.}$$

The approximate formula, given in the remark, gives for this work

$$A = \tfrac{1}{6} \,.\, 237711 \left(\frac{30}{28} + \frac{8 \,.\, 2}{58} - \frac{28}{30}\right) = 39618{,}5 \,.\, 0{,}41387$$
$$= 16396{,}9 \text{ inch-pounds} = 1366{,}4 \text{ foot-pounds.}$$

2) If under the piston of a steam-engine, whose area is $F = \pi \,.\, 8^2 = 201$ square inches, there is a quantity of steam 15 inches high and at a tension of 3 atmospheres, and if this steam, in expanding, moves the piston forward 25 inches, the energy restored and transmitted to the piston is, if we assume Mariotte's law to be true for the expansion of steam,

$$A = 201 \,.\, 3 \,.\, 14{,}70 \,.\, 15 \, l\left(\frac{15 + 25}{15}\right) = 132961{,}5 \, l \tfrac{8}{3}$$
$$= 132961{,}5 \,.\, 0{,}98083 = 130413 \text{ inch-lbs.} = 10866 \text{ foot-lbs.},$$

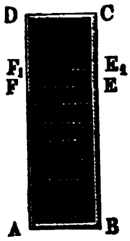
and the mean force upon the piston is, when we neglect the friction and the opposing pressure,

$$P = \frac{180413}{25} = 5217 \text{ pounds.}$$

§ 389. Pressure in the Different Layers of Air.—The air enclosed in a vessel has a different density and tension at different depths; for the upper layers compress those below them, upon which they rest; the density and tension are the same in the same horizontal layer only, and both increase with the depth. In order to find the law of this increase of the density from above downwards, or of the decrease from below upwards, we make use of a method similar to that employed in the foregoing paragraph.

Let us imagine a vertical column $A\,E$, Fig. 662, whose cross-section $A\,B = 1$ and whose height $A\,F = s$. Putting the heaviness of the lowest layer $= \gamma$ and its tension $= p$, and the heaviness of the upper layer $E\,F_1 = \gamma_1$ and its tension $= p_1$, we have $\frac{\gamma_1}{\gamma} = \frac{p_1}{p}$.

FIG. 662.



If σ denotes the height $E\,E_1$ of the layer $E_1\,F_1$ its weight, which is the decrease of the tension corresponding to σ, is

$$v = 1\,.\,\sigma\,.\,\gamma_1 = \frac{\sigma\,\gamma\,p_1}{p},$$

hence by inversion we obtain

$$\sigma = \frac{p}{\gamma}\,.\,\frac{v}{p_1},$$

or, as in the foregoing paragraph,

$$\sigma = \frac{p}{\gamma}\,l\left(1 + \frac{v}{p_1}\right) = \frac{p}{\gamma}\,[l\,(p_1 + v) - l\,p_1].$$

If we substitute in it for p_1 successively $p_1 + v, p_1 + 2\,v, p_1 + 3\,v$, etc., up to $p = p_1 + (n - 1)\,v$ and add the corresponding heights of the layers of air or values of σ, we obtain, exactly as in the foregoing paragraph, the height of the entire column of air

$$s = \frac{p}{\gamma}\,(l\,p - l\,p_1) = \frac{p}{\gamma}\,l\left(\frac{p}{p_1}\right),$$

or also

$$s = \frac{p}{\gamma}\,l\left(\frac{b}{b_1}\right) = 2{,}302\,\frac{p}{\gamma}\,log.\left(\frac{b}{b_1}\right),$$

when b and b_1 denote the tensions and p and p_1 the corresponding heights of the barometer in A and F.

Inversely, if the height s is given, the corresponding tension and density of the air can be calculated. We have

$$\frac{p}{p_1} = \frac{\gamma}{\gamma_1} = e^{\frac{s\,\gamma}{p}}, \text{or } \gamma_1 = \gamma\,e^{-\frac{s\,\gamma}{p}},$$

in which $e = 2{,}71828$ denotes the base of the Naperian system of logarithms.

Remark.—This formula is employed for the measurement of heights by means of the barometer, a subject which is treated in the "Ingenieur," page 273. If we neglect the temperature, etc., we can write as a mean value

$$s = 60346 \; log. \left(\frac{b}{b_1}\right) \text{ feet.}$$

Example 1) If we have found the height of the barometer at the foot of a mountain to be 339 and at the top 315 lines, the height of the mountain given by these observations is

$$s = 60346 \; log. \left(\tfrac{339}{315}\right) = 60346 \,.\, 0{,}031889 = 1924 \text{ feet.}$$

2) For the density of the air at the top of a mountain 10000 feet high, we have

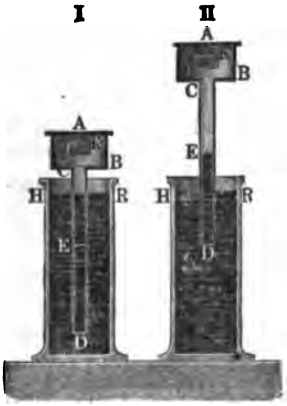
$log. \frac{\gamma}{\gamma_1} = \frac{10000}{60346} = 0{,}165711$, whence $\frac{\gamma}{\gamma_1} = 1{,}465$ and $\frac{\gamma_1}{\gamma} = \frac{1}{1{,}465} = 0{,}683$; its density is therefore $68\frac{1}{3}$ per cent. of that of the air at its foot.

§ 390. Stereometer and Volumeter.

—Mariotte's law finds a practical application in the determination of the volumes of pulverent and fibrous bodies, etc., by means of the so-called stereometer and volumeter.

1) *Say's Stereometer.*—If the glass tube $C\,D$, which is immersed in mercury $H\,D\,R$ and at the same time is in communication with the closed vessel $A\ B$, Fig. 663, I, is raised up without being drawn entirely out of the mercury (II), then, in consequence of the expansion of the enclosed air, a column $C\,E$ of air enters into the tube and a column of mercury $D\,E$ will remain behind in the tube, by the aid of which the diminished tension of the enclosed air balances the pressure of the atmosphere.

FIG. 663.



Now if V_0 is the volume of the space $A\ B\ C$, V_1 the required volume of the body K, which is placed in it, V the volume of the column of air $C\,E$, b the height of the barometer and h that of the column of mercury $D\,E$, we have, according to Mariotte's law, since the same quantity of air occupies the volume $V_0 - V_1$, when the tension is b, and the volume $V_0 - V_1 + V$, when the tension is $b - h$,

$$\frac{V_0 - V_1}{V_0 - V_1 + V} = \frac{b - h}{b};$$

hence the required *volume of the body* is

$$V_1 = V_0 - \left(\frac{b - h}{h}\right) V.$$

If we know the volume V_0, and if, when making the experiment, we draw the tube so far out of the water that the length and consequently the volume V of the column of air in the tube $C\,D$ becomes a certain definite one, and if we observe also the height b of the barometer and that h of the column of mercury $D\,E$, we can calculate by means of this formula the volume V_1 of the body K.

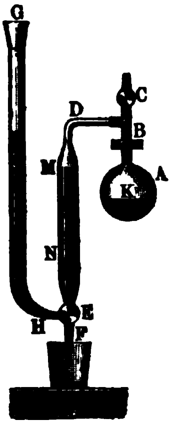
2) *Regnault's Volumeter.*—If the space $A\,B\,C\,D$, Fig. 664, which is filled with atmospheric air and which contains also the body K, whose volume V_1 is to be determined, is shut off by the cock C from the exterior air, and if, by opening the cock E, we let out so much mercury from the tube $D\,E$ that its level descends from M to N, we can again employ (according to Mariotte's law) the above formula

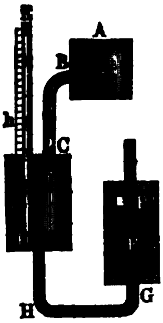
$$\frac{V_0 - V_1}{V_0 - V_1 + V} = \frac{b - h}{b},$$

in which we denote the volume of the space $A\,B\,C\,D$ by V_0, that of the mercury drawn off by V and the height $M\,N$ of the same by h. It follows, exactly as in the above case, that the volume of the body in A is

$$V_1 = V_0 - \left(\frac{b - h}{h}\right) V.$$

Fig. 664.





In order to fill the tube $D\,E$ with mercury again for the purpose of making a new measurement, we put that tube $D\,E$ in communication with the reservoir of mercury $G\,H$ by turning the cock E.

3) *Kopp's Volumeter.*—The pressure of the air enclosed in the space $A\;B\;C\,D$, Fig. 665, is the same as that of the exterior air, when the surface of the mercury $D\,G$ touches the lower opening D of the manometer $D\,E$. If by means of a piston P we press the mercury into $D\,G$, until it rises to a certain height and its surface reaches the point

S, the enclosed air will be compressed and the mercury will rise a certain distance h in the manometer, which distance can be read off upon the scale. If again V_0 is the volume $A\ B\ C\ D$ of the air, V_1 the required volume of the body placed in it and V the volume of the mercury, which has been pressed into the air-vessel, we have in this case

$$\frac{V_0 - V_1}{V_0 - V_1 - V} = \frac{b + h}{b},$$

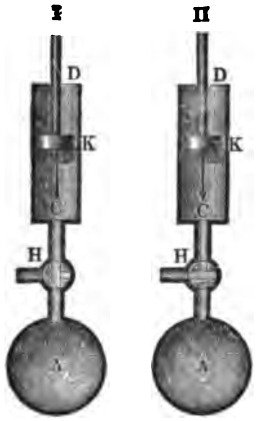
and, therefore, the required volume of the body

$$V_1 = V_0 - \left(\frac{b + h}{h}\right) V.$$

The constant volumes V_0 and V_1 are determined for each particular instrument by filling them with mercury and weighing the quantity which they hold.

§ **391. Air Pump.**—(Fr. machine pneumatique; Ger. Luftpumpe.) If we raise the piston K, Fig. 666, of an air pump when the stop-cock is in the position (I) and push it down when the stop-cock is in position (II), it acts as an *exhausting or rarefying pump;* if, on the contrary, we raise the piston when the stop-cock is in position (II) and depress it when it is in position (I), it acts as a *compressing or condensing pump.* In the first case the air in the receiver A is more and more rarefied by the reciprocating motion of the piston K in the cylinder $C\ D$, and in the latter case it is rendered more and more dense.

FIG. 666.



1) *The Exhaust Pump.*—If V is the volume of the receiver, measured to the cock H, V_1 the *clearance* between H and the lowest position of the piston, and C the volume described by the piston K, which is also measured by the product $F\,s$ of the surface F of the piston and the space s described by it, the pressure b of the air originally contained in the receiver becomes, according to Mariotte's law, at the end of a single stroke of the piston

$$b_1 = \left(\frac{V + V_1}{V + V_1 + C}\right) b.$$

Since upon the return of the piston the clearance remains filled with air at the pressure of the exterior air b, if the pressure of the air in the receiver at the end of the second stroke is denoted by b_2, we will have

$$(V + V_1 + C)\, b_2 = V b_1 + V_1 b$$

$$= \frac{V_2 b}{V + V_1 + C} + \frac{V V_1 b}{V + V_1 + C} + V_1 b, \text{ whence}$$

$$b_2 = \left(\frac{V}{V + V_1 + C}\right)^2 b + \frac{V V_1 b}{(V + V_1 + C)^2} + \frac{V_1 b}{V + V_1 + C}.$$

In like manner for the tension b_3 at the end of the third stroke we find

$$(V + V_1 + C)\, b_3 = V b_2 + V_1 b, \text{ and therefore}$$

$$b_3 = \left(\frac{V}{V + V_1 + C}\right)^3 b + \frac{V^2 V_1 b}{(V + V_1 + C)^3} + \frac{V V_1 b}{(V + V_1 + C)^2}$$

$$+ \frac{V_1 b}{V + V_1 + C} = \left(\frac{V}{V + V_1 + C}\right)^3 b + \left[\left(\frac{V}{V + V_1 + C}\right)\right.$$

$$\left. + \frac{V}{V + V_1 + C} + 1\right] \frac{V_1 b}{V + V_1 + C},$$

and from the foregoing we see that the pressure b_n, after n strokes, will be

$$b_n = \left(\frac{V}{V + V_1 + C}\right)^n b$$

$$+ \left[\left(\frac{V}{V + V_1 + C}\right)^{n-1} + \left(\frac{V}{V + V_1 + C}\right)^{n-2} + \ldots + 1\right] \frac{V_1 b}{V + V_1 + C}.$$

If we denote $\frac{V}{V + V_1 + C}$ by p and $\frac{V_1}{V + V_1 + C}$ by q, we will have

$$b_n = p^n b + (1 + p + p^2 + \ldots + p^{n-1})\, q\, b,$$

or, since the sum of the geometrical series in the parenthesis is $= \frac{p^n - 1}{p - 1} = \frac{1 - p^n}{1 - p}$ (see Ingenieur, page 82), the required *final tension* is simply

$$b_n = \left[p^n + \left(\frac{1 - p^n}{1 - p}\right) q\right] b.$$

For $n = \infty$, p^n becomes $= 0$, and consequently the *smallest possible tension* is

$$b_n = \frac{q\, b}{1 - p} = \frac{V_1 b}{C + V_1}.$$

2) *The Condensing Pump.* If we adopt the same notations as for the exhaust pump, we have here for the tension of the air at the end of the first single stroke,

$$(V + V_1)\, b_1 = (V + V_1 + C)\, b, \text{ whence } b_1 = \left(\frac{V + V_1 + C}{V + V_1}\right) b;$$

and for that b_2 at the end of the second stroke

$$(V + V_1)\, b_2 = V\, b_1 + (V_1 + C)\, b, \text{ whence}$$

$$b_2 = \frac{(V + V_1 + C)\, V b}{(V + V_1)^2} + \frac{V_1 + C}{V + V_1}\, b$$

$$= \left(\frac{V}{V + V_1}\right)^2 b + \left(\frac{V}{V + V_1} + 1\right) \frac{V_1 + C}{V + V_1}\, b.$$

In like manner the tension at the end of the third stroke is found to be

$$(V + V_1)\, b_3 = V\, b_2 + (V_1 + C)\, b, \text{ and therefore}$$

$$b_3 = \left(\frac{V}{V + V_1}\right)^3 b + \left[\left(\frac{V}{V + V_1}\right)^2 + \frac{V}{V + V_1} + 1\right] \frac{V_1 + C}{V + V_1}\, b,$$

or putting

$$\frac{V}{V + V_1} = p_1 \text{ and } \frac{V_1 + C}{V + V_1} = q_1$$

$$b_3 = [p_1^3 + (1 + p_1 + p_1^2)\, q_1]\, b.$$

In general, we have for the tension at the end of the *n*th stroke of the piston

$$b_n = [p_1^n + (1 + p_1 + p_1^2 + \ldots + p_1^{n-1})\, q_1]\, b, \text{ or, since}$$

$$1 + p_1 + p_1^2 + \ldots + p_1^{n-1} = \frac{p_1^n - 1}{p_1 - 1} = \frac{1 - p_1^n}{1 - p_1},$$

$$b_n = \left[p_1^n + \left(\frac{1 - p_1^n}{1 - p_1}\right) q_1\right] b.$$

For $n = \infty, p_1^n = 0$ and

$$b_n = \frac{q_1\, b}{1 - p_1} = \frac{V_1 + C}{V_1}\, b.$$

This is of course the *greatest tension* that can be produced by this condensing pump.

If the clearance V_1 were $= 0$, we would have for the exhaust pump $q = 0$, whence

$$b_n = p^n\, b = \left(\frac{V}{V + C}\right)^n p;$$

and, on the contrary, for the condensing pump $p_1 = 1$ and $\frac{1 - p_1^n}{1 - p_1} = n$, and consequently

$$b_n = (1 + n\, q_1)\, b = \left(1 + n\, \frac{C}{V}\right) b.$$

Example.—If the volume of the receiver of an air pump is $V = 1000$ cubic inches and the clearance is 10 cubic inches, while the volume of the cylinder is 300 cubic inches, the tension of the air after 20 strokes is

1) when rarifying, since

$$p = \frac{1000}{1310} = 0{,}76336 \text{ and}$$

$$q = \frac{10}{1310} = \frac{1}{131} = 0{,}0076336,$$

$$b_n = b_{20} = \left(0,76336^{20} + \frac{1 - 0,76336^{20}}{1 - 0,76336} \cdot 0,0076336\right) b$$
$$= (0,0045148 + 0,0321126)\, b = 0,076269\, b;$$

on the contrary,

2) when condensing, in which case

$$p_1 = \frac{1000}{1010} = 0,99010 \text{ and}$$
$$q_1 = \frac{310}{1010} = 0,30693,$$
$$b_n = b^{20} = \left(0,9901^{20} + \frac{1 - 0,9901^{20}}{1 - 0,9901} \cdot 0,30693\right) b$$
$$= \left(0,81954 + \frac{0,18046}{0,009901} \cdot 0,30693\right) b = 6,414\, b.$$

§ 392. **Gay-Lussac's Law.**—The *heat or temperature* of gases has an important influence upon their density and tension. The more the air enclosed in a vessel is warmed, the greater its tension becomes, and the more the temperature of a gas, contained in a vessel closed by a piston, is raised, the more it will expand and drive the piston before it. *Gay-Lussac's* experiments, repeated more recently by Rudberg, Magnus and Regnault, have shown that for the same density the tensions, and for the same tensions the volume, of one and the same quantity of air increases with the temperature. We can place this law by the side of that of Mariotte and call it *Gay-Lussac's Law.* According to the latest researches the increase of the tension of a given volume of air, when heated from the freezing to the boiling point of water, is 0,367 times the original tension, or if its temperature is raised that much, the volume of a given quantity of air is increased 36,7 per cent., when the tension remains constant. If the temperature is given by the centigrade thermometer, in which the distance between the freezing and boiling points of water is divided into 100 degrees, the expansion for each degree is = 0,00367, and for the temperature $t°$ it is = 0,00367 $t°$, or if, on the contrary, we use Reaumur's division of the same space into 80 degrees, we have the expansion for each degree = 0,00459, or for a temperature of $t°$, = 0,00459 t.

In England and America the Fahrenheit thermometer is generally used, in which the boiling point is 212° and the freezing point is 32°; hence the increase for each degree is = 0,00204, and for $t°$ it is 0,00204 $(t - 32)$.

This ratio or coefficient of expansion δ = 0,00367 or = 0,00204 is strictly correct for atmospheric air alone; its value for other gases is generally smaller, and it varies slightly with the temperature for atmospheric air.

If a mass of air, originally of the volume V_0, is warmed from the freezing point to t degrees without changing its tension, its volume becomes

$$V = (1 + 0{,}00367\, t)\, V_0 = [1 + 0{,}00204\, (t - 32^\circ)]\, V_0,$$

and if it reaches the temperature t_1, the volume becomes

$$V_1 = (1 + 0{,}00367\, t_1)\, V_0 = [1 + 0{,}00204\, (t_1 - 32^\circ)]\, V_0;$$

hence the ratio of the volumes is

$$\frac{V}{V_1} = \frac{(1 + 0{,}00367\, t)}{(1 + 0{,}00367\, t_1)} = \frac{1 + 0{,}00204\, (t - 32^\circ)}{1 + 0{,}00204\, (t_1 - 32^\circ)};$$

on the contrary, the ratio of the densities or heavinesses is

$$\frac{\gamma}{\gamma_1} = \frac{V_1}{V} = \frac{1 + 0{,}00367\, t_1}{1 + 0{,}00367\, t} = \frac{1 + 0{,}00204\, (t_1 - 32^\circ)}{1 + 0{,}00204\, (t - 32^\circ)},$$

or generally

$$\frac{\gamma}{\gamma_1} = \frac{V_1}{V} = \frac{1 + \delta\, t_1}{1 + \delta\, t} = \frac{1 + \delta\, (t_1 - 32^\circ)}{1 + \delta\, (t - 32^\circ)}.$$

When a change in the tension also occurs, if p_0 is the tension at the freezing point, p that at the temperature t and p_1 that at t_1, we have

$$V = (1 + 0{,}00367\, t)\, \frac{p_0}{p}\, V_0,$$

$$V_1 = (1 + 0{,}00367\, t_1)\, \frac{p_0}{p_1}\, V_0,$$

$$\frac{V}{V_1} = \frac{1 + 0{,}00367\, t}{1 + 0{,}00367\, t_1} \cdot \frac{p_1}{p}, \text{ and}$$

$$\frac{\gamma}{\gamma_1} = \frac{1 + 0{,}00367\, t_1}{1 + 0{,}00367\, t} \cdot \frac{p}{p_1}, \text{ or}$$

$$\frac{\gamma}{\gamma_1} = \frac{1 + 0{,}00367\, t_1}{1 + 0{,}00367\, t} \cdot \frac{b}{b_1}, \text{ as well as}$$

$$\frac{p}{p_1} = \frac{b}{b_1} = \frac{1 + 0{,}00367\, t}{1 + 0{,}00367\, t_1} \cdot \frac{\gamma}{\gamma_1}.$$

When t is given in degrees of Fahrenheit's thermometer, we must substitute in the latter formulas for $0{,}00367\, t$, $0{,}00204\, (t - 32^\circ)$.

Example.—If 800 cubic feet of air, at a tension of 15 pounds and at a temperature of 50° Fahrenheit, are brought, by means of the blowing engine and warming apparatus of an iron furnace, to a temperature of 892° and to a tension of 19 lbs, its volume will be

$$V_1 = \frac{1 + 0{,}00204\,.\,(892 - 32)}{1 + 0{,}00204\,.\,(50 - 32)} \cdot \tfrac{15}{19} \,.\, 800 = \frac{1{,}784}{1{,}0367} \cdot \frac{12000}{19} = 1056 \text{ cubic feet.}$$

Remark.—The formula

$$\frac{\gamma}{\gamma_1} = \frac{V_1}{V} = \frac{1 + \delta\, t_1}{1 + \delta\, t} = \frac{1 + \delta\, (t_1 - 32)}{1 + \delta\, (t - 32)}$$

can be employed for solids and for some liquids; but for every solid we must substitute a different coefficient of expansion, E.G.,

Centigrade. Fahrenheit.

for cast iron, $\delta = 0{,}0000386 = 0{,}0000187$,

for glass, $\delta = 0{,}0000258 = 0{,}0000148$,

for mercury, $\delta = 0{,}0001802 = 0{,}0001001$.

§ **393. Heaviness of the Air.**—By the aid of the formula at the end of the last paragraph, we can calculate the *heaviness* γ of the air for a given temperature and tension. Regnault, by his recent weighings and measurements, found the weight of a cubic meter of atmospheric air, at the temperature 0° of the centigrade thermometer and at a tension corresponding to height of 0,76 meters of the barometer, to be = 1,2935 kilograms. Since a cubic foot (English) = 0,02832 cubic meters and 1 kilogram = 2,20460 pounds English, the heaviness of air under the given conditions is

$$= 2{,}20460 \,.\, 0{,}02832 \,.\, 1{,}2935 = 0{,}08076 \text{ pounds English.}$$

If the temperature is $= t°$ centigrade, we have for the French measure

$$\gamma = \frac{1{,}2935}{1 + 0{,}00367\, t} \text{ kilograms,}$$

and for the English system of measures and Fahrenheit's thermometer

$$\gamma = \frac{0{,}08076}{1 + 0{,}00204\,(t - 32°)}.$$

If the tension differs from the mean tension, or if the height of the barometer is not 0,76 meters, but b, we have

$$\gamma = \frac{1{,}2935}{1 + 0{,}00367\, t} \cdot \frac{b}{0{,}76} = \frac{1{,}702 \,.\, b}{1 + 0{,}00367\, t} \text{ kilograms,}$$

or, since in England and America the height of the barometer is generally given in inches, and since 0,76 meters = 29,92 English inches,

$$\gamma = \frac{0{,}08076}{1 + 0{,}00204\,(t - 32°)} \cdot \frac{b}{29{,}92} = \frac{0{,}002699\, b}{1 + 0{,}00204\,(t - 32°)} \text{ lbs.}$$

Very often we express the tension by the pressure p upon the square centimeter or inch, and then we must introduce the factor $\frac{p}{1{,}0336}$ or $\frac{p}{14{,}7}$, by doing which we obtain

$$\gamma = \frac{1{,}2935}{1 + 0{,}00367\, t} \cdot \frac{p}{1{,}0336} = \frac{1{,}2514\, p}{1 + 0{,}00367\, t} \text{ kilograms, or}$$

$$\gamma = \frac{0{,}08076}{1 + 0{,}00204\,(t - 32)} \cdot \frac{p}{14{,}7} = \frac{0{,}005494\, p}{1 + 0{,}00204\,(t - 32)} \text{ lbs.}$$

For the same temperature and tension, the density of steam is about $\frac{5}{8}$ of that of atmospheric air; hence for steam we have

$$\gamma = \frac{0,8084}{1 + 0,00367\,t} \cdot \frac{p}{1,0336} = \frac{0,7821\,p}{1 + 0,00367\,t} \text{ kilograms, or}$$

$$\gamma = \frac{0,050475}{1 + 0,00204\,(t - 32)} \cdot \frac{p}{14,7} = \frac{0,003434\,p}{1 + 0,00204\,(t - 32)} \text{ pounds.}$$

EXAMPLE—1) What is the weight of the air contained in a cylindrical regulator 40 feet long and 6 feet wide, when it is at a temperature of 50° and its tension is 18 pounds? The heaviness of this air is

$$\gamma = \frac{0,005494 \,.\, 18}{1,0367} = \frac{0,098892}{1,0367} = 0,09539 \text{ pounds,}$$

and the capacity of the reservoir is

$$V = \pi \,.\, 3^2 \,.\, 40 = 1131 \text{ cubic feet;}$$

hence the air enclosed in it weighs

$$V\gamma = 0,09539 \,.\, 1131 = 107,9 \text{ pounds.}$$

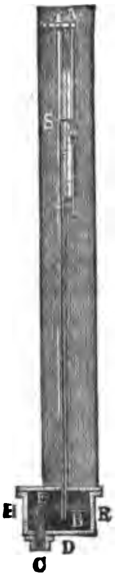
2) A steam-engine uses per minute 500 cubic feet of steam at a temperature of 224,6° F. and at a tension of 39 inches $= 0,4913 \,.\, 39 = 19,161$ pounds; how much water is needed to produce this steam? The heaviness of the steam is

$$= \frac{0,003434 \,.\, 19,161}{1 + 0,00204 \,.\, 192,6} = \frac{0,06580}{1,393} = 0,04724 \text{ pounds;}$$

hence the weight of 500 cubic feet of steam is

$$V\gamma = 500 \,.\, 0,04724 = 23,62 \text{ pounds.}$$

FIG. 667.



§ 394. Air Manometer.—From the results obtained in the last paragraphs, the theory of the *air* or *closed manometer* can be deduced. It is composed of a barometer tube $A\,B$, Fig. 667, of regular calibre, the upper part of which is filled with air and the lower part with mercury, and of a cistern $C\,E\,R$, which also contains mercury and is put in communication with the gas or vapor. From the heights of the columns of air and mercury in $A\,B$, the tension can be calculated in the following manner. The instrument is generally so arranged that the mercury in the tube and in the cistern are upon the same level, when the temperature of the enclosed air is $t = 10°$ Cent. $= 50°$ Fahr. and the tension in the space $E\,R$ is equal to the mean height of the barometer $b = 0,76$ meter $= 29,92$ inches.

If, when the height of the barometer is b, a column of quicksilver rises from the cistern $E\,R$ into the tube to a height h_1, and if the length $A\,S$ of the remaining column of air is $= h_2$, the tension of the latter is

$$z = \left(\frac{h_1 + h_2}{h_2}\right) b,$$

and, therefore, the height of the barometer of the air in $E\ R$

$$b_1 = h_1 + z = h_1 + \left(\frac{h_1 + h_2}{h_2}\right) b.$$

Now if a change of temperature takes place, I.E., if the temperature at the time when h_1 and h_2 were observed, was not as in the beginning $= t$, but $= t_1$, we have for the tension of the column of air $A\ S$

$$z = \frac{1 + 0{,}00204\ (t_1 - 32)}{1 + 0{,}00204\ (t - 32)} \cdot \left(\frac{h_1 + h_2}{h_2}\right) b,$$

and, therefore, the required height of barometer is

$$b_1 = h_1 + \frac{1 + 0{,}00204\ (t_1 - 32)}{1 + 0{,}00204\ (t - 32)} \cdot \frac{h_1 + h_2}{h_2} b.$$

For $b = 29{,}92$ inches and $t = 50°$ Fahr.

$$b_1 = h_1 + 28{,}86\ [1 + 0{,}00204\ (t_1 - 32)]\ \frac{h}{h_2},$$

$h = h_1 + h_2$ denoting the total length of the tube, measured from its upper end A to the surface $H\ R$ of the mercury. From the height of the barometer b inches we obtain the pressure upon each square inch (English)

$$p_1 = \frac{14{,}7}{29{,}92} h_1 + 14{,}7 \cdot \frac{28{,}86}{29{,}92} [1 + 0{,}00204\,(t_1 - 32)] \frac{h}{h_2}$$
$$= 0{,}4913\,h_1 + 14{,}179\,[1 + 0{,}00204\,(t_1 - 32)] \frac{h}{h_2} \text{ lbs.}$$

Putting $\frac{1 + \delta\ (t_1 - 32)}{1 + \delta\ (t - 32)} = \mu$, we have

$(b_1 - h_1)\ (h - h_1) = \mu\ h\ b$, and therefore

$$h_1 = \frac{b_1 + h}{2} + \sqrt{\left(\frac{b_1 + h}{2}\right)^2 + (\mu\ b - b_1)\ h}.$$

By the aid of this formula we can calculate the values of the divisions of a scale, upon which the pressure b can be read off from the height of the manometer.

EXAMPLE.—If a closed manometer 25 inches long, at a temperature of 69,8° Fahr., shows a column of air 12 inches long, the corresponding height of barometer is

$$b_1 = 25 - 12 + 28{,}86\ (1 + 0{,}00204\ .\ 37{,}8)\ \tfrac{25}{12} = 13 + 28{,}86\ .\ 1{,}07707\ .\ \tfrac{25}{12}$$
$$= 13 + 64{,}76 = 77{,}76 \text{ inches, and the pressure on a square inch is}$$
$$p_1 = 0{,}4913\ .\ 77{,}76 = 38{,}20 \text{ pounds.}$$

§ 395. Buoyant Effort or Upward Thrust of the Air.— The law of the *buoyant effort* of water against a body immersed in it, discussed in § 364, can of course be applied to bodies in the air. If V is the volume of the body and γ the heaviness of the air, in which it is placed, the buoyant effort, according to this law, is $P = V\gamma$; if the body has the apparent weight G (in the air), its true weight (*in vacuo*) is

$$G_1 = G + V\gamma.$$

If, further, γ_1 is the heaviness of this body, we have also

$$G_1 = V\gamma_1, \text{ and therefore}$$

$$V = \frac{G_1}{\gamma_1}, \text{ so that we can put}$$

$$G_1 = G + \frac{G_1\gamma}{\gamma_1} \text{ or } G_1(\gamma_1 - \gamma) = G\gamma_1, \text{ whence it follows that}$$

$$G_1 = \left(\frac{\gamma_1}{\gamma_1 - \gamma}\right) G.$$

If the body is weighed upon a scale by a weight G_2, whose heaviness is γ_2, the following equation

$$G_2 = \left(\frac{\gamma_2}{\gamma_2 - \gamma}\right) G$$

holds good; if we divide the last two equations by each other, we obtain the ratio of the weights

$$\frac{G_1}{G_2} = \frac{\gamma_1}{\gamma_2} \cdot \frac{\gamma_2 - \gamma}{\gamma_1 - \gamma} = \frac{1 - \frac{\gamma}{\gamma_2}}{1 - \frac{\gamma}{\gamma_1}},$$

or, approximatively, and generally accurately enough,

$$\frac{G_1}{G_2} = 1 + \frac{\gamma}{\gamma_1} - \frac{\gamma}{\gamma_2} = 1 + \gamma\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right),$$

or also

$$\frac{G_1}{G_2} = 1 + \varepsilon\left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}\right),$$

ε, ε_1, and ε_2 denoting the specific gravities of the air, of the body weighed, and of the weight itself.

In many cases $\frac{\varepsilon}{\varepsilon_1}$ and $\frac{\varepsilon}{\varepsilon_2}$ are such small fractions that they can be neglected and the true weight can be put equal to the apparent one.

Remark.—The law of the buoyancy of the air can be employed to determine the force, with which, and the height, to which an *air-balloon* (Fr. aérostat; Ger. Luftballon) $A\ B$, Fig. 668, will rise. If V is the volume of the balloon, G its total apparent weight, including the car, etc., γ_1 the heaviness of the external and γ_2 that of the enclosed air, we have the buoyant effect

$$P = V\gamma_1 = V\gamma_2 + G, \text{ and therefore}$$

$$V(\gamma_1 - \gamma_2) = G;$$

the necessary volume of the balloon is

$$V = \frac{G}{\gamma_1 - \gamma_2},$$

FIG. 668.

and the heaviness of the external air, when the balloon attains the greatest height, is

$$\gamma_1 = \gamma_2 + \frac{G}{V}.$$

From this heaviness, by means of the formula

$$s = \frac{p}{\gamma}\, l\left(\frac{b}{b_1}\right) = \frac{p}{\gamma}\, l\left(\frac{\gamma}{\gamma_1}\right),$$

found in § 389, we can determine the greatest height s, to which the balloon will rise, by substituting for γ the heaviness of the air at the point of beginning, which must be calculated according to § 393.

EXAMPLE 1.—What is the ratio of the true weight of dry hard wood to its apparent weight, when it is weighed by means of brass weights at a temperature of 32° and when the height of the barometer is 29 inches. The density of the air is, according to § 393,

$\gamma = 0,002699 \,.\, 29 = 0,07827$ pounds, that of the wood

$\gamma_1 = 0,453 \,.\, 62,425$, and that of brass

$\gamma_2 = 8,55 \,.\, 62,425$ (see § 61),

consequently the ratio required is

$$\frac{G_1}{G_2} = 1 + \frac{0,07827}{62,425} \cdot \left(\frac{1}{0,453} - \frac{1}{8,55}\right) = 1 + 0,001254 \,.\, 2,091 = 1,00262.$$

Thus we see that one thousand pounds of wood lose about 2⅝ pounds in consequence of the buoyancy of the air.

EXAMPLE 2.—If the diameter of a spherical balloon is 30 feet and the heaviness of the matter with which it is filled is $\gamma_2 = 0,017$ pounds, and if the weight of the balloon with the car and load is $G = 500$ pounds, the heaviness of the air at the place, where the balloon ceases to rise, is

$$\gamma_1 = \gamma_2 + \frac{G}{V} = \gamma_2 + \frac{6\,G}{\pi\, d^3} = 0,017 + \frac{3000}{\pi\, 30^3} = 0,017 + 0,03537$$
$$= 0.05237 \text{ pounds.}$$

Now if the density of the exterior air at the starting-point is 0,0800 pounds, we have

$$l\left(\frac{\gamma}{\gamma_1}\right) = l\left(\frac{8000}{5237}\right) = 0,4948,$$

and if we assume the ratio of the pressure per square foot to the heaviness of the air, I.E., $\frac{p}{\gamma} = 26210$, we obtain the maximum height to which the balloon will rise

$$s = \frac{p}{\gamma}\, l\left(\frac{\gamma}{\gamma_1}\right) = 26210 \,.\, 0,4948 = 12969 \text{ feet.}$$

SEVENTH SECTION.

DYNAMICS OF FLUIDS.

CHAPTER I.

THE GENERAL THEORY OF THE EFFLUX OF WATER FROM VESSELS.

§ 396. **Efflux.**—The theory of the *efflux* (Fr. écoulement; Ger. Ausfluss) of fluids from vessels forms the first grand division of hydrodynamics. We distinguish, in the first place, the efflux of *water* and the efflux of *air*, and, in the second place, efflux under *constant* and under *variable* pressure. We will begin with the efflux of water under constant pressure. We can regard the pressure of water as constant, when the same quantity of water enters the vessel as is discharged from it, or when the quantity of water discharged is very small, compared with the capacity of the vessel. The principal problem to be solved is to determine the quantity of water or the *discharge* (Fr. dépense; Ger. Wassermenge), which passes through a given *aperture* or *orifice* (Fr. orifice; Ger. Oeffnung) under a given pressure and in a given time.

If the discharge per second $= Q$, we have the discharge in t seconds, when the pressure is constant,

$$V = Q\,t.$$

But if we wish to find the discharge per second, we must know the size of the orifice and the velocity of the effluent molecules of the water. To simplify our researches, we assume that the molecules flow in parallel straight lines, and, consequently, form a prismatic *stream*, *vein* or *jet* of water (Fr. veine, courant de fluide; Ger. Wasserstrahl). If F is the *cross-section* of the stream and v the velocity of the water, or that of every one of its molecules, the discharge Q per second forms a prism, whose base is F and whose height is v, and, therefore, we have

$$Q = F\,v \text{ units of volume}$$

and

$$G = F\,v\,\gamma \text{ units of weight,}$$

γ denoting the heaviness of the effluent water or liquid.

EXAMPLE—1) If water flows through a sluice gate, the cross-section of which is 1,7 square feet, with a velocity of 14 feet, the discharge per second is

$$Q = 14 \,.\, 1{,}7 = 23{,}8 \text{ cubic feet,}$$

and the hourly discharge is

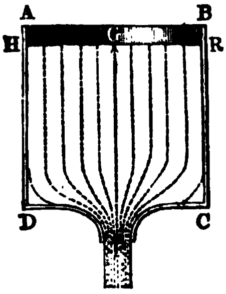
$$= 23{,}8 \,.\, 3600 = 85680 \text{ cubic feet.}$$

2) If 264 cubic feet of water are discharged in 3 minutes and 10 seconds through an orifice, the area of which is 5 square inches, the mean velocity of the liquid is

$$v = \frac{V}{F\,t} = \frac{264}{\frac{5}{144}\,.\,190} = \frac{264\,.\,144}{5\,.\,190} = 40 \text{ feet.}$$

§ 397. Velocity of Efflux.—Let us imagine a vessel $A\,C$, Fig. 669, which is full of water, to be provided with an orifice F, which is rounded upon the inside and is very small, compared to the surface $H\,R$ of the water, and let us put the *head of water* $F\,G$ (Fr. charge d'eau; Ger. Druckhöhe), which is to be regarded as constant during the efflux, $= h$, the velocity of efflux $= v$, and the discharge per second $= Q$, or its weight $= Q\,\gamma$. The work, which this quantity of water can perform while sinking through the distance h, is $= Q\,h\,\gamma$, and the energy stored by the discharge, whose weight is $Q\,\gamma$, in passing from a state of rest to the velocity v, is $\frac{v^2}{2g}\,Q\,\gamma$ (§ 74). If no loss of mechanical effect takes place during the passage through the orifice, the quantities of work are equal to each other, or $h\,Q\,\gamma = \frac{v^2}{2g}\,Q\,\gamma$, i. e.

FIG. 669.



$$h = \frac{v^2}{2g},$$

and inversely

$$v = \sqrt{2\,g\,h},$$

in meters

$$h = 0{,}0510\ v^2 \text{ and } v = 4{,}429\ \sqrt{h},$$

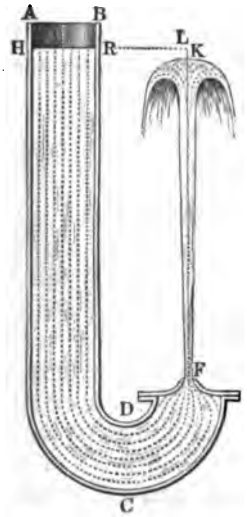
and in feet (English),

$$h = 0{,}0155\ v^2 \text{ and } v = 8{,}025\ \sqrt{h}.$$

The velocity of the effluent water is the same as that of a body which has fallen freely through a height which is equal to the head of water.

The correctness of this law can also be shown by the following experiment. If in the vessel $A\ C\ F$, Fig. 670, we make an orifice directed upwards, the jet $F\ K$ will rise vertically and will nearly reach the level $H\ R$ of the water in the vessel, and we can assume that it would actually reach it, if all impediments (such as the resistance of the air, the friction upon the sides of the vessel, the disturbance caused by the falling back of the water upon itself, etc.) were removed. Since a body which rises vertically to the height h has an initial velocity

FIG. 670.



$$v = \sqrt{2\,g\,h},$$

it follows that the velocity of efflux must be

$$v = \sqrt{2\,g\,h}.$$

For another head of water h_1 the velocity of efflux is

$$v_1 = \sqrt{2\,g\,h_1},$$

hence we have

$$v : v_1 = \sqrt{h} : \sqrt{h_1};$$

the velocities of efflux are, therefore, to each other as the square roots of their heads of water.

EXAMPLE—1) The discharge per second through an orifice whose area is 10 square inches, under a head of water of 5 feet, is

$$Q = F v = 10 \,.\, 12\ \sqrt{2\,g\,h} = 120 \,.\, 8{,}025\ \sqrt{5} = 963 \,.\, 2{,}236 = 2158 \text{ cubic inches.}$$

2) In order that 252 cubic inches of water shall pass in one second through an opening of 6 square inches, the head of water must be

$$h = \frac{v^2}{2\,g} = \frac{1}{2\,g}\left(\frac{Q}{F}\right)^2 = \frac{0{,}0155}{12} \,.\, \left(\frac{252}{6}\right)^2 = \frac{0{,}0155}{12} \,.\, 42^2 = 2{,}28 \text{ inches.}$$

§ 398. Velocities of Influx and Efflux.—If the water flows in with a certain velocity c, we must add to the mechanical effect $h\,Q\,\gamma$ the energy $\frac{c^2}{2g}\,Q\,\gamma$, possessed by the influent water and corresponding to the height $h_1 = \frac{c^2}{2g}$, due to the velocity; hence we must put

$$(h + h_1)\,Q\,\gamma = \frac{v^2}{2g}\,Q\,\gamma, \text{ or } h + h_1 = \frac{v^2}{2g},$$

and the velocity of efflux

$$v = \sqrt{2g(h + h_1)} = \sqrt{2gh + c^2}.$$

If the vessel is maintained constantly full, the quantity of the influent water is equal to the discharge Q, and we can put $G\,c = F\,v$, in which G denotes the area of the cross-section $H\,R$ (Fig. 669) of the water that is flowing in. Putting $c = \frac{F}{G}\,v$, we obtain

$$h = \frac{v^2}{2g} - \left(\frac{F}{G}\right)^2 \frac{v^2}{2g} = \left[1 - \left(\frac{F}{G}\right)^2\right]\frac{v^2}{2g},$$

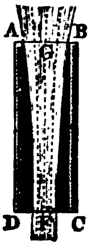
whence

$$v = \frac{\sqrt{2gh}}{\sqrt{1 - \left(\frac{F}{G}\right)^2}}.$$

According to this formula, the velocity increases with the ratio $\frac{F}{G}$ of the cross-sections, and it is a minimum and $= \sqrt{2gh}$, when the cross-section F of the orifice of discharge is very small, compared with that G of the orifice of influx, and it approaches nearer and nearer to infinity, the smaller the difference between the two orifices becomes. If $F = G$ or $\frac{F}{G} = 1$, we have $v = \frac{\sqrt{2gh}}{0} = \infty$, and also $c = \infty$; this infinite value must be understood thus: if a vessel $A\,C$, Fig. 671, is without a bottom, water must flow in and out with an infinitely great velocity or the stream of liquid $G\,F$ will not fill the orifice of exit $C\,D$. Putting $v = \frac{G\,c}{F}$, we obtain

$$h = \left[\left(\frac{G}{F}\right)^2 - 1\right]\frac{c^2}{2g}, \text{ and therefore } F = \frac{G}{\sqrt{1 + \frac{2gh}{c^2}}}.$$

FIG. 671.



which expression shows that the cross-section F of the discharging stream is always smaller for a finite velocity of influx than that G through which the water flows in, and that it therefore does not fill the orifice of efflux, when the latter is larger than $\frac{G}{\sqrt{1 + \frac{2gh}{c^2}}}$.

REMARK.—The correctness of the formula

$$v = \frac{\sqrt{2gh}}{\sqrt{1 - \left(\frac{F}{G}\right)^2}},$$

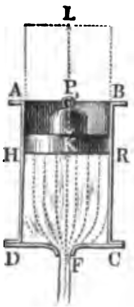
which was first established by Daniel Bernoulli, was afterwards much disputed. I have endeavored to prove in the "Allgemeinen Maschinen-encyclopädie," by Hulsse, in the article "Efflux" (Ausfluss), how unfounded were the representations, which were made.

EXAMPLE.—If water flows from a vessel, whose cross-section is 60 square inches, through a circular orifice in the bottom 5 inches in diameter under a head of water of six feet, its velocity is

$$v = \frac{8,025\sqrt{6}}{\sqrt{1 - \left(\frac{25\,\pi}{4.60}\right)^2}} = \frac{8,025 \,.\, 2,449}{\sqrt{1 - (0,327)^2}} = \frac{19,653}{\sqrt{0,8931}} = \frac{19,653}{0,945} = 20,79 \text{ feet.}$$

§ 399. Velocity of Efflux, Pressure and Heaviness.—The formulas, which we have found, hold good so long only as the pressure of the air upon the surface of the water is the same as that upon the orifice of efflux; but if these pressures differ, these formulas must have an addition made to them. If the surface $H R$, Fig. 672, is pressed upon by a *piston* K with a force P_1, as occurs, E.G., in fire engines, we can imagine this force to be replaced by the pressure of a column of water. If h_1 is the height $L\,K$ of this column and γ the heaviness of the liquid, we can put

FIG. 672.



$$P_1 = G\,h_1\,\gamma.$$

Substituting for h the head of water $h + h_1 = h + \frac{P_1}{G\gamma}$, which has been increased by $h_1 = \frac{P_1}{G\gamma}$, we obtain for the velocity of efflux

$$v = \sqrt{2\,g\left(h + \frac{P_1}{G\,\gamma}\right)},$$

when we assume $\frac{F}{G}$ to be very small. If we denote the pressure upon each unit of the surface G by p_1, we have more simply

$$\frac{P_1}{G} = p_1,$$

and therefore

$$v = \sqrt{2\,g\left(h + \frac{p_1}{\gamma}\right)}.$$

Finally, if we denote the pressure of the water at the orifice of efflux by p, we can put

$$p = \left(h + \frac{p_1}{\gamma}\right)\gamma, \text{ or}$$

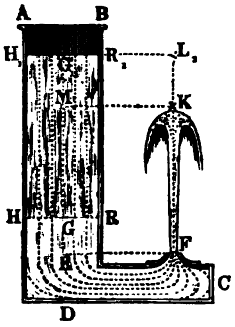
$$h + \frac{p_1}{\gamma} = \frac{p}{\gamma}, \text{ whence}$$

$$v = \sqrt{2\,g\frac{p}{\gamma}}.$$

Hence *the velocity of efflux is directly proportional to the square root of the pressure upon the unit of surface and inversely to the square root of the density or heaviness of the liquid.* When the pressure is the same, a liquid four times as heavy as another discharges one-half as fast as the latter. Since air is 770 times lighter than water, it would, if it were inelastic, flow out under the same pressure $\sqrt{770} = 27\frac{3}{4}$ times faster than water.

This theory is also applicable to the case where the effluent water is subjected to the pressure of a column of another liquid.

FIG. 673.



If above the level $H\,R$ of the water $H\,E\,R$ in a vessel $A\ C\ D$, Fig. 673, there is still a column of liquid $H\ R_1$, whose height $G\ G_1 = h$ and whose heaviness $= \gamma_1$, while that of the water is $= \gamma$, we can replace the latter by a column of water whose height is $\frac{\gamma_1}{\gamma}\,h_1$ without changing the pressure upon $H\ R$ or causing the velocity v of the water, which is passing through the opening F, to vary. Hence if h is the head $E\ G$ of water, I.E., the height of the surface of separation $H\ R$ above the orifice F, we have the height due to velocity

$$\frac{v^2}{2\,g} = h + \frac{\gamma_1}{\gamma}\,h_1,$$

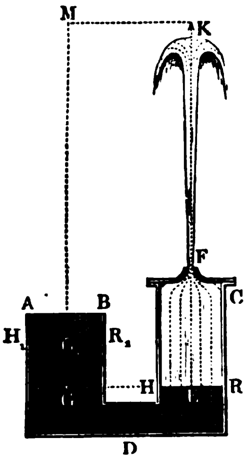
and therefore

$$v = \sqrt{2\,g\left(h + \frac{\gamma_1}{\gamma}\,h_1\right)}.$$

Now if $\gamma_1 < \gamma$ or $h + \frac{\gamma_1}{\gamma}\,h_1 < h + h_1$, the jet $F\,K$, which rises vertically, will not reach the level $H_1\,R_1\,L_1$ of the surface of the liquid.

If the surface of separation $H\,R$, Fig. 674, is not above, but a certain distance $E\,F = h$ below *the orifice* F of the vessel $A\,D\,C$, while the surface $H_1\,R_1$ of the liquid $H_1\,D\,R$ is at the height $G\,G_1 = h_1$ above the surface of separation $H\,R$, we have

FIG. 674.



$$\frac{v^2}{2\,g} = \frac{\gamma_1}{\gamma}\,h_1 - h,$$

and therefore the velocity of the jet

$$v = \sqrt{2\,g\left(\frac{\gamma_1}{\gamma}\,h_1 - h\right)}.$$

This supposes $\frac{\gamma_1}{\gamma}\,h_1 > h$, or $\frac{h_1}{h} > \frac{\gamma}{\gamma_1}$. From this it is easy to see that the jet $F\,K$, which is projected vertically upwards, can rise above the surface $H_1\,R_1$ of the liquid $H_1\,D\,R$. If $G\,M = \frac{\gamma_1}{\gamma}\,h_1$ is the head of the liquid, reduced to that of water, M gives the level to which the jet will nearly reach.

If the water does not discharge freely, but *under water*, a diminution of the velocity of efflux takes place owing to the opposite pressure. If the orifice F of the vessel $A\,C$, Fig. 675, is at a distance $F\,G = h$ below the upper level $H\,R$ of the water and at a distance $F\,G_1 = h_1$ below the lower level $H_1\,R_1$, we have the pressure from above downwards

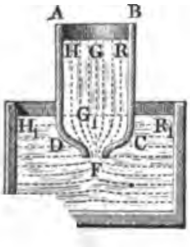
$$p = h\,\gamma,$$

and the opposite pressure from below upwards

$$p_1 = h_1\,\gamma;$$

hence the force, which produces the efflux, is

FIG. 675.



$$p - p_1 = (h - h_1)\,\gamma$$

and the velocity of efflux is

$$v = \sqrt{2g\left(\frac{p - p_1}{\gamma}\right)} = \sqrt{2g(h - h_1)}.$$

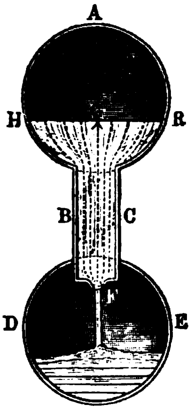
When water discharges under water, we must regard the *difference of level* $h - h_1$ between the surfaces of water as the head of water.

If the water at the orifice of efflux is pressed upon with a force p and at the surface or orifice of influx with a force p_1, we have in general

$$v = \sqrt{2g\left(h + \frac{p_1 - p}{\gamma}\right)}.$$

This case occurs when water flows from one closed vessel $A\ B\ C$ into another closed one $D\ E$, Fig. 676. Here h is the height $F\ G$ of the surface of the water $H\ R$ above the orifice F, p_1 the pressure of the air in $A\ H\ R$ and p the pressure of the air or the steam in $D\ E$.

FIG. 676.



EXAMPLE—1) If the piston of a fire engine is 12 inches in diameter and it is pressed down in the cylinder with a force of 8000 pounds, and if there are no resistances in the pipes and hose, the water will pass through the nozzle of the hose with a velocity

$$v = \sqrt{2g\frac{p_1}{\gamma}} = \sqrt{2g\frac{P_1}{G\gamma}} = 8,025\sqrt{\frac{8000}{\frac{\pi}{4} . 62,5}}$$

$$= 8,025\sqrt{64 . \frac{8}{\pi}} = 62,74 \text{ feet};$$

if the stream is directed vertically upwards, it will reach a height

$$h = 0,0155 . v^2 = 61,007 \text{ feet}.$$

2) If water flows into a space in which the air has been rarified, E.G., into the condenser of a steam engine, while its upper surface is pressed upon by the atmosphere, we must employ the last formula for the velocity of efflux, viz.,

$$v = \sqrt{2g\left(h + \frac{p_1 - p}{\gamma}\right)}.$$

If the head of water is $h = 3$ feet, the height of the barometer of the exterior air 29 inches and that of the enclosed air 4 inches, we have

$$\frac{p_1 - p}{\gamma} = 29 - 4 = 25 \text{ inches} = 2,083 \text{ feet of mercury}$$

$$= 13,6 . 2,083 = 28,33 \text{ feet of water},$$

hence the velocity of the water flowing into the space, which is filled with rarefied air, is

$$v = 8{,}025\sqrt{3 + 28{,}33} = 8{,}025\sqrt{31{,}33} = 44{,}92 \text{ feet.}$$

3) If the water in the feed-pipe of a steam boiler stands 12 feet above the level of the water in the boiler and if the pressure of the steam in the latter is 20 pounds and that of the exterior air is 15 pounds, the velocity with which the water enters the boiler is

$$v = 8{,}025\sqrt{12 + \frac{(15 - 20)\,.\,144}{62{,}5}} = 8{,}025\sqrt{12 - \frac{5\,.\,144}{62{,}5}}$$

$$= 8{,}025\sqrt{12 - 11{,}52} = 5{,}56 \text{ feet.}$$

§ 400. Hydraulic or Hydrodynamic Head.—If the water in a vessel is in motion, it presses less against the sides of the vessel than when it is at rest. We must, therefore, distinguish *the hydraulic or hydrodynamic* from the *hydrostatic head of water.* If p_1 is the pressure upon each unit of the surface of the water $H_1 R_1 = G_1$, Fig. 677, p the pressure at the orifice F and h the head of water $F\,G_1$, we have the velocity of efflux

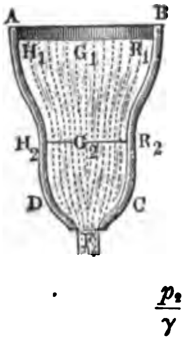
$$v = \sqrt{2g\left(h + \frac{p_1 - p}{\gamma}\right)} : \sqrt{1 - \left(\frac{F}{G_1}\right)^2},$$

or

$$h + \frac{p_1 - p}{\gamma} = \left[1 - \left(\frac{F}{G_1}\right)^2\right]\frac{v^2}{2g};$$

now if in another section $H_2 R_2 = G_2$, which is at a distance $F\,G_2 = h_1$ above the orifice, the pressure is $= p_2$, we have in like manner

FIG. 677.



$$h_1 + \frac{p_2 - p}{\gamma} = \left[1 - \left(\frac{F}{G_2}\right)^2\right]\frac{v^2}{2g}.$$

If we subtract these two equations from each other, we obtain

$$h - h_1 + \frac{p_1 - p_2}{\gamma} = \left[\left(\frac{F}{G}\right)^2 - \left(\frac{F}{G_1}\right)^2\right]\frac{v^2}{2g},$$

or, if we denote the head of water $G_1\,G_2$ of the layer $H_2 R_2 = G_2$ by h_2, we have for the hydrodynamic head at $\overline{H_2 R_2}$

$$\frac{p_2}{\gamma} = h_2 + \frac{p_1}{\gamma} - \left[\left(\frac{F}{G_2}\right)^2 - \left(\frac{F}{G_1}\right)^2\right]\frac{v^2}{2g}.$$

But $\frac{F\,v}{G_1}$ is the velocity v_1 of the water at the upper surface G_1, and $\frac{F\,v}{G_2}$ the velocity v_2 of the water in the cross-section G_2, we can, therefore, put

$$\frac{p_2}{\gamma} = \frac{p_1}{\gamma} + h_2 - \left(\frac{v_2^2}{2g} - \frac{v_1^2}{2g}\right).$$

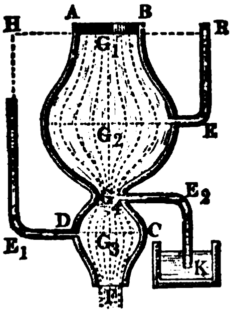
The hydraulic head $\frac{p_2}{\gamma}$ *at any position in the vessel is equal to the hydrostatic head* $\frac{p_1}{\gamma} + h_2$, *diminished by the difference of the heights due to the velocities of the water at this point and at the inlet orifice.* If the free surface G_1 of the water is very great, we can neglect the velocity of influx and put

$$\frac{p_2}{\gamma} = \frac{p_1}{\gamma} + h_2 - \frac{v_2^2}{2g};$$

hence the hydraulic head is less than the hydrostatic head by an amount equal to the height due to the velocity of the water. The quicker the water moves, the less it presses upon the sides of the pipe. For this reason pipes often burst or leak for the first time, when the motion of the water is checked, when the pipes clog, etc.

By means of the apparatus $A\ B\ C\ D$, represented in Fig. 678, the difference between the hydraulic and the hydrostatic head can be ocularly demonstrated. If from the cross-section G_2 we carry a tube $E\ R$ upwards the latter will fill with water, which will rise above the level $H\ R$ of the water, when $G_2 > G_1$ or $v_2 < v_1$; for, since the pressure p_1 upon the surface of the water is balanced by the pressure of the air upon the mouth of the tube, we can put the height, which measures the pressure in G_2,

FIG. 678.



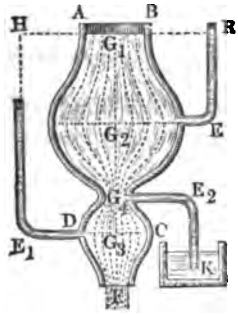
$$x = \frac{p_2}{\gamma} = h_2 - \left(\frac{v_2^2}{2g} - \frac{v_1^2}{2g}\right),$$

and x is $> h_2$ when $\frac{v_2^2}{2g} < \frac{v_1^2}{2g}$. If, on the contrary, the cross-section $G_3 < G_1$, the water flows more rapidly through G_1, and we have for the height of column of water in the tube E_1, inserted at G_3,

$$y = h_3 - \left(\frac{v_3^2}{2g} - \frac{v_1^2}{2g}\right),$$

which is less than h_3, so that the water does not rise to the level $H\ R$ of G_1. If, finally, G_4 is very small and the corresponding velocity very great, $\frac{v_4^2}{2g} - \frac{v_1^2}{2g}$ can be $> h_4$, and the corresponding hydraulic head z will be negative, I.E. the pressure of the air on the outside will be greater than that of the water within. Hence, if a tube is carried downwards and its end placed under water, a column of water E_2 K will rise in it, which, together with the pressure of the water, will balance the atmospheric pressure. If the tube is short, the water in the vessel K, which, in this experiment, should be colored, will rise in the tube, enter the reservoir A B C D and flow, with the other water, out at F.

FIG. 679.

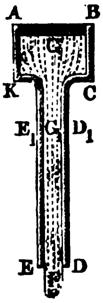


REMARK.—If the vessel A C E, Fig. 680, consists of a reservoir A C and of a narrow vertical tube C E, the hydrodynamic head is negative in all parts of this tube. If we do not regard the pressure of the atmosphere p_1, the pressure of the water at the orifice of efflux is $= 0$; for here the entire head of water is expended in producing the velocity $v = \sqrt{2gh}$; on the contrary, for a position D E, which is at a distance G_1 $G = h_1$ under the water level, the hydraulic head is

$$= h_1 - h = -(h - h_1),$$

or *negative;* if, then, a hole were bored in this tube, no water would escape, but, on the contrary, air would be sucked in and discharged at F. This negative pressure is a maximum directly under the reservoir, since h_2 is here a minimum.

FIG. 680.



§ 401. Rectangular Lateral Orifices.—By the aid of the formula

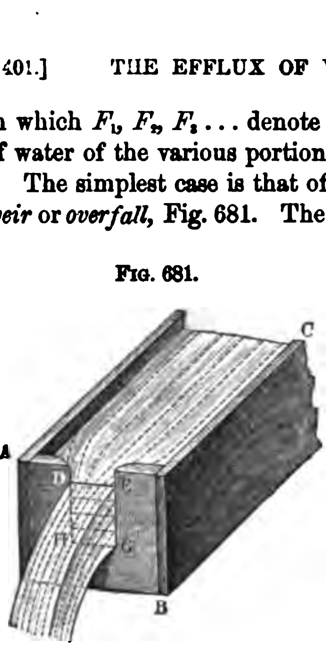
$$Q = F v = F \sqrt{2gh},$$

the discharge per second can be calculated only when the orifice is horizontal, since in that case the velocity is uniform in the whole cross-section F; but if the cross-section is inclined to the horizon, if, E.G., the opening is in the side of the vessel, the molecules of water at different depths flow out with different velocities, and the discharge can no longer be regarded as a prism; hence the formula $Q = Fv = F\sqrt{2gh}$ cannot be applied directly. The general formula is

$$Q = F_1 \sqrt{2gh_1} + F_2 \sqrt{2gh_2} + F_3 \sqrt{2gh_3} + \ldots$$
$$= \sqrt{2g}(F_1 \sqrt{h_1} + F_2 \sqrt{h_2} + F_3 \sqrt{h_3} + \ldots),$$

in which $F_1, F_2, F_3 \ldots$ denote the areas and $h_1, h_2, h_3 \ldots$ the heads of water of the various portions of the orifice.

FIG. 681.



The simplest case is that of efflux through a *notch* in the side, *weir* or *overfall*, Fig. 681. The notch $D\,E\,G\,H$ in the wall, through which the efflux takes place, is rectangular; let us denote its width $D\,E = G\,H$ by b and its height $D\,H = E\,G$ by h. If we decompose this surface $b\,h$, by horizontal lines, into a great number n of horizontal strips of equal width, we can consider the velocity to be constant for each of them. Since, if we proceed from above downwards, the heads of water of these strips are

$$\frac{h}{n}, \frac{2\,h}{n}, \frac{3\,h}{n}, \text{ etc.,}$$

we have for the corresponding velocities

$$\sqrt{2\,g\,.\,\frac{h}{n}}, \sqrt{2\,g\,.\,\frac{2\,h}{n}}, \sqrt{2\,g\,.\,\frac{3\,h}{n}}, \text{ etc.,}$$

and since the area of each of these strips is $= b\,.\,\frac{h}{n} = \frac{b\,h}{n}$, we have the corresponding discharges

$$\frac{b\,h}{n}\sqrt{2\,g\,.\,\frac{h}{n}}, \frac{b\,h}{n}\sqrt{2\,g\,.\,\frac{2\,h}{n}}, \frac{b\,h}{n}\sqrt{2\,g\,.\,\frac{3\,h}{n}}, \text{ etc.;}$$

hence that of the whole section is

$$Q = \frac{b\,h}{n}\left(\sqrt{2\,g\,.\,\frac{h}{n}} + \sqrt{2\,g\,.\,\frac{2\,h}{n}} + \sqrt{2\,g\,.\,\frac{3\,h}{n}} + \ldots\right)$$
$$= \frac{b\,h\sqrt{2\,g\,h}}{n\sqrt{n}}\,(\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n}).$$

Since (as is given in the Ingenieur, page 88)

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n},$$

or

$$1^{\frac{1}{2}} + 2^{\frac{1}{2}} + 3^{\frac{1}{2}} + \ldots + n^{\frac{1}{2}} = \frac{n^{1+\frac{1}{2}}}{1+\frac{1}{2}} = \tfrac{2}{3}\,n^{\frac{3}{2}} = \tfrac{2}{3}\,n\sqrt{n},$$

it follows that the required discharge is

$$Q = \frac{b\,h\sqrt{2\,g\,h}}{n\sqrt{n}} \cdot \tfrac{2}{3}\,n\sqrt{n} = \tfrac{2}{3}\,b\,h\sqrt{2\,g\,h} = \tfrac{2}{3}\,b\sqrt{2\,g\,h^3}.$$

If we understand by the mean velocity v that velocity, which must exist at all points of the overfall, when the same quantity of water passes through the whole cross-section with a uniform velocity as does pass through with the variable velocity, we can put

$Q = b\,h\,v$, whence it follows that

$$v = \tfrac{2}{3}\sqrt{2\,g\,h},$$

I.E. *the mean velocity of water flowing out through a rectangular notch in the side of a vessel is $\tfrac{2}{3}$ the velocity at the sill or lower edge of the notch.*

If the *rectangular orifice* $K\,G$, Fig. 682, with the horizontal base $G\,H$, does not reach to the level of the water, we find the discharge through it by regarding it as the difference between two notches in the side $D\,E\,G\,H$ and $D\,E\,L\,K$. If h_1 is the depth $E\,G$ of the lower and $h_2 = E\,L$ that of the upper edge, we have for the discharges through these notches

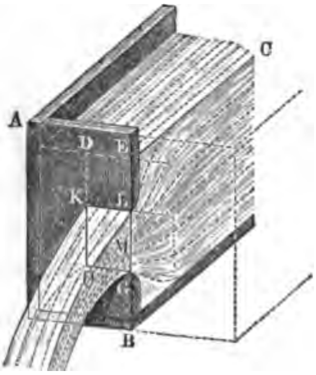
$$\tfrac{2}{3}\,b\sqrt{2\,g\,h_1^3},$$

and

$$\tfrac{2}{3}\,b\sqrt{2\,g\,h_2^3};$$

hence the discharge through the *rectangular opening* $G\,H\,K\,L$ is

FIG. 682.



$$Q = \tfrac{2}{3}\,b\sqrt{2\,g\,h_1^3} - \tfrac{2}{3}\,b\sqrt{2\,g\,h_2^3} = \tfrac{2}{3}\,b\sqrt{2\,g}\,(h_1^{\frac{3}{2}} - h_2^{\frac{3}{2}}),$$

and the mean velocity of efflux is

$$v = \frac{Q}{b\,(h_1 - h_2)} = \tfrac{2}{3}\,b\sqrt{2\,g}\,.\,\frac{h_1^{\frac{3}{2}} - h_2^{\frac{3}{2}}}{h_1 - h_2}.$$

If h is the mean head of water $E\,M = \frac{h_1 + h_2}{2}$, or the depth of the centre of the orifice below the level of the water, and a the height $K\,H = L\,G = h_1 - h_2$ of the orifice, we can put

$$v = \tfrac{2}{3}\sqrt{2\,g}\,.\,\frac{\left(h + \frac{a}{2}\right)^{\frac{3}{2}} - \left(h - \frac{a}{2}\right)^{\frac{3}{2}}}{a}, \text{ or approximatively}$$

$$= \left[1 - \tfrac{1}{96}\left(\frac{a}{h}\right)^2\right]\sqrt{2\,g\,h}.$$

EXAMPLE.—If a rectangular orifice of efflux is 3 feet wide and 1¼ feet high and the lower edge is 2¾ feet below the level of the water, the discharge is

$$Q = \tfrac{2}{3} \,.\, 8{,}025 \,.\, 3\,(2{,}75^{\frac{3}{2}} - 1{,}5^{\frac{3}{2}}) = 16{,}05\,(4{,}560 - 1{,}837)$$
$$= 16{,}05 \,.\, 2{,}723 = 43{,}7 \text{ cubic feet.}$$

According to the approximate formula

$$v = \left[1 - \tfrac{1}{96}\left(\frac{1{,}25}{2{,}125}\right)^2\right] .\, 8{,}025\,\sqrt{2{,}125} = (1 - 0{,}0036)\; 11{,}698$$
$$= 11{,}698 - 0{,}042 = 11{,}656 \text{ feet,}$$

and the discharge is, therefore,

$$Q = 3 \,.\, \tfrac{5}{4} \,.\, 11{,}656 = 43{,}710 \text{ cubic feet.}$$

REMARK.—If the notch in the wall is inclined to the horizon at an angle δ, we must substitute for the height of the orifice $\frac{h_1 - h_2}{sin.\ \delta}$ instead of $h_1 - h_2$, and therefore we must put

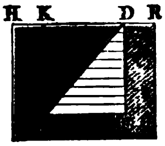
$$Q = \tfrac{2}{3}\,\frac{b\sqrt{2\,g}}{sin.\ \delta}\,(\sqrt{h_1^3} - \sqrt{h_2^3}).$$

If the cross-section of the reservoir, from which the water is discharging, is not much larger than the cross-section of the orifice, we must take into account the velocity of approach $v_1 = \frac{F}{G}\,v$ of the water and put

$$Q = \tfrac{2}{3}\,b\,\sqrt{2\,g}\left[\left(h + \frac{v_1^2}{2\,g}\right)^{\frac{3}{2}} - \left(h_2 + \frac{v_1^2}{2\,g}\right)^{\frac{3}{2}}\right].$$

§ 402. Triangular Lateral Orifice.—Besides rectangular lateral orifices, *triangular* and *circular ones* also occur in practice. We will next discuss the discharge through a triangular orifice $D\,E\,G$, Fig. 683, with a horizontal base $E\,G$ and with its apex D at the level of the water $H\,R$. If we put the base $E\,G = b$ and the height $D\,E = h$ and if we divide the latter into n equal parts and pass through these divisions lines parallel to the base, we divide the entire surface into small strips, whose areas are $\frac{b}{n} \cdot \frac{h}{n}, \frac{2\,b}{n} \cdot \frac{h}{n}, \frac{3\,b}{n} \cdot \frac{h}{n}$, etc., and whose heads of water are $\frac{h}{n}, \frac{2\,h}{n}, \frac{3\,h}{n}$, etc. The discharges through them are

FIG. 683.



$$\frac{b\,h}{n^2}\sqrt{2\,g\,\frac{h}{n}},\ \frac{2\,b\,h}{n^2}\sqrt{2\,g\,\frac{2\,h}{n}},\ \frac{3\,b\,h}{n^2}\sqrt{2\,g\,\frac{3\,h}{n}}, \text{ etc.,}$$

by summing these we obtain the discharge of the whole *triangular orifice*

$$Q = \frac{b\,h}{n^2}\sqrt{2\,g\frac{h}{n}}\,(1 + 2\sqrt{2} + 3\sqrt{3} + \ldots + n\sqrt{n})$$
$$= \frac{b\,h\sqrt{2\,g\,h}}{n^2\sqrt{n}}\,(1^{\frac{3}{2}} + 2^{\frac{3}{2}} + 3^{\frac{3}{2}} + \ldots + n^{\frac{3}{2}}),$$

or, since the series in the parenthesis $= \left(\frac{n^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right) = \frac{2}{5}\,n^{\frac{5}{2}}$,

$$Q = \tfrac{2}{5}\,b\,h\,\sqrt{2\,g\,h} = \tfrac{2}{5}\,\sqrt{2\,g\,h^3}.$$

If the base $D\,K$ of the orifice $D\,G\,K$ lies in the surface of the water and the apex G is at the depth h below it, we have the corresponding discharge, since that through the rectangle $D\,E\,G\,K$ is $\frac{2}{3}\,b\,h\,\sqrt{2\,g\,h}$,

$$Q = \tfrac{2}{3}\,b\,h\,\sqrt{2\,g\,h} - \tfrac{2}{5}\,b\,h\,\sqrt{2\,g\,h} = \tfrac{4}{15}\,b\,h\,\sqrt{2\,g\,h}.$$

The discharge through a trapezium $A\,B\,C\,D$, Fig. 684, whose upper base $A\,B = b_1$ lies in the surface of the water, whose lower base is $C\,D = b_2$ and whose height is $D\,E = h$, is found by combining the discharge through a rectangle with those through two triangles, and it is

$$Q = \tfrac{2}{3}\,b_2\,h\sqrt{2\,g\,h} + \tfrac{4}{15}\,(b_1 - b_2)\,h\sqrt{2\,g\,h}$$
$$= \tfrac{2}{15}\,(2\,b_1 + 3\,b_2)\,h\sqrt{2\,g\,h}.$$

FIG. 684.

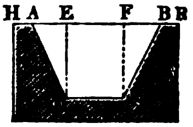
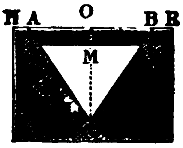


FIG. 685.

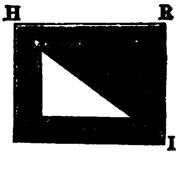


Further, the discharge through a triangle $C\,D\,E$, Fig. 685, whose base is $D\,E = b_1$, whose altitude is $O\,M = h_1$ and whose apex C is situated at a depth $O\,C = h$ below the level $H\,R$ of the water, is Q = discharge through $A\,B\,C$ minus that through $A\,E$

$$= \tfrac{4}{15}\,b\,h\sqrt{2\,g\,h} - \tfrac{2}{15}\,(2\,b + 3\,b_1)\,h_1\sqrt{2\,g\,h_1}$$
$$= \tfrac{2}{15}\sqrt{2\,g}\,[2\,b\,(h^{\frac{3}{2}} - h_1^{\frac{3}{2}}) - 3\,b_1\,h_1^{\frac{3}{2}}].$$

Since the width $A\,B = b$ is determined by the proportion $b : b_1 :: h : (h - h_1)$, it follows that

FIG. 686.



$$Q = \frac{2\sqrt{2\,g}\,.\,b_1}{15}\left(\frac{2\,h\,(h^{\frac{3}{2}} - h_1^{\frac{3}{2}})}{h - h_1} - 3\,h_1^{\frac{3}{2}}\right)$$
$$= \frac{2\sqrt{2\,g}\,.\,b_1}{15}\left(\frac{2\,h^{\frac{5}{2}} - 5\,h\,h_1^{\frac{3}{2}} + 3\,h_1^{\frac{5}{2}}}{h - h_1}\right).$$

Finally, we have for the discharge through a triangle $A\,C\,D$, Fig. 686, whose apex lies above its base,

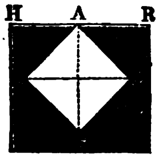
$$Q = \frac{2}{3}\sqrt{2g}\,.\,b_1\,(h^{\frac{3}{2}} - h_1^{\frac{3}{2}}) - \frac{2\sqrt{2g}\,.\,b_1}{15}\left(\frac{2\,h^{\frac{5}{2}} - 5\,h\,h_1^{\frac{3}{2}} + 3\,h_1^{\frac{5}{2}}}{h - h_1}\right)$$

$$= \frac{2\sqrt{2g}\,.\,b_1}{15}\left(\frac{3\,h^{\frac{5}{2}} - 5\,h_1\,h^{\frac{3}{2}} + 2\,h_1^{\frac{5}{2}}}{h - h_1}\right).$$

Example.—What is the discharge through the square orifice *A B C D*, Fig. 687, whose vertical diagonal *A C* = 1 foot, when the corner *A* reaches to the level of the water? The discharge through the upper half of the square is

$Q = \frac{2}{5}\,b\,\sqrt{2\,g\,h^3} = \frac{1}{5}\,.\,1\,.\,8{,}025\,\sqrt{\frac{1}{2}} = 1{,}605\,.\,0{,}7071 = 1{,}135$ cubic feet,
and that through the lower half

Fig. 687.



$$Q_1 = \frac{2\,b\,\sqrt{2\,g}}{15}\left(\frac{2\,h^{\frac{5}{2}} - 5\,h\,h_1^{\frac{3}{2}} + 3\,h_1^{\frac{5}{2}}}{h - h_1}\right)$$

$$= \frac{2\,.\,8{,}025}{15}\left(\frac{2 - 5\,(\frac{1}{2})^{\frac{3}{2}} + 3\,(\frac{1}{2})^{\frac{5}{2}}}{1 - \frac{1}{2}}\right)$$

$$= \frac{32{,}10}{15}\,(2 - 1{,}7678 + 0{,}5303)$$

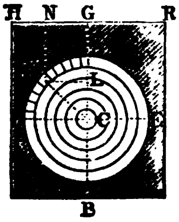
$= 2{,}14\,.\,0{,}7625 = 1{,}632$ cubic feet,

consequently the total discharge is

$$Q = 1{,}135 + 1{,}632 = 2{,}767 \text{ cubic feet.}$$

§ 403. Circular Lateral Orifices.—The discharge for a *circular aperture A B*, Fig. 688, can only be determined by means of approximate formulas obtained in the following manner. Let us decompose the circular orifice by concentric circles into small rings of equal width and let us consider each ring to be composed of elements, which may be regarded as parallelograms. If r is the radius, b the width and n the number of elements of one of these rings, $\frac{2\,\pi\,r}{n}$ is the length of one of these elements and its area is

Fig. 688.



$$K = \frac{2\,\pi\,r\,b}{n}.$$

Now if h is the depth *C G* of the centre *C* below the level of the water *H R* and ϕ the angle *A C K*, which measures the distance of the element *K* from the highest point *A* of the ring, we have for the head of water of this element

$$K\,N = C\,G - C\,L = h - r\cos.\phi,$$

and therefore the discharge through this element

$$= \frac{2\,\pi\,r\,b}{n}\sqrt{2\,g\,(h - r\cos.\phi)}.$$

But

$$\sqrt{h - r\, cos.\, \phi}$$

$$= \sqrt{h}\left[1 - \tfrac{1}{2}\frac{r}{h}\, cos.\, \phi - \tfrac{1}{8}\left(\frac{r}{h}\right)^2 cos.^2\, \phi + \ldots\right]$$

$$= \sqrt{h}\left[1 - \tfrac{1}{2}\frac{r}{h}\, cos.\, \phi - \tfrac{1}{16}\left(\frac{r}{h}\right)^2 (1 + cos.\, 2\, \phi) + \ldots\right],$$

and therefore the discharge through this element is

$$= \frac{2\, \pi\, r\, b}{n}\sqrt{2\, g\, h}\left[1 - \tfrac{1}{2}\cdot\frac{r}{h}\, cos.\, \phi - \tfrac{1}{16}\left(\frac{r}{h}\right)^2 (1 + cos.\, 2\, \phi) + \ldots\right].$$

The discharge through the whole ring is found by substituting in the parenthesis instead of 1, $n \, . \, 1 = n$, and instead of *cos.* ϕ the sum of all the cosines of ϕ from $\phi = 0$ to $\phi = 2\, \pi$, and instead of *cos.* $2\, \phi$ the sum of all the cosines of $2\, \phi$ from $2\, \phi = 0$ to $2\, \phi = 4\, \pi$. Since the sum of all the cosines of a full circle is equal to 0, these cosines disappear, and we have the discharge through the ring

$$= \frac{2\, \pi\, r\, b}{n}\sqrt{2\, gh}\left[n - \tfrac{1}{16}\left(\frac{r}{h}\right)^2 . \, n - \ldots\right]$$

$$= 2\, \pi\, r\, b\, \sqrt{2\, g\, h}\left[1 - \tfrac{1}{16}\left(\frac{r}{h}\right)^2 - \ldots\right].$$

If, instead of b, we substitute $\frac{r}{m}$, and instead of r, $\frac{r}{m}$, $\frac{2\, r}{m}$, $\frac{3\, r}{m}$ to $\frac{m\, r}{m}$, we obtain the discharge through each of the rings, which form the entire circle, and finally the discharge through the *entire circular aperture* is

$$Q = 2\pi r\sqrt{2gh}\left(\frac{r}{m^2}(1+2+3+\ldots+m) - \tfrac{1}{16}\frac{r^3}{m^4\, h^2}(1^3+2^3+3^3+\ldots+m^3)\right)$$

$$= 2\, \pi\, r\sqrt{2\, g\, h}\, . \left(\frac{r}{m^2}\cdot\frac{m^2}{2} - \tfrac{1}{16}\cdot\frac{r^3}{m^4\, h^2}\cdot\frac{m^4}{4}\right)$$

$$= \pi\, r^2\sqrt{2\, g\, h}\left[1 - \tfrac{1}{32}\left(\frac{r}{h}\right)^2 - \ldots\right],$$

or more exactly

$$Q = \pi\, r^2\sqrt{2\, g\, h}\left[1 - \tfrac{1}{32}\left(\frac{r}{h}\right)^2 - \tfrac{5}{1024}\left(\frac{r}{h}\right)^4 - \ldots\right].$$

If the circle reaches to the level of the water, we have

$$Q = \tfrac{987}{1024}\, \pi\, r^2\sqrt{2\, g\, h} = 0{,}964\, F\sqrt{2\, g\, h},$$

when $F = \pi\, r^2$ denotes the area of the circle.

Moreover, it is easy to understand that in all cases, where the head of water at the centre is equal to or greater than the diameter of the orifice, we can put the value of the entire series $= 1$ and

$$Q = F\sqrt{2\, g\, h}.$$

This rule can also be applied to other orifices and also to all cases, where the depth of the centre of gravity of the orifice below the level of the water is as great as the height of the aperture; we can then regard the depth h of this point as the head of water and put $$Q = F\sqrt{2gh}.$$

If we consider that the mean of all the cosines of the first quadrant is $= \frac{2}{\pi}$ and that of all those of the second quadrant is $= -\frac{2}{\pi}$, or that the mean of the first and second quadrant $= 0$, the discharge for the *upper semicircle*, determined in the manner shown above, is

$$Q_1 = \frac{\pi r^2}{2}\sqrt{2gh}\left[1 - \frac{2}{3\pi}\left(\frac{r}{h}\right) - \frac{1}{32}\left(\frac{r}{h}\right)\right]$$
$$= F\sqrt{2gh}\left[1 - \frac{2}{3\pi}\left(\frac{r}{h}\right) - \frac{1}{32}\left(\frac{r}{h}\right)^2\right],$$

and that through the *lower semicircle* is

$$Q_2 = \frac{\pi r^2}{2}\sqrt{2gh}\left[1 + \frac{2}{3\pi}\left(\frac{r}{h}\right) - \frac{1}{32}\left(\frac{r}{h}\right)^2 + \dots\right]$$
$$= F\sqrt{2gh}\left[1 + \frac{2}{3\pi}\left(\frac{r}{h}\right) - \frac{1}{32}\left(\frac{r}{h}\right)^2 + \dots\right],$$

in which F denotes the area of the aperture.

The formulas for Q, Q_1 and Q_2 hold good also for elliptical orifices with horizontal axes; for the discharges, when the other circumstances are the same, are proportional to the widths of the apertures and the width of an ellipse is proportional to the width of an equally high circle (see Introduction to the Calculus, Art. 12).

Example.—What is the hourly discharge through a circular orifice 1 inch in diameter, when the level of the water is one line above the top of it? Here we have

$$\frac{r}{h} = \frac{6}{7}; \text{ hence } \left(\frac{r}{h}\right)^2 = \frac{36}{49} = 0,735,$$

and $$1 - \frac{1}{32}\left(\frac{r}{h}\right)^2 = 1 - 0,023 = 0,977,$$

and consequently the discharge per second is

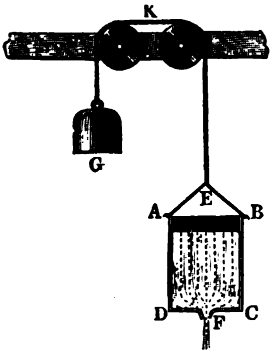
$$Q = \frac{\pi \,.\, 1^2}{4} \,.\, 12 \,.\, 8,025\sqrt{\frac{7}{144}} \,.\, 0,0977 = \frac{\pi}{4} \,.\, 8,025 \,.\, 0,977\sqrt{7} = 16,29 \text{ c. inches},$$

per minute $= 977,4$ cubic inches, and per hour $= 33,94$ cubic feet.

§ 404. Efflux from a Vessel in Motion.—The *velocity of efflux* changes when a vessel, originally at rest or moving uniformly, is set in motion, or when a change in its condition of motion takes place, since in this case every molecule of the water acts upon those surrounding it not only by its weight, but also by its inertia.

If the vessel $A\ C$, Fig. 689, is moved *with an accelerated motion vertically upwards*, while the water flows through an opening F in the bottom, the velocity of efflux is augmented, and if it descends with an accelerated motion, the velocity is diminished. If the acceleration is p, every molecule M of the water presses not only with its weight $M\ g$, but also with its inertia $M\ p$, and in the first case we must put the force of each molecule equal to $(g + p)\ M$, and in the second case equal to $(g - p)\ M$, or instead of g, $g \pm p$. Hence it follows that

FIG. 689.



$$\frac{v^2}{2} = (g \pm p)\, h,$$

and that the *velocity of efflux* is

$$v = \sqrt{2\,(g \pm p)\,h}.$$

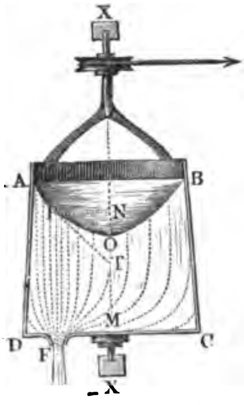
If the vessel rises with the velocity g, we have

$$v = \sqrt{2 \,.\, 2\,g\,h} = 2\,\sqrt{g\,h},$$

and the velocity of efflux is 1,414 times as great as it would be if the vessel stood still. If the vessel falls by its own weight or with the acceleration g, v is $= \sqrt{0} = 0$ and no water runs out. If the vessel moves uniformly upwards or downwards, v remains $= \sqrt{2\,g\,h}$, but if its rise is retarded, v becomes $= \sqrt{2\,(g - p)\,h}$, and if its fall is retarded, v is $= \sqrt{2\,(g + p)\,h}$.

If the vessel, from which the water flows, is moved horizontally or at an acute angle to the horizon, the surface (see § 354) becomes oblique to the horizon and a variation of the velocity of efflux is the result.

FIG. 690.



If a vessel $A\ C$, Fig. 690, is caused *to revolve about its vertical axis* $X\ \overline{X}$, its surface will assume, according to § 354, the shape of a parabolic funnel $A\ O\ B$, and at the centre M of the bottom the head of water $M\ O$ is smaller than near the edge, and the water will flow more slowly through an orifice at the centre than through any other equally large aperture in the bottom. If h denotes the head of water $M\ O$ at the centre M, the velocity of efflux through an aperture at that point will be $= \sqrt{2gh}$; but if y denotes the distance $MF = NP$ of an aperture F from the axis $X\bar{X}$ and ω the angular velocity, we have, since the subtangent TN of the arc OP of the parabola is equal to twice the abscissa ON, the corresponding elevation of the water above the centre O

$$ON = \tfrac{1}{2}\, TN = \tfrac{1}{2}\, PN \,.\, tang.\, NPT,$$

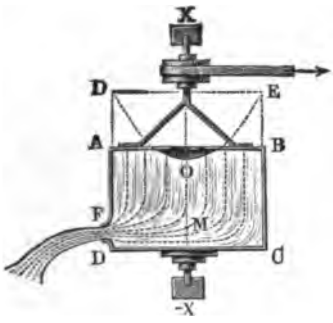
consequently if we substitute $tang.\, NPT = tang.\, \phi = \frac{\omega^2 y}{g}$ (see § 354) and denote the angular velocity ωh of F by w, we can put

$$ON = x = \tfrac{1}{2}\, y \,.\, \frac{\omega^2 y}{g} = \frac{\omega^2 y^2}{2g} = \frac{w^2}{2g}.$$

Hence the velocity of efflux through the orifice F is

$$v = \sqrt{2g\left(h + \frac{w^2}{2g}\right)} = \sqrt{2gh + w^2}.$$

FIG. 691.



This formula holds good for a vessel of any shape, even when it is *closed on top*, like $A\,C$, Fig. 691, in such a manner that the funnel $D\,O\,C$ cannot be completely formed. Here also h is the depth $M\,O$ of the orifice below the vertex O of the funnel and v the velocity of rotation of the aperture. It will be employed repeatedly in the discussion of reaction wheels and turbines in another part of the work.

EXAMPLE—1) If the vessel $A\,C$, Fig. 689, which when filled with water weighs 350 pounds, is drawn upwards by a weight G of 450 pounds by means of a cord passing over a pulley, it rises with an acceleration

$$p = \frac{450 - 350}{450 + 350} \,.\, g = \frac{100}{800} \,.\, g = \tfrac{1}{8}\, g,$$

and the velocity of efflux is

$$v = \sqrt{2\,(g + p)\,h} = \sqrt{2 \,.\, \tfrac{9}{8}\, g h} = \sqrt{\tfrac{9}{4}\, g h}.$$

Now if the head of water were $h = 4$ feet, the velocity of efflux would be

$$v = \sqrt{9 \,.\, g} = 3\,\sqrt{32{,}2} = 17{,}02 \text{ feet.}$$

2) If the vessel $A\,C$, Fig. 691, which is filled with water, makes 100 revolutions per minute and if the orifice F is 2 feet below the level of the water at the centre and at a distance from the axis $X\bar{X}_1 = 3$ feet, the velocity of efflux is

$$v = \sqrt{2gh + w^2} = \sqrt{64{,}4 \,.\, 2 + \left(\frac{3 \,.\, \pi \,.\, 100}{30}\right)^2} = \sqrt{128{,}8 + 100 \,.\, \pi^2}$$
$$= \sqrt{128{,}8 + 987} = \sqrt{1115{,}8} = 33{,}4 \text{ feet.}$$

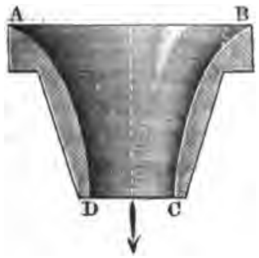
If the vessel stands still, we have $v = \sqrt{128{,}8} = 11{,}35$ feet.

CHAPTER II.

OF THE CONTRACTION OF THE VEIN OR JET OF WATER WHEN ISSUING FROM AN ORIFICE IN A THIN PLATE.

§ **405. Coefficient of Velocity.**—The laws of efflux, deduced in the last chapter, coincide almost exactly with the results obtained in practice, so long as the head of water is not very small, compared to the width of the aperture, if the orifice of efflux is gradually widened inwards and joins bottom or sides without forming an angle or edge. The experiments made with polished metal mouth-pieces by *Michelotti*, *Eytelwein* and others, and also by the author, have shown that the real effective discharge is from 96 to 99 per cent. of the theoretical one. The mouth-piece $A\ D$, Fig. 692, which is represented in one-half its natural size, gave under a pressure of 10 feet 98 per cent., under a pressure of 5 feet 97 per cent., and under a pressure of 1 foot 96 per cent. of the discharge calculated theoretically (Experiments with large orifices, see Untersuchungen in dem Gebiete der Mechanik und Hydraulik, Zweite Abtheil.). If the efflux through such a mouth-piece is to be as free from disturbance as possible, the rounding must not be in the form of a circle, but in that of a curve $A\ D = B\ C$, the curvature of which gradually decreases from within outwards (from A towards D). Since in this case the stream has the same cross-section F as the orifice, we can assume that the diminution of the discharge is caused by a loss of velocity arising from the friction of the water upon, or its adhesion to, the inner surface of the mouth-piece and from the viscosity of the water. Hereafter we will call the ratio of the real or effective velocity to the theoretical velocity $v = \sqrt{2gh}$ *the coefficient of velocity* (Fr. coefficient de vitesse; Ger. Geschwindigkeitscoefficient) and we will denote it by ϕ. Thus the effective velocity of efflux in the simplest case is

FIG. 692.



$$v_1 = \phi\, v = \phi \sqrt{2\, g\, h},$$

and the *effective discharge* is

$$Q = F v_1 = \phi F v = \phi F \sqrt{2 g h}.$$

Substituting for ϕ its mean value 0,975, we obtain (in English feet)

$$Q = 0{,}975 \,.\, F \sqrt{2 g h} = 0{,}975 \,.\, 8{,}025\ F \sqrt{h} = 7{,}824\ F \sqrt{h}.$$

The vis viva of a quantity Q of water, issuing with the velocity v_1, is $\frac{Q \gamma}{g} \,.\, v_1^2$, by virtue of which it can perform the mechanical effect $Q \gamma \,.\, \frac{v_1^2}{2 g}$. But since the weight $Q \gamma$ in descending from the height $h = \frac{v^2}{2 g}$ performs the work $Q \gamma \,.\, h = Q \gamma \frac{v^2}{2 g}$, it follows that the loss of mechanical effect of the water during the efflux is

$$L = Q \gamma \left(\frac{v^2}{2 g} - \frac{v_1^2}{2 g}\right) = (1 - \phi^2)\, Q \gamma \,.\, \frac{v^2}{2 g} = (1 - 0{,}975^2)\, Q \gamma \,.\, \frac{v^2}{2 g},$$

i.e.,

$$L = 0{,}049 \,.\, \frac{v^2}{2 g}, \text{ or } 4{,}9 \text{ per cent.}$$

The water, which issues from the vessel, will therefore perform 4,9 per cent. less work by virtue of its vis viva than by virtue of its weight, when falling from the height h.

REMARK.—The author has tested the law of efflux, expressed by the formula $v = \sqrt{2 g h}$, under very different heads, viz., from the very great head of 100 meters to the very small one of 0,02 meters. A well rounded mouth-piece 1 centimeter wide gave for the heads

h = 0,02 meters . . .	0,50 meters	3,5 meters	17 meters	103 meters
ϕ = 0,959	0,967	0,975	0,994	0,994

See Civilingenieur, New Series, Vol. 5, first and second numbers.

§ 406. Coefficient of Contraction.—If the water issues from an orifice in a thin plate (Fr. orifice en mince paroi; Ger. Mündung in der dünnen Wand), and if the other circumstances are the same, a considerable diminution in the discharge takes place. This diminution is due to the fact that the directions of the molecules of the water, which are passing through the orifice, converge and produce a contracted *stream or vein* (Fr. veine contractée; Ger. contrahirter Wasserstrahl). The measurements of the stream, made by several experimenters and more recently by the author himself, have shown that the stream, at a distance from the orifice equal to half its width, experiences its maximum contraction, and that its thickness is 0,8 of the diameter of the orifice. If F_1 is the cross-section of the contracted vein and F that of the orifice, we have therefore

$$F = 0{,}8^2\, F = 0{,}64\, F.$$

The ratio $\frac{F_1}{F}$ of these cross-sections is called the *coefficient of contraction* (Fr. coefficient de contraction; Ger. Contractionscoefficient), and is denoted by a; from what precedes we see that its mean value for the efflux of water through an orifice in a thin plate is $a = 0{,}64$.

So long as we have no more accurate knowledge of the law of the contraction of the stream, we can assume that the stream flowing through a circular orifice $A\,A$, Fig. 693, forms a solid of rotation $A\,E\,E\,A$, whose surface is generated by the revolution of the arc $A\,E$ of a circle about the axis $C\,D$ of the stream. Putting the diameter $A\,A$ of the orifice $= d$ and the distance $C\,D$ of the contracted section $E\,E$ from the orifice $= \frac{1}{2}\,d$, we obtain the radius

$$M\,A = M\,E = r$$

of the generating arc $A\,E$ by means of the equation

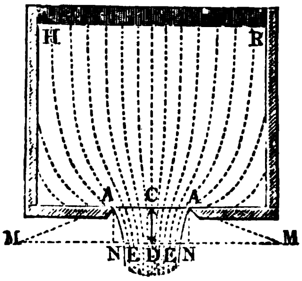
$$\overline{A\,N}^2 = E\,N\,(2\,M\,E - E\,N)$$

or $$\frac{d^2}{4} = \frac{d}{10}\left(2\,r - \frac{d}{10}\right),$$

from which we obtain

$$r = 1{,}3\,d.$$

FIG. 693.



The velocity of efflux through orifices of this kind is about

$$v_1 = 0{,}97\,v.$$

The contraction of the stream of water owes its origin to the fact that not only the water immediately above the orifice flows out, but also that the water all around flows in and is discharged with it. The filaments of water begin to converge within the vessel, as is shown in the figure, and the contraction of the stream is caused by the prolongation of this convergence. We can convince ourselves of this fact by employing a glass vessel and putting into the water small bodies, such as saw-dust, bits of sealing-wax, etc., of nearly the same specific gravity as the water, and allowing them to flow out with it.

§ **407. Contracted Vein of Water.**—If the water flows through triangular, quadrangular, etc., orifices in a thin plate, the stream assumes particular forms. The most striking phenomenon is the inversion of the stream or the change in position of its cross-section in reference to the cross-section of the orifice, in consequence of which a corner of the former cross-section comes into the same position as the middle of one of the sides of the orifice. Thus the cross-section of the stream, issuing from a triangular orifice *A B C*, Fig. 694, is, at a certain distance from the latter, a three-pointed star *D E F*; that from a square orifice *A B C D*, Fig. 695, is a four-pointed star *E F G H*; that from a pentagonal orifice *A B C D E*, Fig. 696, is a five-pointed star *E G H K L*, etc. The cross-sections are very different at different distances from the orifice; they decrease for a certain distance and then increase again, etc.; the stream consists, therefore, of ribs of variable width and forms, as can be best observed when the pressure is very great, bulges and nodes similar in form to the cactus plant. If the orifice *A B C D*, Fig. 697, is *rectangular*, the cross-section at a small distance from the aperture forms also a star or cross, but at a greater distance it assumes more the form of an inverted rectangle *E F*.

FIG. 694.

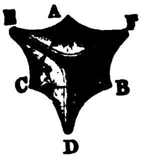


FIG. 695.

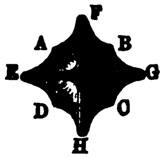


FIG. 696.

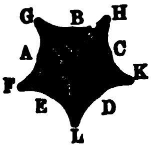
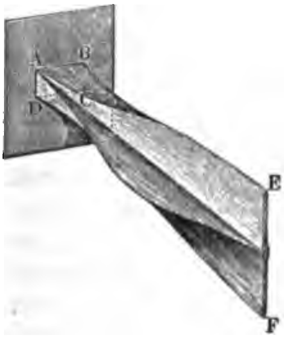


FIG. 697.



Bidone observed the discharge from various kinds of orifices; Poncelet and Lesbros have made the only accurate measurements of the stream issuing from square orifices (see the Allgemeine Maschinenencyklopädie, article "Ausfluss"). The last measurements have led to a small coefficient of contraction 0,563.

Measurements of the water discharged through smaller openings have given greater coefficients of contraction; they indicate that the coefficients are greater for oblong rectangles than for rectangles, which approach the square in form.

§ **408. Coefficient of Efflux.**—If the effective velocity of water issuing from an opening in a thin plate was equal to the theoretical $v = \sqrt{2 g h}$, we would have for the effective discharge

$$Q = a F v = a F \sqrt{2 g h},$$

$a F$ denoting the cross-section of the stream at the point of maximum contraction, where the molecules of water move in parallel lines; but this is by no means true. It appears, from experiment, that Q is smaller than $a F \sqrt{2 g h}$ and that we must multiply the theoretical discharge $F \sqrt{2 g h}$ by a coefficient smaller than the coefficient of contraction, in order to obtain the real discharge. We must therefore assume that, when water issues from an orifice in a thin plate, a certain loss of velocity takes place, and consequently a coefficient of velocity ϕ must also be introduced; hence the effective velocity of efflux is

$$v_1 = \phi v = \phi \sqrt{2 g h}.$$

The *effective discharge* is

$$Q_1 = F_1 . v_1 = a F . \phi v = a \phi F v = a \phi F \sqrt{2 g h}.$$

Let us call the ratio of the real discharge Q_1 to the theoretical or hypothetical discharge Q the *coefficient of efflux* (Fr. coefficient de dépense; Ger. Ausflusscoefficient) and let us denote it hereafter by μ; then we have

$$Q_1 = \mu Q = \mu F v = \mu F \sqrt{2 g h},$$

and therefore

$$\mu = a \phi,$$

I.E. *the coefficient of efflux is the product of the coefficient of velocity and the coefficient of contraction.*

Repeated observations, and particularly the measurements of the author, have led to the conclusion that the coefficient of efflux is not constant for all orifices in a thin plate, that it is greater for small orifices and small velocities of efflux than for large orifices and great velocities and that it is much greater for long, narrow orifices than for those whose forms are regular or circular.

For square orifices, whose areas are from 1 to 9 square inches, under a head of from 7 to 21 feet, according to the experiments of

Bossut and Michelotti, the mean coefficient of efflux is $\mu = 0,610$; for circular orifices from ½ to 6 inches in diameter and under a head of from 4 to 21 feet, it is $\mu = 0,615$ or about $\frac{8}{13}$. The values, which were obtained by Bossut and Michelotti from their observations, differ materially from each other; but they do not appear to depend upon the size of the orifice or upon the head. According to the experiments of the author, under a head of 0,6 meters, the coefficient of efflux is for a circular orifice

1 centimeter in diameter . . . . $\mu = 0,628$
2 centimeters " . . . . $= 0,621$
3 " " . . . . $= 0,614$
4 " " . . . . $= 0,607$.

On the contrary, under a head of 0,25 meters, with the same orifice,

1 centimeter in diameter, he found. . . . $\mu = 0,637$
2 centimeters " " . . . . $= 0,629$
3 " " " . . . . $= 0,622$
4 " " " . . . . $= 0,614$.

We see from these results of experiment that the coefficient of efflux increases when the size of the orifice and the head of water diminish. If we assume as mean values $\mu = 0,62$ and $a = 0,64$, we obtain the coefficient of velocity for efflux through an orifice in a thin plate

$$\phi = \frac{\mu}{a} = 0,97,$$

or about the same as for efflux through mouth-pieces rounded internally.

REMARK—1) Experiments made by Buff (see Poggendorff's Annalen, Vol. XLVI) show that the coefficients of velocity for small orifices and small heads or velocities are considerably greater than for large or medium orifices and velocities. An orifice of 2,084 lines in diameter gave, under a head of 1⅓ inches, $\mu = 0,692$ and, under a head of 35 inches, $\mu = 0,644$. On the contrary, an orifice 4,848 lines wide, under a head of 4⅓ inches, gave $\mu = 0,682$ and, under a head of 29 inches, $\mu = 0,658$. The author also obtained similar results.

2) For efflux under water, ~~according~~ to ~~the experiments of~~ the author, the coefficients of velocity are nearly 1⅓ per cent. smaller than for efflux into the air.

§ 409. Experiments.—The coefficient of efflux μ corresponding to a certain mouth-piece can be determined, when we know the discharge V, which passes through the known cross-section F of the orifice under a head of water h in a certain time t; here we have

$$V = \mu\, F\sqrt{2gh}\,.\,t,$$

and inversely
$$\mu = \frac{V}{F t\,.\,\sqrt{2gh}}.$$

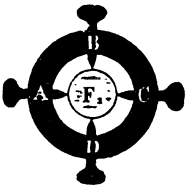
In order to find its two factors, viz.: the coefficient of contraction and that of velocity, it is necessary to measure either the cross-section $F_1 = a\,F$ of the stream or to determine the velocity of efflux $v_1 = \phi\, v = \phi\, \sqrt{2gh}$ by means of the range of the jet. Neither measurement can be made with sufficient accuracy unless the stream is thin and the cross-section is circular.

The circular cross-section F_1 of a jet can be determined very simply by means of the apparatus represented in Fig. 698. It is composed of a ring and four sharp-pointed set-screws A, B, C, D, which screw in towards each other. The screws are directed towards the centre of the cross-section of the stream and are turned until their points touch its surface; the ring is then removed from the stream and the distance between the opposite points of the screws is measured; the mean d_1 of these two distances is assumed to be the diameter of the stream. Now if d is the diameter of the cross-section of the orifice, we have

$$a = \frac{F_1}{F} = \left(\frac{d_1}{d}\right)^2,$$

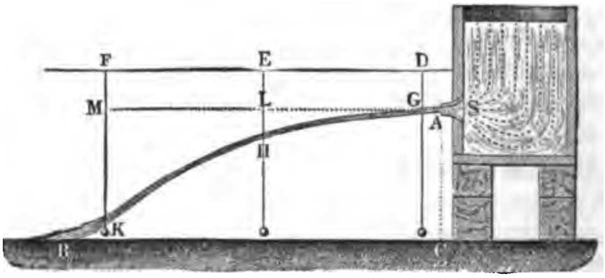
and therefore
$$\phi = \frac{\mu}{a}.$$

Fig. 698.



If we measure the range $B\,C = b$ of a jet $A\,B$, Fig. 699, which issues horizontally from the mouth-piece $S\,A$, which is at a certain height $A\,C = a$ above the ground, we have, according to § 36, the velocity of efflux

Fig. 699.



$$v_1 = \sqrt{\frac{g\,b^2}{2\,a}},$$

and since $v_1 = \phi\, v = \phi\, \sqrt{2\,g\,h}$, we obtain

$$\phi = \frac{v_1}{v} = \sqrt{\frac{b^2}{4\,a\,h}} = \frac{b}{2\sqrt{a\,h}},$$

whence

$$a = \frac{\mu}{\phi} = \frac{2\sqrt{a\,h}}{\mu\, b}.$$

The determination of v is more certain when, instead of a and b, we measure the horizontal and vertical co-ordinates of three points of the parabolic axis of the stream; for the axis of the mouth-piece may have an unknown inclination to the horizon. The most simple method of proceeding is to stretch a horizontal thread $D\,F$ above the stream and to hang three plumb-lines from three points D, E, and F, which are at equal distances from each other; we then measure the distances $D\,G$, $E\,H$, and $F\,K$ of the axis of the stream from $D\,F$. If $D\,F = x$ is the horizontal distance of the extreme points from each other, if the vertical distances $D\,G$, $E\,H$, and $F\,K = z$, z_1, and z_2, and if we take G as the origin of co-ordinates, we have the co-ordinates for the point H

$$x_1 = G\,L = D\,E = \tfrac{1}{2}\,D\,F = \frac{x}{2} \text{ and } y_1 = L\,H = E\,H - D\,G = z_1 - z,$$

and for the point K

$$x_2 = G\,M = D\,F = x \text{ and } y_2 = M\,K = F\,K - D\,G = z_2 - z.$$

According to § 39, if a denotes the angle of inclination of the axis of the stream at G,

$$y_1 = x_1\, tang.\, a + \frac{g\, x_1^2}{2\, v_1^2\, cos.^2\, a}, \text{ and also}$$

$$y_2 = x_2\, tang.\, a + \frac{g\, x_2^2}{2\, v_1^2\, cos.^2\, a}, \text{ or}$$

$$y_1 - x_1\, tang.\, a = \frac{g\, x_1^2}{2\, v_1^2\, cos.^2\, a}, \text{ and}$$

$$y_2 - x_2\, tang.\, a = \frac{g\, x_2^2}{2\, v_1^2\, cos.^2\, a};$$

whence, by division, we obtain, since $x_2 = 2\, x_1$,

$$\frac{y_1 - x_1\, tang.\, a}{y_2 - x_2\, tang.\, a} = \tfrac{1}{4}, \text{ and therefore } tang.\, a = \frac{4\, y_1 - y_2}{x}.$$

If in one of the foregoing formulas, instead of $\frac{1}{cos.^2\, a}$, we put $1 + tang.^2\, a$, and for $tang.\, a$ we substitute the last expression, we obtain the required formula for the velocity of efflux

$$v_1 = \sqrt{\frac{g\,x^2}{2\,(y_2 - x\,\mathit{tang.}\,a)\,\mathit{cos.}^2\,a}} = \sqrt{\frac{(1 + \mathit{tang.}\,a^2)\,g\,x^2}{2\,(2\,y_2 - 4\,y_1)}}$$

$$= \sqrt{\frac{g\,[x^2 + (4\,y_1 - y_2)^2]}{4\,(y_2 - 2\,y_1)}}.$$

Hence the coefficient of velocity is

$$\phi = \frac{v_1}{v} = \frac{v_1}{\sqrt{2\,g\,h}} = \sqrt{\frac{x^2 + (4\,y_1 - y_2)^2}{8\,h\,(y_2 - 2\,y_1)}}.$$

EXAMPLE 1) The following measurements of an uncontracted stream, which issued from a well-rounded orifice 1 centimeter wide, were made:

$x = 2{,}480$ meters,

$y_1 = z_1 - z = 0{,}267 - 0{,}1135 = 0{,}1535$ meters,

$y_2 = z_2 - z = 0{,}669 - 0{,}1135 = 0{,}5555$ "

and the head of water was $h = 3{,}359$ meters. From these data we find the coefficient of velocity to be

$$\phi = \sqrt{\frac{2{,}48^2 + 0{,}059^2}{8\,.\,3{,}359\,.\,0{,}2485}} = \sqrt{\frac{6{,}185}{26{,}872\,.\,0{,}2485}} = 0{,}963.$$

Since no contraction took place, $a = 1$ and therefore $\mu = \phi$. The results of measurements given in the remark to § 405 agree well with this value.

2) The measurements of a perfectly contracted stream, which passed through a circular orifice in a thin plate, were, for a head of water $h = 3{,}396$ meters, the following:

$x = 2{,}70$ meters,

$y_1 = z_1 - z = 0{,}2465 - 0{,}1115 = 0{,}1350$ meters,

$y_2 = z_2 - z = 0{,}6620 - 0{,}1115 = 0{,}5505$ "

whence it follows that

$$\phi = \sqrt{\frac{2{,}70^2 + 0{,}01^2}{8\,.\,3{,}396\,.\,0{,}2805}} = \sqrt{\frac{7{,}2901}{27{,}168\,.\,0{,}2805}} = 0{,}978.$$

From the measurement of the discharge μ was calculated to be $= 0{,}617$; hence the coefficient of contraction was $a = \frac{\mu}{\phi} = 0{,}631$, which agreed very well with the measurement of the cross-section of the stream.

§ 410. **Rectangular Lateral Orifices.**—The most accurate experiments upon efflux through large lateral rectangular orifices are those made at Metz by Poncelet and Lesbros. The width of these apertures were 2 and in some cases 6 decimeters and their heights were different, varying from 1 centimeter to 2 decimeters. In order to produce a perfect contraction, the orifice was made in a brass plate 4 millemeters thick. From the results of these experiments, these savants have calculated, by interpolation, the tables, which are given at the end of this paragraph, and which can be employed for the measurement or calculation of discharges.

If b is the width of the orifice $K\,L$, Fig. 700, and if h_1 and h_2

are the heights $E\,G$ and $E\,L$ of the level of the water above the lower and upper horizontal edge of the orifice, we have, according to § 401, the discharge

$$Q = \tfrac{2}{3}\, b \sqrt{2g}\, (h_1^{\frac{3}{2}} - h_2^{\frac{3}{2}}).$$

If we introduce the height of the orifice $G\,L = a = h_1 - h_2$ and the mean head of water $E\,M = h = \frac{h_1 + h_2}{2}$, we have approximatively

$$Q = \left(1 - \frac{a^2}{96\,h^2}\right) a\, b \sqrt{2\,g\,h},$$

and, therefore, the effective discharge is

$$Q_1 = \mu\, Q = \left(1 - \frac{a^2}{96\,h^2}\right) \mu\, a\, b \sqrt{2\,g\,h}.$$

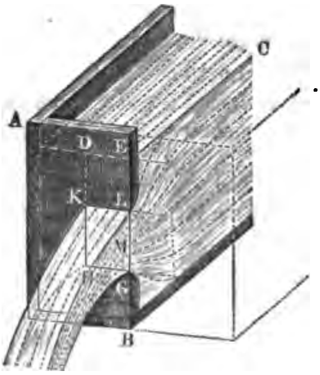


FIG. 700.

If we put

$$\left(1 - \frac{a^2}{96\,h^2}\right) \mu = \mu_1,$$

we have more simply

$$Q_1 = \mu_1\, a\, b \sqrt{2\,g\,h},$$

and as it is more convenient to employ this simple formula for the discharge, the values of μ_1, and not those of μ are given.

Since the water in the neighborhood of the orifice is in motion, it stands higher immediately in front of the wall, in which the aperture is made; for this reason two tables are given, one to be used, when the heads of water are measured at a distance from the orifice, and the other, when they are measured directly at the wall of orifice. We see from both these tables that, with some exceptions, the less the height of the orifice and the head of water is, the greater the coefficient of efflux is.

If the width of an orifice is different from those given, we must employ these coefficients to calculate the discharge, as we have no other experiments to base our calculations upon. That we are not liable to great error can be seen by comparing the coefficients for the orifices, whose widths are 0,6 meters, with those, whose widths are 0,2 meters, for the same head of water. If the apertures are not rectangular, we determine their mean height and width and substitute in the calculation the coefficient corresponding to these dimensions. It is always better to measure the head of water at a great distance from the orifice and to employ the first table than to measure it immediately at the orifice, where the surface of the water is curved and less tranquil than at a distance from it.

EXAMPLE—1) What is the discharge through an orifice 2 decimeters wide and 1 decimeter high, when the surface of the water is 1½ meters above the upper edge? Here we have

$$b = 0{,}2,\ a = 0{,}1,\ h = \frac{h_1 + h_2}{2} = \frac{1{,}6 + 1{,}5}{2} = 1{,}55 \text{ meters},$$

and, therefore, the theoretical discharge is

$$Q = 0{,}1 \,.\, 0{,}2\,\sqrt{2g}\,\sqrt{1{,}55} = 0{,}02 \,.\, 4{,}429 \,.\, 1{,}245 = 0{,}1103 \text{ cubic meters.}$$

But Table I gives for $a = 0{,}1$ and $h_2 = 1{,}5$, $\mu_1 = 0{,}611$, hence the real discharge is

$$Q = 0{,}611 \,.\, 0{,}1103 = 0{,}0674 \text{ cubic meters.}$$

2) What is the discharge through a rectangular orifice in a thin plate, whose height is 8 inches and whose width 2 inches, under a head of water of 15 inches above the upper edge? The theoretical discharge is

$$Q = \tfrac{2}{3} \,.\, \tfrac{1}{6} \,.\, 8{,}025\,\sqrt{\tfrac{4}{3}} = 0{,}8917 \,.\, 1{,}1547 = 1{,}0296 \text{ cubic feet.}$$

But two inches is about 0,05 meters and 15 inches about 0,4 meters, we can therefore take the value $\mu_1 = 0{,}628$, corresponding to $a = 0{,}05$ and $h_2 = 0{,}4$, and put the required discharge

$$Q_1 = 0{,}628 \,.\, 1{,}0296 = 0{,}647 \text{ cubic feet.}$$

3) If the width is 0,25 meter, the height 0,15 and the head of water $h_2 = 0{,}045$, we have

$$Q = 0{,}25 \,.\, 0{,}15 \,.\, 4{,}429\,\sqrt{0{,}12} = 0{,}166 \,.\, 0{,}3464 = 0{,}0575 \text{ cubic meters;}$$

the height 0,15 corresponds, for $h_2 = 0{,}04$, to the mean value

$$\mu_1 = \frac{0{,}582 + 0{,}603}{2} = 0{,}5925,$$

and, for $h_2 = 0{,}05$, to

$$\mu_1 = \frac{0{,}585 + 0{,}605}{2} = 0{,}595.$$

Now since $h_2 = 0{,}045$ is given, we substitute the new mean

$$\frac{0{,}5925 + 0{,}5950}{2} = 0{,}594$$

as coefficient of efflux, and we obtain the required discharge

$$Q_1 = 0{,}594 \,.\, 0{,}0575 = 0{,}03415 \text{ cubic meters.}$$

REMARK.—The coefficients of velocity do not change sensibly for a rectangular orifice, when we change the height into the width or vice versa, as is demonstrated by the following experiments of Lesbros (see his "Experiences Hydrauliques, Paris, 1851").

An orifice 0,60 meters wide and 0,02 meters high, under a head of water from $h = 0{,}30$ to 1,50 meters, gave

$$\mu_1 = \mu = 0{,}635 \text{ to } 0{,}622,$$

and, on the contrary, when it was set on edge, or when the height was 0,60 meters and the width 0,02 meters,

$$\mu_1 = 0{,}610 \text{ to } 0{,}626 \text{ and}$$
$$\mu = 0{,}638 \text{ to } 0{,}627.$$

TABLE I.

The coefficients of efflux of water issuing from rectangular orifices in a thin vertical plate, according to Poncelet and Lesbros.

(The heads of water are measured above the orifice at a point where the water can be considered as still. The values below the asterisk (*) are determined only by interpolation.)

	Head of water or distance of the level of the water above the upper edge of the orifice, in meters.	HEIGHT OF THE ORIFICE, IN METERS.							
		Width of the orifice = 0,2 meters.						Width of the orifice = 0,6 meters.	
		0,20	0,10	0,05	0,03	0,02	0,01	0,20	0,02
~~.0154~~	0,000		"	"	"	"	"	"	"
.0154	0,005	"	"	"	"	"	0,705	"	"
.0308	0,010	"	"	0,607	0,630	0,660	0,701	"	0,644
.0462	0,015	"	0,593	0,612	0,632	0,660	0,697	"	0,644
.0616	0,020	0,572	0,596	0,615	0,634	0,659	0,694	"	0,643
.0984	0,030	0,578	0,600	0,620	0,638	0,659	0,688	0,598	0,642
.1312	0,040	0,582	0,603	0,623	0,640	0,658	0,683	0,595	0,642
.1640	0,050	0,585	0,605	0,625	0,640	0,658	0,679	0,597	0,641
.1969	0,060	0,587	0,607	0,627	0,640	0,657	0,676	0,599	0,641
.2297	0,070	0,588	0,609	0,628	0,639	0,656	0.673	0,600	0,640
.2625	0,080	0,589	0,610	0,629	0.638	0,656	0,670	0,601	0.640
.2953	0,090	0,591	0,610	0,629	0,637	0,655	0.668	0,601	0,639
.3281	0,100	0,592	0,611	0,630	0.637	0,654	0.666	0,602	0,639
.3937	0,120	0,593	0,612	0,630	0,636	0,653	0,663	0.603	0,638
.4593	0,140	0,595	0,613	0,630	0,635	0,651	0,660	0,603	0,637
.5249	0,160	0,596	0,614	0,631	0,634	0,650	0,658	0,604	0,637
.5906	0,180	0,597	0,615	0,630	0,634	0,649	0,657	0,605	0,636
.6562	0,200	0.598	0,615	0,630	0,633	0,648	0,655	0.605	0,635
.8202	0,250	0,599	0,616	0,630	0,632	0.646	0,653	0,606	0.634
.9843	0.300	0,600	0,616	0,629	0,632	0,644	0,650	0 607	0,633
1.3124	0,400	0,602	0,617	0,628	0,631	0,642	0,647	0,607	0,631
1.6405	0,500	0,603	0,617	0,628	0,630	0,640	0,644	0,607	0,630
1.9685	0,600	0,604	0,617	0,627	0,630	0,638	0,642	0,607	0,629
2.2966	0,700	0,604	0,616	0,627	0,629	0,637	0,640	0,607	0.628
2.6247	0,800	0,605	0,616	0,627	0,629	0,636	0,637	0,606	0,628
2.9528	0,900	0,605	0,615	0,626	0,628	0,634	0,635	0,606	0,627
3.2809	1,000	0,605	0,615	0,626	0,628	0,633	0,632	0 605	0.626
3.6089	1,100	0,604	0,614	0,625	0,627	0,631	0,629	0.604	0.626
3.9370	1,200	0,604	0,614	0,624	0,626	0,628	0,626	0,604	0,625
4.2651	1,300	0,603	0,613	0,622	0,624	0,625	0,622	0.603	0,624
4.5933	1,400	0,603	0,612	0,621	0,622	0,622	0,618	0,603	0,624
4.9213	1,500	0,602	0,611	0,620	0,620*	0,619*	0,615*	0.602	0,623
5.2494	1,600	0,602	0,611	0,618	0,618	0,617	0,613	0,602*	0,623
5.5775	1,700	0,602*	0,610*	0,617	0,616	0,615	0,612	0,602	0,622
5.9056	1,800	0,601	0,609	0,615*	0,615	0,614	0,612	0,602	0,621*
6.2336	1,900	0,601	0,608	0,614	0,613	0,612	0,611	0,602	0,621
6.5617	2,000	0,601	0,607	0,613	0,612	0,612	0.611	0,602	0,620
9.8427	3,000	0,601	0,603	0,606	0,608	0,610	0,609	0,601	0,615

Similar tables for the Prussian system of measures are to be found in the Ingenieur, page 482.

TABLE II.

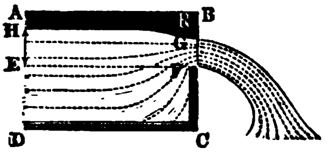
The coefficients of efflux of water issuing from rectangular orifices in a thin vertical plate, according to Poncelet and Lesbros.

(The heads of water were measured directly at the orifice. The values above and below the asterisks (*) are determined by interpolation only.)

Head of water or distance of the surface of the water above the upper edge of the orifice, in meters.	HEIGHT OF THE ORIFICE, IN METERS.						
	Width of the orifice = 0,2 meters.						Width of the orifice = 0,6 meters.
	0,20	0,10	0,05	0,03	0,02	0,01	0,20
0,000	0,619	0,667	0,713	0,776	0,788	0,795	0,586
0,005	0,597	0,630*	0,668*	0,725*	0,750*	0,778*	0,587
0,010	0,595	0,618	0,642	0,687	0,720	0,762	0,589
0,015	0,594	0,615	0,639	0,674	0,707	0,745	0,590
0,020	0,594*	0,614	0,638	0,668	0,697	0.729	0,591
0,030	0,593	0,613	0,637	0,659	0,685	0,708	0,592
0,040	0,593	0,612	0,636	0,654	0,678	0,695	0,594*
0,050	0,593	0,612	0,636	0,651	0,672	0,686	0,595
0,060	0,594	0,613	0,635	0,647	0,668	0,681	0,596
0,070	0,594	0,613	0,635	0,645	0,665	0,677	0,597
0,080	0,594	0,613	0,635	0,643	0,662	0,675	0,598
0,090	0,595	0,614	0,634	0,641	0,659	0,672	0,599
0,100	0,595	0,614	0,634	0,640	0,657	0,669	0,600
0,120	0,596	0,614	0,633	0,637	0,655	0,665	0,601
0,140	0,597	0,614	0,632	0,636	0,653	0,661	0,602
0,160	0,597	0,615	0,631	0,635	0,651	0,659	0,602
0,180	0,598	0,615	0,631	0,634	0,650	0,657	0,603
0,200	0,599	0,615	0,630	0,633	0,649	0,656	0,603
0,250	0,600	0,616	0,630	0,632	0,646	0,653	0,604
0,300	0,601	0,616	0,629	0,632	0,644	0,651	0,605
0,400	0,602	0,617	0,629	0,631	0,642	0,647	0,606
0,500	0,603	0,617	0,628	0,630	0,640	0,645	0,607
0,600	0,604	0,617	0,627	0,630	0,638	0,643	0,607
0,700	0,604	0,616	0,627	0,629	0,637	0,640	0,607
0,800	0,605	0,616	0,627	0,629	0,636	0,637	0,607
0,900	0,605	6,615	0,626	0,628	0,634	0,635	0,607
1,000	0,605	0,615	0,626	0,628	0,633	0,632	0,606
1,100	0,604	0,614	0,625	0,627	0,631	0,629	0,606
1,200	0,604	0,614	0,624	0,626	0,628	0,626	0,605
1,300	0,603	0,613	0,622	0,624	0,625	0,622	0,604
1,400	0,603	0,612	0,621	0,622	0,622	0,618	0,603
1,500	0,602	0,611	0,620	0,620*	0,619*	0,615*	0,603
1,600	0,602	0,611	0,618	0,618	0,617	0,613	0,602
1,700	0,602*	0,610*	0,617	0,616	0,615	0,612	0,602
1,800	0,601	0,609	0,615*	0,615	0,614	0,612	0,602
1,900	0,601	0,608	0,614	0,613	0,613	0,611	0,602
2,000	0,601	0,607	0,614	0,612	0,612	0,611	0,602
3,000	0,601	0,603	0,606	0,608	0,610	0,609	0,601

§ **411. Overfalls.**—If the water flows through an overfall, weir or notch (Fr. déversoirs; Ger. Ueberfälle) in a thin wall, as, E.G., *F B*, Fig. 701, the stream is contracted on three sides and a diminution of the discharge is produced. The discharge through this orifice is

FIG. 701.



$$Q = \tfrac{2}{3}\,\mu\, b\, h \sqrt{2 g h}.$$

Here the head of water $E H = h$ is to be measured, not at the edge, but at least three feet from the wall in which the notch is cut; for the surface of the water is depressed immediately behind the orifice, and the depression increases continually towards the orifice, and in the plane of the orifice its value $G R$ is from 0,1 to 0,25 of the head of water $F R$, so that the thickness $F G$ of the stream is but 0,9 to 0,75 of the head of water.

Many experiments have been made upon efflux of water *through notches in a thin plate*, and the results, although very multifarious, do not agree as well as could be desired. The following tables contain the results of the experiments of Poncelet and Lesbros.

1. TABLE OF COEFFICIENTS OF EFFLUX FOR OVERFALLS TWO DECIMETERS WIDE, ACCORDING TO PONCELET AND LESBROS.

Head of water h in meters.	0,01	0,02	0,03	0,04	0,06	0,08	0,10	0,15	0,20	0,22
Coefficient of efflux $\mu_1 = \frac{2}{3}\mu$.	0,424	0,417	0,412	0,407	0,401	0,397	0,395	0,393	0,390	0,385

2. TABLE OF THE COEFFICIENTS OF EFFLUX FOR OVERFALLS SIX DECIMETERS WIDE.

Head of water h in meters.	0,06	0,08	0,10	0,12	0,15	0,20	0,30	0,40	0,50	0,60
Coefficient of efflux $\mu_1 = \frac{2}{3}\mu$.	0,412	0,409	0,406	0,403	0,400	0,395	0,391	0,391	0,391	0,390

Hence for approximate determinations we can put $\mu_1 = 0,4$.

Eytelwein found, by his experiments with overfalls of great width, the mean value of μ_1 to be $= \frac{2}{3}\,\mu = 0{,}42$, and Bidone $\mu_1 = \frac{2}{3}\,0{,}62 = 0{,}41$, etc. The most extensive experiments were made by d'Aubuisson and Castel. From these d'Aubuisson concludes that for overfalls, whose width is not greater than $\frac{1}{3}$ that of the canal or of the wall in which the weir is placed, we can put $\mu = 0{,}60$ or $\frac{2}{3}\,\mu \doteq 0{,}40$; that, on the contrary, when the overfall extends across the whole wall or has the same width as the canal, we must take $\mu = 0{,}665$ or $\mu_1 = 0{,}444$; that, finally, when the relations between the width of the notch and that of the canal differ from the above, the coefficient of efflux is very varied, the extremes being 0,58 and 0,66. The experiments made in 1853 and 1854, at Hanswyk, upon overfalls 3 to 6 meters wide under a head of 0,1 to 1,0 meters gave $\mu = 0{,}64$ to $0{,}65$ or $\frac{2}{3}\,\mu = 0{,}427$ to $0{,}433$ (see the "Zeitschrift des Archit- und Ingen-Vereins für Hanover, 1857"). The researches made by the author upon the efflux of water through overfalls refer the variation of these coefficients of efflux to certain laws, which will be noticed further on (§ 417).

EXAMPLE—1) The discharge per second of an overfall, 0,25 meters wide under a head of water of 0,15 meters is

$$Q = 0393\,.\,b\,h\,\sqrt{2\,g\,h} = 0{,}393\,.\,4{,}429\,.\,0{,}25\,(0{,}15)^{\frac{3}{2}} = 0{,}435\,.\,0{,}0581$$
$$= 0{,}02527 \text{ cubic meters.}$$

2) What must be the width of an overfall, which under a head of water of 8 inches will discharge 6 cubic feet of water? Here we have

$$b = \frac{Q}{\mu_1\sqrt{2\,g\,h^3}} = \frac{6}{0{,}4\,.\,8{,}025\,\sqrt{(\frac{2}{3})^3}} = \frac{6}{3{,}210\,.\,0{,}5443} = 3{,}434 \text{ feet.}$$

If according to Eytelwein we take $\mu_1 = 0{,}42$, we have

$$b = \frac{6}{3{,}37\,.\,0{,}5443} = 3{,}271.$$

§ **412. Maximum and Minimum Contraction.**—When water flows through an orifice in a *plane surface*, the axis of the orifice is at right angles to the wall of the vessel and we have a medium contraction; if, however, the axis of the orifice or of the stream forms an acute angle with the portion of the wall of the vessel containing the aperture, the contraction is smaller, and if the angle between this axis and the inner surface of the vessel is obtuse, the contraction is greater. The first case is represented in Fig. 702 and the second in Fig. 703. This difference of contraction is, of course, due to the fact that in the former case the molecules of the water, which are flowing towards the orifices, are deviated less, and in the latter case more, from their primitive direction, while passing through this aperture and forming the vein.

The contraction is a *minimum*, I.E., null, if, by gradually contracting the wall surrounding the orifice, the water is prevented from flowing in upon the side and, on the contrary, a maximum when the direction of the wall is opposite to that of the stream, so that certain molecules must describe an angle of 180 degrees in order to reach the orifice. Both cases are represented in Figures 704 and 705. In the first case the coefficient of efflux is nearly 1, viz.: 0,96 to 0,98, and in the second case, according to the measurements of Borda, Bidone and of the author, its mean value is = 0,53.

FIG. 702.

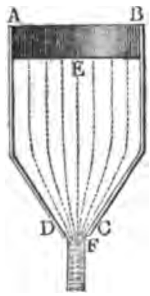


FIG. 703.

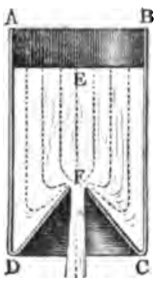


FIG. 704.

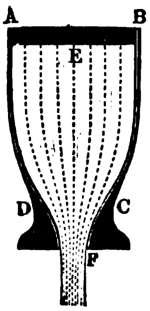
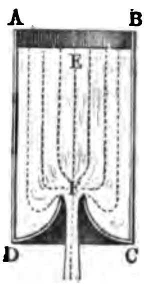


FIG. 705.



In practice, variations of the coefficients of efflux, produced by convergent walls, often occur, particularly in the case of sluices, which are inclined to the horizon, as is shown in Fig. 706. Poncelet found for such an orifice the coefficient of efflux $\mu = 0,80$, when the gate was inclined at an angle of 45°, and, on the contrary, μ is only $= 0,74$, when the inclination is $63\frac{1}{2}$ degrees, I.E., for a slope of one-half to one. For the overfall, represented in Fig. 707, where, as in Poncelet's sluice, contraction takes place upon one side only, the author found $\mu = 0,70$ or $\mu_1 = \frac{2}{3}\mu = 0,467$ for an inclination of 45°, and $\mu = 0,67$ or $\mu_1 = 0,447$ for an inclination of $63\frac{1}{2}$ degrees.

FIG. 706.

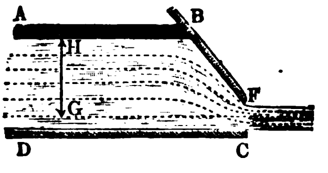
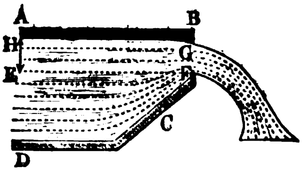


FIG. 707.



According to M. Boileau (see his Traité de la mesure des eaux courantes) we can put for an overfall, which is inclined upwards in such a way that the horizontal projection is $\frac{1}{3}$ the vertical, or that the angle of inclination is $71\frac{1}{2}$ degrees, the coefficient of efflux $= 0,973$ times the coefficient of efflux for an overfall with a vertical wall. We also find from the experiments of Boileau that, for vertical overfalls placed at an angle to the direction of the stream, we must put, when the angle is 45°, the coefficient of efflux $= 0,942$ and, when the angle is 65°, only 0,911 times the coefficient of efflux for the normal overfall; the whole length of the edge, over which the water flows, being of course considered as the length of the orifice.

EXAMPLE.—If a sluice gate, which is inclined at an angle of 50 degrees and closes a trough $2\frac{1}{4}$ feet wide, is raised $\frac{1}{2}$ foot and if the surface of the water then stands permanently 4 feet above the bottom of the trough, the height of the orifice is

$$a = \frac{1}{2} \sin. 50^\circ = 0,3830,$$

the head of water is

$$h = 4 - \frac{1}{2} \,.\, 0,3830 = 3,8085 \text{ feet},$$

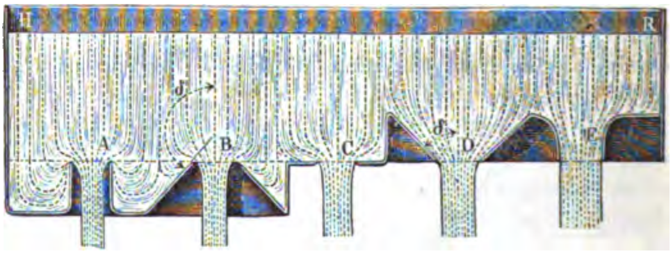
and the coefficient of velocity is $\mu = 0,78$, hence the discharge is

$$Q = 0,78 \,.\, 2,25 \,.\, 0,3830 \,.\, 8,025 \sqrt{3,8085} = 10,52 \text{ cubic feet.}$$

§ 413. Scale of Contraction.—*The more the direction of the water which flows in from the sides differs from that of the stream, the greater is the contraction of the vein.*

When a stream flows through the orifice *C*, Fig. 708, in a plane thin plate, the angle δ, formed by its axis or direction of motion

FIG. 708.



with that of the molecules of water which flow in from the side, is a right angle $\left(\frac{\pi}{2}\right)$; when the orifice *A* is formed by the thin edge of a tube, this angle δ is two right angles (π); when we have a conical divergent mouth-piece *B*, δ is between $\frac{1}{2}\,\pi$ and π; when the discharge takes place through a conical convergent mouth-piece, δ is between 0 and $\frac{\pi}{2}$; and when a cylindrical mouth-piece E well rounded off internally is used, it is $= 0$.

In order to discover the law, according to which the contraction diminishes with the angle δ, the author made a series of experiments with a great number of mouth-pieces 2 centimeters wide and under different pressures (from 1 to 10 feet); the results of these experiments are given in the following table:

δ	180°	157½°	135°	112½°	90°	67½°	45°	22½°	11¼°	5⅝°	0°
μ	0,541	0,546	0,577	0,606	0,682	0,684	0,753	0,882	0,924	0,949	0,966

This table gives, it is true, only the coefficients of efflux μ corresponding to different angles of deviation δ; the coefficients of contraction are from 1 to 2 per cent. greater, since a small loss of velocity always takes place during the efflux. In order to prevent any loss of vis viva, when the water enters the mouth-pieces D and E, the latter are rounded off at the entrance. The friction, to be overcome by the water in passing along the walls of the mouth-piece, will be determined in the following chapter.

REMARK.—According to the calculations of Prof. Zeuner (see Civilingenieur, Vol. 2d, page 55) of the results of the above experiments, we can put

$$\mu_\delta = \mu_{\frac{1}{2}\pi} (1 + 0,83214\, (\cos.\ \delta)^2 + 0,16672\, (\cos.\ \delta)^4)$$

$\mu_{\frac{1}{2}\pi}$ denoting the coefficient of efflux for an orifice in a plane thin plate, for which the maximum deviation of the elements of the water during efflux is $= \frac{1}{2}\pi = 90°$, and μ_δ, on the contrary, denoting the coefficient of efflux for an orifice in a conical thin plate, where the maximum deviation of the elements of the water upon entering is δ.

§ 414. Partial or Incomplete Contraction.—We have as yet studied only the case, where the water flows in from all sides of the opening and forms a stream contracted upon all sides; we must now consider the case, where the water flows in from but one or more sides to the orifice, and consequently produces a stream which is incompletely contracted. In order to distinguish these conditions of contraction from each other, we will call the case, where the stream is contracted on all sides, *complete contraction*, and the case, where the stream is contracted upon a part only of its periphery, *partial or incomplete contraction* (Fr. contraction incomplète; Ger. unvollständige or partielle Contraction). Incomplete contraction occurs whenever an orifice in a thin plane plate is surrounded upon one or more sides by a plate placed in the direction of the stream. In Fig. 709 there are represented four orifices *a*, *b*, *c*, *d* of equal magnitude in the bottom *A C* of a vessel. The contraction of the water flowing through the orifice *a* in the middle of the bottom is complete, for in this case the water can flow in from all sides; the contraction of the stream in passing through *b*, *c* or *d* is incomplete, for the water in these cases can flow in from only three, two or one side. In like manner, when a rectangular lateral orifice extends to the bottom of the vessel, the contraction is incomplete; for that upon the side of the base is wanting; if further the opening extend to the bottom and sides of the trough, there will be contraction upon one side only.

Incomplete contraction manifests itself in two ways. First, it gives an inclined direction to the stream; and secondly, it causes a greater discharge.

FIG. 709.

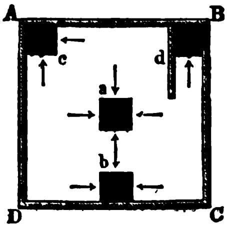
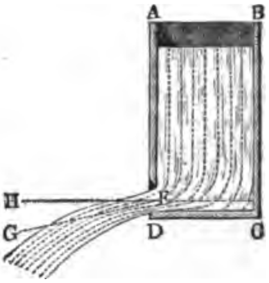


FIG. 710.



If, E.G., the lateral orifice *F*, Fig. 710, reaches to the bottom *C D*, so that no contraction can take place there, the axis *F G* of the stream will form an angle *H F G* of about 9 degrees with the normal *F H* to the plane of the orifices. This deviation of the stream becomes much greater when two adjoining sides are confined. If the orifice has a border upon two opposite sides, the contraction at those points is thereby prevented, and this deviation of the stream does not take place, but at a certain distance from the orifice the stream becomes wider than it would have done, if it had not been confined upon those sides. Although a greater discharge is obtained when the contraction is incomplete, yet it is generally to be avoided, since it is always accompanied by a deviation in the direction and by a great increase in the width of the stream. Experiments upon the efflux of water, when the contraction is incomplete, have been made by Bidone and by the author.

Their results show that the coefficient of efflux increases very nearly with the ratio of the length of the border to the entire periphery of the orifice; but it is easy to perceive that this relation is different, when the periphery is nearly or entirely surrounded by a border, in which case the contraction is almost or totally done away with. If we put the ratio of the portion with a rim to the entire periphery $= n$ and denote by κ an empirical quantity, we can put, approximatively, the ratio of the coefficient μ_n of efflux for incomplete contraction to the coefficient μ_0 for complete contraction

$$\frac{\mu_n}{\mu_0} = 1 + \kappa\, n, \text{ and consequently } \mu_n = (1 + \kappa\, n)\, \mu_0.$$

Bidone's experiments gave for small circular orifices $\kappa = 0{,}128$, and for square ones $\kappa = 0{,}152$; those of the author gave for small rectangular orifices $\kappa = 0.134$, and for larger ones (Poncelet's mouthpieces) 0,2 meter wide and 0,1 meter high $\kappa = 0{,}157$ (see the Magazine "der Ingenieur," vol. 2d). In practice rectangular orifices with rims are almost the only ones employed; we will assume for them, as a mean value, $\kappa = 0{,}155$, and consequently put

$$\mu_n = (1 + 0{,}155\, n)\, \mu_0.$$

For a rectangular lateral orifice, whose height is a and whose width is b, we have $n = \frac{b}{2\,(a + b)}$, when there is no contraction upon the side b, if, E.G., this side is upon the bottom; $n = \frac{1}{2}$, when one side a and one side b are provided with rims; and $n = \frac{2\,a + b}{2\,(a + b)}$, when the contraction is prevented upon the side b and upon the two sides a, the latter case occurs, when the orifice occupies the entire width of the reservoir and extends to the bottom.

EXAMPLE.—What is the discharge through a vertical sluice 3 feet wide and 10 inches high, when the head of water is 1½ feet above the upper edge of the orifice and the lower edge is at the bottom of the trough, so that there is no contraction upon that side? The theoretical discharge is

$$Q = \tfrac{10}{12}\,.\,3\,.\,8{,}025\,\sqrt{1{,}5 + \tfrac{5}{12}} = \tfrac{5}{2}\,.\,8{,}025\,\sqrt{1{,}9166} = 27{,}77 \text{ cubic feet.}$$

According to Poncelet's table for perfect contraction $\mu = 0{,}604$, but we have

$$n = \frac{3}{2\,(3 + \frac{10}{12})} = \frac{9}{18 + 5} = \tfrac{9}{23},$$

hence for the present case of incomplete contraction

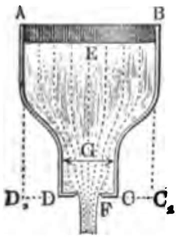
$$\mu_n = (1 + 0{,}155\,.\,\tfrac{9}{23})\,.\,0{,}604 = 1{,}060\,.\,0{,}604 = 0{,}640$$

and the effective discharge is

$$Q = 0{,}640\ Q = 0{,}640\,.\,27{,}77 = 17{,}77 \text{ cubic feet}$$

§ **415. Imperfect Contraction.**—The contraction of the vein depends also upon this fact, viz.: whether *the water is sensibly at rest in front of the orifice or whether it arrives there with a certain velocity;* the faster the water approaches the orifice of efflux, the less the stream is contracted, and consequently the greater is the discharge. The various relations of contraction and efflux, given and discussed in what precedes, are applicable only where the orifice is in a large wall, in which case we can assume that the water arrives at the orifice with a very small velocity; we must now investigate the relations of contraction and efflux, when the cross-section of the orifice is not much smaller than that of the approaching water, in which case the water arrives with a velocity, which is not negligable. In order to distinguish these two cases from each other, let us call the contraction which occurs, when the water above the orifice is at rest, *perfect contraction* and that which occurs, when the water is in motion, *imperfect contraction* (Fr. contraction imparfaite; Ger. unvollkommene Contraction). The contraction during efflux from the vessel $A\ C$, Fig. 711, is imperfect; for the cross-section F of the orifice is not much smaller than that G of the water approaching it or the area of the wall $C\ D$, in which this orifice is placed. If the vessel was of the form $A\ B\ C_1\ D_1$ and the area of the base $C_1\ D_1$ was much greater than that of the orifice F, the efflux would take place with perfect contraction. The imperfectly contracted stream is distinguished from the perfectly contracted one not only by its size, but also by the fact that it is not so transparent and crystalline as the latter is.

FIG. 711.



If we denote the ratio of the area F of the orifice to that G of the wall in which it is situated, or $\frac{F}{G}$, by n, the coefficient of efflux for perfect contraction by μ_0 and that for imperfect contraction by μ_n, we can put with great accuracy, according to the experiments and calculations of the author,

1) for circular orifices

$$\mu_n = \mu_0 [1 + 0{,}04564 (14{,}821^n - 1)],$$

2) and for rectangular orifices

$$\mu_n = \mu_0 [1 + 0{,}0760 (9^n - 1)].*$$

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* Experiments upon the imperfect contraction of water, etc., Leipzig, 1843.

In order to facilitate the calculations which are required in practice, the corrections $\frac{\mu_n - \mu_0}{\mu_0}$ of the coefficient of efflux in consequence of the imperfect contraction have been arranged in the following tables:

TABLE I.

The corrections of the coefficients of efflux for circular orifices.

n	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50
$\frac{\mu_n - \mu_0}{\mu_0}$	0,007	0,014	0,023	0,034	0,045	0,059	0,075	0,092	0,112	0,134
n	0,55	0,60	0,65	0,70	0,75	0,80	0,85	0,90	0,95	1,00
$\frac{\mu_n - \mu_0}{\mu_0}$	0,161	0,189	0,223	0,260	0,308	0,351	0,408	0,471	0,546	0,631

TABLE II.

The corrections of the coefficients of efflux for rectangular orifices.

n	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50
$\frac{\mu_n - \mu_0}{\mu_0}$	0,009	0,019	0,030	0,042	0,056	0,071	0,088	0,107	0,128	0,152
n	0,55	0,60	0,65	0,70	0,75	0,80	0,85	0,90	0,95	1,00
$\frac{\mu_n - \mu_0}{\mu_0}$	0,178	0,208	0,241	0,278	0,319	0,365	0,416	0,478	0,537	0,608

The upper lines in these tables contain various values of the ratio $\frac{F}{G}$ of the cross-sections, and immediately below are the corresponding additions to be made to the coefficient of efflux on account of the imperfect contraction, E.G., for the ratio $n = 0,35$, I.E., for the case, where the area of the orifice is 35 hundredths of the area of the entire wall, in which the orifice is made, we have for a circular orifice

$$\frac{\mu_n - \mu_0}{\mu_0} = 0,075,$$

and for a rectangular one = 0,088; the coefficient of efflux for perfect contraction must be increased in the first case 75 thousandths and in the second 88 thousandths, when we wish to obtain the corresponding coefficient of efflux for imperfect contraction. If the coefficient of efflux were = 0,615, we would have in the first case

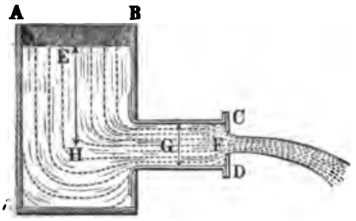
$$\mu_{0\,25} = 1,075 \,.\, 0,615 = 0,661$$

and in the second case

$$\mu_{0,25} = 1,088 \,.\, 0,615 = 0,669.$$

EXAMPLE.—What is the discharge through a rectangular lateral orifice F, which is $1\frac{1}{4}$ feet wide and $\frac{1}{2}$ foot high, when it is cut in a rectangular wall $C\,D$, Fig. 712, 2 feet wide and 1 foot high, and when the head of water $E\,H = h$, where the water is at rest, is 2 feet. The theoretical discharge is

FIG. 712.



$$Q = 1,25 \,.\, 0,5 \,.\, 8.025\, \sqrt{2}$$
$$= 5,0156 \,.\, 1,414 = 7,092 \text{ cubic feet,}$$

and the coefficient of efflux for perfect contraction is, according to Poncelet,

$$\mu_0 = 0,610,$$

but the ratio of the cross-sections is

$$n = \frac{F}{G} = \frac{1,25 \,.\, 0,5}{2 \,.\, 1} = 0,312,$$

and for $n = 0,312$ we have, according to Table II, page 841,

$$\frac{\mu_n - \mu_0}{\mu_0} = 0,071 + \tfrac{12}{50}\,(0,088 - 0,071) = 0,071 + 0,004 = 0,075;$$

hence the coefficient of efflux for the present case is

$$\mu_{0,312} = 1.075 \,.\, \mu_0 = 1,075 + 0,610 = 0,6557,$$

and the effective discharge is

$$Q_1 = 0,6557 \,.\, Q = 0,6557 \,.\, 7,092 = 4,65 \text{ cubic feet.}$$

§ **416. Efflux of Moving Water.**—We have heretofore assumed that the *head of water* was *measured in still water;* we must now discuss the case where the head of water can be measured only in water, which is approaching the orifice with a certain velocity. If we assume the orifice to be rectangular and denote the width by b, the head of water in reference to the two horizontal edges by h_1 and h_2 and the height due to the velocity of approach c of the water by k, we have the theoretical discharge

$$Q = \tfrac{2}{3}\, b \sqrt{2g}\, [(h_1 + k)^{\frac{3}{2}} - (h_2 + k)^{\frac{3}{2}}].$$

This formula cannot be directly employed for the determination of the discharge, since the height due to the velocity

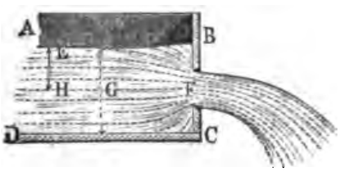
$$k = \frac{c^2}{2g} = \frac{1}{2g}\left(\frac{Q}{G}\right)^2$$

depends also upon Q, and, if we transform it, we obtain a complicated equation of a high degree; it is much simpler, therefore, to put the effective discharge

$$Q_1 = \mu_1\, a\, b \sqrt{2 g h}$$

and to understand by μ_1 not a simple coefficient of efflux, but a coefficient depending principally upon the ratio of the cross-sections. This case is often met with in practice, E.G., when we wish to measure the quantity of water which passes through a ditch or canal; for we can seldom dam up the water by means of a transverse wall $B\ C$, Fig. 713, to such a height that the area F of the orifice, through which the water is discharged, will be but a small fraction of the cross-section of the stream which approaches it, and it is only in the latter case that the velocity of approach is very small compared to the mean velocity of efflux.

FIG. 713.



In the experiments made by the author with Poncelet's orifices the head of water was measured 1 meter back from the plane of the orifice; they gave

$$\frac{\mu_n - \mu_0}{\mu_0} = 0{,}641 \left(\frac{F}{G}\right) = 0{,}641\,.\,n^2,$$

$n = \frac{F}{G}$ denoting the ratio of the cross-sections, which should not be much greater than $\frac{1}{2}$, μ_0 denoting the coefficient of efflux for perfect contraction, taken from Poncelet's table, and μ_n the coefficient of efflux for the present case. Let b be the width and a the height of the orifice, b_1 the width and a_1 the depth of the stream of water and h the depth of the upper edge of the orifice below the level of the water, then we have the effective discharge

$$Q_1 = \mu_0\,.\,a\,b\left[1 + 0{,}641\left(\frac{a\,b}{a_1\,b_1}\right)^2\right]\sqrt{2\,g\left(h + \frac{a}{2}\right)}.$$

The following table is useful in abridging calculations in practice.

n	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50
$\frac{\mu_n - \mu}{\mu_0}$	0,002	0,006	0,014	0,026	0,040	0,058	0,079	0,103	0,180	0,160

EXAMPLE.—In order to find the amount of water brought by a ditch 3 feet wide, a transverse wall, containing a rectangular orifice 2 feet wide and

1 foot high is put in it, and the water is thus raised so that, when its level becomes constant, it is at a distance of 2¼ above the bottom and 1¼ feet above the lower edge of the orifice. The corresponding theoretical discharge is

$$Q = a\,b\sqrt{2\,g\,h} = 1\,.\,2\,.\,8{,}025\,\sqrt{1{,}25} = 16{,}05\,.\,1{,}118 = 17{,}94 \text{ cubic feet.}$$

As the coefficient of efflux for perfect contraction is 0,602 and the ratio of the cross-sections is

$$n = \frac{F}{G} = \frac{a\,b}{a_1\,b_1} = \frac{1\,.\,2}{2{,}25\,.\,3} = 0{,}296,$$

we have the coefficient of efflux in the present case

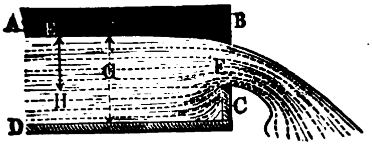
$$\mu_n = (1 + 0{,}641\,.\,0{,}296^2)\,\mu_0 = 1{,}056\,.\,0{,}602 = 0{,}6357,$$

and the effective discharge

$$Q_1 = 17{,}94\,.\,0{,}6357 = 11{,}4 \text{ cubic feet.}$$

§ 417. The contraction is also imperfect when water is discharged through overfalls (like that in Fig. 714), if the cross-section F of the stream passing over the sill C is a notable fraction of the cross-section G of the approaching water. The overfall may extend over but a portion or over the whole of the canal or ditch. In the latter case, as there is no contraction upon the sides of the orifice, the discharge is greater than through orifices of the first kind. The author has made experiments upon these cases of efflux and deduced from the results obtained formulas, by means of which the coefficient of efflux can be calculated with sufficient accuracy, when the ratio $n = \frac{F}{G}$ of the cross-sections is known.

FIG. 714



Let h be the head of water $E\,H$ above the sill of the overfall, a_1 the total depth of water, b the width of the overfall, and b_1 that of the approaching water; we have then

$$n = \frac{F}{G} = \frac{h\,b}{a_1\,b_1}, \text{ and}$$

1) for *Poncelet's overfall*

$$\frac{\mu_n - \mu_0}{\mu_0} = 1{,}718\left(\frac{F}{G}\right)^4 = 1{,}718\,n^4;$$

on the contrary,

2) for an *overfall occupying the whole width of the ditch or trough*

$$\frac{\mu_n - \mu_0}{\mu_0} = 0{,}041 + 0{,}3693\,n^2;$$

hence the discharge in the first case is

$$Q_1 = \tfrac{2}{3}\,\mu_0 \,.\, b\left[1 + 1{,}718\left(\frac{h\,b}{a_1\,b_1}\right)^4\right]\sqrt{2\,g\,h^3},$$

and in the second case,

$$Q_1 = \tfrac{2}{3}\,\mu_0 \,.\, b\left[1{,}041 + 0{,}3693\left(\frac{h}{a_1}\right)^2\right]\sqrt{2\,g\,h^3},$$

h denoting the head of water $E\,H$ above the sill F of the overfall, measured at a point about one meter back of it.

In the following tables the corrections $\frac{\mu_n - \mu_0}{\mu_0}$ for the simplest values of n are given.

TABLE I.

Corrections of the coefficients of efflux for Poncelet's overfalls.

n	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50
$\frac{\mu_n - \mu_0}{\mu_n}$	0,000	0,000	0,001	0,003	0,007	0,014	0,026	0,044	0,070	0,107

TABLE II.

Corrections for overfalls extending over the entire width, or without lateral contraction.

n	0,00	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50
$\frac{\mu_n - \mu_0}{\mu_0}$	0,041	0,042	0,045	0,049	0,056	0,064	0,074	0,086	0,100	0,116	0,133

EXAMPLE.—In order to determine the amount of water carried by a canal 5 feet wide, we place in it a transverse partition with the upper edge beveled outwards and we allow the water to flow over this. After the upper water had ceased to rise, the height of its surface above the bottom of the canal was 3½ feet and above the sill 1½ feet; the theoretical discharge was therefore

$$Q = \tfrac{2}{3} \,.\, 5 \,.\, 8{,}025\left(\frac{3}{2}\right)^{\frac{3}{2}} = 49{,}14 \text{ cubic feet.}$$

The coefficient of efflux is in this case, since $\frac{h}{a_1} = \frac{1{,}5}{3{,}5} = \frac{3}{7}$ and $\mu_0 = 0{,}577$,

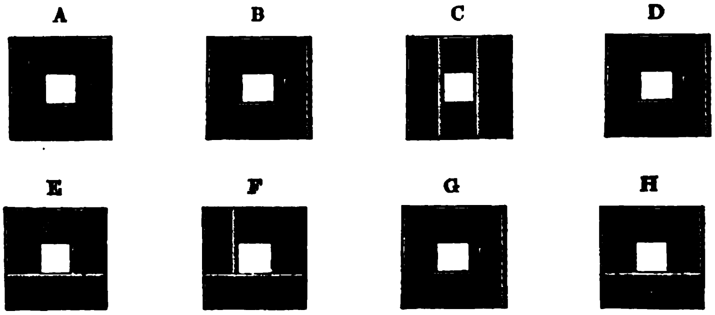
$$\mu_{\frac{3}{7}} = [1{,}041 + 0{,}3693\,(\tfrac{3}{7})^2] \,.\, 0{,}577 = 1{,}110 \,.\, 0{,}577 = 0{,}64,$$

and therefore the effective discharge is

$$Q_1 = 0{,}64 \,.\, Q = 0{,}64 \,.\, 49{,}14 = 31{,}45 \text{ cubic feet.}$$

§ **418. Lesbros's Experiments.**—We are indebted to Mons. Lesbros for a great number of experiments upon the efflux of water through rectangular orifices in a thin plate; the orifices, being provided internally and externally with rims, afforded examples of both partial and incomplete contraction (see his "Experiences hydrauliques sur les lois de l'écoulement de l'eau"). We will give here only the principal results of his experiments with a rectangular orifice 2 decimeters wide. The orifices, which were surrounded with borders of different kinds, are distinguished from each other in Fig. 715 by the letters *A*, *B*, *C*, etc.

FIG. 715.



A denotes the ordinary mouth-piece without any rim or border (as in § 410);

B denotes a similar mouth-piece with a vertical wall upon the inside perpendicular to the plane of the orifice and at a distance of 2 centimeters from one side of it;

C denotes the first mouth-piece enclosed on the inside by two such walls;

D the orifice *A*, provided on the inside with two vertical walls, which converge towards each other at an angle of 90° and cut the plane of the orifice at an angle of 45° and at a distance of 2 centimeters from the side of it;

E the orifice *A* with a horizontal wall, which passes across the reservoir and reaches exactly to the lower edge of the orifice;

F the orifice *B*,

G the orifice *C*, and

H the orifice *D* with a horizontal rim or wall, as in *E*, which completely prevents the contraction at the lower edge of the orifice.

L

TABLE OF THE COEFFICIENTS OF EFFLUX FOR *FREE EFFLUX* THROUGH THE ORIFICES *A*, *B*, *C*, ETC.

Head of water above the upper edge of the orifice, measured back from the plane of the orifice.	Height of orifice.	Coefficient of efflux for the orifices.							
		A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Meters.	Meters.								
0,020	0,200	0,572	0,587	—	0,589	0,599	—	—	—
0,050		0,585	0,593	0,631	0,595	0,608	0,622	—	0,636
0,100		0,592	0,600	0,631	0,601	0,615	0,628	—	0,639
0,200		0,598	0,606	0,632	0,607	0,621	0,633	0,708	0,643
0,500		0,603	0,610	0,631	0,611	0,623	0,636	0,680	0,644
1,000		0,605	0,611	0,628	0,612	0,624	0,637	0,676	0,642
1,500		0,602	0,611	0,627	0,611	0,624	0,637	0,672	0,641
2,000		0,601	0,610	0,626	0,611	0,619	0,636	0,668	0,640
3,000		0,601	0,609	0,624	0,610	0,614	0,634	0,665	0,638
0,020	0,050	0,616	0,627	0,647	0,631	0,664	0,663	—	0,678
0,050		0,625	0,630	0,646	0,632	0,667	0,669	0,690	0,677
0,100		0,630	0,633	0,645	0,633	0,669	0,674	0,688	0,677
0,200		0,631	0,635	0,642	0,633	0,670	0,676	0,687	0,675
0,500		0,628	0,634	0,637	0,632	0,668	0,676	0,682	0,671
1,000		0,625	0,628	0,635	0,627	0,666	0,672	0,680	0,670
1,500		0,619	0,622	0,634	0,621	0,665	0,670	0,678	0,670
2,000		0,613	0,616	0,634	0,615	0,664	0,670	0,674	0,669
3,000		0,606	0,609	0,632	0,608	0,662	0,669	0,673	0,668

II.

TABLE OF THE COEFFICIENTS OF EFFLUX THROUGH THE ORIFICES *A*, *B*, *C*, ETC.,

With *external shoots* or *uncovered canals of the same dimensions as the orifice* (Fr. canaux de fuite; Ger. äussere Ansatzgerinnen).

The shoots fitted the orifice exactly, and consequently the beveling of the sides and bottom of the mouth-piece was done away with. They were either horizontal and 3 meters long or (in the experiments marked with *) inclined $\frac{1}{10}$ of their length, which was but 2,5 meters.

Head of water above the upper edge of the orifice measured back from the plane of the orifice.	Height of orifice.	Coefficients of efflux for the orifices.									
		A	*B*	*C*	*E*	*E**	*F*	*F**	*G*	*G**	*H*
Meters.	Meters.										
0,020	0,200	0,480	0,489	0,496	0,480	0,527	—	—	—	—	0,488
0,050		0,511	0,517	0,531	0,510	0,553	0,509	0,546	0,528	—	0,520
0,100		0,542	0,545	0,563	0,538	0,574	0,534	0,569	0,560	0,593	0,552
0,200		0,574	0,576	0,591	0,566	0,592	0;562	0,589	0,589	0,617	0,582
0,500		0,599	0,602	0,621	0,592	0:607	0,591	0,608	0,591	0,632	0,613
1,000		0,601	0,609	0,628	0,600	0,610	0,601	0,615	0,601	0,638	0,623
1,500		0,601	0,610	0,627	0,602	0,610	0,604	0,617	0,604	0,641	0,624
2,000		0,601	0,610	0,626	0,602	0,609	0,604	0,617	0,604	0,642	0,624
3,000		0,601	0,609	0,624	0,601	0,608	0,602	0.616	0,602	0,641	0,622
0,020	0,050	0,488	0,555	0,557	0,487	0,585	0,483	0,579	0,512	—	0,494
0,050		0,577	0,600	0,603	0,571	0,614	0,570	0,611	0,582	0,625	0,577
0,100		0,624	0,625	0,628	0,605	0,632	0,609	0,628	0,621	0,639	0,616
0,200		0,631	0,633	0,637	0,617	0,645	0,623	0,643	0,637	0,649	0,629
0,500		0,625	0,630	0,635	0,626	0,652	0,630	0,650	0,647	0,656	0,636
1,000		0,624	0,627	0,635	0,628	0,651	0,633	0,651	0,649	0,656	0,638
1,500		0,619	0,622	0,634	0,627	0,650	0,632	0,651	0,647	0,656	0,637
2,000		0,613	0,616	0,634	0,623	0,650	0,631	0,651	0,644	0,656	0,635
3,000		0,606	0,609	0,632	0,618	0,649	0,628	0,651	0,639	0,656	0,632

EXAMPLE.—What is the discharge through an orifice 2 decimeters wide and 1 decimeter high, when the lower edge is 0,35 meters below the level of the water and upon a level with the bottom of the vessel, 1) for free efflux, and 2) for efflux through a short horizontal shoot? We have in this case the orifice *E*, and the head of water above the upper edge is $= 0,35 - 0,10 = 0,25$ meters. Table I gives, when the head is $= 0,20$ and the height of orifice $= 0, 20$, the coefficient of efflux $\mu = 0,621$, and, on the contrary, when the height of the orifice is $= 0,05$ meters, $\mu = 0,670$; hence for the first case of the problem we can put

$$\mu = \frac{0,621 + 0,670}{2} = 0,645.$$

Table II gives, on the contrary, by interpolation, for a head of water 0,25 meters above the upper edge of the orifice, the following values for μ.

$$0,566 + \tfrac{5}{30}\,(0,592 - 0,566) = 0,570, \text{ and}$$
$$0,617 + \tfrac{5}{30}\,(0,626 - 0,617) = 0,619;$$

hence in the second case we can put

$$\mu = \frac{0,570 + 0,619}{2} = 0,594.$$

The cross-section of the orifice is

$$F = a\,b = 0,20\,.\,0,10 = 0,020 \text{ square meters};$$

the mean head of water is

$$h = 0,350 - 0,050 = 0,300 \text{ meters};$$

and, consequently, the theoretical discharge is

$$Q = F\sqrt{2\,g\,h} = 0,02\,\sqrt{2\,.\,9,81\,.\,0,3} = 0,02\,\sqrt{5,886}$$
$$= 0,02\,.\,2,425 = 0,0485 \text{ cubic meters.}$$

The effective discharge is in the first case

$$Q_1 = \mu_1\,Q = 0,645\,.\,0,0485 = 0,0313 \text{ cubic meters,}$$

and, on the contrary, in the second case, I.E., when a shoot is added,

$$Q = \mu_2\,Q = 0,594\,.\,0,0485 = 0,0288 \text{ cubic meters.}$$

According to the formula $\mu_n = (1 + 0,155\,n)\,\mu_0$ of § 414, we can put for efflux with partial contraction $\mu_n = \mu_{\frac{1}{3}} = (1 + 0,52)\,\mu_0 = 1,052\,\mu_0$, since $\frac{2}{6} = \frac{1}{3}$ of the periphery of the orifice is surrounded by a border. But for such an orifice with complete contraction we have, according to Table I, page 831, $\mu_0 = 0,616$; hence

$$\mu_{\frac{1}{3}} = 1,052\,.\,0,616 = 0,648,$$

and the discharge is

$$Q_1 = \mu_{\frac{1}{3}}\,Q = 0,648\,.\,0,0485 = 0,0314 \text{ cubic meters,}$$

I.E., a little greater than that obtained by employing Lesbros's table.

§ 419. M. Lesbros has also experimented upon efflux through *overfalls*, employing the same orifices *A*, *B*, *C*, etc., but not allowing the water to rise to the upper edge of the orifice. The principal results of these experiments are to be found in the following tables.

TABLE I.

Table of the coefficients of efflux ($\frac{2}{3}\mu$) for free efflux through overfalls or notches.

Head of water above the sill, measured where the water is still.	Coefficients of efflux for the orifices.						
	A	*B*	*C*	*D*	*E*	*F*	*G*
Meter.							
0,015	0,421	0,450	0,450	0,441	0,395	0,371	0,305
0,020	0,417	0,446	0,444	0,437	0,402	0,379	0,318
0,030	0,412	0,437	0,435	0,430	0,410	0,388	0,337
0,040	0,407	0,430	0,429	0,424	0,411	0,394	0,352
0,050	0,404	0,425	0,426	0,419	0,411	0,398	0,362
0,070	0,398	0,416	0,422	0,412	0,409	0,402	0,375
0,100	0,395	0,409	0,420	0,405	0,408	0,405	0,382
0,150	0,393	0,406	0,423	0,403	0,407	0,407	0,383
0,200	0,390	0,402	0,424	0,403	0,405	0,408	0,383
0,250	0,379	0,396	0,422	0,401	0,404	0,407	0,381
0,300	0,371	0,390	0,418	0,398	0,403	0,406	0,378

TABLE II.

Table of the coefficients of efflux ($\frac{2}{3}\mu$) for efflux through weirs with short shoots or open canals.

Head of water above the sill, measured where the water is still.	Coefficients of efflux for the orifices.							
	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Meter.								
0,015	—	0,375	0,388	0,400	—	—	—	—
0,020	0,196	0,368	0,383	0,395	0,208	0,201	0,175	0,190
0,030	0,234	0,358	0,373	0,385	0,232	0,228	0,205	0,222
0,040	0,263	0,351	0,365	0,379	0,251	0,250	0,234	0,250
0,050	0,278	0,346	0,360	0,375	0,268	0,267	0,260	0,272
0,070	0,292	0,343	0,352	0,371	0,288	0,289	0,285	0 296
0,100	0,304	0,340	0,345	0,369	0,302	0,304	0,299	0,313
0,150	0,315	0,335	0,340	0,367	0,314	0,316	0,313	0,327
0,200	0,319	0,331	0,338	0,366	0,323	0,322	0,322	0,335
0,250	0,321	0,328	0,336	0,364	0,329	0,326	0,329	0,341
0,300	0,324	0,326	0,334	0,361	0,332	0,329	0,332	0,345

A comparison of the coefficients in Table I and Table II shows that the discharge through orifices provided with shoots is smaller than that through those without them, and that the difference is greater, the smaller the head of water is; we also see, by comparing the columns C and C^*, E and E^*, F and F^*, and G and G^* in the tables of the last paragraph, that the inclined shoot creates less disturbance in the efflux than the horizontal one.

Remark 1.—A different theory of the efflux of water is advanced by G. Boileau in his "Traité sur la mesure des eaux courantes." According to it the velocity of the effluent water is the same at all parts of the cross-section and depends upon the depth of the upper limiting line of the vein at the plane of the orifice below the level of the water in the reservoir. Boileau employs the same formula for overfalls, in which case he must know of course the height of the stream in the plane of the orifice. Later, in the 12th volume of the 5th series of the Annales des Mines, 1857, M. Clarinval has given another formula for efflux through overfalls in which no empirical number μ appears, but instead of $\frac{2}{3}\mu$ he substitutes the factor $\frac{a\sqrt{1-\frac{a}{h}}}{\sqrt{2(h^2-a^2)}}$, in which h denotes the head of water and a the thickness of the stream above the sill of the overfall. See the "Civilingenieur," Vol. 5th. I consider the hypothesis upon which this formula is based to be incorrect.

Remark 2.—Mr. J. B. Francis gives in his work "The Lowell Hydraulic Experiments, Boston, 1855," the following formula for efflux through a wide overfall or weir.

$$Q = 3{,}33\,(l - 0{,}1\,n\,h)\,h^{\frac{3}{2}} \text{ English cubic feet,}$$

in which h denotes the head of water above the sill of the weir, l its length, and n either 0 or 1 or 2, according as the contraction of the vein is prevented upon both, one or none of the sides. Since for the English system of measures

$$\sqrt{2g} = 8{,}025,$$

we have

$$\frac{2}{3}\mu = \frac{3{,}33}{8{,}025} = 0{,}415.$$

The experiments, upon which this formula is based, were made with weirs 10 feet wide and under heads of water from 0,6 to 1,6 feet. The edge of the weir was formed of an iron plate beveled down stream, the reservoir was 13,96 feet wide, and the sill was 4,6 feet above its bottom. See the Civilingenieur, Vol. 2, 1856.

Bakewell's experiments upon efflux through weirs or overfalls give results differing in some respects from the above. (See Polytech. Central Blatt, 18th year, 1852.)

Remark 3.—At the sluice-gate of the wheel at Remscheid, Herr Röntchen found $\mu = 0{,}90$ to $0{,}98$. See Dingler's Journal, Vol. 158.

A new edition of Mr. J. B. Francis' work has been recently published by D. Van Nostrand, New York.—[Tr.]

CHAPTER III.

OF THE FLOW OF WATER THROUGH PIPES.

§ 420. **Short Tubes.**—If we allow the water to discharge through a *short tube*, or *pipe*, called also an ajutage, (Fr. tuyau additionel; Ger. kurze Ansatzröhre), the condition of affairs is entirely different from that existing, when the water issues from an orifice in a thin plate or from an orifice in thick wall, which is rounded off on the outside. If the short tube is prismatic and $2\frac{1}{2}$ to 3 times as long as wide, the stream is uncontracted and non-transparent and its range and consequently its velocity is smaller than when it issues, under the same circumstances, from an orifice in a thin plate. If, therefore, the tube *K L* has the same cross-section as the orifice *F*, Fig. 716, and if the head of water is the same for both, we obtain at *R L* a troubled and uncontracted or thicker stream and at *F H* a clear and contracted or thinner one; we can also see that the range *E R* is smaller than the range *D H*. This condition of efflux exists only when the length of the tube is the given one; if the tube is shorter, E.G. as long as wide, the vein *K R*, Fig. 717, does not touch the sides of the tube, the latter has then no influence upon the efflux, and the stream issues from it as from an orifice in a thin plate.

FIG. 716.

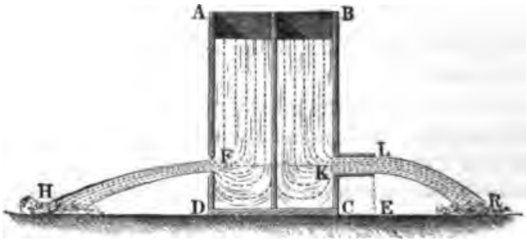
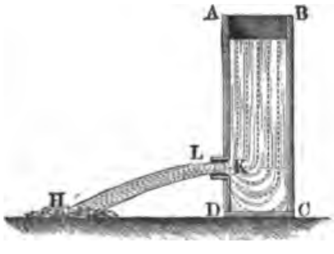
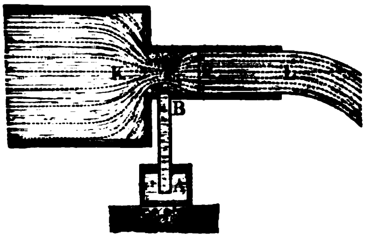


FIG. 717.



Sometimes it happens, when the length of the tube is greater, that the stream does not fill it; this occurs when the water has no opportunity of coming in contact with the sides of the tube; if in this case we close for an instant the outside end of the tube with the hand or with a board, the stream will fill the tube and we have the so-called *discharge of a filled tube* (Fr. à gueule bée; Ger. voller Ausfluss). The vein is contracted in this case also, but the contracted portion is within the tube. We can satisfy ourselves of this by employing glass tubes like $K\ L$, Fig. 718, and by throwing small light bodies into the water. Upon so doing, we observe that near the entrance K there is a motion of translation in the middle of the cross-section F_1, but that, on the contrary, at the periphery of the same the water forms an eddy. It is, however, the capillarity or adhesion of the water to the walls of the tube, which causes it to fill the end $F\ L$ of the tube completely. The pressure of the water discharging from the tube is that of the atmosphere, but the contracted cross-section F_1 is only a times as great as that F of the tube; the velocity v_1 at that point is therefore $\frac{1}{a}$ times as great as the velocity of efflux v and the pressure of the water at F_1 is smaller than that at the end of the tube, which is equal to the pressure of the atmosphere. If we bore a small hole in the pipe near F_1 no water will run out, but air will be sucked in and the discharge with a filled tube ceases, when the hole is enlarged or when several of them are made. We can also cause the water in the tube $A\ B$ to rise and flow through the tube $K\ L$ by making it enter the latter at F_1. The discharge with a filled tube ceases for cylindrical tubes, when the head of water attains a certain magnitude (see § 439, Chap. IV).

FIG. 718.



§ **421. Short Cylindrical Tubes.**—Many experiments have been made upon the efflux of water through *short cylindrical tubes*, but the results obtained differ quite sensibly from each other. It is particularly Bossut's coefficients of efflux which differ most from those of others by their smallness (0,785). The results of the experiments Michelotti with tubes $1\frac{1}{4}$ to 3 inches in diameter, under a head of water varying from 3 to 20 feet, gave as a mean value

$\mu = 0,813$. The results of the experiments of Bidone, Eytelwein and d'Aubuisson differ but little from those of the latter. But, according to the experiments of the author, we can adopt for ***short cylindrical tubes*** as a mean value $\mu = 0,815$. Since we found this coefficient for an orifice in a thin plate $= 0,615$, it follows that, when the other circumstances are the same, $\frac{815}{615} = 1,325$ times as much water is discharged through a short pipe as through an orifice in a thin plate. These coefficients increase, when the diameter of the tube becomes greater and decrease a little, when the head of water or the velocity of efflux increases. According to some experiments of the author's, made under heads varying from 0,23 to 0,6 meters, we have for tubes 3 times as long as wide

When the width is	1	2	3	4 centimeters.
$\mu =$	0,843	0,832	0,821	0,810

According to this table the coefficients of efflux decrease sensibly as the width of the tube increases. In like manner Buff found with a tube 2,79 lines wide and 4,3 lines long that the coefficient of efflux increased gradually from 0,825 to 0,855, when the head of water decreased from 33 to $1\frac{1}{2}$ inches.

For the efflux of water through ***short parallelopipedical tubes*** the author found the coefficient to be 0,819.

If the short tube $K\,L$, Fig. 719, is ***partially surrounded by a border*** or rim in the inside of the vessel, if, E.G., one of its sides is flush with the bottom $C\,D$ of the vessel and if partial contraction is thus produced, according to the experiments of the author, the coefficient of efflux is not sensibly increased, but the water

FIG. 719.

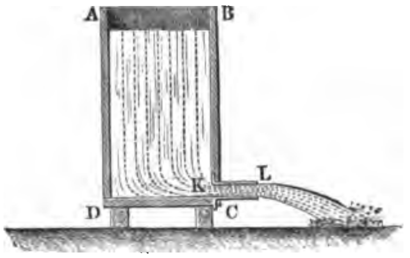
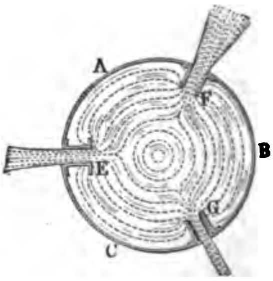


FIG. 720.



moves with different velocities in different parts of the cross-section, viz., upon the side C more quickly than upon the opposite one.

If the face of the tube is not in the surface of the plate but projects into the vessel, like E, F, G, Fig. 720, it is then called an *interior short tube.* If the face of the tube is at the least 5 times as wide as the bore of the tube, as at E, the coefficient of efflux remains the same as if the face were in the plane of the wall, but if the face of the tube is smaller, as at F and G, the coefficient of efflux is smaller. According to the experiments of Bidone and of the author, if the face is very small, it is 0,71, when the stream fills the tube; on the contrary, it is = 0,53 (compare § 113), when it does not touch the internal surface of the tube. In the first case (F) the stream is troubled and divergent like a broom, but in the second (G) it is compact and crystalline.

§ 422. **Coefficient of Resistance.**—Since the stream of water issues from a short prismatical tube without being contracted, it follows that the coefficient of contraction of this mouth-piece a = unity and that its coefficient of velocity ϕ = its coefficient of efflux μ. The vis viva of a quantity of water Q, which issues with a velocity v, is $\frac{Q\gamma}{g} v^2$, and its energy is $\frac{v^2}{2g} Q \gamma$ (see § 74). But the theoretical velocity of efflux is $\frac{v}{\phi}$, and therefore the theoretical energy of the water discharged is $\frac{1}{\phi^2} \cdot \frac{v^2}{2g} \cdot Q\gamma$. Hence the loss of energy of the quantity Q of water during the efflux is

$$\left(\frac{1}{\phi^2} \cdot \frac{v^2}{2g} - \frac{v^2}{2g}\right) Q \gamma = \left(\frac{1}{\phi^2} - 1\right) \frac{v^2}{2g} Q \gamma.$$

For efflux through orifices in a thin plate, the mean value of ϕ is 0,975; hence the loss of energy is

$$\left[\left(\frac{1}{0{,}975}\right)^2 - 1\right] \frac{v^2}{2g} Q \gamma = 0{,}052 \frac{v^2}{2g} Q \gamma;$$

for efflux through a short cylindrical pipe, on the contrary, ϕ = 0,815, and the corresponding loss of energy is

$$= \left[\left(\frac{1}{0{,}815}\right)^2 - 1\right] \frac{v^2}{2g} Q \gamma = 0{,}505 \frac{v^2}{2g} Q \gamma,$$

I.E., nearly 10 times as much as for efflux through an orifice in a thin plate. Consequently if the vis viva of the water is to be made use of, it is better to allow it to flow through an orifice in a thin plate than through a short prismatical tube. If, however, we round off the edge of the tube, where it is united to the interior surface of the vessel, so as to produce a gradual passage from the vessel into the tube, the coefficient of efflux is increased to 0,96 and at the same time the loss of energy is reduced to 8½ per cent. For short tubes or ajutages, which are rounded off or shaped internally like the contracted vein, we have $\mu = \phi = 0,975$, and the loss of mechanical effect is the same as it is for an orifice in a thin plate, viz., 5 per cent.

The loss of mechanical effect $\left(\frac{1}{\phi^2} - 1\right) \frac{v^2}{2g} Q \gamma$ corresponds to a head of water $\left(\frac{1}{\phi^2} - 1\right) \frac{v^2}{2g}$; we can therefore consider that the loss of head due to the resistance to efflux is $\left(\frac{1}{\phi^2} - 1\right) \frac{v^2}{2g}$ and we can assume that, when this loss has been subtracted, the remaining portion of the head is employed in producing the velocity.

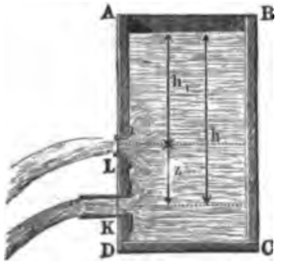
This loss $z = \left(\frac{1}{\phi^2} - 1\right) \frac{v^2}{2g}$, which increases with the square of the velocity, is known as the *height of resistance* (Fr. hauteur de résistance; Ger. Widerstandshöhe) and the coefficient $\frac{1}{\phi^2} - 1$, by which the head of water must be multiplied in order to obtain the height of resistance, is called the *coefficient of resistance*. Hereafter we will denote this coefficient, which also gives the ratio of the height of resistance to the head of water, by ζ or the height of resistance itself by $z = \zeta \cdot \frac{v^2}{2g}$. By means of the formulas

$$\zeta = \frac{1}{\phi^2} - 1 \text{ and}$$

$$\phi = \frac{1}{\sqrt{1 + \zeta}}$$

we can calculate from the coefficient of velocity the coefficient of resistance, or the latter from the former.

FIG. 721.



If the velocity of efflux v is the same, the head of water of an orifice K, Fig. 721, whose coefficient of resistance is ϕ, is $h = \frac{v^2}{2g\phi^2}$, and the head of water of the orifice L, through which the water flows with this theoretical velocity, is $h_1 = \frac{v^2}{2g}$, consequently the first orifice must lie at a distance $K L = z = h - h_1 = \left(\frac{1}{\phi^2} - 1\right) \frac{v^2}{2g} = \zeta \frac{v^2}{2g}$ below the second one. This distance z is called the height of resistance. If they have the same cross-section F and there is no contraction at either orifice, the discharge $Q = F v$ is the same for both.

EXAMPLE—1) What is the discharge under a head of water of 3 feet through a tube 2 inches in diameter, whose coefficient of resistance is $\zeta = 0{,}4$. Here

$$\phi = \frac{1}{\sqrt{1{,}4}} = 0{,}845\text{; hence}$$

$$v = 0{,}845 \,.\, 8{,}025 \sqrt{3} = 11{,}745 \text{ feet;}$$

$$F = (\tfrac{1}{12})^2 \, \pi = 0{,}02182 \text{ square feet,}$$

and consequently the required discharge is

$$Q = 0{,}02182 \,.\, 11{,}745 = 0{,}256 \text{ cubic feet.}$$

2) If a tube 2 inches wide discharges under a head of 2 feet 10 cubic feet of water in a minute, the coefficient of efflux or velocity is

$$\phi = \frac{Q}{F\sqrt{2gh}} = \frac{10}{60 \,.\, 0{,}02182 \,.\, 8{,}025 \sqrt{2}} = \frac{1}{1{,}05 \sqrt{2}} = 0{,}673,$$

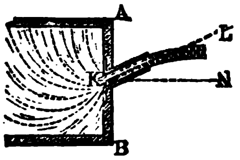
the coefficient of resistance $\zeta = \left(\frac{1}{0{,}673}\right)^2 - 1 = 1{,}208$,

and the loss of head, caused by the resistance of the tube, is

$$z = \zeta \frac{v^2}{2g} = 1{,}208 \,.\, \frac{v^2}{2g} = 1{,}208 \,.\, 0{,}0155 \left(\frac{Q}{F}\right)^2 = 0{,}0187 \,.\, \frac{1}{0{,}1309^2} = 1{,}092 \text{ feet.}$$

§ 423. **Inclined Short Tubes or Ajutages.**—When the tubes are applied to the vessel in an inclined position or when they are cut off obliquely to the axis, the discharge is less than when they are inserted into the vessel at right angles or cut off at right angles to their axis; for in this case the direction of the water is changed. The author's extended experiments upon this subject have led to the following conclusions. If δ denotes the angle $L K N$, formed by the axis of the tube $K L$, Fig. 722, with the normal $K N$ to the plane $A B$ of the orifice, and if ζ denotes the coefficient of resistance for tubes cut off at right angles, we have for the coefficient of resistance of inclined tubes

FIG. 722.



$$\zeta_1 = \zeta + 0{,}303 \, sin. \, \delta + 0{,}226 \, sin.^2 \, \delta.$$

Assuming for ζ the mean value 0,505, we obtain

for δ° =	0	10	20	30	40	50	60 deg.
the coefficients of resistance ζ_1 =	0,505	0,565	0,635	0,713	0,794	0,870	0,937
the coefficient of efflux μ_1 =	0,815	0,799	0,782	0,764	0,747	0,731	0,719

Hence, E.G., the coefficient of resistance of a short tube, the angle of deviation of whose axis is 20°, is $\zeta_1 = 0,635$ and the coefficient of efflux is

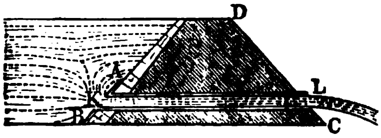
$$\mu_1 = \frac{1}{\sqrt{1,635}} = 0,782,$$

and, on the contrary, when the deviation is 35°, the former is $= 0,753$ and the latter $= 0,755$.

These inclined tubes are generally longer than those we have previously considered, and they must be longer when they are to be completely filled with water. The foregoing formula gives only that part of the resistance due to the short tube at the inlet orifice, that is, three times as long as the tube is wide. The resistance of the remaining part of the tube will be given further on.

EXAMPLE.—If the plane of the orifice $A\,B$ of the discharge-pipe $K\,L$, Fig. 723, as well as the inside slope of the dam, is inclined at an angle of 40° to the horizon, the axis of the tube will form an angle of 50° with that plane; hence the coefficient of resistance for efflux through the entrance of this pipe is $\zeta = 0,870$, and if the coefficient of resistance for the remaining longer portion is 0,650, we have the coefficient of resistance for the entire tube

FIG. 723.



$$\zeta = 0,870 + 0,650 = 1,520,$$

and therefore the coefficient of efflux is

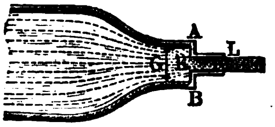
$$\mu = \frac{1}{\sqrt{1 + 1,520}} = \frac{1}{\sqrt{2,520}} = 0,630.$$

If the head of water is 10 feet and the width of the pipe 1 foot, the discharge is

$$Q = 0,630 \,.\, \frac{\pi}{4} \,.\, 8,025 \sqrt{10} = 12,56 \text{ cubic feet.}$$

§ **424. Imperfect Contraction.**—If a short tube $K\,L$, Fig. 724, is inserted in a plane wall, whose area G is but little larger than the cross-section F of the tube, the water will approach the mouth of the short tube with a velocity, which we cannot neglect, and the stream which enters it is imperfectly contracted; hence the velocity of efflux is greater than when the water can be considered to be at rest at the mouth of the tube. Now if $\frac{F}{G} = n$ is the ratio of the cross-section of the tube to that of the wall and μ_0 the coefficient of efflux for perfect contraction, in which case we can put $\frac{F}{G} = 0$, we have, according to the experiments of the author, for the coefficient of efflux with imperfect contraction, when we put the ratio of the cross-sections $= n$,

FIG. 724.



$$\frac{\mu_n - \mu_0}{\mu_0} = 0{,}102\, n + 0{,}067\, n^2 + 0{,}046\, n^3, \text{ or}$$

$$\mu_n = \mu_0 (1 + 0{,}102\, n + 0{,}067\, n^2 + 0{,}046\, n^3).$$

If, E.G., we assume the cross-section of the tube to be one-sixth of that of the wall, we have

$$\mu_{\frac{1}{6}} = \mu_0 (1 + 0{,}102 \,.\, \tfrac{1}{6} + 0{,}067 \,.\, \tfrac{1}{36} + 0{,}046 \,.\, \tfrac{1}{216})$$
$$= \mu_0 (1 + 0{,}017 + 0{,}0019 + 0{,}0002) = 1{,}019\, \mu_0,$$

or putting $\mu_0 = 0{,}815$

$$\mu_{\frac{1}{6}} = 0{,}815 \,.\, 1{,}019 = 0{,}830.$$

The values $\frac{\mu_n - \mu_0}{\mu_0}$ of the correction are given in the following tables, which are more convenient for use.

TABLE OF THE CORRECTIONS OF THE COEFFICIENTS OF EFFLUX, ON ACCOUNT OF IMPERFECT CONTRACTION, FOR EFFLUX THROUGH *SHORT CYLINDRICAL TUBES.*

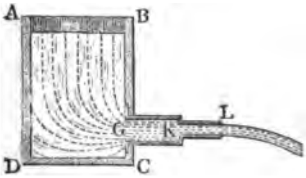
n	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50
$\frac{\mu_n - \mu}{\mu_0}$	0,006	0,013	0,020	0,027	0,035	0,043	0,052	0,060	0,070	0,080

n	0,55	0,60	0,65	0,70	0,75	0,80	0,85	0,90	0,95	1,00
$\frac{\mu_n - \mu_0}{\mu_0}$	0,090	0,102	0,114	0,127	0,138	0,152	0,166	0.181	0,198	0,227

When the water is discharged through *short parallelopipedical tubes*, these corrections are about the same.

The principal applications of these corrections are to the efflux of water through compound tubes, as, E.G., in the case represented in Fig. 725, where the short tube $K\,L$ is inserted into another short tube $G\,K$, and the latter into the vessel $A\,C$. Here, when the water enters the smaller from the larger tube, the stream is imperfectly contracted, and the coefficient of efflux is determined by the last rule. If we put the coefficient of resistance corresponding to this coefficient of efflux $= \zeta_1$, the coefficient of resistance for its entrance into the larger tube from the reservoir $= \zeta$, the head of water $= h$, the velocity of efflux $= v$ and the ratio $\frac{F}{G}$ of the cross-sections of the tube $= n$, or the velocity of the water in the larger tube $= n\,v$, we have the formula

FIG. 725.



$$h = \frac{v^2}{2\,g} + \zeta \,.\, \frac{(n\,v)^2}{2\,g} + \zeta_1 \,.\, \frac{v^2}{2\,g}, \text{ I.E.}$$

$$h = (1 + n^2\,\zeta + \zeta_1)\,\frac{v^2}{2\,g}, \text{ and therefore}$$

$$v = \frac{\sqrt{2\,g\,h}}{\sqrt{1 + n^2\,\zeta + \zeta_1}}.$$

EXAMPLE.—What is the discharge from the vessel represented in Fig. 725, when the head of water is $h = 4$ feet, the width of the narrow tube 2 inches and that of the larger one 3 inches? Here

$$n = (\tfrac{2}{3})^2 = \tfrac{4}{9}, \text{ whence } \mu_{\frac{4}{9}} = 1{,}069 \,.\, 0{,}815 = 0{,}871$$

and the corresponding coefficient of resistance

$$\zeta_1 = \left(\frac{1}{0{,}871}\right)^2 - 1 = 0{,}318; \text{ but we have}$$

$$\zeta = 0{,}505 \text{ and } n^2\,\zeta = \tfrac{16}{81} \,.\, 0{,}505 = 0{,}099,$$

whence it follows that

$$1 + n^2\,\zeta + \zeta_1 = 1 + 0{,}099 + 0{,}318 = 1{,}417,$$

and the velocity of efflux

$$v = \frac{8{,}025 \,.\, \sqrt{4}}{\sqrt{1{,}417}} = \frac{16{,}05}{\sqrt{1{,}417}} = 13{,}48.$$

Finally, since the cross-section of the tube is $F = \frac{\pi}{144} = 0{,}02182$ square feet, it follows that the discharge is

$$Q = 13{,}48 \,.\, 0{,}02182 = 0{,}294 \text{ cubic feet.}$$

Fig. 726.

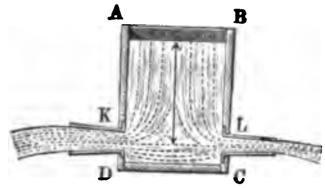
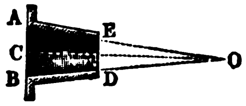
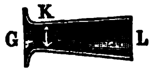


Fig. 727.



§ 425. Conical Short Tubes or Ajutages.—The discharges from *conical mouth-pieces or short conical tubes* are different from those obtained from cylindrical or prismatic ones. They are either *conically convergent* or *conically divergent.* In the first case the outlet orifice is smaller than the inlet, and in the second case the inlet is smaller than the outlet. The coefficients of efflux through the former tubes are greater and those of efflux through the latter smaller than for cylindrical tubes. The same conical tube discharges more water when we make the wider end the orifice of discharge, as in K, Fig. 726, than when we put it in the wall of the reservoir, as is represented at L in the same figure; but the ratio of the discharge is not as great as that of the openings. When authors such as B. Venturi and Eytelwein give greater coefficients of efflux for conically divergent than for conically convergent tubes, it must be remembered that the smaller cross-section is always considered as the orifice. The influence of the conicalness of the tubes upon the discharge is shown by the following experiments, made under heads of from 0,25 to 3,3 meters, with a tube $A\ D$, Fig. 727, 9 centimeters long. The width of this tube at one end was $D\ E = 2{,}468$, at the other $A\ B = 3{,}228$ centimeters, and the *angle of convergence,* I.E. the angle $A\ O\ B$, formed by the prolongation of the opposite sides $A\ E$ and $B\ D$ of a section through the axis of the tube, was $= 40° 50'$. When the water issued from the narrow opening, the coefficient of efflux was $= 0{,}920$; but when it issued from the wider opening, it was $= 0{,}553$. If we substitute in the calculation the narrower orifice as cross-section, we find it $= 0{,}946$. The stream, in the first case, when the tube was conically convergent, was but little contracted, dense and smooth; in the second case, where the mouth-piece was conically divergent, the stream was very divergent and torn and pulsated violently. Venturi and Eytelwein have experimented upon efflux through conically divergent tubes. Both these experimenters also attached to these conical tubes cylindrical and conical mouth-pieces, shaped like the contracted vein. With a compound mouth-piece, like the one represented in Fig. 728, the diverging portion $K\ L$ of which was 12 lines in diameter in the narrowest place and $21\frac{1}{2}$ lines at the widest, and $8\frac{1}{1}\frac{3}{8}$ inches long, and whose angle of convergence was 5° 9′, Eytelwein found $\mu =$ 1,5526, when he treated the narrow end as the orifice, and, on the contrary, $\mu = 0{,}483$ when, as was proper, he treated the larger end as the orifice. However, $\frac{1{,}5526}{0{,}615} = 2{,}5$ times as much water is discharged through this compound mouth-piece as through a simple orifice in a thin plate, and $\frac{1{,}5526}{0{,}815} = 1{,}9$ times as much as through a short cylindrical pipe. When the velocities and the angle of divergence are great, it is not possible to produce a complete efflux, even by at first closing the end of the mouth-piece.

FIG. 728.



The author found with a short conically divergent mouth-piece 4 centimeters long, whose minimum and maximum widths were 1 and 1,54 centimeters and whose angle of divergence was 8° 4′, under a head of 0,4 meters, $\mu = 0{,}738$ when the internal edge was rounded off, and $\mu = 0{,}395$ when it was not.

§ **426**. The most extensive experiments upon the efflux of water through *conically convergent tubes* are those made by d'Aubuisson and Castel. A great variety of tubes, which differed in length, width and in the angle of convergence, were employed. The most extensive were the experiments with tubes 1,55 centimeters wide at the orifice of efflux and 2,6 times as long, I.E., 4 centimeters long; for this reason we give their results in the following table. The head of water was always 3 meters. The discharge was measured by a gauged vessel, but in order to determine not only the coefficient of efflux, but also the coefficients of velocity and contraction, the ranges of the jet corresponding to the given heights were measured, and from them the velocities of efflux were calculated.

The ratio $\frac{v}{\sqrt{2gh}}$ of the effective velocity v to the theoretical one $\sqrt{2gh}$ gave the coefficient of velocity ϕ, the ratio $\frac{Q}{F\sqrt{2gh}}$ of the effective discharge Q to the theoretical discharge $F\sqrt{2gh}$ the coefficient of efflux μ, and, finally, the ratio of the two coefficients, I.E., $\frac{\mu}{\phi}$, determined the coefficient of contraction α.

This determination is not accurate enough, when the velocities of efflux are great; for in that case the resistance of the air is too great.

TABLE OF THE COEFFICIENTS OF EFFLUX AND VELOCITY FOR EFFLUX THROUGH CONICALLY CONVERGENT TUBES.

Angle of convergence.	Coefficient of efflux.	Coefficient of velocity.	Angle of convergence.	Coefficient of efflux.	Coefficient of velocity.
0° 0′	0,829	0,829	13° 24′	0,946	0,963
1° 36′	0,866	0,867	14° 28′	0,941	0,966
3° 10′	0,895	0,894	16° 36′	0,938	0,971
4° 10′	0,912	0,910	19° 28′	0,924	0,970
5° 26′	0,924	0,919	21° 0′	0,919	0,972
7° 52′	0,930	0,932	23° 0′	0,914	0,974
8° 58′	0,934	0,942	29° 58′	0,895	0,975
10° 20′	0,938	0,951	40° 20′	0,870	0,980
12° 4′	0,942	0,955	48° 50′	0,847	0,984

According to this table, the coefficient of efflux attains its maximum value 0,946 for a tube, whose sides converge at an angle of 13½°, that, on the contrary, the coefficients of velocity increase continually with the angle of convergence. How the foregoing table is to be employed in practice, is shown by the following example.

EXAMPLE.—What is the discharge through a short conical mouth-piece 1½ inches wide at the orifice of efflux and converging at an angle of 10°, when the head of water is 16 feet? According to the author's experiments, a cylindrical tube of this width gives $\mu = 0,810$, d'Aubuisson tube, however, gave $\mu = 0,829$, or $0,829 - 0,810 = 0,019$ more; now, according to the table, for a tube converging at 10°, $\mu = 0,937$; it is therefore better to put for the given tube $\mu = 0,937 - 0,019 = 0,918$; whence we obtain the discharge

$$Q = 0,918 \cdot \frac{\pi}{4 \cdot 8^2} \cdot 0,825 \sqrt{16} = \frac{0,918 \cdot 8,025\, \pi}{64} = 0,3616 \text{ cubic feet.}$$

§ 427. **Resistance of Friction.**—The longer prismatical or cylindrical pipes are, the greater is the diminution of the discharge through them; we must therefore assume that the walls of the pipes by friction, adhesion or by the water's sticking to them resist the motion of the water. As we might suppose, and in accordance with many observations and measurements, we can assume that this resistance of friction is entirely independent of the pressure, that it is directly proportional to the length l and inversely to the diameter d of the pipe, I.E., it is proportional to the ratio $\frac{l}{d}$. It has also been proved that this resistance is greater when the velocities are great and less when they are small, and that it increases, very nearly, with the square of the velocity v. If we measure this resistance by a column of water, which must afterwards be subtracted from the total head h, in order to obtain the height necessary to produce the velocity, we can put this height, which we will hereafter call the *height of resistance of friction*,

$$h = \zeta \cdot \frac{l}{d} \cdot \frac{v^2}{2g},$$

ζ denoting here an empirical number, which we can style the *coefficient of friction*. Hence the loss of head or of pressure in consequence of the friction of the water in the pipe is greater, the greater the ratio $\frac{l}{d}$ of the length to the width and the greater the height due to the velocity $\frac{v^2}{2g}$ is. From the discharge Q and the cross-section of the tube

$$F = \frac{\pi d^2}{4}$$

we obtain the velocity

$$v = \frac{4Q}{\pi d^2},$$

and, therefore, the *height of resistance of friction*

$$h = \zeta \cdot \frac{l}{d} \cdot \frac{1}{2g} \left(\frac{4Q}{\pi d^2}\right)^2 = \zeta \cdot \frac{1}{2g} \cdot \left(\frac{4}{\pi}\right)^2 \cdot \frac{lQ^2}{d^5}.$$

If we wish to conduct a certain quantity Q of water through a pipe with as little loss of head or fall as possible, we must make the pipe as short and as wide as we can. If the width of the pipe is double that of another, the friction in the former is $(\frac{1}{2})^5 = \frac{1}{32}$ that in the latter.

If the cross-section of the pipe is a rectangle, whose height is a and whose width is b, we must substitute

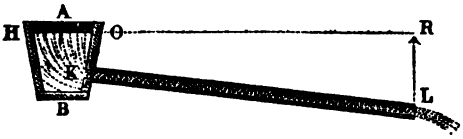
$$\frac{1}{d} = \frac{1}{4} \cdot \frac{\pi d}{\frac{1}{4}\pi d^2} = \frac{1}{4} \cdot \frac{\text{periphery}}{\text{area}} = \frac{1}{4} \cdot \frac{2(a+b)}{ab} = \frac{a+b}{2ab},$$

whence we have

$$h = \zeta \cdot \frac{l(a+b)}{2ab} \cdot \frac{v^2}{2g}.$$

By the aid of these formulas for the resistance of friction in pipes, we can find the discharge and the velocity of efflux of the water conveyed by a pipe of a given length and width, under a given pressure. It is also of no consequence whether the tube $K L$, Fig. 729, is horizontal or inclined upwards or downwards, so long as we understand by the head of water the depth $R L$ of the centre L of the mouth of the pipe below the level $H O$ of the water in the reservoir.

FIG. 729.



If h is the head of water, h_1 the height of resistance for the orifice of influx, and h_2 the height of resistance for the remaining part of the tube, we have

$$h - (h_1 + h_2) = \frac{v^2}{2g}, \text{ or } h = \frac{v^2}{2g} + h_1 + h_2.$$

If ζ_0 denotes the coefficient of resistance for the orifice of influx and ζ the coefficient of resistance of friction of the rest of the tube, we can put

$$h = \frac{v^2}{2g} + \zeta_0 \cdot \frac{v^2}{2g} + \zeta \cdot \frac{l}{d} \cdot \frac{v^2}{2g},$$

or

$$1)\ h = \left(1 + \zeta_0 + \zeta \frac{l}{d}\right) \frac{v^2}{2g},$$

and

$$2)\ v = \frac{\sqrt{2gh}}{\sqrt{1 + \zeta_0 + \zeta \cdot \frac{l}{d}}}.$$

From the latter formula we obtain the discharge $Q = F v$.

For very long tubes $1 + \zeta_0$ is very small, compared with $\zeta \frac{l}{d}$, and we can write more simply

$$h = \zeta \frac{l}{d} \cdot \frac{v^2}{2g}, \text{ or inversely,}$$

$$v = \sqrt{\frac{1}{\zeta} \cdot \frac{d}{l} \cdot 2gh}.$$

§ **428**. The *coefficient of friction*, like the coefficient of efflux, is not perfectly constant; it is greater for low velocities than for high ones, I.E. the resistance of friction of the water in tubes does not increase exactly with the square, but with another power of the velocities. Prony and Eytelwein have assumed that the head lost by the resistance of friction increases with the simple velocity and with the square of the same, and have established for it the formula

$$h = (\alpha v + \beta v^2) \frac{l}{d}$$

in which α and β denote constants determined by experiment. In order to determine these constants, these authors availed themselves of 51 experiments made at different times by Couplet, Bossut, and du Buat upon the flow of water through long tubes. Prony deduced from them

$$h = (0{,}0000693\, v + 0{,}0013932\, v^2) \frac{l}{d},$$

Eytelwein,

$$h = (0{,}0000894\, v + 0{,}0011213\, v^2) \frac{l}{d},$$

d'Aubuisson assumes

$$h = (0{,}0000753\, v + 0{,}001370\, v^2) \frac{l}{d} \text{ meters.}$$

The following formula, proposed by the author, coincides better with the results of observation; it is

$$h = \left(\alpha + \frac{\beta}{\sqrt{v}}\right) \frac{l}{d} \frac{v^2}{2g}$$

and is founded upon the assumption that the resistance of friction increases at the same time with the square and with the square root of the cube of the velocity. We have, therefore, for the coefficient of resistance

$$\zeta = \alpha + \frac{\beta}{\sqrt{v}},$$

and for the height of resistance of friction simply

$$h = \zeta \cdot \frac{l}{d} \frac{v^2}{2g}.$$

For the determination of the coefficient of resistance ζ or of the auxiliary constants α and β the author availed himself of not only the 51 experiments of Couplet, Bossut, and du Buat, employed by Prony and Eytelwein, but also of 11 experiments made by himself and one by a M. Gueymard, of Grenoble. The older experiments were made with velocities of from 0,043 to 1,930 meters, but by the experiments of the author this limit has been extended to 4,648 meters. The widths of the pipes in the older experiments were 27, 36, 54, 135, and 490 millimeters, and the newer experiments were made with pipes 33, 71, and 275 millimeters in diameter. By the aid of the method of least squares, the author found from the 63 experiments

$$\zeta = 0{,}01439 + \frac{0{,}0094711}{\sqrt{v}},$$

or

$$h = \left(0{,}01439 + \frac{0{,}0094711}{\sqrt{v}}\right)\frac{l}{d}\cdot\frac{v^2}{2g} \text{ meters},$$

or for the English system of measure

$$h = \left(0{,}01439 + \frac{0{,}017155}{\sqrt{v}}\right)\frac{l}{d}\cdot\frac{v^2}{2g}.$$

REMARK—1) If we take into consideration some other experiments made by Professor Zeuner with a zinc tube $2\frac{1}{2}$ centimeters wide, and with a velocity of from 0,1856 to 0,4287 meters, we obtain

$$\zeta = 0{,}014812 + \frac{0{,}010327}{\sqrt{v}},$$

v being given in meters.

2) Newer experiments upon the flow of water with great and very great velocities were made by the author in 1856 and 1858 (see the "Civilingenieur," Vol. V, Nos. 1 and 3, as well as Vol. IX, No. 1). The results of these experiments are contained in the following table:

Nature of the tubes.	Width of the tubes (d).	Mean velocity of the water in the tubes (v).	Coefficient of friction ζ.
Narrow glass tubes	1,03 ctm.	8,51 meters.	0,01815
Wider glass tubes	1,43 "	10,18 "	0,01865
Narrow brass tubes	1,04 "	8,64 "	0,01869
The same made shorter	1,04 "	12,32 "	0,01784
The same under very great pressure .	1,04 "	20,99 "	0,01690
Wider brass tubes	1,43 "	8,66 "	0,01719
The same made shorter	1,43 "	12,40 "	0,01736
The same under very great pressure .	1,43 "	21,59 "	0,01478
Wider zinc tubes	2,47 "	3,19 "	0,01962
The same shorter	2,47 "	4,73 "	0,01838
The same still shorter	2,47 "	6,24 "	0,01790
The same still shorter	2,47 "	9,18 "	0,01670

The values in the last column again show that the coefficient of resistance ζ for the friction of water in tubes decreases not only as the velocity (v) increases, but also, although more slowly, as the width (d) of the pipe becomes greater. However, for high velocities, the formula

$$\zeta = 0{,}01439 + \frac{0{,}0094711}{\sqrt{v}}$$

agrees tolerably well with the numbers found by experiment, E.G., for $v = 9$ meters

$$\zeta = 0{,}01439 + 0{,}00316 = 0{,}01755$$

and for $v = 16$ meters

$$\zeta = 0{,}01439 + 0{,}00237 = 0{,}01676.$$

These coincide very well with the values in the last table, which correspond most nearly to them.

REMARK 3.—M. de Saint-Venant found that the well-known formula for the resistance of water in tubes agrees better with the results of experiment, when we assume the height due to the friction to increase not with v^2 or $\frac{v^2}{2g}$, but with $v^{\frac{21}{11}}$. (See his "Mémoire sur des formules nouvelles pour la solution des problèmes relatifs aux eaux courantes.") According to him we must put

$$h = \frac{4\,l}{d} \,.\, 0{,}00029557\, v^{\frac{21}{11}} = 0{,}00118228\, \frac{l}{d} \,.\, v^{\frac{21}{11}} = 0{,}023197\, v^{-\frac{1}{11}} \,.\, \frac{l}{d}\,\frac{v^2}{2g}.$$

The assumption of a fractional exponent for v is not at all new; Woltmann put $v^{\frac{7}{4}}$ instead of v^2 and Eytelwein proposed $v^{\frac{15}{8}}$ instead of v^2 (see the author's article upon Efflux [Ausfluss] in the "allgemeine Maschinenencyclopädie" of Hülsse.

REMARK 4.—New and very extended experiments upon the motion of water in pipes have been made by Monsieur H. Darcy (see the report to the Academy of Sciences at Paris in the Comptes rendus, etc., Tom. 38, 1854, "sur des recherches expérimentales relatives au mouvement des eaux dans les tuyaux"). Mons. Darcy deduces from these experiments, where the velocity is not less than 2 decimeters, the formula

$$h = \left(0{,}000507 + \frac{0{,}00000647}{r}\right)\frac{l}{r} \,.\, v^2$$

$$= \left(0{,}01989 + \frac{0{,}0005078}{d}\right)\frac{l}{d}\,\frac{v^2}{2g} \text{ meters;}$$

hence the coefficient of resistance should be

$$\zeta = 0{,}01989 + \frac{0{,}0005078}{d}.$$

This formula, however, is not sufficiently accurate for small velocities.

§ 429. To facilitate calculation the following table of the *coefficients of resistance* has been arranged. We see from it that the variation of this coefficient is not insignificant, since for a velocity of 0,1 meter it is = 0,0443, for one of 1 meter, = 0,0239 and for one of 5 meters, = 0,0186.

TABLE OF THE COEFFICIENTS OF FRICTION OF WATER.

		Decimeters.									
	v	0	1	2	3	4	5	6	7	8	9
Meters.	0	∞	0,0443	0,0356	0,0317	0,0294	0,0278	0,0266	0,0257	0,0250	0,0244
	1	0,0239	0,0234	0,0230	0,0227	0,0224	0,0221	0,0219	0,0217	0,0215	0,0213
	2	0,0211	0,0209	0,0208	0,0206	0,0205	0,0204	0,0203	0,0202	0,0201	0,0200
	3	0,0199	0,0198	0,0197	0,0196	0,0195	0,0195	0,0194	0,0193	0,0193	0,0192
	4	0,0191	0,0191	0,0190	0,0190	0,0189	0,0189	0,0188	0,0188	0,0187	0,0187

We find in this table the coefficients of resistance corresponding to a certain velocity by searching for the whole meters in the vertical columns and for the tenths of a meter in the horizontal column and then moving horizontally from the first number and vertically from the last, until we arrive at the point where the two motions meet. E.G. for $v = 1,3$ meters, $\zeta = 0,0227$; for $v = 2,8$, $\zeta = 0,0201$.

For the English foot we can put

v	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9
ζ	0,0686	0,0527	0,0457	0,0415	0,0387	0,0365	0,0349	0,0336	0,0325

v	1	1¼	1½	2	3	4	6	8	12	20
ζ	0,0315	0,0297	0,0284	0,0265	0,0243	0,0230	0,0214	0,0205	0,0193	0,0182

REMARK.—A more extensive and more convenient table is to be found in the Ingenieur, pages 442 and 443.

§ 430. Long Pipes.—In considering the motion of water in long pipes or combinations of pipes, the three principal questions to be solved are the following.

1) The length l and the width d of the pipe and the quantity Q of water to be conducted may be given and we may be required to find the necessary head. In this case we must first calculate the velocity

$$v = \frac{Q}{F} = \frac{4\,Q}{\pi\,d^2} = 1,2732\,.\,\frac{Q}{d^2},$$

and then search in one of the last tables for the value of the coefficient of friction ζ, corresponding to this value, and finally we must substitute the values d, l, v, ζ and ζ_0 (ζ_0 denoting the coefficient for the orifice of influx) in the first principal formula

$$h = \left(1 + \zeta_0 + \zeta \frac{l}{d}\right) \frac{v^2}{2g}.$$

2) The length and width of the pipe and the head of water may be given and the discharge may be required. The velocity must be found by means of the formula

$$v = \frac{\sqrt{2gh}}{\sqrt{1 + \zeta_0 + \zeta \cdot \frac{l}{d}}}.$$

Now as the coefficient of resistance is not perfectly constant, but varies somewhat with v, we must first find v approximatively in order to be able to calculate ζ from it.

From v we determine

$$Q = \frac{\pi d^2}{4} v = 0{,}7854\, d^2 v.$$

3) The discharge, the head of water and the length of the pipe may be given, and we may be required to determine the necessary width of the pipe.

Since $v = \frac{4\,Q}{\pi d^2}$ or $v^2 = \left(\frac{4\,Q}{\pi}\right)^2 \cdot \frac{1}{d^4}$, we have

$$2\,g\,h = \left(1 + \zeta_0 + \zeta \frac{l}{d}\right) \left(\frac{4\,Q}{\pi}\right)^2 \cdot \frac{1}{d^4}, \text{ or}$$

$$2\,g\,h \cdot \left(\frac{\pi}{4\,Q}\right)^2 = (1 + \zeta_0) \frac{1}{d^4} + \zeta \frac{l}{d^5}, \text{ or}$$

$$2\,g\,h \cdot \left(\frac{\pi}{4\,Q}\right)^2 d^5 = (1 + \zeta_0)\, d + \zeta\, l;$$

hence the width of the pipe is

$$d = \sqrt[5]{\frac{(1 + \zeta_0)\, d + \zeta\, l}{2\,g\,h} \cdot \left(\frac{4\,Q}{\pi}\right)^2}.$$

But since $\left(\frac{4}{\pi}\right)^2 = 1{,}6212$ and $1 + \zeta_0$ as a mean $= 1{,}505$ and for the English system of measures $\frac{1}{2\,g} = 0{,}0155$, we can put

$$d = 0{,}4787 \sqrt[5]{(1{,}505 \,.\, d + \zeta\, l)\, \frac{Q^2}{h}} \text{ feet.}$$

This formula can only be used to obtain approximative values;

for not only the unknown quantity d, but also the coefficient ζ, which depends upon the velocity $v = \frac{4\,Q}{\pi\,d^2}$, occurs in it.

EXAMPLE 1) What must the head of water be, when a set of pipes 150 feet long and 5 inches in diameter is required to deliver 25 cubic feet of water per minute? Here we have

$$v = 1,2732\,\frac{25\,.\,12^2}{60\,.\,5^2} = 3,056 \text{ feet},$$

and therefore we can make $\zeta = 0,0243$; hence the head of water or total fall of the pipes must be

$$h = \left(1,505 + 0,0243\,.\,\frac{150\,.\,12}{5}\right).\,0,0155\,.\,3,056^2$$
$$= (1,505 + 8,748)\;0,0155\,.\,9,339 = 1,484 \text{ feet}.$$

2) What is the discharge through a set of pipes 48 feet long and 2 inches in diameter, under a head of 5 feet? Here

$$v = \frac{8,025\,\sqrt{5}}{\sqrt{1,505 + \zeta\,.\,\frac{48\,.\,12}{2}}} = \frac{17,945}{\sqrt{1,505 + 288\,.\,\zeta}}.$$

For the present, assuming $\zeta = 0,020$, we obtain

$$v = \frac{17,945}{\sqrt{7,26}} = \frac{17,945}{2,7} = 6,6\,;$$

but $v = 6,6$ gives more correctly $\zeta = 0,0211$, and therefore we have

$$v = \frac{17,945}{\sqrt{1,505 + 288\,.\,0,0211}} = \frac{17,945}{\sqrt{7,582}} = 6,52 \text{ feet},$$

and the discharge

$$Q = 0,7854\left(\frac{2}{12}\right)^2 6,52 = 0,142 \text{ cubic feet} = 245,4 \text{ cubic inches}.$$

3) What must be the diameter of a set of pipes 100 feet long, which are to discharge one half of one cubic foot of water per second under a head of 5 feet? Here

$$d = 0,4787\,\sqrt[5]{(1,505\,d + 100\,\zeta)\,.\,\tfrac{1}{5}\,.\,(\tfrac{1}{2})^2} = 0,4787\,\sqrt[5]{0,075\,d + 5\,\zeta}.$$

Assuming for the present $\zeta = 0,02$, we obtain

$$d = 0,4787\,\sqrt[5]{0,075\,d + 0,100}, \text{ or approximatively}$$
$$d = 0,4787\,\sqrt[5]{0,100} = 0,30\,;\text{ hence we have more accurately}$$
$$d = 0,4787\,\sqrt[5]{0,0225 + 0,100} = 0,4787\,\sqrt[5]{0,1225}$$
$$= 0,3145 \text{ feet} = 3,774 \text{ inches}.$$

This diameter corresponds to the cross-section

$$F = 0,7854\,.\,0,3145^2 = 0,0777 \text{ square feet};$$

the velocity is consequently

$$v = \frac{Q}{F} = \frac{0,5}{0,0777} = 6,435 \text{ feet},$$

and the coefficient of resistance $\zeta = 0,212$. Substituting the latter more correct value, we obtain

$$d = 0,4787\,\sqrt[5]{0,1285} = 0,318 \text{ feet} = 3,82 \text{ inches}.$$

Remark 1.—Experiments made by the author with ordinary wooden pipes 2⅓ and 4⅓ inches in diameter gave coefficients of resistance 1,75 times greater than those for metal pipes, given in the tables in the foregoing paragraph. While we have, when the velocity is 3 feet, for metal pipes $\zeta = 0,0243$, for wooden pipes its value is $= 0,0243 \,.\, 1,75 \,.\, 0,042525$; in example 1 we found for a metal pipe 150 feet long the head to be 1,484 feet, but for a wooden pipe under the same circumstances it would be

$$h = (1,505 + 0,042525 \,.\, 360)\; 0,0155 \,.\, 9,839 = 16,81 \,.\, 0,1448 = 2,43 \text{ feet.}$$

According to D'Arcy's Experiments, the coefficient of resistance ζ increases very considerably with the roughness of the walls of the pipe, and if the walls are very rough it is doubled or even trebled. The author found more recently the same result.

Remark 2.—The temperature also has an important influence upon the resistance of water in pipes. Experiments have been made upon this subject by Gerstner (see his "Handbuch der Mechanic," Vol. II), and more recently by Geh. Rath Hagen (see his "Abhandlungen über den Einfluss der Temperatur auf die Bewegung des Wassers in Röhren," Berlin, 1854). The experiments of the latter, made, it is true, with very narrow tubes ($d = 0,108$ to $0,227$ inches), have shown that under the same circumstances the velocity of the water in pipes does not decrease indefinitely with the temperature, but that for every tube there is a certain temperature for which this velocity is a maximum. For the experiments without this maximum, Hagen finds the following formula:

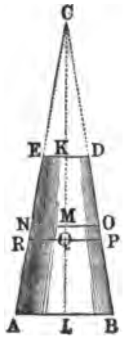
$$h = m\, l\, r^{-1,25} \,.\, v^{1,75}, \text{ and}$$

$$m = 0,000038941 - 0,0000017185 \sqrt{t},$$

in which the temperature t is expressed in degrees of the Reaumur thermometer, and the head h, the length l, the radius of the tube r and the velocity v in inches (Prussian).

(**§ 431.**) **Conical Pipes.**—The resistance of friction in a conical pipe $A\,D$, Fig. 730, can be found in the following manner. Let us denote the semi-angle of convergence of the walls of the pipe $A\,C\,L = B\,C\,L$ by δ, the diameter of the inlet orifice by d_1, that of the outlet by d_2, the length $K\,L$ of the pipe by l, and the velocity of efflux at $D\,E$ by v.

FIG. 730.



At a distance $K\,M = x$ from the outlet of the tube the diameter of the tube is

$$N\,O = y = D\,E + 2\,K\,M\, tang.\, \delta = d_2 + 2\,x\, tang.\, \delta,$$

hence for the velocity w at that point, since

$$\frac{w}{v} = \frac{d_2^2}{y^2}, \text{ we can put}$$

$$w = \frac{d_2^2}{y^2}\, v = \frac{v}{\left(1 + \dfrac{2\,x}{d_2}\, tang.\, \delta\right)^2}.$$

For an element $N\,O\,P\,R$ of the tube, whose length is

$$O\,P = N\,R = \frac{M\,Q}{cos.\,\delta} = \frac{d\,x}{cos.\,\delta},$$

the height of resistance of the friction is

$$d\,h = \zeta \cdot \frac{d\,x}{y\,cos.\,\delta} \cdot \frac{w^2}{2g} = \zeta \cdot \frac{d\,x}{y\,cos.\,\delta\left(1 + \frac{2x}{d_2}\,tang.\,\delta\right)^4} \cdot \frac{v^2}{2g}$$

$$= \zeta \cdot \frac{d\,x}{d_2\,cos.\,\delta\left(1 + \frac{2x}{d_2}\,tang.\,\delta\right)^5} \cdot \frac{v^2}{2\,g};$$

hence the height of resistance of friction for the whole tube is

$$h = \zeta \cdot \frac{v^2}{2g\,d_2} \int_0^l \frac{d\,x}{\left(1 + \frac{2x}{d_2}\,tang.\,\delta\right)^5 cos.\,\delta}.$$

But

$$\int \frac{d\,x}{\left(1 + \frac{2x}{d_2}\,tang.\,\delta\right)^5 cos.\,\delta}$$

$$= \frac{d_2}{2\,sin.\,\delta} \int \left(1 + \frac{2\,x}{d_2}\,tang.\,\delta\right)^{-5} d\left(\frac{2\,x}{d_2}\,tang.\,\delta\right)$$

$$= -\frac{d_2}{8\,sin.\,\delta}\left(1 + \frac{2\,x}{d_2}\,tang.\,\delta\right)^{-4},$$ whence we obtain

$$\int_0^l \frac{d\,x}{\left(1 + \frac{2\,x}{d_2}\,tang.\,\delta\right)^5 cos.\,\delta}$$

$$= \frac{d_2}{8\,sin.\,\delta}\left[1 - \left(1 + \frac{2\,l}{d_2}\,tang.\,\delta\right)^{-4}\right],$$ or

$$= \frac{d_2}{8\,sin.\,\delta}\left[1 - \left(\frac{d_1}{d_2}\right)^{-4}\right] = \frac{d_2}{8\,sin.\,\delta}\left[1 - \left(\frac{d_2}{d_1}\right)^4\right],$$

since $d_2 + 2\,l\,tang.\,\delta$ expresses the diameter d_1 of the inlet orifice.

Consequently the required height of resistance is

$$h = \zeta \cdot \frac{v^2}{2\,g\,d_2} \cdot \frac{d_2}{8\,sin.\,\delta}\left[1 - \left(\frac{d_2}{d_1}\right)^4\right]$$

$$= \tfrac{1}{8}\,\frac{\zeta}{sin.\,\delta}\left[1 - \left(\frac{d_2}{d_1}\right)^4\right] \cdot \frac{v^2}{2\,g} = \tfrac{1}{8}\,\zeta\,cosec.\,\delta\left[1 - \left(\frac{d_2}{d_1}\right)^4\right]\frac{v^2}{2\,g}.$$

If the inlet orifice is much larger than the outlet orifice, we can put $\left(\frac{d_2}{d_1}\right)^4 = 0$, and consequently

$$h = \tfrac{1}{8}\,\frac{\zeta}{sin.\,\delta} \cdot \frac{v^2}{2\,g} = \tfrac{1}{8}\,\zeta\,cosec.\,\delta\,.\,\frac{v^2}{2\,g};$$

the resistance of friction in this case does not depend at all upon the length of the tube.

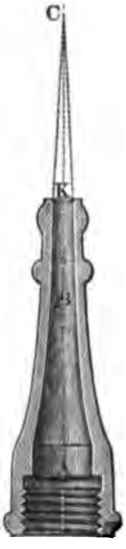
EXAMPLE.—If the angle of convergence of the outlet portion of the nozzle $A\,K$, Fig. 731, of a fire-engine is $2\,\delta = 5°$, that of the inlet portion $A\,B$, $2\,\delta_1 = 18°$, the width of the outlet $d_2 = 7$ lines, and the width of the inlet $d_1 = 1\frac{1}{2}$ inches $= 18$ lines, and if its whole length $A\,K = l = 6$ inches $= 72$ lines, what is its coefficient of resistance? Putting the length of the outlet portion $B\,K = l_1$ and that of the inlet portion $A\,B = l_2$, we have

$$l = l_1 + l_2 \text{ and } l_1 \, tang. \, \delta + l_2 \, tang. \, \delta_1 = \frac{d_1 - d_2}{2},$$

or in figures

$$l_1 + l_2 = 72 \text{ and } l_1 \, tang. \, 2\tfrac{1}{2}° + l_2 \, tang. \, 9° = \tfrac{11}{2}, \text{ or}$$
$$0{,}04362 \, l_1 + 0{,}15838 \, l_2 = 5{,}5.$$

FIG. 731.



Hence $l_1 = 51{,}54$ and $l_2 = 20{,}46$ lines and the width at B, where the conical surfaces meet each other, is
$d_3 = d_2 + 2\,l_1 \, tang. \, \delta = 7 + 2 \,.\, 51{,}54 \,.\, 0{,}04362 = 11{,}53$ lines. Since this place is rounded off, we can put $d_3 = 13$ lines; hence for the outlet piece

$$\left[1 - \left(\frac{d_2}{d_3}\right)^4\right] . \frac{1}{sin. \, \delta} = [1 - (\tfrac{7}{13})^4] \,.\, cosec. \, 2\tfrac{1}{2}°$$
$$= 0{,}9159 \,.\, 22{,}926 = 21{,}08,$$

and for the inlet portion

$$\left[1 - \left(\frac{d_3}{d_1}\right)^2\right] cosec. \, \delta_1 = [1 - (\tfrac{13}{18})^4] \,.\, cosec. \, 9°$$
$$= 0{,}7795 \,.\, 6{,}392 = 4{,}98.$$

Therefore the height of resistance for the entire nozzle is

$$h = \frac{\zeta}{8}\left[21{,}08 + 4{,}98 \,.\, \left(\frac{d_2}{d_3}\right)^4\right] . \frac{v^2}{2\,g}$$
$$= \frac{\zeta}{8}\left[21{,}08 + 4{,}98 \,.\, \left(\frac{7}{13}\right)^4\right] \frac{v^2}{2\,g} = 21{,}5 \,.\, \frac{\zeta}{8} \,.\, \frac{v^2}{2\,g}$$
$$= 2{,}7\,\zeta \,.\, \frac{v^2}{2\,g};$$

if we substitute $\frac{1}{2\,g} = 0{,}0155$ and assume $\zeta = 0{,}02$, we have

$$h = 0{,}054 \,.\, \frac{v^2}{2\,g},$$

i.e. about $\frac{1}{18}$ the height due to the velocity, which result coincides very well with the results of experiments with such a nozzle.

§ 432. Conduit Pipes.—The outlet at the end of a system of pipes is either under water or in the air. Both cases are represented in Figures 732 and 733. In the first case we must regard as the head h the difference of level $R\,C$ of the two surfaces of water, and in the second case the vertical distance $R\,O$ of the outlet orifice O below the level H of the water in the reservoir. If the tube is everywhere of the same width d, the formulas found in § 430 can be applied directly; but if the tube is enlarged or narrowed at any point, we will have several different velocities in the pipe, and therefore the resistance of friction for each portion of the pipe must be calculated separately. Such a case is presented by the pipes in Fig. 733, which lead to a fountain or jet d'eau, in which case the mouth-piece O is narrower than the pipe $B\,L\,M$, which conveys the water. If we put, as we generally do, the velocity of efflux $= v$, the width of the orifice O of efflux $= d$, the width of the pipe $= d_1$, we have the velocity of the water in the pipe

FIG. 732.

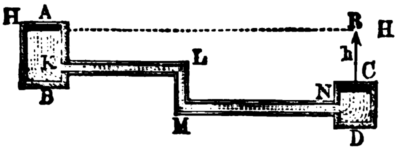
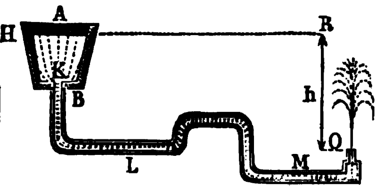


FIG. 733.



$$v_1 = \left(\frac{d}{d_1}\right)^2 v,$$

and if we denote by l the length of the pipe $B\,L\,M$ and by ζ_1 the coefficient of friction, we have for the corresponding height of friction

$$h_1 = \zeta_1 \frac{l_1}{d_1} \frac{v_1^2}{2g} = \zeta_1 \frac{l_1}{d_1} \left(\frac{d}{d_1}\right)^4 \cdot \frac{v^2}{2g}.$$

Now if ζ_0 is the coefficient of friction for the *inlet orifice* K and ζ that for the *outlet orifice* O, it follows that the loss of head caused by the first is

$$h_0 = \zeta_0 \frac{v_1^2}{2g} = \zeta_0 \left(\frac{d}{d_1}\right)^4 \cdot \frac{v^2}{2g},$$

and, on the contrary, that occasioned by passing through the second is

$$h_2 = \zeta \frac{v^2}{2g};$$

hence we have the entire head

$$h = \frac{v^2}{2g} + h_0 + h_1 + h_2 = \left[1 + \zeta_0 \left(\frac{d}{d_1}\right)^4 + \zeta_1 \frac{l_1}{d_1} \left(\frac{d}{d_1}\right)^4 + \zeta\right] \frac{v^2}{2g},$$

and inversely the velocity of efflux

$$v = \sqrt{\frac{2gh}{1 + \left(\zeta_0 + \zeta_1 \frac{l_1}{d_1}\right) \left(\frac{d}{d_1}\right)^4 + \zeta}}.$$

If we wish the jet to rise to the greatest height, the *orifice or mouthpiece* must not only cause as little resistance as possible, but also allow the water to issue from it with its fibres nearly parallel, so that they may form, while rising, a stream which will hold together as long as possible, and consequently be less disturbed by the air than a stream which was more or less torn when it left the orifice. For this reason we prefer a short, cylindrical or slightly conical mouth-piece, with the orifice of influx rounded off, to an orifice in a thin plate or to the orifices of the form of the contracted stream, although the former cause a greater loss of velocity than the latter. The nodes and bulges, which a stream which has passed through the latter orifices forms or tends to form, give the air a much better chance to penetrate it than a cylindrical stream.

§ **433. Jets of Water.**—So long as the stream *K L N*, which flows vertically downwards through a horizontal orifice *K*, Fig. 734, remains continuous and is not broken up by the air, its cross-section *L* decreases more and more as the distance $K L = x$ from the orifice increases. If c is the velocity of efflux and v the velocity at *L*, we have

$$v^2 = 2 g x + c^2,$$

denoting by F the cross-section of the orifice of efflux and by Y that of the stream at *L*, we have the following equation

$$F c = Y v \text{ or } F^2 c^2 = Y^2 v^2,$$

from which we deduce the equation

$$Y^2 (c^2 + 2 g x) = F c^2, \text{ or}$$

$$Y^2 = \frac{F^2 c^2}{c^2 + 2 g x}$$

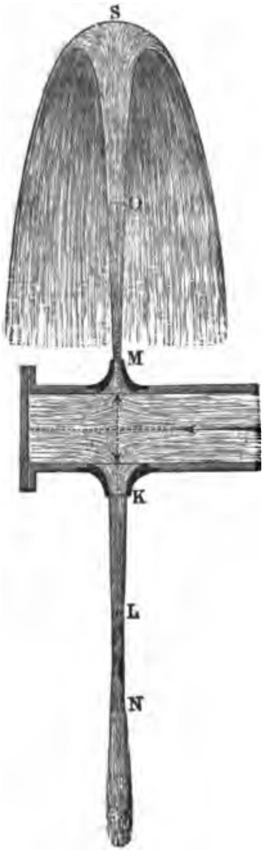
for the form of the *cataract* of Newton (see Newton's Principia Philosophiæ, Vol. II, Sect. VII). If the cross-section of the orifice *K* is a circle, whose diameter is *d*, the cross-section at *L* forms a circle, whose diameter is *y* and for which we can put

$$y^4 = \frac{c^2 d^4}{c^2 + 2 g x}, \text{ or}$$

$$y = \frac{d}{\sqrt[4]{1 + \frac{2 g x}{c^2}}}.$$

Experiments upon the internal constitution of falling streams of water have

FIG. 734.



been made by Savart. See Poggendorff's Annalen der Physik, Vol. 33.

The cross-section O of a stream MS, which rises vertically from a horizontal orifice M, increases gradually with its distance $MO = x$ from the orifice M; for here the velocity of the water at O is

$$v = \sqrt{c^2 - 2gx}, \text{ and therefore}$$

$$Y^2 = \frac{F^2 c^2}{c^2 - 2gx};$$

hence we have for the diameter of the cross-section at O

$$y^4 = \frac{c^2 d^4}{c^2 - 2gx}, \text{ or } y = \frac{d}{\sqrt[4]{1 - \frac{2gx}{c^2}}}.$$

Denoting the height due to the velocity $\frac{c^2}{2g}$ by h, we have simply and generally

$$y_1 = \frac{d}{\sqrt[4]{1 \pm \frac{x}{h}}}.$$

This formula becomes incorrect at its limits; according to it, E.G. in the rising stream for $x = h$ or at the apex S, the diameter of the stream would be

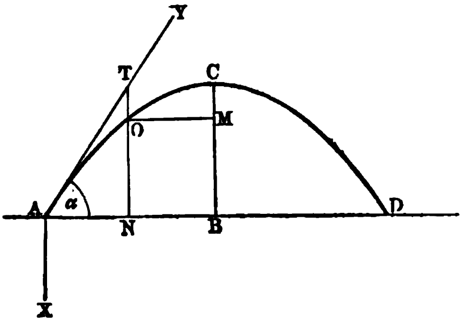
$$y = \frac{d}{\sqrt{1 - 1}} = \frac{d}{0} = \infty.$$

This, however, is not the case; for the various fibres of water, of which the stream is composed, are not really at rest at the highest point, but possess a small velocity radially outwards. If the stream of water $A\ O\ C$, Fig. 735, is inclined to the horizon, this formula

$$y = \frac{d}{\sqrt[4]{1 \pm \frac{x}{h}}}$$

is still applicable, when we substitute instead of x the vertical projection NO of the stream AO. If the jet flows

FIG. 735.



out of the orifice at an angle v to the horizon, its maximum height $B\,C$ is

$$a = \frac{c^2\,(sin.\ v)^2}{2\,g} = h\,(sin.\ v)^2 \text{ (see § 39).}$$

Therefore its diameter (at the vertex C) is

$$y = \frac{d}{\sqrt[4]{1 - \frac{a}{h}}} = \frac{d}{\sqrt[4]{1 - (sin.\ v)^2}} = \frac{d}{\sqrt{cos.\ v}}.$$

In the descending portion $C\,D$ of the stream, y becomes gradually smaller and smaller, and when the stream reaches the horizontal plane $A\ D$, from which it started, y becomes again $= d$, if the air has produced no disturbance in the motion of the stream.

§ **434.** The height s, to which a vertical jet of water will rise, is approximatively equal to height due to the velocity $h = \frac{c^2}{2\,g}$, only when the velocity of efflux (c) is small. From the experiments made by the author (see the experiments upon the height of rise of jets of water with different mouth-pieces in the 5th vol. of the Zeitschrift des Vereins deutscher Ingenieure), the following facts concerning jets of water were ascertained.

1) The resistance of the air for small velocities of efflux, viz., from 5 to 20 feet, or for heights of rise of from 1 to 6 feet, is so small that the height of rise of the jet may in this case without appreciable error be put equal to the height due to the velocity $\frac{c^2}{2\,g}$.

2) If the height due to the velocity does not exceed 75 feet or the velocity of efflux 56 feet, the ratio of the height of rise to the height due to the velocity can be expressed by the formula

$$\frac{s}{h} = \frac{1}{\alpha + \beta\,h + \gamma\,h^2},$$

in which α, β and γ denote empirical coefficients to be determined for each mouth-piece.

3) For jets, which issue from orifices in a thin plate, the constant α can be put $= 1$; hence we can assume that the resistance during the passage through the orifice is almost null, when the velocities are small, and that it is measurable only when the velocities are great. The coefficient of resistance for these orifices is therefore not constant, but increases from zero gradually with

the velocity; the value $\zeta = 0,97$, given in § 408, can only be considered as a mean one.

4) For the same velocity of efflux the height of rise increases with the thickness of the stream, or with the width of the orifice; consequently the resistance of the air is smaller for thick than for thin streams. The height of rise increases, therefore, not only with the head, but also with the thickness of the stream.

5) Under the same circumstance a stream, issuing from a circular orifice, rises higher than one discharged from an aperture of a different shape (square, etc.)

6) If the velocities of efflux and the widths of the orifices are the same, those streams which are not contracted rise higher than those which are, not only because the former are thinner, but also because the latter, in consequence of their contractions and expansions, oppose less resistance to the penetration of the air.

If the other circumstances and relations are the same and if the velocities of efflux are not very small, the jets issuing from short cone-shaped and longer conical tubes or ajutages with an internal rounding off attain the greatest height.

Mariotte concluded from his experiments upon the height of rise of jets of water (see Meining's Translation of Mariotte's Principles of Hydrostatics and Hydraulics) with orifices in a thin plate 4 to 6 lines in diameter and under heads of from 5½ to 35 feet that the head or height due to the velocity, necessary to produce the rise s, must be

$$h = s + \frac{s^2}{300} \text{ Paris feet,}$$

whence

$$\frac{h}{s} = 1 + \frac{s}{300} = 1 + 0,003333\, s.$$

The very extensive and varied experiments of the author, made under heads of from 3 to 70 feet, give, on the contrary, for *circular orifices in a thin plate*, when their diameter was

1) 1 centimeter

$$\frac{h}{s} = 1 + 0,0035305\, h + 0,00005406\, h^2, \text{ and when it was}$$

2) 1,41 centimeters

$$\frac{h}{s} = 1 + 0,00237191\, h + 0,00005609\, h^2,$$

h being given in English feet.

FIG. 736.

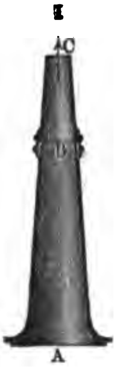
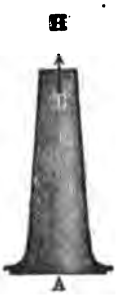
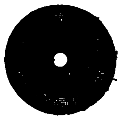


FIG. 737.





With a conical mouth-piece $A\ B\ C$, Fig. 736, 15 centimeters long and 1 centimeter wide at the outlet C and 3 centimeters wide at the inlet orifice A, which was well rounded off, the following result was obtained:

$$3)\ \frac{h}{s} = 1{,}0453 + 0{,}0001137\,h + 0{,}00007981\,h^2,$$

and, on the contrary, with the truncated mouth-piece $A\ B$, Fig. 737, whose width was 1,41 centimeters at the outlet B, the result was

$$4)\ \frac{h}{s} = 1{,}0216 + 0{,}0007294\,h + 0{,}00003036\,h^2.$$

By the aid of these formulas the following table of the *heights of jets* has been calculated.

Height due to velocity h =	10	20	30	40	50	60	70
Height of jet according to (1)..	9,61	18,31	25,98	32,58	38,12	42,66	46,30
" " " (2)..	9,715	18,69	26,75	33,77	39,72	44,68	48,58
" " " (3)..	9,48	18,53	26,77	33,97	39,98	44,79	48,47
" " " (4)..	9,69	19,08	28,02	36,39	44,09	51,08	57,31

EXAMPLE.—If the pipe conducting the water to a fountain is 350 feet long and 2 inches in diameter, and if the conical orifice is ½ inch wide, how high would the jet rise under a head of 40 feet, provided all the resistances, except the friction, are small enough to be neglected?

Here if we put

$$\zeta_1 = 0{,}025,\ \zeta_0 = 0{,}5,\ \left(\frac{d}{d_1}\right)^4 = (\tfrac{1}{4})^4 = \tfrac{1}{256} \text{ and } \frac{l_1}{d_1} = \frac{350}{\tfrac{2}{12}} = 2100,$$

the height due to the velocity of efflux is

$$h = \frac{v_2}{2g} = \frac{h_0}{1 + \left(\zeta_0 + \zeta_1\frac{l}{d}\right)\left(\frac{d}{d_1}\right)^4} = \frac{40}{1 + (0{,}5 + 0{,}025\,.\,2100)\,.\,\frac{1}{256}}$$

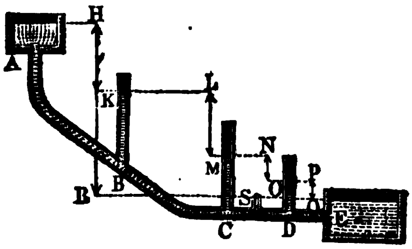
$$= \frac{40}{1{,}207} = 33{,}14 \text{ feet},$$

and therefore the height to which the jet will rise in still air is

$$z = \frac{h}{1{,}0216 + 0{,}0007294\,h + 0{,}00003036\,h^2} = \frac{33{,}14}{1{,}0216 + 0{,}02417 + 0{,}03334}$$
$$= \frac{33{,}14}{1{,}0791} = 30{,}71 \text{ feet.}$$

§ **435. Piezometer.**—The *head*, lost by the water which is passing through a set of pipes *A B C D E*, Fig. 738, in consequence of contractions in the conduit, friction, etc., can be measured by means of the columns of water maintained in the vertical tubes *B K*, *C M*, *D O* which are attached to the pipe; when they serve for this purpose only, they are called *piezometers* (see § 386).

Fig. 738.



If v is the velocity of the water at the point B, Fig. 738, where a piezometer is inserted, l the length and d the width of the portion $A\ B$ of the pipe, h the head of water or depth of the point B below the level of the water, ζ_0 the coefficient of resistance for the entrance of the water from the reservoir into the pipe and ζ the coefficient of friction, we have the height of the piezometer, which measures the pressure in B,

$$h - z = \cancel{h} + \left(1 + \zeta_0 + \zeta\frac{l}{d}\right)\frac{v^2}{2g}.$$

On the contrary, if the length of the portion $B\ C$ of the pipe is l_1 and the fall is h_1, we have the height of the piezometer at C

$$z_1 = h + h_1 - \left(1 + \zeta_0 + \zeta\frac{l}{d} + \zeta\frac{l_1}{d}\right)\frac{v^2}{2g}.$$

Hence the difference of the heights of the piezometer is

$$z_1 - z = h_1 - \zeta\frac{l_1}{d}\cdot\frac{v^2}{2g},$$

and, inversely, the *height of resistance of the portion B C of the pipe is*

$\zeta\frac{l_1}{d}\cdot\frac{v^2}{2g} = h_1 + z - z_1 =$ *fall of this portion of the pipe plus the difference of the heights of the piezometers.*

We see from this example that the piezometer can be employed to measure the resistances, which the water has to overcome in passing through the pipes. If any obstacle, if, E.G., a small body sticks fast in the pipe, its presence will be shown immediately by the sinking of the column of water in the piezometer, and the distance it sinks will indicate the amount of this resistance. The resistances occasioned by regulating apparatuses, such as cocks, valves, etc. (a subject which will be treated in the following chapter), can also be expressed by the height of the piezometer. Thus the piezometer at D is lower than at C not only on account of the friction of the water in the portion $C\,D$ of the tube, but also on account of contraction in the pipe produced by the valve gate S. If, when the valve-gate is completely open, the difference $N\,O$ of the heights of the piezometers $= h_1$ and if, when the gate is pushed in a certain distance, it is $= h_2$, the difference, or sinking, $h_1 - h_2$, gives the height of resistance due to the passage of the water through the valve gate.

Finally, the velocity of efflux of the water can be calculated from the height of the piezometer. If the height of the piezometer $P\,Q = z$, the length of the last portion of the tube $D\,E = l$ and the width of the same $= d$, we have

$$z = \zeta \frac{l}{d} \cdot \frac{v^2}{2g}$$ and therefore the velocity of efflux is

$$v = \sqrt{\frac{2gz}{\zeta \frac{l}{d}}} = \sqrt{\frac{d}{l} \cdot \frac{2gz}{\zeta}}.$$

EXAMPLE.—If the height of the piezometer $P\,Q = z$ upon the system of pipes in Fig. 738 is ¾ feet, if the length of the pipe $D\,E$, measured from the piezometer to the outlet orifice, is $l = 150$ feet and if the diameter of the tube is 3½ inches, it follows, when the coefficient of resistance $\zeta = 0{,}025$, that the velocity of efflux is

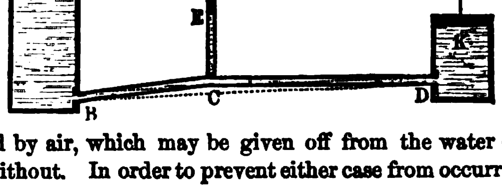
$$v = 8{,}025 \sqrt{\frac{3{,}5}{150\,.\,12} \cdot \frac{0{,}75}{0{,}025}} = 8{,}025\,.\,0{,}2415 = 1{,}94 \text{ feet,}$$

and the discharge

$$Q = \frac{\pi}{4} \cdot \left(\frac{3{,}5}{12}\right)^2 . \; 1{,}94 = 0{,}129 \text{ cubic feet.}$$

REMARK.—The motion of water in a pipe $B\,C\,D$, Fig. 739, can easily

FIG. 739.



be disturbed by air, which may be given off from the water or enter the pipe from without. In order to prevent either case from occurring, we must take care that the pressure at every point shall be positive, or rather that it shall exceed the atmospheric pressure, or that there shall be a column of water $C\,E$ in every piezometer. The height of this column is

$$s = h_1 - \left(1 + \zeta_0 + \zeta\frac{l_1}{d}\right)\frac{v^2}{2\,g},$$

h_1 denoting the head $C\,O$ at C, l_1 the length of the portion $B\,C$ of the pipe and v the velocity of the water in the tube. It is, therefore, necessary that

$$h_1 > \left(1 + \zeta_0 + \zeta\frac{l_1}{d}\right)\frac{v^2}{2\,g},$$

that, E.G., the head of water in the receiving reservoir shall at least exceed the height due to the velocity of the water in the pipe. Otherwise the pipe may suck in air in an eddy.

We can also put $h_1 > \dfrac{1 + \zeta_0 + \zeta\dfrac{l_1}{d}}{1 + \zeta_0 - \zeta\dfrac{l}{d}}\,h$, h denoting the entire fall $H\,K$ of the pipe and l its entire length $B\,C\,D$.

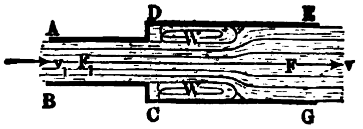
If we wish to prevent the air from accumulating in the pipe, we may lay the pipe in such a position that it will rise slightly in the direction in which the water is moving. The air will then be carried along with the water.

CHAPTER IV.

RESISTANCE TO THE MOTION OF WATER WHEN THE CONDUIT IS SUDDENLY ENLARGED OR CONTRACTED

§ 436. **Sudden Enlargement.**—*Changes in the cross-section* of a pipe or any other conduit produce a change of velocity. The velocity is inversely proportional to the cross-section of the stream; the wider the vessel is, the smaller is the velocity, and the narrower the vessel is, the greater is the velocity of the water flowing through it. If the cross-section of a vessel changes suddenly, as, E.G., that of the tube $A\ C\ E$, Fig. 740, does, a sudden change of velocity, accompanied by a loss of vis viva and a corresponding diminution of pressure, takes place. This loss is calculated in exactly the same manner as the loss of mechanical effect occasioned by the impact of inelastic bodies (see § 335). Every element of the water,

FIG. 740.



which passes out of the narrower tube $B\ D$ into the wider one $D\ G$, impinges against the more slowly moving current in this pipe and after the impact moves forward with it. Exactly the same phenomena occur when solid inelastic bodies collide; these bodies also move forward after the impact with a common velocity. Now we have found that the loss of mechanical effect occasioned by the impact of inelastic bodies is

$$L = \frac{(v_1 - v_2)^2}{2\,g} \cdot \frac{G_1\,G_2}{G_1 + G_2},$$

and since in this case the impinging element G_1 is infinitely small compared to the mass of water G_2 impinged upon, we can put

$$L = \frac{(v_1 - v_2)^2}{2\,g} G_1,$$

and consequently the corresponding *loss of head* is

$$h = \frac{(v_1 - v_2)^2}{2\,g}.$$

Hence, by the sudden change of velocity, a loss of head is caused, which is measured by the height due to this change of velocity.

Now if the cross-section of the one pipe $A\ C$, $= F_1$, that of the other pipe $C\ E$, which is united to it, $= F$, the velocity of the water in the first tube $= v_1$ and that in the other $= v$, we have

$$v_1 = \frac{F\,v}{F_1}$$

and therefore the *loss of head* in passing from one tube to the other is

$$h_1 = \left(\frac{F}{F_1} - 1\right)^2 \cdot \frac{v^2}{2\,g},$$

and the corresponding *coefficient of resistance*, which was first found by Borda, is

$$\zeta = \left(\frac{F}{F_1} - 1\right)^2.$$

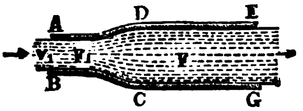
The head

$$h_1 = \left(\frac{F}{F_1} - 1\right)^2 \cdot \frac{v^2}{2\,g},$$

which we have just found, cannot of course be lost without producing any effect; we must rather assume that the mechanical effect corresponding to it is employed in separating and communicating a vibratory motion to the elements of the water, which before formed a continuous mass, and in forming the eddies W, W.

The experiments made by the author confirm this theory. If the tube $D\ G$ is to be maintained full of water it must not be very short or much wider than the tube $A\ C$. The loss is done away with when, as in Fig. 741, the edges are rounded off so as to cause a gradual passage from one tube into the other.

Fig. 741.



Example.—If the diameter of one of the portions of the compound pipe, Fig. 740, is twice that of the other, then $\frac{F}{F_1} = (\frac{2}{1})^2 = 4$, the coefficient of resistance $\zeta = (4 - 1)^2 = 9$ and the corresponding height of resistance for the passage from the narrower to the wider tubes is $= 9 . \frac{v^2}{2g}$. If the velocity of the water in the latter pipe is $= 10$ feet, it follows that the height of resistance is $= 9 . 0{,}0155 . 10^2 = 13{,}95$ feet.

§ **437. Contraction.**—A *sudden change of velocity* also takes place, when the water passes from a vessel $A\ B$, Fig. 742, into a narrower pipe $D\ G$, particularly if at the place of inlet $C\ D$ there is a diaphragm with an opening, whose cross-section is smaller than the cross-section of the pipe $D\ G$. If the area of this orifice $= F_1$ and if a is the coefficient of contraction, we have the cross-section of the contracted stream $F_2 = a\ F_1$; and if, on the contrary, F is the cross-section of the pipe and v the velocity of efflux, we find the velocity of the water at the contracted cross-section F_2 by means of the formula

$$v_2 = \frac{F}{a\ F_1}\, v;$$

hence the loss of head in passing from F_2 to F or from v_2 to v is

$$h = \frac{(v_2 - v)^2}{2\ g} = \left(\frac{F}{a_1\ F_1} - 1\right)^2 \frac{v^2}{2\ g},$$

and the corresponding coefficient of resistance is

$$\zeta = \left(\frac{F}{a_1\ F_1} - 1\right)^2.$$

Fig. 742.

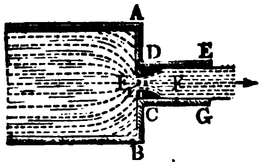
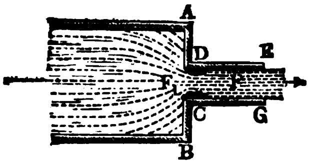


Fig. 743.



If the diaphragm is absent, we have a common short pipe, Fig. 743, and then $F = F_1$ and

$$\zeta = \left(\frac{1}{a} - 1\right)^2,$$

or inversely

$$a = \frac{1}{1 + \sqrt{\zeta}}.$$

Assuming $a = 0,64$, we obtain

$$\zeta = \left(\frac{1 - 0,64}{0,64}\right)^2 = (\tfrac{9}{16})^2 = 0,316.$$

ζ is increased by the resistance at the entrance into the tube and by the friction of the water in the exterior portion of the tube to 0,505 (§ 422).

From experiments made with a *short tube, the inlet orifice of which was contracted* as is represented in Fig. 742, the author has been led to the following conclusion: The coefficient of resistance for the passage of the water through the diaphragm and into the wider tube can be expressed by the following formula:

$$\zeta = \left(\frac{F}{a\,F_1} - 1\right)^2;$$

but we must put

for $\frac{F_1}{F}$ =	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
a =	0,616	0,614	0,612	0,610	0,607	0,605	0,603	0,601	0,598	0,596

and consequently

ζ =	231,7	50,99	19,78	9,612	5,256	3,077	1,876	1,169	0,734	0,480

If, E.G., the narrow cross-section is half that of the pipe, the coefficient of resistance is $\zeta = 5,256$, I.E. the passage through this contracted orifice occasions a loss of head 5¼ times as great as the height due to the velocity.

EXAMPLE.—What is the discharge through the apparatus represented in Fig. 742, when the head is 1½ feet, the diameter of the contracted circular orifice 1½, and that of the pipe $C\,E_1 = 2$ inches? Here we have

$$\frac{F_1}{F} = \left(\frac{1\frac{1}{2}}{2}\right)^2 = (\tfrac{3}{4})^2 = \tfrac{9}{16} = 0,56 \text{ and therefore } a = 0,606, \text{ and}$$

$$\zeta = \left(\frac{16}{9\,.\,0,606} - 1\right)^2 = \left(\frac{16 - 5,454}{5,454}\right)^2 = \left(\frac{10,546}{5,454}\right)^2 = 3,74.$$

Now if we put $h = (1 + \zeta)\frac{v^2}{2g}$, we obtain the velocity of efflux

$$v = \frac{\sqrt{2gh}}{\sqrt{1+\zeta}} = \frac{8{,}025\sqrt{1{,}5}}{\sqrt{4{,}74}} = 4{,}51,$$

and consequently the discharge is

$$Q = \frac{\pi d^2}{4} v = \frac{\pi}{4} \,.\, 4 \,.\, 12 \,.\, 4{,}51 = 54{,}12 \,.\, \pi = 170 \text{ cubic inches.}$$

§ 438. Influence of Imperfect Contraction.—In the case considered in the last paragraph, where the water comes from a large vessel, the contraction can be considered as perfect; but if the cross-section of the vessel, or that of the stream which arrives at the narrow orifice, is not very great compared to the cross-section F_1, Fig. 744, of that orifice, the *contraction is imperfect*, and the coefficient of resistance is consequently smaller than in the case just considered. If the notations previously employed are retained, we have again the height of resistance or the head lost in passing through F_1

$$h = \left(\frac{F}{a_1 F_1} - 1\right)^2 \frac{v^2}{2g},$$

but we must substitute variable values for a, which increase with the ratio $\frac{F_1}{G}$ of the cross-section of the narrow orifice to that of the pipe, which conducts the water to it. If a diaphragm is placed in a pipe $A\,G$, Fig. 745, of constant diameter, the same reasoning holds good; but the coefficient a depends upon $\frac{F_1}{F}$

FIG. 744.

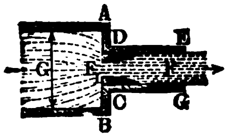
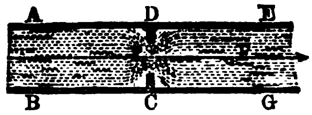


FIG. 745.



According to the author's experiments, we must substitute in the formula for the coefficient of resistance

$$\zeta = \left(\frac{F}{a_1 F_1} - 1\right)^2.$$

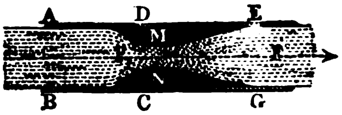
for $\frac{F_1}{F}$ =	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
a_1 =	0,624	0,632	0,643	0,659	0,681	0,712	0,755	0,813	0,892	1,000

whence it follows that

ζ =	225,9	47,77	30,83	7,801	3,753	1,796	0,797	0,290	0,060	0,000

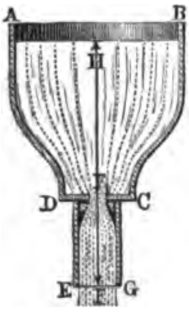
If *by rounding off the edges* the contraction is diminished or prevented, the loss of head becomes smaller, and it can be done away with, almost entirely, by introducing into the pipe a piece, which widens gradually and is shaped like $M\,N$, Fig. 746.

FIG. 746.



EXAMPLE.—What head is necessary, if the apparatus represented in Fig. 747 is required to deliver 8 cubic feet of water per minute? Let the width of the diaphragm F_1 be = 1½ inches, the width of the discharge-pipe $D\,G$, = 2 inches, and the width of the vessel $A\,C$, = 3 inches, then we have

FIG. 747.



$$\frac{F}{G} = \left(\frac{1\frac{1}{2}}{3}\right)^2 = \frac{1}{4}, \text{ whence } a = 0,637; \text{ now}$$

$$\frac{F}{F_1} = \left(\frac{2}{1\frac{1}{2}}\right)^2 = (\tfrac{4}{3})^2 = \tfrac{16}{9},$$

and the coefficient of resistance

$$\zeta = \left(\frac{16}{9\,.\,0,637} - 1\right)^2 = \left(\frac{10,267}{5,733}\right)^2 = 3,207.$$

Hence it follows that the velocity of efflux is

$$v = \frac{4\,Q}{\pi\,d^2} = \frac{4\,.\,8}{60\,.\,\pi\,(\frac{1}{6})^2} = \frac{19,2}{\pi} = 6,112 \text{ feet},$$

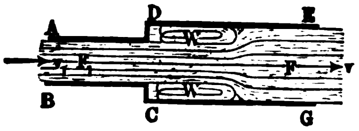
and, therefore, the required head is

$$h = \left(1 + \zeta\right)\frac{v^2}{2\,g} = 4,207\,.\,0,0155\,.\,6,112^2 = 2,43 \text{ feet.}$$

§ 439. Relations of Pressure in Cylindrical Pipes.

—By the aid of Borda's formula we can calculate the various relations of the pressure in a discharge pipe, the diameter of which is not constant. Let p_1 be the pressure and v_1 the ve-

FIG. 748.



locity of the water at F_1, and p the pressure and v the velocity of the same at F, then we have

$$\frac{p}{\gamma} + \frac{v^2}{2g} + \frac{(v_1 - v)^2}{2g} = \frac{p_1}{\gamma} + \frac{v_1^2}{2g}, \text{ and therefore}$$

$$\frac{p_1}{\gamma} = \frac{p}{\gamma} + \frac{v^2 - v_1^2 + (v_1 - v)^2}{2g} = \frac{p}{\gamma} - \frac{(v_1 - v)\,v}{g}, \text{ or}$$

$$\frac{p_1}{\gamma} = \frac{p}{\gamma} - \left(\frac{F}{F_1} - 1\right)\frac{v^2}{g}.$$

But the total head is

$$h = \frac{v^2}{2g} + \frac{(v_1 - v)^2}{2g} = \left[1 + \left(\frac{F}{F_1} - 1\right)^2\right]\frac{v^2}{2g};$$

hence we have also

$$\frac{p_1}{\gamma} = \frac{p}{\gamma} - \frac{2\,(v_1 - v)\,v}{v^2 + (v_1 - v)^2}\,h,$$

$$= \frac{p}{\gamma} - \frac{2\left(\frac{F}{F_1} - 1\right)h.}{1 + \left(\frac{F}{F_1} - 1\right)^2}$$

When a stream of water, whose cross-section is F, flows into the free air, $\frac{p}{\gamma}$ is $=$ to the height b of the water barometer, and therefore the height of the piezometer at F_1 is

$$z_1 = \frac{p_1}{\gamma} = b - \frac{2\left(\frac{F}{F_1} - 1\right)h}{1 + \left(\frac{F}{F_1} - 1\right)^2}.$$

So long as p remains positive, the water will discharge at $E\,G$ with the cross-section F filled; if, on the contrary, p becomes negative, the supposed condition of efflux ceases to exist and the water flows through the exterior tube $C\,E$, as if it were not there, with the theoretical velocity $v_1 = \sqrt{2\,g\,h}$.

In order to have a full discharge at $E\,G$, it is necessary that

$$\frac{2\left(\frac{F}{F_1} - 1\right)h}{1 + \left(\frac{F}{F_1} - 1\right)^2} < b, \text{ or that}$$

$$\frac{h}{b} < \frac{1 + \left(\frac{F}{F_1} - 1\right)^2}{2\left(\frac{F}{F_1} - 1\right)}.$$

If, then, the limits of the head h, given by this formula, are surpassed, the discharge with a full cross-section ceases.

These formulas are also applicable to the case of the pipe $C\,E$, Fig. 742, with a diaphragm; but here we must substitute instead F_1, $a_1\,F_1$; hence, for efflux with a filled tube, we must have

$$\frac{h}{b} < \frac{1 + \left(\frac{F}{a_1\,F_1} - 1\right)^2}{2\left(\frac{F}{a_1\,F_1} - 1\right)}$$

If we remove the diaphragm, we have a *simple short pipe* $C\,E$, Fig. 743, and then $F_1 = F$; hence we must put

$$\frac{h}{b} < \frac{1 + \left(\frac{1}{a} - 1\right)^2}{2\left(\frac{1}{a} - 1\right)}.$$

If we substitute $a = 0{,}64$ or $\frac{1}{a} - 1 = 0{,}5625$, we obtain the limit of discharge with filled cross-section through these pipes

$$\frac{h}{b} < \frac{1 + 0{,}3164}{2\,.\,0{,}5625}, \text{ I.E. } \frac{h}{b} < 1{,}17.$$

If we assume $b = 34$ feet, it follows that when the head is greater than $1{,}17\,.\,34 = 39{,}8$ feet, the efflux with a full cross-section through a short pipe ceases.

The results of the author's experiments coincide perfectly with the above conclusions (see the article upon the efflux of water under great pressure in the 9th volume of the "Civilingenieur").

This limit is reached more quickly, when the water discharges into rarefied air; for in that case b is less than 34 feet. If, E.G., the height of the water barometer in this space was three feet, the efflux with filled cross-section for a short pipe would cease when the head became $h = 1{,}17\,.\,3 = 3{,}51$ feet.

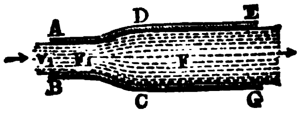
If the water flows through a pipe $A\;C\,E$, Fig. 749, which is gradually enlarged, the height of the piezometer at the inlet portion $A\,B$ is

$$\frac{p_1}{\gamma} = \frac{p}{\gamma} - \frac{v_1^2 - v^2}{2\,g} = \frac{p}{\gamma} - \left[\left(\frac{F}{F_1}\right)^2 - 1\right]\frac{v^2}{2\,g} = \frac{p}{\gamma} - \left[\left(\frac{F}{F_1}\right)^2 - 1\right] h,$$

consequently, if we put $\frac{p}{\gamma} = b$,

$$z_1 = \frac{p_1}{\gamma} = b - \left[\left(\frac{F}{F_1}\right)^2 - 1\right] h.$$

FIG. 749.



We must have, therefore,

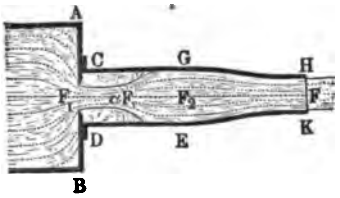
$$\frac{h}{b} < \frac{1}{\left(\frac{F}{F_1}\right)^2 - 1},$$

when the efflux takes place with a filled cross-section. If we put $\frac{h}{b} = 1{,}17$ or $\frac{b}{h} = 0{,}8547$, we obtain the ratio of the cross-section, for which, under a head $h = 39{,}8$ feet, the efflux with filled cross-section ceases, viz.

$$\frac{F}{F_1} = \sqrt{1 + 0{,}8547} = 1{,}362.$$

§ **440. The Relations of Pressure in Conical Pipes.**—The relations of efflux and pressure in a cylindrical pipe $C\,E$, with or without diaphragm, undergo the following modifications, when another mouth-piece or another tube $E\;G\;H\;K$, Fig. 750, is added to the former. Let F denote the cross-section, v the velocity and p the pressure of the water at the outlet $H\,K$, F_1 the cross-section of the inlet, $a\,F_1$ that of the contracted stream of water, v_1 the velocity and p_1 the pressure of the water in the latter; in like manner let F_2 be the cross-section of the tube, where the stream of water again touches the wall, v_2 the velocity and p_2 the pressure of the water at that point. Then we have

Fig. 750.



$$\frac{p_2}{\gamma} = \frac{p}{\gamma} + \frac{v^2 - v_2^2}{2\,g}, \text{ and therefore}$$

$$\frac{p_1}{\gamma} = \frac{p_2}{\gamma} - \frac{v_2\,(v_1 - v_2)}{g} = \frac{p}{\gamma} + \frac{v^2 - v_2^2}{2\,g} - \frac{v_2\,(v_1 - v_2)}{g}$$

$$= \frac{p}{\gamma} + \frac{v^2 + v_2^2}{2\,g} - \frac{v_1\,v_2}{g} = \frac{p}{\gamma} + \frac{v^2 - 2\,v_1\,v_2 + v_2^2}{2\,g},$$

or, since we can put $a\,F_1\,v_1 = F_2\,v_2 = F\,v$, or

$$v_1 = \frac{F\,v}{a\,F_1} \text{ and } v_2 = \frac{F\,v}{F_2},$$

$$\frac{p_1}{\gamma} = \frac{p}{\gamma} + \left[1 - \frac{2\,F^2}{a\,F_1\,F_2} + \left(\frac{F}{F_2}\right)^2\right]\frac{v^2}{2\,g}.$$

Now the head necessary to produce the required velocity of efflux is

$$h = \frac{v^2}{2\,g} + \frac{(v_1 - v_2)^2}{2\,g} = \left[1 + \left(\frac{F}{a\,F_1} - \frac{F}{F_2}\right)^2\right]\frac{v^2}{2\,g},$$

from which it follows that

$$\frac{p_1}{\gamma} = \frac{p}{\gamma} + \frac{1 - \frac{2F^2}{aF_1F_2} + \left(\frac{F}{F_2}\right)^2}{1 + \left(\frac{F}{aF_1} - \frac{F}{F_2}\right)^2}\, h = \frac{p}{\gamma} + \frac{\frac{1}{F_2} - \frac{2}{aF_1F_2} + \frac{1}{F_2^2}}{\frac{1}{F^2} + \left(\frac{1}{aF_1} - \frac{1}{F_2}\right)^2}\, h,$$

$$\text{I.E. } z_1 = \frac{p}{\gamma} - \frac{\frac{2}{aF_1F_2} - \left(\frac{1}{F_2} + \frac{1}{F_2^2}\right)}{\frac{1}{F^2} + \left(\frac{1}{aF_1} - \frac{1}{F_2}\right)^2}\, h,$$

or, when the water is discharged into free air,

$$z_1 = b - \frac{\frac{2}{aF_1F_2} - \left(\frac{1}{F_1} + \frac{1}{F_2^2}\right)}{\frac{1}{F_2} + \left(\frac{1}{aF_1} - \frac{1}{F_2}\right)^2}\, h.$$

If the efflux takes place with full cross-section, we must have, according to what precedes,

$$\frac{h}{b} < \frac{\frac{1}{F^2} + \left(\frac{1}{aF_1} - \frac{1}{F_2}\right)^2}{\frac{2}{aF_1F_2} - \left(\frac{1}{F^2} + \frac{1}{F_2^2}\right)}, \text{ or}$$

$$\frac{1 + \frac{h}{b}}{F^2} > \left(\frac{2}{aF_1F_2} - \frac{1}{F_2^2}\right)\frac{h}{b} - \left(\frac{1}{aF_1} - \frac{1}{F_2}\right)^2.$$

By the aid of the foregoing formula the relations of the efflux through the conical tubes $A\,B\,D\,E$, Figs. 751 and 752, can be given by substituting for F_2 the cross-section of the pipe, where the stream touches the wall. If δ denotes the semi-angle of divergence $A\ C\ B$ of one, or the semi-angle of convergence of the other tube, and if we assume that the length $F_1\ F_2$ of the eddy is equal to the width $A\ B = d$ of the orifice, we have the width of pipe, where the stream reaches its wall,

FIG. 751.

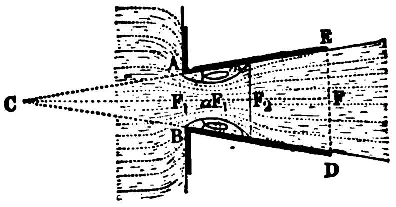
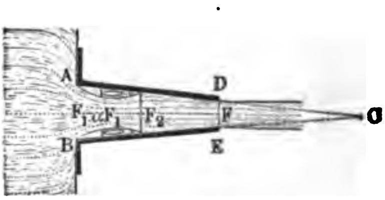


FIG. 752.



$$d_2 = d_1 \pm 2\,d_1\, tang.\, \delta = (1 \pm 2\, tang.\, \delta)\, d_1,$$

and therefore the ratio of the cross-sections

$$\frac{F_2}{F_1} = \left(\frac{d_2}{d_1}\right)^2 = (1 \pm 2\, tang.\, \delta)^2$$

in which the positive sign is to be employed for the divergent pipe in Fig. 751 and the negative sign for the convergent one in Fig. 752, E.G. for $\delta = 2\frac{1}{2}$ degrees, $2\, tang.\, \delta = 0{,}0875$ and

$$\frac{F_2}{F_1} = (1 \pm 0{,}0875)^2 \text{ either} = 1{,}1827 \text{ or } 0{,}8327;$$

hence the velocity of efflux in the first case is

$$v = \sqrt{\frac{2\,g\,h}{1 + \left(\frac{1}{a} - \frac{1}{1{,}1827}\right)^2 \left(\frac{F}{F_1}\right)^2}} = \sqrt{\frac{2\,g\,h}{1 + 0{,}514 \left(\frac{F}{F_1}\right)^2}},$$

and, on the contrary, in the second

$$v = \sqrt{\frac{2\,g\,h}{1 + \left(\frac{1}{a} - \frac{1}{0{,}8327}\right)^2 \left(\frac{F}{F_1}\right)^2}} = \sqrt{\frac{2\,g\,h}{1 + 0{,}1308 \left(\frac{F}{F_1}\right)^2}}.$$

The corresponding coefficient of efflux

$$\mu = \frac{1}{\sqrt{1 + 0{,}514 \left(\frac{F}{F_1}\right)^2}}$$

for the divergent tube is, of course, considerably smaller than the coefficient of efflux

$$\mu = \frac{1}{\sqrt{1 + 0{,}1308 \left(\frac{F}{F_1}\right)^2}}$$

of the convergent tube.

If, E.G., the tubes were three times as long as wide at the inlet orifice, we would have in the first case

$\left(\frac{F}{F_1}\right)^2 = (1 + 6\, tang.\, \delta)^4 = 1{,}2625^4 = 2{,}5405$ and

$\mu = \frac{1}{\sqrt{2{,}306}} = 0{,}659$, and, on the contrary, in the second case

$\left(\frac{F}{F_1}\right)^2 = (1 - 6\, tang.\, \delta)^4 = 0{,}7375^4 = 0{,}2958$ and

$\mu = \frac{1}{\sqrt{1{,}0387}} = 0{,}981$ (compare § 425).

If the efflux through these pipes takes place with filled cross-section, we must have

$$\frac{h}{b} < \frac{1 + \left(\frac{F}{a\,F_1} - \frac{F}{F_2}\right)^2}{\frac{2\,F}{a\,F_1}\,\frac{F}{F_2} - \left[1 + \left(\frac{F}{F_2}\right)^2\right]},$$

or in the first case, when

$$\frac{F}{a\,F_1} = \frac{1,5939}{0,64} = 2,4906 \text{ and } \frac{F}{F_2} = \frac{1,5939}{1,1827} = 1,3477,$$

$$\frac{h}{b} < \frac{1 + 1,1429^2}{6,7112 - 2,8163} = \frac{2,3062}{3,8949} = 0,592,$$

and the head h must be less than $34 \,.\, 0,592 = 20,1$ feet.

§ **441. Elbows.**—A particular kind of impediment is opposed to the motion of water in pipes, when the latter are *bent* or form *elbows*. These resistances cannot be determined with safety by theory and must, therefore, like so many of the phenomena of efflux, be studied by experiment. If a pipe *A C B*, Fig. 753, forms an elbow, the stream separates itself from the inner surface of the second branch of the pipe in consequence of the centrifugal force; when this piece is short, the efflux with full cross-section ceases, and the discharge is, therefore, smaller than from an equally long straight pipe. If the exterior portion *C B* of the elbow *A C B*, Fig. 754, is longer, an eddy *S* is formed beyond *C*, and, when the tube is again filled, the velocity of efflux *v* is smaller. This diminution of the velocity of efflux must be treated exactly in the same manner as the resistance produced by a contraction in the pipe. If F is the cross-section of the tube and F_1 that of the contracted vein, we have the coefficient of contraction of the latter

FIG. 753.

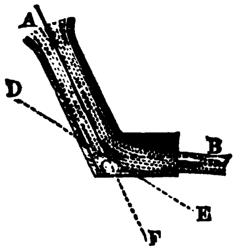
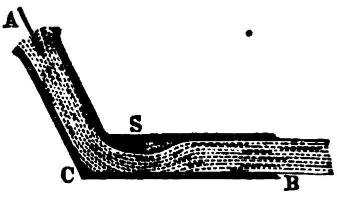


FIG. 754.



$$a = \frac{F}{F_1},$$

and, therefore, the corresponding coefficient of resistance

$$\zeta = \left(\frac{F}{F_1} - 1\right)^2 = \left(\frac{1}{a} - 1\right)^2.$$

The coefficient of contraction a, and consequently the corresponding coefficient of resistance ζ, depends upon *the semi-angle of deviation* $\delta = A\,C\,D = B\,C\,E = \frac{1}{2}\,B\,C\,F$, Fig. 753, and according to the experiments of the author, made with a tube 3 centimeters in diameter, we can put

$$\zeta = 0{,}9457\ sin.^2\,\delta + 2{,}047\ sin.^4\,\delta.$$

The following table contains a series of coefficients of resistance, calculated for different angles of deviation.

$\delta^\circ =$	10	20	30	40	45	50	55	60	65	70
$\zeta =$	0,046	0,139	0,364	0,740	0,984	1,260	1,556	1,861	2,158	2,431

We see from this table that the vis viva of water in pipes is considerably diminished by the elbows. If, E.G., the elbow makes a right angle or $\delta = 45°$, we have the loss of head occasioned by it

$$h = \zeta \cdot \frac{v^2}{2g} = 0{,}984 \cdot \frac{v^2}{2g},$$

or nearly as much as the height due to the velocity.

When the pipes are narrower, ζ becomes considerably greater, E.G., for an elbow 1 centimeter in diameter and with an angle of deviation of 90°, ζ was found $= 1{,}536$. See the author's "*Experimentalhydraulik.*"

If to one elbow $A\ C\ B$, Fig. 755, another elbow is joined, as is shown in Fig. 756, and Fig. 757, a peculiar, but at the same time easily explicable, phenomenon of efflux is observed. The second elbow $B\ D\ E$, Fig. 756, which turns the stream to the same side as the first one $A\ C\ B$, produces no further contraction of the

FIG. 755.

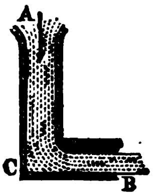


FIG. 756.

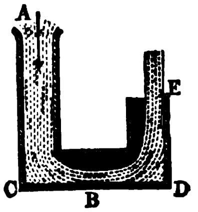
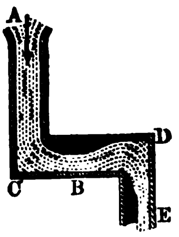


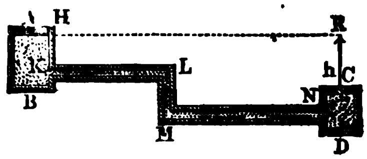
FIG. 757.



stream, and, therefore, for efflux with full cross-section ζ is no greater than for a simple elbow *A C B*. But if the elbow *B D E*, Fig. 757, turns the stream to the opposite side, the contraction is a double one, and the coefficient of resistance is consequently twice as great as for a single elbow. If, finally, *B D E* is so joined to *A C B* that *D E* stands at right-angles to the plane *A B D*, ζ then becomes about $1\frac{1}{2}$ times as great as for the single elbow *A C B*.

EXAMPLE.—If a system of pipes *K L N*, Fig. 758, 150 feet long and 5 inches in diameter, which should discharge 25 cubic feet of water, contains two elbows, the required head will be

FIG. 758.



$$h = (1{,}505 + 8{,}712 + 2 \,.\, 0{,}984)\,\frac{v^2}{2g}$$

$$= 12{,}185 \,.\, 0{,}1448 = 1{,}76 \text{ feet.}$$

(Compare Example in § 430.)

§ **442. Bends.**—*Curved pipes*, when the other circumstances are the same, cause much less resistance than elbows. They also cause, in consequence of the centrifugal force of the water, a partial contraction of the stream *A B D*, Fig. 759, so that, when the bend is not terminated by a long straight pipe, the cross-section F_1 of the stream at its outlet is smaller than that F of the pipe. But if the bend *A B D*, Fig. 760, is terminated by a long straight pipe *D E*, an eddy *F* is formed and an efflux with filled cross-section again takes place at the expense of the vis viva of water. If the coefficient of contraction $\frac{F_1}{F} = \alpha$, we have for the coefficient of resistance of the bend.

FIG. 759.

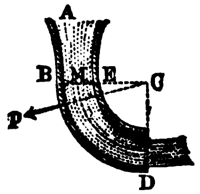
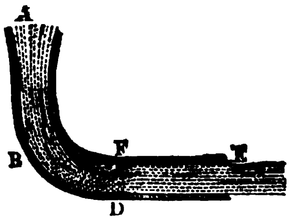


FIG. 760.



$$\zeta = \left(\frac{1}{\alpha} - 1\right)^2.$$

The coefficient of contraction α depends upon the ratio $\frac{a}{r}$ of the radius $B M = E M = a$, Fig. 759, of the pipe to its radius of curvature $C M = r$, and it can be determined approximatively in the following manner. If v is the velocity of the water upon entering the bend and v_1 that of the contracted vein, we have $v_1 F_1 = v F$, whence $v_1 = \frac{F}{F_1} v$, and, therefore, the head which measures the pressure in $B E$ is

$$h = \frac{v_1^2 - v^2}{2 g} = \left[\left(\frac{F}{F_1}\right)^2 - 1\right] \frac{v^2}{2 g}.$$

This height, multiplied by 1 and γ, gives the pressure of the stream of water in all directions upon the unit of surface at E

$$p = h \gamma = \left[\left(\frac{F}{F_1}\right)^2 - 1\right] \frac{v^2}{2 g} \gamma = \left[\left(\frac{1}{a}\right)^2 - 1\right] \frac{v^2}{2 g} \gamma.$$

Since the centrifugal force of the water acts upon the convex side in opposition to the pressure p, it is possible that it may balance the latter completely. But in this case the exterior air would enter and separate the stream entirely from the convex side, as is shown in Figs. 759 and 760. The centrifugal force of a prism of water, whose length is $B E = 2 a$ and whose cross-section is 1, is, when the radius of curvature is $C M = r$,

$$q = \frac{v^2}{g r} . 2 a \gamma.$$

Now if we put $p = q$, we have the condition of separation of the stream from the wall of the pipe

$$\frac{1}{a^2} - 1 = \frac{4 a}{r},$$

and consequently the *coefficient of contraction*

$$a = \sqrt{\frac{r}{r + 4 a}};$$

hence the *coefficient of resistance* for efflux with a full pipe is

$$\zeta = \left(\sqrt{\frac{r + 4 a}{r}} - 1\right)^2.$$

As this calculation is based upon a mean velocity and a mean radius of curvature, it will, of course, give but an approximate value of a and ζ.

From his own experiments and from the results of some observations made by Du Buat, the author has deduced the following empirical formulas for the coefficients of resistance of water in passing through bent pipes:

1) for bends with *circular cross-sections*

$$\zeta = 0,131 + 1,847 \left(\frac{a}{r}\right)^{\frac{7}{2}};$$

2) for bends with *rectangular cross-sections*

$$\zeta = 0{,}124 + 3{,}104 \left(\frac{a}{r}\right)^{\frac{7}{2}}.$$

The following tables are calculated according to these formulas:

TABLE I.

Coefficients of the resistance due to the curvature of pipes with circular cross-sections.

$\frac{a}{r} =$	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
$\zeta =$	0,131	0,138	0,158	0,206	0,294	0,440	0,661	0,977	1,408	1,978

TABLE II.

Coefficients of the resistance due to the curvature of pipes with rectangular cross-sections.

$\frac{a}{r} =$	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
$\zeta =$	0,124	0,135	0,180	0,250	0,398	0,643	1,015	1,546	2,271	3,228

From the above tables we see that for a circular pipe, whose radius of curvature is twice the radius of its cross-section, the coefficient of resistance = 0,294, and that for a pipe, whose radius of curvature is at least 10 times the radius of the cross-section, the coefficient = 0,131.

In order to check the contraction of the stream of water in a bend $A\,B\,D$, Fig. 761, the cross-section of the pipe must be gradually diminished in such a manner that the ratio of the cross-section $D\,H = F_1$ of the outlet orifice to that $B\,E = F$ of the inlet shall be $a = \frac{1}{\sqrt{\zeta + 1}}$.

FIG. 761.

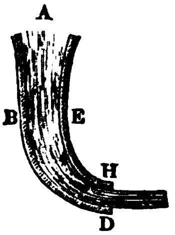


FIG. 762.

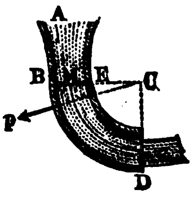
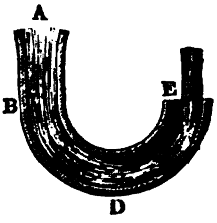


FIG. 763.



If one bend *B D*, Fig. 762, is terminated by another, which turns the stream further in the same direction, if, E.G., the axis of the pipe forms a semicircle, like *B D E*, Fig. 763, the contraction is not changed and a and ζ have the same values as for the pipe in Fig. 762, which forms but a quadrant. If, on the contrary, a bend *D E*, Fig. 764, which turns the stream in the opposite direction, is attached to the first one, an eddy *F* is formed between the two and a second contraction of the stream takes place, by which the resistance (ζ) is nearly doubled.

FIG. 764.

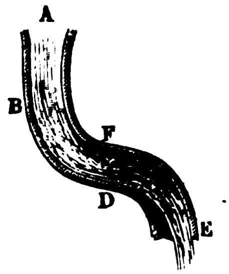


FIG. 765.

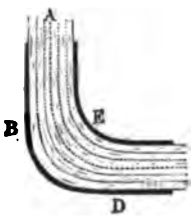
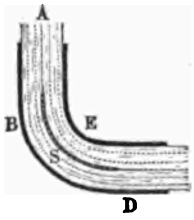
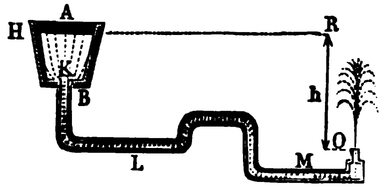


FIG. 766.



The resistance to water flowing through bends can be ***diminished*** by enlarging the cross-section of the pipe, as in *B D E*, Fig. 765, or by inserting in it a ***thin partition***, like *S* in *B D E*, Fig. 766; for in the first case the velocity v, and in the second the ratio $\frac{a}{r}$ is smaller, and consequently the coefficient of resistance ζ is rendered less.

FIG. 767.



EXAMPLE.—If the system of pipes *B L M*, Fig. 767, in the second example of § 430, contains 5 bends each of 90°, and if the radius of curvature of each is 2 inches, we have

$$\frac{a}{r} = \frac{1}{2},$$

and according to the first of the foregoing tables, the corresponding coefficient of resistance $\zeta = 0{,}294$; consequently for the 5 bends $5\,\zeta = 1{,}47$; hence the velocity of the water issuing from the pipe, instead of

$$v = \frac{17{,}945}{\sqrt{7{,}582}} = 6{,}52 \text{ feet, is}$$

$$v = \frac{17{,}945}{\sqrt{7{,}582 + 1{,}47}} = \frac{17{,}945}{\sqrt{9{,}052}} = 5{,}964 \text{ feet,}$$

so that the discharge per second is now

$$Q = 0,7854 \,.\, \tfrac{1}{36} \,.\, 5,964 = 0,1301 \text{ cubic feet} = 224,81 \text{ cubic inches.}$$

§ **443. Valve-Gates, Cocks, Valves.**—In order to regulate the discharge of water from pipes and vessels, we employ various kinds of apparatus, such as *cocks, valve-gates*, and *valves*, by means of which we produce a contraction in the pipe, which occasions a resistance to the passage of the water, the value of which is determined in the same manner as the losses of head in the foregoing paragraph. As the stream of water is subjected to changes of direction, is divided, etc., the coefficients a and ζ can only be determined by experiments made for that purpose. Such experiments have been made by the author,* the principal results of which are given in the following tables:

TABLE I.

The coefficients of resistance for the passage of water through valve-gates or slide valves (Fr. tiroirs; Ger. Schieber or Schubventile) in *parallelopipedical pipes.*

Ratio of the cross sections $\frac{F_1}{F}$ =	1,0	0,9	0,8	0,7	0,6	0,5	0,4	0,3	0,2	0,1
Coefficient of resistance ζ =	0,00	0,09	0,39	0,95	2,08	4,02	8,12	17,8	44,5	193

TABLE II.

The coefficients of resistance for the passage of water through valve-gates or slide-valves in *cylindrical pipes.*

Relative height of opening s =	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$
Ratio of the cross-sections =	1,000	0,948	0,856	0,740	0,609	0,466	0,315	0,159
Coefficient of resistance ζ =	0,00	0,07	0,26	0,81	2,06	5,52	17,0	97,8

---

* Experiments upon the efflux of water through valve-gates, cocks, clacks, and valves, made and calculated by Julius Weisbach, or under the title "Untersuchungen im Gebiete der Mechanik und Hydraulik, etc.," Leipzig, 1842.

TABLE III.

The coefficients of resistance for the passage of water through a *cock* (Fr. robinet; Ger. Hahn) in *parallelopipedical pipes.*

Angle that the cock is turned δ =	5°	10°	15°	20°	25°	30°	35°	40°	45°	50°	55°	66⅔
Ratio of the cross-sections =	0,926	0,849	0,769	0,687	0,604	0,520	0,436	0,352	0,269	0,188	0,110	0
Coefficient of resistance =	0,05	0,31	0,88	1,84	3,45	6,15	11,2	20,7	41,0	95,3	275	∞

TABLE IV.

The coefficients of resistance for the passage of water through a cock in a *cylindrical pipe.*

Angle that the cock is turned δ =	5°	10°	15°	20°	25°	30°	35°
Ratio of the cross-sections =	0,926	0,850	0,772	0,692	0,613	0,535	0,458
Coefficient of resistance =	0,05	0,29	0,75	1,56	3,10	5,47	9,68
Angle that the cock is turned δ =	40°	45°	50°	55°	60°	65°	82½°
Ratio of the cross-sections =	0,385	0,315	0,250	0,190	0,137	0,091	0
Coefficient of resistance =	17,3	31,2	52,6	106	206	486	∞

TABLE V.

The coefficients of resistance for the passage of water through *throttle-valves* (Fr. valves; Ger. Drehklappen or Drosselventile) in *parallelopipedical pipes.*

Angle that the valve is turned δ =	5°	10°	15°	20°	25°	30°	35°
Ratio of the cross-sections =	0,913	0,826	0,741	0,658	0,577	0,500	0,426
Coefficients of resistance =	0,28	0,45	0,77	1,34	2,16	3,54	5,7

Angle that the valve is turned δ =	40°	45°	50°	55°	60°	65°	70°	90°
Ratio of the cross-sections =	0,357	0,293	0,234	0,181	0,134	0,094	0,060	0
Coefficients of resistance =	9,27	15,07	24,9	42,7	77,4	158	368	∞

TABLE VI

Coefficients of resistance for the passage of water through *throttle-valves in cylindrical pipes.*

Angle that the valve is turned δ=	5°	10°	15°	20°	25°	30°	35°
Ratio of the cross-sections =	0,918	0,826	0,741	0,658	0,577	0,500	0,426
Coefficient of resistance =	0,24	0,52	0,90	1.54	2,51	3,91	6,22

Angle that the valve is turned δ =	40°	45°	50°	55°	60°	65°	70°	90°
Ratio of the cross-sections =	0,357	0,293	0,234	0,181	0,134	0,094	0,060	0
Coefficient of resistance =	10,8	18,7	32,6	58,8	118	256	751	∞

§ **444**. With the aid of the coefficients of resistance, given in the above tables, we can find not only the loss of head for a certain position of the valve-gate, cock or valve, but also the position we must give to these apparatus in order to produce a certain velocity of efflux or a certain resistance. Of course, such a determination will be more accurate, the more the regulating apparatus resembles that used in the experiments. Besides, the values given in the above tables are not correct, when the water, after passing the contracted orifice produced by the apparatus, does not fill the pipe again. In order that the efflux with a filled cross-section shall take place, it is necessary, when the contraction is great, that the pipe shall have a certain length. The cross-section of the parallelopipedical pipe was 5 centimeters wide and 2½ centimeters high, and the diameter of the cylindrical pipe was 4 centimeters. With the *slide-*

valve or *valve-gate*, Fig. 768, the cross-section is merely narrowed, and its shape in one pipe is a simple rectangle F_1, Fig. 769, and in the other a crescent F_1, Fig. 770. When *cocks* are employed, as in Fig. 771, there are two contractions and two changes of direction, and the resistance is therefore in this case very great. The cross-sections of the maximum contractions have very peculiar forms. The stream is divided by the *throttle-valve* (or disc and pivot valve), Fig. 772, into two parts, each of which passes through a contracted orifice. The cross-sections of the contracted openings are rectangular in parallelopipedical pipes and crescent-shaped in cylindrical ones. The following examples will sufficiently explain the use of the foregoing tables.

FIG. 768.

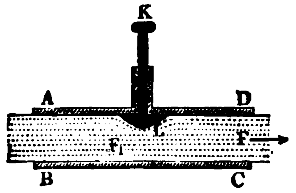


FIG. 769.

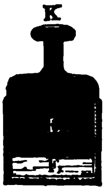


FIG. 770.

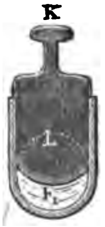


FIG. 771.

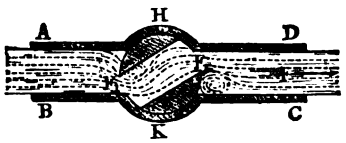
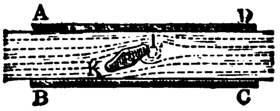


FIG. 772.



EXAMPLE—1) If in a system of cylindrical pipes 3 inches in diameter and 500 feet long a valve-gate is introduced, and if it is raised $\frac{3}{8}$ of the entire height, so as to close $\frac{5}{8}$ of the diameter, what will be the discharge through it under a head of 4 feet? According to what precedes we can put the coefficient of resistance for the entrance of the water into the pipe $\zeta_0 = 0.505$ and the coefficient ζ_1 of resistance of the pipe according to Table II, § 443, $= 5{,}52$, whence it follows that the velocity of efflux is

$$v = \frac{8{,}025\sqrt{4}}{\sqrt{1{,}505 + 5{,}52 + \zeta\frac{l}{d}}} = \frac{8{,}025\,.\,2}{\sqrt{7{,}025 + 500\,.\,4\,\zeta}} = \frac{16{,}05}{\sqrt{7{,}025 + 2000\,\zeta}}.$$

If we put the coefficient of friction $\zeta = 0{,}025$, we obtain

$$v = \frac{16{,}05}{\sqrt{57{,}025}} = 2{,}125 \text{ feet.}$$

But the velocity $v = 2,125$ feet corresponds more accurately to $\zeta = 0,0265$, hence we have more correctly

$$v = \frac{16,05}{\sqrt{60,025}} = 2,07 \text{ feet},$$

and the discharge per second is

$$Q = \frac{\pi}{4} \,.\, 9 \,.\, 12 \,.\, 2,07 = 55,89\,\pi = 176 \text{ cubic inches.}$$

2) A system of pipes 4 inches in diameter discharges under a head of 5 feet 10 cubic feet of water per minute; at what angle must a throttle valve, placed in them, be turned to cause a discharge of 8 cubic feet per minute? The initial velocity is

$$= \frac{10 \,.\, 4}{60 \,.\, \pi \left(\frac{1}{3}\right)^2} = \frac{6}{\pi} = 1,91 \text{ feet},$$

and that after turning the valve

$$= \tfrac{8}{10} \,.\, 1,91 = 1,528 \text{ feet.}$$

The coefficient of efflux in the first case is

$$\frac{v}{\sqrt{2\,g\,h}} = \frac{1,91}{8,025\ \sqrt{5}} = 0,107;$$

hence the coefficient of resistance is

$$= \frac{1}{\mu^2} - 1 = \frac{1}{0,107^2} - 1 = 86,34,$$

and the coefficient of efflux in the second case is

$$= \tfrac{8}{10} \,.\, 0,107 = 0,0856;$$

hence the coefficient of resistance is

$$= \frac{1}{0,0856^2} - 1 = 135,5,$$

and the coefficient of resistance of the throttle valve

$$\zeta = 135,5 - 86,34 = 49,16.$$

Now Table VI, § 443, gives for the angle $\delta = 50°$, $\zeta = 32,6$ and for the angle $\delta = 55°$, $\zeta = 58,8$; we can, therefore, assume that, when the valve is placed at an angle of $50° + \frac{16,56}{26,2} \,.\, 5° = 53°\ 10'$, the required quantity of water will be discharged. If we take into consideration the fact that the coefficient of friction changes from 0,0268 to 0,0288 when the velocity decreases from 1,91 to 1,528, we have more correctly

$$\zeta = 135,5 - 86,34\,\frac{288}{268} = 135,5 - 91,2 = 44,3,$$

and consequently the angle that the valve must be turned

$$\delta = 50° + \frac{11,7}{26,2}\,5° = 52°\ 14'.$$

§ **445. Valves.**—The knowledge of the resistance produced by *valves* (Fr. soupapes; Ger. Ventile) is of the greatest importance. Experiments have also been made by the author with them. Those most commonly employed are the *puppet valve* and the *clack valve*, which are represented in Figs. 773 and 774. In both cases the water passes through an aperture in a ring $R\ G$, which

FIG. 773.

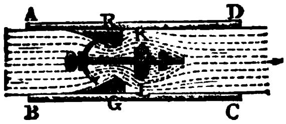
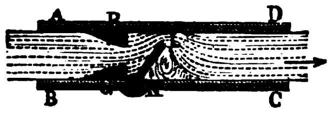


FIG. 774.



is called the *seat*. The puppet valve $K\ L$, Fig. 773, is provided with a spindle, which runs in guides and which permits the valve to move only in the direction of its axis; the clack $K\ L$, Fig. 774, on the contrary, opens by turning like a door. We see that in both apparatuses not only the ring, but also the valve are obstacles to the motion of the water.

The ratio of the aperture in the seat of the *puppet valve*, with which the experiments were made, to that of the pipe was 0,356, and, on the contrary, the ratio of ring-shaped surface around the open valve to the cross-section of the pipe was = 0,406, hence we can put as a mean $\frac{F_1}{F} = 0{,}381$. By observing the efflux for different positions of the valve it was found that the coefficient of resistance decreased with the lift of the valve, but that this decrease was very inconsiderable, when the lift exceeded one-half the width of the orifice. Its value for this position was = 11, and the *height of resistance* or loss of head was

$$z = \zeta \frac{v^2}{2g} = 11 \cdot \frac{v^2}{2g},$$

v denoting the velocity of the water in the full pipe. This number can be used to find the coefficients of resistance corresponding to other relations of cross-section. If we put in general

$$\zeta = \left(\frac{F}{a F_1} - 1\right)^2,$$

we obtain for the case observed

$$\frac{F_1}{F} = 0{,}381 \text{ and } \zeta = \left(\frac{1}{0{,}381\, a} - 1\right)^2 = 11,$$

and therefore

$$a = \frac{1}{0{,}381\,(1 + \sqrt{11})} = \frac{1}{4{,}317 \cdot 0{,}381} = 0{,}608,$$

and finally the general expression for the coefficient of resistance

$$\zeta = \left(\frac{F}{0{,}608\, F_1} - 1\right)^2 = \left(1{,}645\,.\,\frac{F}{F_1} - 1\right)^2.$$

If, E.G., the cross-section of the aperture is one half that of the pipe, the coefficient of resistance becomes

$$= (1{,}645\,.\,2 - 1)^2 = 2{,}29^2 = 5{,}24.$$

In the experiments with clack-valves the ratio of the cross-section of the aperture to that of the pipe, I.E., $\frac{F}{F_1}$, was $= 0{,}535$. The following table shows how the coefficients of resistance decrease as the opening increases.

TABLE OF THE COEFFICENTS OF RESISTANCE FOR *CLACK-VALVES.*

Angle of opening	15°	20°	25°	30°	35°	40°	45°	50°	55°	60°	65°	70°
Coefficient of resistance..	90	62	42	30	20	14	9,5	6,6	4,6	3,2	2,3	1,7

By the aid of this table the coefficient of resistance for clack-valves can be calculated approximatively, when the relations of the cross-sections are different. We must adopt the same method as we did for puppet valves.

Example.—A force-pump delivers every time the plunger descends in 4 seconds 5 cubic feet of water, the diameter of the column pipe in which the puppet-valve is placed is 6 inches, the interior diameter of the valve-ring is 3½ inches, and the maximum diameter of the valve is 4½ inches; what resistance is to be overcome by the water in passing through this valve? The ratio of the cross-sections for these apertures is

$$\left(\frac{3{,}5}{6}\right)^2 = (\tfrac{7}{12})^2 = 0{,}34,$$

and the ratio of the ring-shaped contraction to the cross-section of the tube is

$$= 1 - \left(\frac{4{,}5}{6}\right)^2 = 1 - (\tfrac{3}{4})^2 = 0{,}44;$$

hence the mean ratio of the cross-sections is

$$\frac{F_1}{F} = \frac{0{,}34 + 0{,}44}{2} = 0{,}39,$$

and the corresponding coefficient of resistance

$$\zeta = \left(\frac{1{,}645}{0{,}39} - 1\right)^2 = 3{,}22^2 = 10{,}4.$$

The velocity of the water is

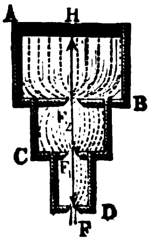
$$v = \frac{5}{4\,.\,\frac{\pi}{4}\,.\,(\frac{1}{2})^2} = \frac{20}{\pi} = 6{,}37 \text{ feet},$$

the height due to the velocity is = 0,680 feet, and consequently the height of resistance is = 10,4 . 0,680 = 6,55 feet. The amount of water raised in a second weighs $\frac{5}{4}$. 62,5 = 78,125 lbs.; the mechanical effect consumed by the passage of the water through the valve in that time is therefore

$$= 6{,}55 \,.\, 78{,}125 = 511{,}72 \text{ foot-pounds.}$$

§ 446. Compound Vessels.—The foregoing theory of the resistance due to the passage of water through contractions, is also applicable to the discharge from compound vessels. The apparatus *A D*, represented in Fig. 775, is divided by two walls, which contain the orifices F_1 and F_2, into three communicating vessels. If the dividing walls were absent and the edges at the passage from one vessel to the other rounded off, we would have, as in the case of a simple vessel, the velocity of efflux

Fig. 775.



$$v = \frac{\sqrt{2gh}}{\sqrt{1 + \zeta_0}},$$

in which h denotes the depth of the orifice below the level of the water and ζ_0 the coefficient of resistance for the passage of the water through the orifice F.

But since when the water has passed through the orifices F_1 and F_2 the cross-sections $a\,F_1$ and $a\,F_2$ change suddenly into the cross-sections G_1 and G_2 of the vessels *C D* and *B C*, and according to § 437 the resistances thus produced are

$$\zeta_1 \frac{v_1^2}{2g} = \left(\frac{G}{a\,F_1} - 1\right)^2 \left(\frac{a\,F}{G}\right)^2 \frac{v^2}{2g} = \left(\frac{F}{F_1} - \frac{a\,F}{G}\right)^2 \cdot \frac{v^2}{2g}$$

and

$$\zeta_2 \frac{v_2^2}{2g} = \left(\frac{G_1}{a\,F_2} - 1\right)^2 \left(\frac{a\,F}{G_1}\right)^2 \frac{v^2}{2g} = \left(\frac{F}{F_2} - \frac{a\,F}{G_1}\right)^2 \cdot \frac{v^2}{2g},$$

we have

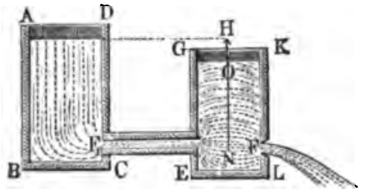
$$(1+\zeta_0)\frac{v^2}{2g} + \zeta_1 \frac{v_1^2}{2g} + \zeta_2 \frac{v_2^2}{2g} = \left[1 + \zeta_0 + \left(\frac{F}{F_1} - \frac{a\,F}{G}\right)^2 + \left(\frac{F}{F_2} - \frac{a\,F}{G_1}\right)^2\right]\frac{v^2}{2g},$$

whence we obtain the velocity of efflux

$$v = \frac{\sqrt{2gh}}{\sqrt{1 + \zeta_0 + \left(\frac{F}{F_1} - \frac{a\,F}{G}\right)^2 + \left(\frac{F}{F_2} - \frac{a\,F}{G_1}\right)^2}}.$$

In the compound vessel, represented in Fig. 776, from which the water is discharging, the same conditions exist, but perhaps we must here consider the friction of the water in the communicating tube $C E$. Let l be the length, d the diameter, ζ the coefficient of friction of this tube, and v_1 the velocity of the water in it, then we have for the head lost by the water in passing from $A\ C$ to $G\ L$

FIG. 776.



$$h_1 = \left[1 + \left(\frac{1}{a} - 1\right)^2 + \zeta \frac{l}{d}\right] \frac{v_1^2}{2g},$$

or, since the velocity $v_1 = \frac{a F}{F_1} v$,

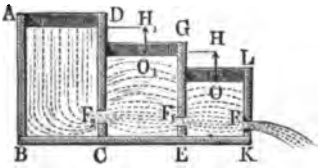
$$h_1 = \left[1 + \left(\frac{1}{a} - 1\right)^2 + \zeta \frac{l}{d}\right] \left(\frac{a F}{F_1}\right)^2 \frac{v^2}{2g}.$$

If we subtract this height from the total head h, there remains the head in the second vessel $h_2 = h - h_1$; hence the velocity of efflux is

$$v = \frac{\sqrt{2 g h_2}}{\sqrt{1 + \zeta_0}} = \frac{\sqrt{2 g h}}{\sqrt{1 + \zeta_0 + \left[1 + \left(\frac{1}{a} - 1\right)^2 + \zeta \frac{l}{d}\right] \left(\frac{a F}{F_1}\right)^2}}.$$

This determination becomes very simple when the apparatus is like the one represented in Fig. 777; for in this case we can assume the cross-sections G, G_1, G_2 to be infinitely large, compared to the cross-sections of the orifices F, F_1, F_2. The first difference of level $O\ H$, or the height of resistance for the passage through F_1, is

FIG. 777.



$$h_1 = \frac{1}{2g}\left(\frac{v_1}{a_1}\right)^2 = \left(\frac{a F}{a_1 F_1}\right)^2 \cdot \frac{v^2}{2g},$$

and in like manner the second difference of level $O_1\ H_1$ or the height of resistance for the passage through F_2 is

$$h_2 = \left(\frac{a F}{a_2 F_2}\right)^2 \cdot \frac{v^2}{2g},$$

in which a, a_1 and a_2 denote the coefficients of contraction for the orifices F, F_1 and F_2. Hence

$$v = \frac{\sqrt{2gh}}{\sqrt{1 + \left(\frac{aF}{a_1 F_1}\right)^2 + \left(\frac{aF}{a_2 F_2}\right)^2}},$$

and the discharge is

$$Q = \frac{aF\sqrt{2gh}}{\sqrt{1 + \left(\frac{aF}{a_1 F_1}\right)^2 + \left(\frac{aF}{a_2 F_2}\right)^2}}$$

$$= \frac{\sqrt{2gh}}{\sqrt{\left(\frac{1}{aF}\right)^2 + \left(\frac{1}{a_1 F_1}\right)^2 + \left(\frac{1}{a_2 F_2}\right)^2}}.$$

It is easy to see that under the same circumstances compound vessels, or reservoirs, discharge less water than simple ones.

EXAMPLE.—If in the apparatus, Fig. 776, the total head or the depth of the centre of the orifice F below the level of the water in the first vessel is $= 6$ feet, if the orifice is 8 inches wide and 4 inches high and if the reservoirs are united by a pipe 10 feet long, 12 inches wide and 6 inches high, what will be the discharge?

The mean width of the trunk is

$$d = \frac{4 \,.\, 1 \,.\, 0{,}5}{2 \,.\, 1{,}5} = \tfrac{2}{3} \text{ feet, hence } \frac{l}{d} = \frac{3 \,.\, 10}{2} = 15.$$

Putting the coefficient of friction $\zeta = 0{,}025$, we obtain

$$\zeta \frac{l}{d} = 0{,}025 \,.\, 15 = 0{,}375,$$

and adding $\zeta_0 = 0{,}505$, the coefficient for the entrance into prismatical pipes, we have

$$1 + \left(\frac{1}{a} - 1\right)^2 + \zeta \frac{l}{d} = 1 + 0{,}505 + 0{,}375 = 1{,}88.$$

Since $\frac{aF}{F_1} = \frac{0{,}64 \,.\, 8 \,.\, 4}{12 \,.\, 6} = 0{,}2845$, it follows that the coefficient of resistance for the entire pipe is $= 1{,}88 \,.\, 0{,}2845^2 = 0{,}152$, and if we put the coefficient of resistance for the passage through $F_1 = 0{,}07$, we obtain the velocity of efflux

$$v = \frac{8{,}025\sqrt{6}}{\sqrt{1{,}07 + 0{,}152}} = \frac{8{,}025\sqrt{6}}{\sqrt{1{,}222}} = 17{,}78 \text{ feet}.$$

The contracted cross-section is $0{,}64 \,.\, 1 \,.\, \tfrac{1}{2} = 0{,}32$ square feet, and therefore the discharge is

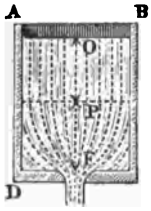
$$Q = 0{,}32 \,.\, 17{,}78 = 5{,}69 \text{ cubic feet}.$$

CHAPTER V.

OF THE EFFLUX OF WATER UNDER VARIABLE PRESSURE.

§ 447. **Prismatic Vessels.**—If a vessel, from which water is issuing through an orifice in the side or bottom, receives no supply of water from any other source, the level of the water will gradually sink, and the vessel will finally become empty. Now if the discharge Q into the vessel is greater or less than that $\mu F \sqrt{2 g h}$ from it, the water level will rise or sink, until the head becomes $h = \frac{1}{2 g}\left(\frac{Q}{\mu F}\right)^2$, and afterwards the head and velocity of efflux will remain constant. Our problem now is to determine *the dependence upon each other of the time, of the rising or sinking of the surface of the water and, if it occurs, of the emptying of the vessel, when the latter has a given form and size.* The most simple case is that of efflux through an orifice in the bottom of a prismatic vessel, which receives no supply of water. Let x be the variable head $F P$, F the area of the orifice and G the cross-section of the vessel $A\ C$, Fig. 778, then the theoretical velocity of efflux is

FIG. 778.



$$v = \sqrt{2 g x},$$

and the theoretical velocity of the sinking surface of the water is

$$= \frac{F}{G} v = \frac{F}{G} \sqrt{2 g x},$$

and the effective velocity

$$v_1 = \frac{\mu F}{G} \sqrt{2 g x}.$$

In the beginning $x = F O = h$, and at the end of the efflux $x = 0$, the initial velocity is therefore

$$c = \frac{\mu F}{G} \sqrt{2 g h},$$

and the final velocity

$$c_1 = 0.$$

We see from the formula

$$v_1 = \sqrt{2 \left(\frac{\mu F}{G}\right)^2 g x},$$

that the motion of the surface of the water is *uniformly retarded,* and that the retardation is $p = \left(\frac{\mu F}{G}\right)^2 g$; we also know (§ 14) that this velocity will be $= 0$ and that the efflux will cease after a time.

$$t = \frac{v_1}{p} = \frac{\mu F}{G} \sqrt{2 g h} : \left(\frac{\mu F}{G}\right)^2 g = \frac{G}{\mu F} \sqrt{\frac{2 g h}{g^2}},$$

I.E.

$$t = \frac{2 G \sqrt{h}}{\mu F \sqrt{2 g}}.$$

We can also put

$$t = \frac{2 G h}{\mu F \sqrt{2 g h}} = \frac{2 G h}{Q} = \frac{2 V}{Q},$$

and consequently we can assume that a volume $V = G h$ of water will be discharged through an opening F in the bottom under a head decreasing from h to 0 in double the time that it would, if the head were constant and equal to h.

As the coefficient of efflux μ is not perfectly constant, but increases when the head diminishes, we must employ a mean value of this coefficient in our calculations.

EXAMPLE.—In what time will a parallelopipedical box, whose cross-section is 14 square feet, empty itself through an orifice in the bottom, which is circular and 2 inches in diameter, when the initial head is 4 feet? Theoretically the time required would be

$$t = \frac{2 . 14 \sqrt{4}}{8{,}025 . \frac{\pi}{4} . \left(\frac{1}{6}\right)^2} = \frac{2 . 14 . 144 . 2}{8{,}025\, \pi} = \frac{8064}{8{,}025\, \pi} = 319'',9 = 5 \text{ min. } 19{,}9 \text{ sec.}$$

At the end of half the duration of the efflux the head is $= (\frac{1}{2})^2 h = \frac{1}{4} . 4 = 1$ foot, but the coefficient of efflux for an orifice in a thin plate, corresponding to a head $= 1$ foot, is $\mu = 0{,}618$; hence the real duration of the efflux is

$$= \frac{319'',9}{0{,}613} = 521'',8 = 8 \text{ minutes } 41{,}8 \text{ seconds.}$$

§ 448. Communicating Vessels.—Since for an initial head h_1 the duration of efflux is

$$t_1 = \frac{2 G \sqrt{h_1}}{\mu F \sqrt{2 g}}$$

and for an initial head h_2 the duration is

$$t_2 = \frac{2 G \sqrt{h_2}}{\mu F . \sqrt{2 g}},$$

it follows that by subtraction we obtain the time during which the head changes from h_1 to h_2, or the level of the water sinks a distance $h_1 - h_2$; its value is

$$t = \frac{2\,G}{\mu\,F\,.\,\sqrt{2\,g}}\,(\sqrt{h_1} - \sqrt{h_2}),$$

or, when the dimensions are expressed in feet,

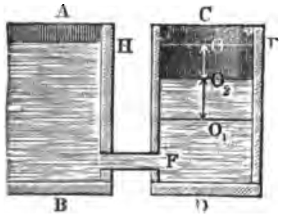
$$t = 0{,}249\,\frac{G}{\mu\,F}(\sqrt{h_1} - \sqrt{h_2}).$$

On the contrary, when the duration of the efflux is given, we determine the distance $s = h_1 - h_2$ that the surface of the water sinks by means of the formula

$$h_2 = \left(\sqrt{h_1} - \frac{\mu\,\sqrt{2\,g}\,.\,F}{2\,G}\,t\right)^2, \text{or}$$

$$s = \frac{\mu\,\sqrt{2\,g}\,.\,F\,t}{G}\left(\sqrt{h_1} - \frac{\mu\,\sqrt{2\,g}}{4\,G}\,.\,F\,t\right).$$

FIG. 779.



The same formulas are applicable to the case of a vessel $C\,D$, Fig. 779, filled from another vessel $A\,B$, in which the level of the water is constant. If the cross-section of the communicating pipe or orifice $= F$, that of the vessel to be filled $= G$ and initial difference of level $O\,O_1$ of the two surfaces of water $= h$, we have, since in this case the level of the water in the second vessel rises with a uniformly retarded motion, the time required to fill it or the time in which the second surface of the water rises to the level $H\,R$ of the first

$$t = \frac{2\,G\,\sqrt{h}}{\mu\,F\,.\,\sqrt{2\,g}},$$

and in like manner the time in which the distance $O\,O_1 = h_1$ between the surfaces of the water becomes $O_2\,O = h_2$, or during which the level of the water rises a distance $O_1\,O_2 = s = h_1 - h_2$,

$$t = \frac{2\,G}{\mu\,F\,.\,\sqrt{2\,g}}\,(\sqrt{h_1} - \sqrt{h_2}).$$

EXAMPLE 1) How much does the surface of the water in the last example (§ 447) sink in 2 minutes? Here we have

$$h_1 = 4,\ t = 2\,.\,60 = 120,\ \frac{F}{G} = \frac{\pi}{14\,.\,144},$$

and if we assume also $\mu = 0{,}605$, it follows that

$$h_2 = \left(\sqrt{h_1} - \mu\,.\,\sqrt{2\,g}\,.\,\frac{F\,t}{2\,G}\right)^2 = \left(2 - \frac{0{,}605\,.\,8{,}025\,.\,\pi\,.\,120}{2\,.\,14\,.\,144}\right)^2$$

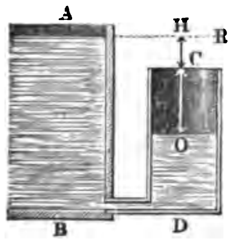
$$= \left(2 - 0,605 \,.\, 8,025 \,.\, \frac{5\,\pi}{168}\right)^2 = 1,546^2 = 2,3901 \text{ feet,}$$

and that the required distance that it sinks is

$$s = 4 - 2,3901 = 1,6099 \text{ feet.}$$

2) What time does the water require to rise in a pipe $C\,D$, Fig. 780, 18 inches in diameter, so as to overflow, when the pipe communicates with a vessel $A\,B$ by means of a short pipe $1\frac{1}{2}$ inches in diameter, and when the surface of the water G is in the beginning at a distance $O\,H = 6$ feet below the constant level of the water A and at a distance $O\,C = 4\frac{1}{2}$ feet below the top C of the pipe. We must substitute in the formula

FIG. 780.



$$t = \frac{2\,G}{\mu \sqrt{2\,g} \,.\, F}\left(\sqrt{h_1} - \sqrt{h_2}\right),$$

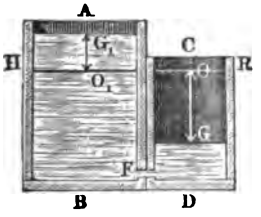
$h_1 = 6,\ h_2 = 6 - 4,5 = 1,5,$

$\frac{G}{F} = \left(\frac{18}{1,5}\right)^2 = 144$ and $\mu = 0,81$; thus we obtain

$$t = \frac{2 \,.\, 144}{0,81 \,.\, 8,025}(\sqrt{6} - \sqrt{1,5}) = \frac{288 \,.\, 1,2248}{0,81 \,.\, 8,025} = 54,8 \text{ seconds.}$$

§ **449.** If the first vessel $A\,B$, Fig. 781, from which the water passes into the second, receives no water, and if its cross-section G_1 cannot be considered *as infinitely great*, compared to the cross-section G of the other vessel $C\,D$, we must modify our calculation. If the variable distance $G_1\,O_1$ of the first surface of the water above the level $H\,R$, at which the water in both vessels stands after the efflux, $= x$ and the distance $G\,O$ of the other surface of the water below the same plane $= y$, the variable head will be $x + y$ and the corresponding velocity of efflux will be $v = \sqrt{2\,g\,(x + y)}$, or, since the quantity of water $G_1\,x = G\,y$,

FIG. 781.



$$v = \sqrt{2\,g\left(1 + \frac{G}{G_1}\right)y}.$$

The velocity with which the surface of the water rises in the second vessel is

$$v_1 = \frac{\mu\,F}{G}v = \frac{\mu\,F}{G}\sqrt{2\,g\left(1 + \frac{G}{G_1}\right)y};$$

hence the corresponding retardation is

$$p = \left(\frac{\mu\,F}{G}\right)^2\left(1 + \frac{G}{G_1}\right)g,$$

and the duration of efflux is

$$t = \frac{\mu F}{G}\sqrt{2g\left(1+\frac{G}{G_1}\right)y} : \left(\frac{\mu F^2}{G}\right)\left(1+\frac{G}{G_1}\right)g$$

$$= \frac{2\,G\sqrt{y}}{\mu F\sqrt{2g\left(1+\frac{G}{G_1}\right)}}.$$

Substituting instead of x and y the initial difference of level h, that is, putting $x + y = h$ or $\left(1+\frac{G}{G_1}\right)y = h$, we obtain

$$y = \frac{h}{1+\frac{G}{G_1}},$$

and the time in which the two surfaces of water come to one level is

$$t = \frac{2\,G\sqrt{h}}{\mu F\left(1+\frac{G}{G_1}\right)\sqrt{2g}} = \frac{2\,G\,G_1\sqrt{h}}{\mu F(G+G_1)\sqrt{2g}}$$

The time during which the difference of level changes from h to h_1 is, on the contrary,

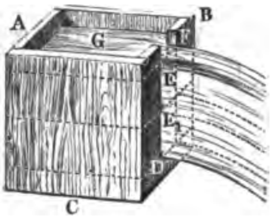
$$t = \frac{2\,G\,G_1\,(\sqrt{h}-\sqrt{h_1})}{\mu F(G+G_1)\sqrt{2g}}.$$

EXAMPLE.—If the cross-section G_1 of the vessel, from which the water flows into the other, is 10 square feet and the cross-section G of the vessel receiving the water is 4 square feet, if the initial difference of level between the two surfaces of water is 8 feet, and if the cylindrical pipe which forms the communication is 1 inch in diameter, the time in which the two surfaces of water will reach the same level is

$$t = \frac{2\,.\,10\,.\,4\,.\,\sqrt{8}}{0{,}82\,.\,8{,}025\,.\,\frac{\pi}{4}\,.\,\frac{14}{144}} = \frac{320\,.\,72\,.\,\sqrt{3}}{0{,}82\,.\,8{,}025\,.\,7\,\pi} = 276 \text{ seconds.}$$

§ **450. Notch in the Side.**—If the water issues through a *notch, overfall* or *weir* $D\,E$ from a prismatic vessel $A\,B\,C$, Fig. 782, into which there is no water discharged, the duration of the efflux is found in the following manner. Let us denote the cross-section of the vessel by G, the width $E\,F$ of the notch by b, and the height $D\,E$ by h, and let us divide the whole orifice of efflux into small strips, the length of each being

FIG. 782.



b and the height $\frac{h}{n}$. If the head is constant, the discharge per second is

$$Q = \tfrac{2}{3}\,\mu\, b \sqrt{2\, g\, h^3};$$

dividing the contents $\frac{G\,h}{n}$ of a layer of water by the latter, we obtain the corresponding duration of the efflux

$$\tau = \frac{G\,h}{\tfrac{2}{3}\,\mu\, n\, b \sqrt{2\, g\, h^3}},$$

for which we will write $\frac{3\,G\,h}{2\,\mu\, n\, b\sqrt{2\,g}}\,.\,h^{-\frac{3}{2}}$.

In order to obtain the duration t of the efflux of a quantity $G\,(h - h_1)$ or to determine the time during which the level of the water above the sill sinks from $D\,E = h$ to $D\,E_1 = h_1$, let us put $h_1 = \frac{m}{n}\,h$, I.E. let h_1 consist of m parts, and let us substitute in the last equation, instead of $h^{-\frac{3}{2}}$, successively

$$\left(\frac{m}{n}\,h\right)^{-\frac{3}{2}}, \left(\frac{m+1}{n}\,h\right)^{-\frac{3}{2}}, \left(\frac{m+2}{n}\,h\right)^{-\frac{3}{2}}, \ldots \left(\frac{n\,h}{n}\right)^{-\frac{3}{2}},$$

and then add the results found. In this manner we obtain the required time

$$t = \frac{3\,G\,h}{2\,\mu\, n\, b\sqrt{2\,g}}\left[\left(\frac{m\,h}{n}\right)^{-\frac{3}{2}} + \left(\frac{m+1}{n}\,h\right)^{-\frac{3}{2}} + \ldots + \left(\frac{n\,h}{n}\right)^{-\frac{3}{2}}\right]$$

$$= \frac{3\,G\,h}{2\,\mu\, n\, b\sqrt{2\,g}}\,.\,\frac{h^{-\frac{3}{2}}}{n^{-\frac{3}{2}}}\left[m^{-\frac{3}{2}} + (m+1)^{-\frac{3}{2}} + \ldots + n^{-\frac{3}{2}}\right]$$

$$= \frac{3\,G\,h^{-\frac{1}{2}}}{2\,\mu\, n^{-\frac{1}{2}}\, b\sqrt{2g}}\left[\left(1^{-\frac{3}{2}} + 2^{-\frac{3}{2}} + 3^{-\frac{3}{2}} + \ldots + n^{-\frac{3}{2}}\right)\right.$$
$$\left. - \left(1^{-\frac{3}{2}} + 2^{-\frac{3}{2}} + 3^{-\frac{3}{2}} + \ldots + m^{-\frac{3}{2}}\right)\right],$$

or, according to the Ingenieur, page 88,

$$t = \frac{3\,G\,h^{-\frac{1}{2}}}{2\,\mu\, n^{-\frac{1}{2}}\, b\sqrt{2\,g}}\left(\frac{n^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} - \frac{m^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right)$$

$$= \frac{3\,G\,n^{\frac{1}{2}}}{2\,\mu\, b\sqrt{2\,g\,h}}\,.\,2\left(m^{-\frac{1}{2}} - n^{-\frac{1}{2}}\right) = \frac{3\,G}{\mu\, b\sqrt{2\,g\,h}}\left[\left(\frac{m}{n}\right)^{-\frac{1}{2}} - 1\right]$$

$$= \frac{3\,G}{\mu\, b\sqrt{2\,g}}\left[\left(\frac{m}{n}\,h\right)^{-\frac{1}{2}} - h^{-\frac{1}{2}}\right] = \frac{3\,G}{\mu\, b\sqrt{2\,g}}\left(\frac{1}{\sqrt{h_1}} - \frac{1}{\sqrt{h}}\right).$$

If we put $h_1 = 0$, we obtain $\frac{1}{\sqrt{h_1}}$ and also $t = \infty$; hence the time required for the water to sink to the level of the sill will be infinitely great.

Example.—If the water issues from a reservoir 110 feet long and 40 feet wide, through an overfall 8 inches wide, in what time will the level of the water fall from 15 inches above the sill to 6 inches above it? Here we have

$$t = \frac{8\,.\,110\,.\,40}{\mu\,.\,\frac{2}{3}\,.\,8,025}\left(\frac{1}{\sqrt{0,5}} - \frac{1}{\sqrt{1,25}}\right) = \frac{19800}{\mu\,.\,8,025}\left(\sqrt{2} - \sqrt{\tfrac{4}{5}}\right)$$

$$= \frac{19800}{8,025\,\mu}(1,4142 - 0,8944) = \frac{19800\,.\,0,5198}{8,025\,\mu} = \frac{1282,5}{\mu} \text{ seconds.}$$

If we assume as the coefficient of efflux $\mu = 0,60$, we have for the real time of discharge

$$t = \frac{1282,5}{0,6} = 2137,5 \text{ seconds} = 35 \text{ minutes } 37,5 \text{ seconds.}$$

Remark.—For a rectangular orifice in the side we can put approximatively

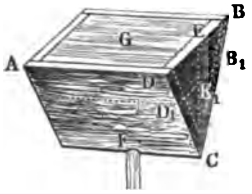
$$t = \frac{2\,G}{\mu\,F\sqrt{2\,g}}\left((\sqrt{h_1} - \sqrt{h_2}) - \frac{a^2}{288}(\sqrt{h_1^{-3}} - \sqrt{h_2^{-3}})\right),$$

in which F and G denote the cross-sections of the orifice and of the vessel, a the height of the orifice, h_1 the initial head, and h_2 the head when the discharge ceases. If $h_2 = \frac{a}{2}$, the orifice becomes a notch and the formula for overfalls must be employed.

§ 451. Wedge-Shaped and Pyramidal Vessels.

—If the vessel $A\,B\,F$, Fig. 783, from which the water is discharged, forms a horizontal *triangular prism*, the time in which it will empty itself is found in the following manner. If we divide the height $C\,E = h$ into n equal parts and pass horizontal planes through the points of division, the whole mass of water will be divided into equally thick horizontal layers, whose common length is $A\,D = l$ and whose width diminishes from the surface downwards. If the width $D\,B$ of the upper layer $= b$, the width $D_1\,B_1$ of another layer situated at the distance $C\,E_1 = x$ above the orifice F, which is located at the lower edge of the prism, is $y = \frac{x}{h}\,b$, and its volume is $y\,l\,.\,\frac{h}{n} = \frac{b\,l\,x}{n}$. But the discharge in the unit of time is

Fig. 783.



$$Q = \mu\,F\sqrt{2\,g\,x};$$

hence the small time, during which the water sinks a distance $\frac{h}{n}$,

$$\tau = \frac{b\,l\,x}{n} : \mu\,F\sqrt{2\,g\,x} = \frac{b\,l}{n\,\mu\,F\sqrt{2\,g}}\,.\,x^{\frac{1}{2}}.$$

Finally, since the sum of all the $x^{\frac{1}{2}}$ from $x = \frac{h}{n}$ to $x = \frac{n\,h}{n}$ is

$$= \left(\frac{h}{n}\right)^{\frac{1}{2}} \cdot \frac{n^{\frac{3}{2}}}{\frac{3}{2}} = \tfrac{2}{3}\, n\, h^{\frac{1}{2}},$$

we have the time of discharge of the whole prism of water

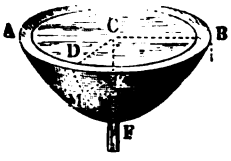
$$= \frac{b\,l}{n\,\mu\,F\sqrt{2g}} \cdot \tfrac{2}{3}\,n\,h^{\frac{1}{2}} = \tfrac{2}{3}\,\frac{b\,l}{\mu\,F\sqrt{2g}} \cdot h^{\frac{1}{2}} = \tfrac{4}{3}\,\frac{\frac{1}{2}\,b\,l\,h}{\mu\,F\sqrt{2g\,h}}, \text{ I.E.}$$

$$t = \tfrac{4}{3} \cdot \frac{V}{\mu\,F\,c},$$

in which $V = \frac{1}{2}\,b\,l\,h$ denotes the total volume of the water and $c = \sqrt{2\,g\,h}$ the initial velocity. In this case it requires $\frac{1}{3}$ more time to empty the vessel than if the velocity c were constant.

If the vessel $A\ B\ F$, Fig. 784, forms an upright *paraboloid*, we have the ratio of the radii $K\,M = y$ and $C\,D = b$

FIG. 784.



$$\frac{y}{b} = \frac{\sqrt{x}}{\sqrt{h}};$$

hence the ratio of the horizontal section G_1 through K to the base $A\ D\ B = G$ is

$$\frac{G_1}{G} = \frac{y^2}{b^2} = \frac{x}{h}, \text{ and therefore}$$

$$G_1 = \frac{G\,x}{h};$$

the volume of the layer of water is

$$= G_1 \cdot \frac{h}{n} = \frac{G\,x}{n}.$$

As this expression coincides exactly with that found for the triangular prism, we can put here also

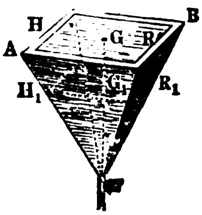
$$t = \tfrac{4}{3} \cdot \frac{\frac{1}{2}\,G\,h}{\mu\,F\sqrt{2\,g\,h}},$$

or, since $V = \frac{1}{2}\,G\,h$ (§ 124, Example),

$$t = \tfrac{4}{3}\,\frac{V}{\mu\,F\,c}.$$

This formula can also be employed in many other cases for the approximate determination of the duration of efflux, E.G., for determining the time required *to empty a dam*. It is applicable, whenever the horizontal sections increase in the same ratio as the distances from the bottom.

FIG. 785.



If, finally, we have a *pyramidal* vessel $A\,B\,F$, Fig. 785, to deal with, then $G_1 : G = x^2 : h^2$, and, therefore, $G_1 = \frac{G\,x^2}{h^2}$; the volume of the layer $H_1\,R_1$ is

$$\frac{G_1\,h}{n} = \frac{G\,x^2}{n\,h},$$

and the time necessary to discharge the latter is

$$\tau = \frac{G\,x^2}{n\,h} : \mu\,F\sqrt{2\,g\,x} = \frac{G}{n\,\mu\,F\,h\sqrt{2\,g}} \,.\, x^{\frac{3}{2}}.$$

But since the sum of all the $x^{\frac{3}{2}}$ from $x = \frac{h}{n}$ to $x = \frac{n\,h}{n}$ is

$$= \left(\frac{h}{n}\right)^{\frac{3}{2}} . \frac{n^{\frac{5}{2}}}{\frac{5}{2}} = \tfrac{2}{5}\,n\,h^{\frac{3}{2}},$$

we have the time necessary to empty the entire pyramid

$$t = \frac{G}{n\,\mu\,F\,h\sqrt{2\,g}} . \tfrac{2}{5}\,n\,h^{\frac{3}{2}} = \tfrac{2}{5}\frac{G\,h^{\frac{1}{2}}}{\mu\,F\sqrt{2\,g}} = \tfrac{6}{5} . \frac{\frac{1}{3}\,G\,h}{\mu\,F\sqrt{2\,g\,h}},$$

or, putting $\frac{1}{3}\,G\,h = V$,

$$t = \tfrac{6}{5} . \frac{V}{\mu\,F\,c}.$$

Since in this case the initial velocity gradually diminishes from c to 0, the duration of the efflux is $\frac{1}{5}$ greater than if the velocity remained invariable and equal to c.

EXAMPLE.—What time will a dam, the area of whose surface is 765000 square feet, require to empty itself, when the discharge pipe enters at the deepest place and is 15 inches in diameter and 50 feet long, and when the depth is 15 feet? Theoretically

$$t = \tfrac{6}{5}\frac{V}{F\sqrt{2\,g\,h}} = \tfrac{6}{5} . \tfrac{1}{3} . \frac{765000\,.\,15}{\frac{\pi}{4} . \left(\frac{5}{4}\right)^2 . 8{,}025\sqrt{15}} = \frac{19584000}{\pi\,.\,8{,}025\sqrt{15}}$$
$$= 200568 \text{ seconds.}$$

But the coefficient of resistance for the entrance of the water into the pipe, which is cut off at an angle of 45°, is

$$\zeta = 0{,}505 + 0{,}327 \text{ (see § 428)} = 0{,}832,$$

and the resistance of friction for the pipe is

$$= 0{,}025\,\frac{l}{d} . \frac{v^2}{2\,g} = 0{,}025 . \frac{50}{\frac{5}{4}} . \frac{v^2}{2\,g} = \frac{v^2}{2\,g};$$

hence the complete coefficient of efflux for the same is

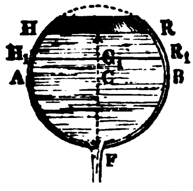
$$\mu = \frac{1}{\sqrt{1 + 0{,}832 + 1}} = \frac{1}{\sqrt{2{,}832}} = 0{,}594,$$

and the required duration of efflux is

$t = 200568 : 0{,}594 = 337655$ seconds = 93 hours 47 minutes 35 seconds.

§ 452. Spherical and Obelisk Shaped Vessels.—By the aid of the formulas, deduced in the foregoing paragraph, we can find the duration of the efflux from spherical, obelisk shaped, pyramidal, etc., vessels.

Fig. 786.



1) The time required to empty a *segment of a sphere* $A\,F\,B$, Fig. 786, which is filled with water, whose radius $C\,A = C\,F = r$ and whose height $F\,G = h$, is

$$t = \tfrac{4}{3}\,\frac{\pi\, r\, h^2}{\mu\, F\sqrt{2\,g\,h}} - \tfrac{2}{5}\cdot\frac{\pi\, h^3}{\mu\, F\sqrt{2\,g\,h}} = \tfrac{2}{15}\,\pi\,\frac{(10\,r - 3\,h)\,h^{\frac{3}{2}}}{\mu\, F\sqrt{2\,g}},$$

or, if an entire sphere is to be emptied, in which case $h = 2\,r$,

$$t = \frac{16\,\pi\, r^2\sqrt{2\,r}}{15\,\mu\, F\sqrt{2\,g}},$$

and for a hemisphere, where $h = r$, $t = \dfrac{14\,\pi\, r^2\sqrt{r}}{15\,\mu\, F\sqrt{2\,g}}$.

Here the horizontal layer $H_1\,R_1 = G_1$, corresponding to the depth $F\,G_1 = x$, is

$$= \pi\, x\,(2\,r - x)\,\frac{h}{n} = \frac{2\,\pi\, r\, h\, x}{n} - \frac{\pi\, h\, x^2}{n};$$

hence, if the velocity of efflux is $v = \sqrt{2\,g\,x}$, the duration of the discharge will be

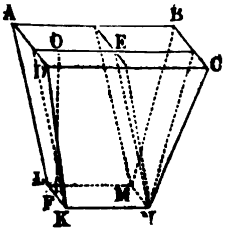
$$\tau = \frac{2\,\pi\, r\, h}{n\,\mu\, F\sqrt{2\,g}}\,.\,x^{\frac{1}{2}} - \frac{\pi\, h}{n\,\mu\, F\sqrt{2\,g}}\,.\,x^{\frac{3}{2}}.$$

Since the first part of this expression coincides with the formula for the emptying of prismatic vessels and the second part with that for the emptying of pyramidal vessels, if we in the first case substitute $2\,\pi\, r\, h$ for $b\,l$ and in the second $\pi\, h^2$ instead of G, we obtain by the aid of the difference of times required to empty a prismatic and a pyramidal vessel

$$t = \tfrac{2}{3}\,.\,\frac{b\,l\,h}{\mu\, F\sqrt{2\,g\,h}} \text{ and } t = \tfrac{2}{5}\,.\,\frac{G\,h}{\mu\, F\sqrt{2\,g\,h}}$$

the time, in which a segment of a sphere will empty itself, as was found above.

Fig. 787.



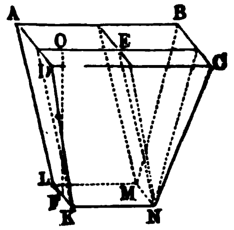
2) For a vessel $A\;C\;K$, Fig. 787, shaped like an *obelisk* or a *pontoon*, we can employ the above formulas; for we can consider it to be composed of a parallelopipedon $A\,E\,K$, of two prisms $B\,E\,N$ and $D\,E\,N$ and of a pyramid $C\,E\,N$ (compare § 121). Let b be

the width $A\,D$ of the top, b_1 that $K\,L$ of the bottom, l the length $A\,B$ of top, l_1 that $K\,N$ of the bottom and h the height $O\,F$ of the vessel. Then we have the surface $A\,C$ of the water

$$b\,l = b_1\,l_1 + b_1\,(l - l_1) + l_1\,(b - b_1) + (l - l_1)\,(b - b_1),$$

in which $b_1\,l_1$ is the base of the parallelopipedon $A\,E\,K$, $b_1\,(l - l_1)$ and $l_1\,(b - b_1)$ the bases of the prisms $B\,E\,N$ and $D\,E\,K$ and $(l - l_1)\,(b - b_1)$ that of the pyramid $C\,E\,N$. Now the time required to empty the parallelopipedon is

FIG. 788.



$$t_1 = \frac{2\,b_1\,l_1\,\sqrt{h}}{\mu\,F\,\sqrt{2\,g}},$$

that required for the triangular prisms is

$$t_2 = \tfrac{2}{3}\,\frac{[b_1\,(l - l_1) + l_1\,(b - b_1)]\,\sqrt{h}}{\mu\,F\,\sqrt{2\,g}}$$

and finally that required to empty the pyramid is

$$t_3 = \tfrac{2}{5}\,\frac{(l - l_1)\,(b - b_1)\,\sqrt{h}}{\mu\,F\,\sqrt{2\,g}}.$$

hence the time required to empty the *entire vessel* is

$$t = t_1 + t_2 + t_3$$

$$= [30\,b_1\,l_1 + 10\,b_1(l - l_1) + 10\,l_1\,(b - b_1) + 6\,(l - l_1)\,(b - b_1)]\,\frac{\sqrt{h}}{15\,\mu\,F\,\sqrt{2\,g}}$$

$$= [3\,b\,l + 8\,b_1\,l_1 + 2\,(b\,l_1 + b_1\,l)]\,\frac{2\,\sqrt{h}}{15\,\mu\,F\,\sqrt{2\,g}}.$$

When $\frac{b_1}{l_1} = \frac{b}{l}$ the vessel is a *truncated pyramid*. Putting in this case the base $b\,l = G$ and the base $b_1\,l_1 = G_1$, we obtain

$$t = (3\,G + 8\,G_1 + 4\,\sqrt{G\,G_1})\,\frac{2\,\sqrt{h}}{15\,\mu\,F\,\sqrt{2g}}.$$

It is easy to see that this formula will hold good for any triangular or polygonal pyramid.

EXAMPLE.—An obelisk-shaped reservoir is 5 feet long and 3 feet wide on top and at a depth of 4 feet, where a short pipe 1 inch in diameter and 3 inches long is inserted in it, it is 4 feet long and 2 feet wide; how long a time will be required for the water to sink $2\frac{1}{2}$ feet? The time required to empty it is, assuming $\mu = 0{,}815$,

$$t = [8\,.\,4\,.\,2 + 3\,.\,5\,.\,3 + 2\,(3\,.\,4 + 5\,.\,2)]\,\frac{2\,\sqrt{4}}{15\,.\,0{,}815\,.\,\frac{\pi}{4}\,.\left(\frac{1}{12}\right)^2.\,8{,}025}$$

$$= \frac{153 \,.\, 4 \,.\, 4 \,.\, 144}{15 \,.\, 0{,}815 \,.\, 8{,}025 \,.\, \pi} = 153 \,.\, \frac{2304}{12{,}225 \,.\, 8{,}025\,\pi} = 153 \,.\, 7{,}475 = 1144 \text{ sec.}$$

At the level $4 - 2\frac{1}{2} = 1\frac{1}{2}$ feet above the tube $l = l_1 + \frac{3}{8} = 4\frac{3}{8}$ and $b = b_1 + \frac{3}{8} = 2\frac{3}{8}$ feet; hence the time required to empty the vessel, when it is filled to that height, is

$$t_1 = [3 \,.\, 4 \,.\, 2 + 3 \,.\, \tfrac{35}{8} \,.\, \tfrac{19}{8} + 2\,(2 \,.\, \tfrac{35}{8} + 4 \,.\, \tfrac{19}{8})] \,.\, \frac{1152\,\sqrt{1{,}5}}{15 \,.\, 0{,}815 \,.\, 8{,}025\,\pi}$$
$$= 603 \text{ seconds.}$$

The difference of these times gives the time (541 seconds) in which the surface of the water will sink $2\frac{1}{2}$ feet from the top.

§ 453. Irregularly-shaped Vessels.—If we are required to find the duration of efflux for an *irregularly-shaped vessel H F R*,

FIG. 789.

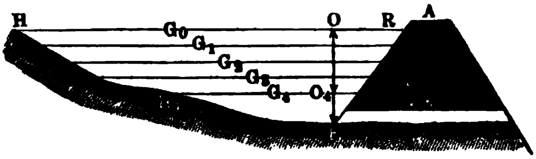


Fig. 789, we must employ some method of approximation, such as Simpson's Rule. If we divide the whole quantity of water into 4 equally thick layers and denote the heads corresponding to the different horizontal sections G_0, G_1, G_2, G_3, G_4, by h_0, h_1, h_2, h_3, h_4, we obtain, according to Simpson's Rule, the duration of the efflux

$$t = \frac{h_0 - h_4}{12\,\mu\,F\sqrt{2g}}\left(\frac{G_0}{\sqrt{h_0}} + \frac{4\,G_1}{\sqrt{h_1}} + \frac{2\,G_2}{\sqrt{h_2}} + \frac{4\,G_3}{\sqrt{h_3}} + \frac{G_4}{\sqrt{h_4}}\right).$$

If we divide it into six layers, we have

$$t = \frac{h_0 - h_6}{18\,\mu\,F\sqrt{2g}}\left(\frac{G_0}{\sqrt{h_0}} + \frac{4\,G_1}{\sqrt{h_1}} + \frac{2\,G_2}{\sqrt{h_2}} + \frac{4\,G_3}{\sqrt{h_3}} + \frac{2\,G_4}{\sqrt{h_4}} + \frac{4\,G_5}{\sqrt{h_5}} + \frac{G_6}{\sqrt{h_6}}\right).$$

The discharge in the first case is

$$V = \frac{h_0 - h_4}{12}\,(G_0 + 4\,G_1 + 2\,G_2 + 4\,G_3 + G_4), \text{ and in the second}$$

$$V = \frac{h_0 - h_6}{18}\,(G_0 + 4\,G_1 + 2\,G_2 + 4\,G_3 + 2\,G_4 + 4\,G_5 + G_6).$$

If the form and size of the reservoir is not known, we can calculate the discharge V by observing the heights h_0, h_1, etc., of the water at equal intervals of time. If t is the whole duration of the efflux, we have for orifices in the side and bottom

$$V = \frac{\mu\,F\,t\,\sqrt{2g}}{12}\,(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}),$$

and for overfalls or notches

$$V = \tfrac{2}{3}\,\frac{\mu\,b\,t}{12}\,\sqrt{2g}\,(\sqrt{h_0^3} + 4\sqrt{h_1^3} + 2\sqrt{h_2^3} + 4\sqrt{h_3^3} + \sqrt{h_4^3}).$$

EXAMPLE.—In what time will the surface of the water of a dam sink 6 feet, when the discharge-pipe is a semi-cylinder 18 inches wide, 9 inches high, and 60 feet long, and when the cross-sections of the surfaces of the water are

for a head of 20 feet, $G_0 = 600000$ square feet,
" " 18,5 " $G_1 = 495000$ "
" " 17,0 " $G_2 = 410000$ "
" " 15,5 " $G_3 = 325000$ "
" " 14,0 " $G_4 = 265000$ " ?

Now $F = \frac{\pi}{8} \cdot (\frac{3}{2})^2 = \frac{9\pi}{32} = 0,8836$ square feet. If we put, as in the Example of § 451, the coefficient of resistance for the entrance of the water in the pipe $= 0,832$ and that of the friction, $= 0,025 \frac{l}{d} = 0,025 \,.\, 60 \,.\, 1,091 = 1,6356$, we obtain the coefficient of efflux

$$\mu = \frac{1}{\sqrt{1 + 0,882 + 1,6356}} = \frac{1}{\sqrt{3,4685}} = 0,537, \text{ and}$$

$$\mu F \sqrt{2g} = 0,537 \,.\, 0,8836 \,.\, 8,025 = 3,808.$$

Now we have

$$\frac{G_0}{\sqrt{h}} = \frac{600000}{\sqrt{20}} = 134170, \quad \frac{G_1}{\sqrt{h_1}} = \frac{495000}{\sqrt{18,5}} = 115090,$$

$$\frac{G_2}{\sqrt{h_2}} = \frac{410000}{\sqrt{17}} = 99440, \quad \frac{G_3}{\sqrt{h_3}} = \frac{325000}{\sqrt{15,5}} = 82550,$$

$$\frac{G_4}{\sqrt{h_4}} = \frac{265000}{\sqrt{14}} = 70830;$$ hence the duration of the efflux is

$$t = \frac{6}{12 \,.\, 3,808} (134170 + 4 \,.\, 115090 + 2 \,.\, 99440 + 4 \,.\, 82550 + 70830)$$

$$= \frac{1194440}{7,616} = 156833 \text{ seconds} = 43 \text{ hours } 33 \text{ minutes } 53 \text{ seconds.}$$

The discharge is

$$V = \tfrac{6}{12} \,.\, (600000 + 4 \,.\, 495000 + 2 \,.\, 410000 + 4 \,.\, 325000 + 265000)$$

$$= \frac{4965000}{2} = 2882500 \text{ cubic feet.}$$

§ 454. **Influx and Efflux.**—If, while water is flowing out of the vessel, other water is flowing into it, the determination of the time in which the level of the water will rise or sink a certain distance becomes much more complicated, and we are generally obliged to content ourselves with an approximate result. If the discharge per second into the vessel $Q_1 > \mu F \sqrt{2 g h}$, the water will rise, and if $Q_1 < \mu F \sqrt{2 g h}$, it will sink. But the level of the water becomes constant, when the head is increased or diminished, until it becomes equal to $k = \frac{1}{2g} \left(\frac{Q_1}{\mu F}\right)^2$. The time τ, during which the

variable head x is increased a small quantity ξ, is determined by the equation $G_1 \xi = Q_1 \tau - \mu F \sqrt{2 g x} \, . \, \tau$,
and, on the contrary, the time, in which the surface of the water sinks a distance ξ, is determined by the equation

$$G_1 \xi = \mu F \sqrt{2 g x} \, . \, \tau - Q_1 \tau.$$

Hence we have in the first case

$$\tau = \frac{G_1 \xi}{Q_1 - \mu F \sqrt{2 g x}}, \text{ and in the second}$$

$$\tau = \frac{G_1 \xi}{\mu F \sqrt{2 g x} - Q_1}.$$

By employing Simpson's Rule, we obtain the time of discharge, during which G_0 becomes successively G_1, G_2, etc., and the head h_0 becomes h_1, $h_2 \ldots$,

$$t = \frac{h_0 - h_4}{12} \left[\frac{G_0}{\mu F \sqrt{2 g h_0} - Q_1} + \frac{4 G_1}{\mu F \sqrt{2 g h_1} - Q_1} + \frac{2 G_2}{\mu F \sqrt{2 g h_2} - Q_1} \right.$$
$$\left. + \frac{4 G_3}{\mu F \sqrt{2 g h_3} - Q_1} + \frac{G_4}{\mu F \sqrt{2 g h_4} - Q_1} \right],$$

or if we denote $\frac{Q_1}{\mu F \sqrt{2 g}}$ by $\sqrt{k}$, we have more simply

$$t = \frac{h_0 - h_4}{12 \mu F \sqrt{2 g}} \left[\frac{G_0}{\sqrt{h_0} - \sqrt{k}} + \frac{4 G_1}{\sqrt{h_1} - \sqrt{k}} + \frac{2 G_2}{\sqrt{h_2} - \sqrt{k}} \right.$$
$$\left. + \frac{4 G_3}{\sqrt{h_3} - \sqrt{k}} + \frac{G_4}{\sqrt{h_4} - \sqrt{k}} \right].$$

If the vessel is *prismatic* and its cross-section is constant and $= G$, we have (see the author's "Experimentalhydraulik," § 9, XII)

$$t = \frac{2 G}{\mu F \sqrt{2 g}} \left[\sqrt{h} - \sqrt{h_1} + \sqrt{k} \, . \, l \left(\frac{\sqrt{h} - \sqrt{k}}{\sqrt{h_1} - \sqrt{k}} \right) \right]$$

for the time, in which the head h changes to h_1.

Since for $h_1 = k$, $\frac{\sqrt{h} - \sqrt{k}}{\sqrt{h_1} - \sqrt{k}} = \frac{\sqrt{h} - \sqrt{k}}{0} = \infty$,

it follows that the level of the water becomes permanent after an infinite time has elapsed.

For a notch in the side we have the following formula

$$t = \frac{G k}{3 Q_1} \left[l \, \frac{(\sqrt{h} - \sqrt{k})^2 (h_1 + \sqrt{h_1 k} + k)}{(\sqrt{h_1} - \sqrt{k})^2 (h + \sqrt{h k} + k)} \right.$$
$$\left. + \sqrt{12} \, tang.^{-1} \frac{(\sqrt{h} - \sqrt{h_1}) \sqrt{12 k}}{3 k + (2 \sqrt{h} + \sqrt{k})(2 \sqrt{h_1} + \sqrt{k})} \right],$$

in which $k = \left(\frac{Q_1}{\frac{2}{3}\,\mu\, b \sqrt{2\,g}}\right)^{\frac{2}{3}}$ and l denotes the Naperian logarithm and $tang.^{-1}\, y$ the arc whose tangent is y.

According as $k \lesseqgtr h$ or the discharge into the vessel

$$Q_1 \gtreqless \tfrac{2}{3}\,\mu\, b \sqrt{2\,g\,h^3},$$

a rising or a sinking of the water in the vessel takes place. The state of permanency occurs, when $h_1 = k$, but in this case the corresponding time t becomes $= \infty$.

Example.—In what time will the water in a parallelopipedical box 12 feet long and 6 feet wide rise 2 feet above the sill of a notch in the side ½ foot wide, when the discharge into it is 5 cubic feet per second? Here we have $h = 0$ and consequently more simply

$$t = \frac{G\,k}{3\,Q_1}\left[l\,\frac{h_1 + \sqrt{h_1\,k} + k}{(\sqrt{h} - \sqrt{k})^2} + 12\; tang.^{-1}\,\frac{-\sqrt{3\,h_1}}{2\sqrt{k} + \sqrt{h_1}}\right].$$

Now $G = 12\,.\,6 = 72$ feet, $Q_1 = 5$, $h_1 = 2$, $b = \frac{1}{2}$, $\mu = 0{,}6$, and

$$k = \left(\frac{5}{\frac{2}{3}\,.\,0{,}6\,.\,\frac{1}{2}\,.\,8{,}025}\right)^{\frac{2}{3}} = 2{,}133\,;$$

hence the time required is

$$t = \frac{72\,.\,2{,}1330}{3\,.\,5}\left[l\,\frac{4{,}1330 + \sqrt{4{,}2660}}{(1{,}4142 - 1{,}4605)^2} - \sqrt{12}\; tang.^{-1}\left(\frac{\sqrt{6}}{1{,}4142 + 2{,}9210}\right)\right]$$

$$= 10{,}238\left[l\,\frac{6{,}1984}{0{,}002144} - \sqrt{12}\; tang.^{-1}\left(\frac{\sqrt{6}}{4{,}3352}\right)\right]$$

$$= 10{,}238\,(7{,}969 - 1{,}776) = 10{,}238\,.\,6{,}193 = 63\tfrac{4}{10} \text{ seconds.}$$

§ 455. Locks and Sluices.—We can make a useful application of the principles just enunciated to the *filling* and *emptying* of *locks and sluices* (Fr. écluses; Ger. Schleusen). We distinguish two kinds of locks, namely, the single and the double. The *single lock* consists of a chamber B, Fig. 790, which is separated from the water in the *head bay* A by the gate $H\,F$ and from that in the *tail bay* C by the gate $R\,S$. The *double lock*, Fig. 791, on the contrary, consists of two chambers with an upper gate $K\,L$, a middle one $H\,F$, and a lower one $R\,S$.

Fig. 790.

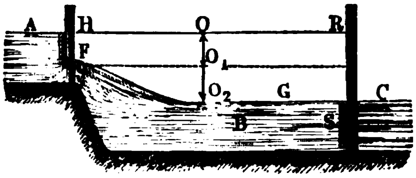
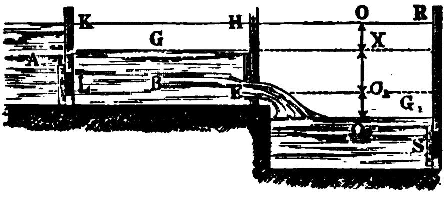


Fig. 791.



1) If we put the mean horizontal cross-section of the chamber of a single lock $= G$, the distance $O\,O_1$ of the centre of the opening in the upper gate below the surface $H\,R$ of the water in the head bay $= h_1$, its distance $O_1\,O_2$ above the water in the tail bay $= h_2$ and the cross-section of the orifice in the gate $= F$, we have the time necessary to fill the lock to the middle of the orifice, during which the head is constant,

$$t_1 = \frac{G\,h_2}{\mu\,F\,\sqrt{2\,g\,h_1}}$$

and the time necessary to fill the remaining space, during which there is a gradual diminution of head,

$$t_2 = \frac{2\,G\,h_1}{\mu\,F\,\sqrt{2\,g\,h_1}},$$

hence the time required to fill the whole lock is

$$t = t_1 + t_2 = \frac{(2\,h_1 + h_2)\,G}{\mu\,F\,\sqrt{2\,g\,h_1}}.$$

If the orifice in the lower gate is entirely submerged, the head decreases gradually during the emptying from $O\,O_2 = h_1 + h_2$ to zero, and the time of emptying the lock is, therefore,

$$t = \frac{2\,G\,\sqrt{h_1 + h_2}}{\mu\,F\,\sqrt{2\,g}}.$$

But if, on the contrary, a part of the orifice lies above the lower water level, we have to consider two quantities of water, one discharged above and the other below the water. Putting the height of the portion of the orifice above the water $= a_1$, the height of that below the water $= a_2$ and the width of the orifice $= b$, we obtain the duration of the efflux by means of the formula

$$t = \frac{2\,G\,(h_1 + h_2)}{\mu\,b\,\sqrt{2\,g}\left(a_1\,\sqrt{h_1 + h_2 - \frac{a_1}{2}} + a_2\,\sqrt{h_1 + h_2}\right)}.$$

2) In the double lock (Fig. 791) the head in the upper chamber which is cut off from the head bay gradually diminishes during the efflux into the second chamber. If G is the horizontal cross-section of the first chamber, and if the initial head $O\,O_1 = h_1$ is diminished to $X\,O_1 = x$, while the water in the lower chamber rises to the middle of the orifice in the gate a distance $O_1\,O_2 = h_2$, we have the time corresponding to it

$$t_1 = \frac{2\,G}{\mu\,F\,\sqrt{2\,g}}\,(\sqrt{h_1} - \sqrt{x}).$$

But the discharge is

$$G\,(h_1 - x) = G_1\,h_2;\ \text{hence}$$

$$x = h_1 - \frac{G_1}{G}\,h_2\ \text{and}$$

$$t_1 = \frac{2\,G}{\mu\,F\,\sqrt{2\,g}}\left(\sqrt{h_1} - \sqrt{h_1 - \frac{G_1\,h_2}{G}}\right)$$

$$= \frac{2\,\sqrt{G}}{\mu\,F\,\sqrt{2\,g}}\,(\sqrt{G\,h_1} - \sqrt{G\,h_1 - G_1\,h_2}).$$

The time in which the water in the second chamber rises to a level with that in the first, or in which the water in the two chambers assumes a common level, is, according to § 449,

$$t_2 = \frac{2\,G\,G_1\,\sqrt{x}}{\mu\,F\,(G+G_1)\,\sqrt{2\,g}} = \frac{2\,G_1\,\sqrt{G}\,\sqrt{G\,h_1 - G_1\,h_2}}{\mu\,F\,(G + G_1)\,\sqrt{2\,g}},$$

and the whole time required to fill it is

$$t = t_1 + t_2 = \frac{2\,\sqrt{G}}{\mu\,F\,\sqrt{2\,g}}\left(\sqrt{G\,h_1} - \frac{G}{G + G_1}\,\sqrt{G\,h_1 - G_1\,h_2}\right).$$

EXAMPLE.—What time is necessary to empty and fill a single lock of the following dimensions: mean length of the lock = 200 feet, mean width = 24 feet or $G = 200\,.\,24 = 4800$ square feet; distance of the centre of the orifice in the upper gate from both surfaces of water = 5 feet, width of both orifices = 2½ feet, height of the orifice in the upper gate = 4 feet, and height of the orifice (entirely submerged) of the lower gate = 5 feet? Substituting in the formula

$$t = \frac{(2\,h_1 + h_2)\,G}{\mu\,F\sqrt{2\,g\,h}},$$

$h_1 = 5, h_2 = 5,\ G = 4800, \mu' = 0{,}615,\ F = 4\,.\,2\tfrac{1}{2} = 10$ and $\sqrt{2\,g} = 8{,}025$, we have for the time required to fill it

$$t = \frac{3\,.\,5\,.\,4800}{6{,}15\,\,.\,\,8{,}025\,\sqrt{5}} = \frac{14400}{1{,}23\,.\,8{,}025\,\sqrt{5}} = 652\tfrac{1}{2}\ \text{séc.} = 10\ \text{min.}\ 52\tfrac{1}{2}\ \text{sec.}$$

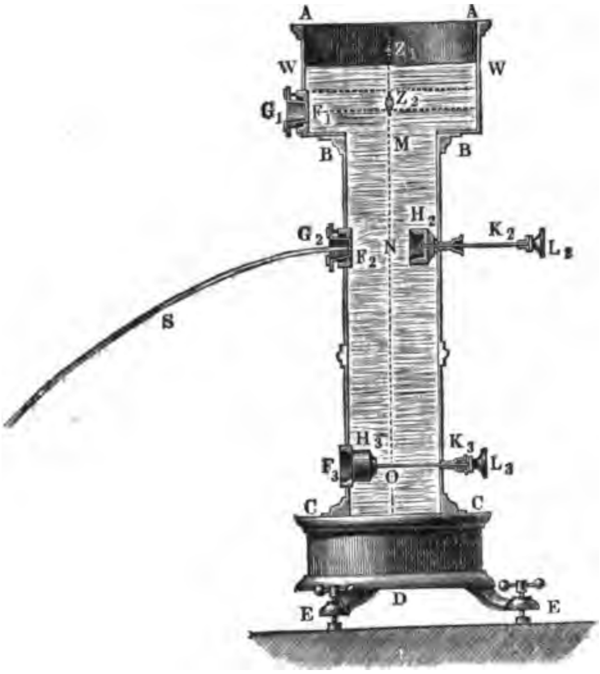
If we substitute in the formula

$t = \dfrac{2\,G\,\sqrt{h_1 + h_2}}{\mu\,F\,\sqrt{2\,g}}$, $G = 4800,\ h_1 + h_2 = 10,\ F = 5\,.\,2\tfrac{1}{2} = 12{,}5$, we have the time necessary to empty the lock

$$t = \frac{2\,.\,4800\,\sqrt{10}}{0{,}615\,.\,12{,}5\,.\,8{,}025} = 492\ \text{seconds} = 8\ \text{minutes}\ 12\ \text{seconds}.$$

§ 456. Apparatus for Hydraulic Experiments.—By means of the apparatus represented in Fig. 792, we can not only show by more than 100 experiments the most important phenomena of efflux, but also prove in figures the most important of its laws. The apparatus consists of a discharging vessel *A B C*, which is provided with three orifices F_1, F_2, F_3, whose centres are at distances from the mean level of the water, which are to each other as the squares 1, 4, 9. To these orifices various mouth-pieces and pipes can be applied, and in order to do this without being disturbed by the water, we close the orifice by means of a particular kind of valve H_2, H_3, to which is attached a rod passing through a stuffing box in the back of the apparatus. In the upper and wider part A B of the apparatus two pointers Z_1 and Z_2, which are directed upwards, are placed. These serve as fixed points, the one marking the beginning and the other the end of the experiment. The water which is discharged is caught in a vessel, which before each experiment is placed on top the discharging reservoir, into which its contents are emptied by opening an orifice that is generally closed by a stopper.

FIG. 792.



In order to find by the aid of this apparatus the coefficient of efflux μ for different mouth-pieces and tubes, we must observe by means of a good stop-watch the time t, in which the water-level sinks from one pointer to the other, or within which the head h_1 becomes h_2; if, then, F is the cross-section of the orifice and G the area of the sinking surface of the water, we have the coefficient of efflux (see § 448)

$$\mu = \frac{2\,G\,(\sqrt{h_1} - \sqrt{h_2})}{F\,t\,\sqrt{2\,g}},$$

and the corresponding mean head

$$h = \left(\frac{\sqrt{h_1} + \sqrt{h_2}}{2}\right)^2.$$

This apparatus is provided with a collection of *mouth-pieces* and tubes, viz., square, rectangular, circular and triangular orifices in a thin plate with or without an internal rim, short cylindrical and conical tubes, long straight tubes of different diameters, elbows, bends, etc., which can be inserted in the different openings F_1, F_2, F_3. By means of an apparatus with the above accessories we can show in a few hours all the phenomena and laws of efflux; with it we can study not only the perfect and imperfect and complete and incomplete contraction, but also the different degrees of the contraction of the jet, and we can make ourselves acquainted with the resistance of friction, with that of elbows and bends, and also, by observing jets of water and the sucking up of water, with the positive and negative pressure of water. We will always find results which agree pretty well, and sometimes extraordinarily well, with the coefficients given by experiment (μ, ϕ, α, ζ). In our apparatus $G = 0{,}125$ square meters, the usual diameter of the orifices and tubes is 1 centimeter, and for the lower orifice $h_1 = 0{,}96$ meters and $h_2 = 0{,}84$ meters. (A detailed description of this apparatus and of the experiments, etc., which can be made with it, is given in the author's "Experimentalhydraulik.")

The following example shows how well observations with this apparatus agree with the well-known experiments on a large scale. With a short cylindrical tube placed in the lower aperture, t was $= 33$, and with a long glass tube, for which the ratio $\frac{l}{d} = 124$, t was found to be $= 56$; from this we deduce in the one case

$$\mu_1 = 0{,}815 \text{ and } \zeta_1 = \frac{1}{\mu_1^2} - 1 = 0{,}504,$$

and in the other

$$\mu_2 = 0{,}480 \text{ and } \zeta_2 = \frac{1}{\mu_2^2} - 1 = 3{,}332;$$

hence

$$\zeta_2 - \zeta_1 = 3{,}332 - 0{,}504 = 2{,}828,$$

and therefore the coefficient of friction for the tube is

$$\zeta = \frac{d}{l}(\zeta_2 - \zeta_1) = \frac{2,828}{124} = 0,0228.$$

According to the first table in § 429, for the mean velocity v = 1,84 meters, with which the water was discharged from the tube, ζ = 0,0215; the results agree, therefore, very well. By means of these experiments, we can satisfy ourselves that the velocity of efflux of the water does not depend at all upon the inclination of the tube, but upon the head of water above the orifice of discharge. The duration of efflux is the same, no matter whether the long tube is inserted in the lower or middle opening, provided its orifice of discharge is at the same depth below the surface of the water in the reservoir.

This apparatus has recently received many additions, so that we can now make with it experiments upon the efflux of water under constant pressure, upon the efflux of air, and also upon the pressure, impact, and reaction of water.

CLOSING REMARK.—A very complete list of the works upon the subject of efflux of water and upon the motion of water in tubes is given in the "Allgemeine Machinenencyclopädie," Vol. I, Art. "Ausfluss." We will mention here, among the later works, Gerstner's "Handbuch der Mechanik," Vol. 2, Prague, 1832; d'Aubuisson's "Traité d'Hydraulique à l'usage des Ingénieurs," II édit. 1840; Eytelwein's "Handbuch der Mechanik fester Körper und der Hydraulik," 3d edition, 1842; Scheffler's "Principien der Hydrostatik und Hydraulik," Braunschweig, 1847. The older works of Bossut and du Buat upon hydraulics are always of value on account of their practical treatment of the subject. "Die Experimentalhydraulik, eine Anleitung zur Ausführung hydraulischer Versuche im kleinen," by J. Weisbach, Freiberg, 1855, is particularly adapted for teaching and for the practical study of hydraulics. Rühlmann's "Hydromechanik" is also to be recommended. The more recent works of Lesbros, Boileau, Francis, etc., have been mentioned before (§§ 378, 380 and 387). We can also recommend Rankine's "Manual of Applied Mechanics," as well as Bresse's "Cours de Mécanique Appliquée," II. But two parts of the hydraulic experiments of the author have as yet appeared, and they are

1) "Experiments upon the efflux of water through valve-gates, cocks, clacks, and valves;" and

2) "Experiments upon the incomplete contraction of water during efflux, etc., Leipzig, 1843."

Several new treatises by the author upon hydraulics are contained in the "Civilingenieur," the "Zeitschrift des Deutschen Ingenieurvereines," etc.

CHAPTER VI.

OF THE EFFLUX OF THE AIR AND OTHER FLUIDS FROM VESSELS AND PIPES.

§ 457. Efflux of Mercury and Oil.—The general formula

$$v = \sqrt{2gh} \text{ (see § 397)}$$

for the velocity v of efflux of water under a pressure, measured by the head h, holds good (see § 399) also for other liquids, such as quicksilver, oil, alcohol, etc., and can also be employed for the efflux of air and other aeriform fluids, when the pressure is not very great. If γ denotes the heaviness of the fluid and p its pressure upon the unit of surface, we have in like manner $h = \frac{p}{\gamma}$, and therefore

$$v = \sqrt{2g\frac{p}{\gamma}}.$$

If we measure the pressure by means of a piezometer, filled with a liquid whose density is γ_1, the height of the column of liquid is

$$h_1 = \frac{p}{\gamma_1};$$

hence $p = h_1 \gamma_1$, and therefore

$$v = \sqrt{2g\frac{\gamma_1}{\gamma}h_1} = \sqrt{2g\varepsilon_1 h_1},$$

in which $\varepsilon_1 = \frac{\gamma_1}{\gamma}$ denotes the ratio of the heaviness of the liquid in the piezometer to that of the fluid which is being discharged.

This agreement of the laws of efflux for different fluids is not confined to the velocity alone, but extends to the contraction of the fluid vein. Streams of mercury, oil, air, etc., when passing through an orifice in a thin plate, are contracted in almost exactly the same manner as a stream of water. Some experiments made by the author upon the efflux of mercury, oil and air, have shown conclusively this agreement (see the Polytechn. Centralblatt, year 1851, page 386). These experiments gave

1) With a *circular* orifice in a *thin plate* 6,5 millimeters in di-

ameter, under heads of 91,5 millimeters and 329 millimeters, the coefficients of efflux

For water.	Mercury.	Rape-seed oil.
$\mu = 0{,}709$	0,670	0,674

From the above table it appears that the contraction of streams of mercury and rape-seed oil is a little greater than that of a stream of water.

2) With a *short, well-rounded, conoidal mouth-piece*, whose diameter d was 6,6 m. m. and whose length was double the diameter ($l = 2\, d$), the following values were found

For water.	Mercury.	Rape-seed oil.	
		At a temp. 12½° C.	At a temp. 39° C.
$\mu = 0{,}942$	0,989	0,430	0,665

3) A *short cylindrical pipe*, which was not rounded off inside, whose diameter was $d = 6{,}76$ millimeters and which was three times as long as wide ($l = 3\, d$), gave the following values:

For water.	Mercury.	Rape-seed oil.	
		At a temp. 12½° C.	At a temp. 39° C.
$\mu = 0{,}885$	0,900	0,363	0,604

From these experiments we find that mercury flows through short mouth-pieces and pipes but little faster than water, and that, on the contrary, the velocity of rape-seed oil increases visibly with the temperature and is less than that of water. The great difference between the velocity of water and oil is due to the greater adhesion of the oil to the walls of the pipe.

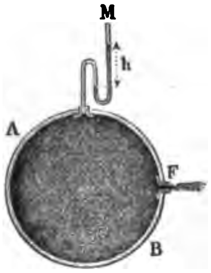
4) The following values of the coefficient of resistance ζ were obtained with *a glass tube* 6,64 millimeters in diameter and 86 times as long as wide (I) and with an *iron tube* 6,78 millimeters in diameter and 85 times as long as wide (II).

	For water.	Mercury.	Rape-seed oil.	
			At a temp. 12½° C.	At a temp. 39° C.
I.	$\zeta = 0{,}0271$	0,0277	39,21	2,722
II.	$\zeta = 0{,}0403$	0,0461	54,90	5,24

According to this last experiment the coefficient of resistance of mercury in an iron or glass tube is a little greater, and, on the contrary, that of rape-seed oil many times greater than that of water. We also see from these tables that the coefficient of resistance of the rape-seed oil diminishes as the temperature or degree of fluidity increases. These experiments also show that the coefficient of resistance for the iron tube is much greater than for the glass tube, which is due to the greater smoothness of the latter.

§ **458. Velocity of Efflux of Air.**—If we assume that *the air does not change its density during the efflux*, the well-known formula for the efflux of water from vessels can also be applied to the efflux of air. If p is the pressure of the exterior air and p_1 and γ_1 the pressure and heaviness of the air inside the vessel $A\ B$, Fig. 793, we can put for the velocity of efflux of the latter (see § 399)

FIG. 793.



$$v = \sqrt{2\,g\,\frac{(p_1 - p)}{\gamma_1}}$$

$$= \sqrt{2\,g\,\frac{p_1}{\gamma_1}\left(1 - \frac{p}{p_1}\right)}.$$

But (according to § 393), if p is the pressure in kilograms upon a square centimeter of surface, γ the weight of a cubic meter of air, and τ its temperature

$$\frac{p}{\gamma} = \frac{1 + 0{,}00367\,.\,\tau}{1{,}2514},$$

or, if p is referred to a surface of one square meter,

$$\frac{p}{\gamma} = \frac{10000}{1{,}2514}\,(1 + 0{,}00367\,\tau) = 7991\,(1 + 0{,}00367\,\tau)\,;$$

hence it follows that

$$\sqrt{\frac{p_1}{\gamma_1}} = \sqrt{\frac{p}{\gamma}} = \sqrt{7991}\sqrt{1 + 0{,}00367\,\tau},$$

or replacing 0,00367 by δ

$$\sqrt{\frac{p}{\gamma}} = 89{,}39\sqrt{1 + \delta\,\tau}, \text{and } v = 89{,}39\sqrt{2g(1 + \delta\,\tau)\left(1 - \frac{p}{p_1}\right)}$$

$$= 396\sqrt{(1 + \delta\,\tau)\left(1 - \frac{p}{p_1}\right)},$$

or for the English system of measures

$$v = 161{,}9\sqrt{2g\,(1 + \delta\,\tau)\left(1 - \frac{p}{p_1}\right)}$$

$$= 1299\sqrt{(1 + \delta\,\tau)\left(1 - \frac{p}{p_1}\right)},$$

τ being expressed in degrees of the centigrade thermometer.

If b is the height of the barometer and h that of the manometer (M), we have also

$$\frac{p}{p_1} = \frac{b}{b + h}, \text{or } 1 - \frac{p}{p_1} = \frac{h}{b + h},$$

and consequently the velocity of the issuing air

$$v = 396\sqrt{(1 + \delta\,\tau)\frac{h}{b + h}} \text{ meters}$$

$$= 1299\sqrt{(1 + \delta\,\tau)\frac{h}{b + h}} \text{ feet,}$$

or approximatively, when the height of the manometer is small, by putting

$$\frac{1}{\sqrt{1 + \frac{h}{b}}} = 1 - \frac{h}{2\,b},$$

$$v = 396\left(1 - \frac{h}{2\,b}\right)\sqrt{(1 + \delta\,\tau)\frac{h}{b}} \text{ meters}$$

$$= 1299\left(1 - \frac{h}{2\,b}\right)\sqrt{(1 + \delta\,\tau)\frac{h}{b}} \text{ feet.}$$

REMARK.—On account of the ordinary humidity of the atmosphere, it is advisable in practice to take $\delta = 0{,}004$.

§ 459. Discharge.—If F is the cross-section of the orifice, we have the effective discharge, measured at the pressure in the reservoir, p_1 or $b + h$,

$$Q_1 = Fv = F\sqrt{2\,g\,\frac{p_1}{\gamma_1}\left(1 - \frac{p}{p_1}\right)} = F\sqrt{2\,g\frac{p}{\gamma}}\sqrt{1 - \frac{p}{p_1}}$$

$$= F\,\sqrt{2\,g\frac{p}{\gamma}}\sqrt{\frac{b}{b + h}},$$

E.G., for atmospheric air

$$Q_1 = 396\ F\sqrt{\frac{(1 + \delta\,\tau)\,h}{b + h}} \text{ cubic meters}$$

$$= 1299\ F\sqrt{\frac{(1. + .\delta\,\tau)\,h}{b + h}} \text{ cubic feet.}$$

If we reduce this quantity of air to the pressure of the exterior air p or b, we obtain

$$Q = \frac{p_1}{p}\,Q_1 = \frac{b + h}{b}\,Q_1$$

$$= F\sqrt{\frac{2\,g\,p}{\gamma}}\,.\,\frac{p_1}{p}\sqrt{1 - \frac{p}{p_1}}$$

$$= F\sqrt{\frac{2\,g\,p}{\gamma}}\,.\,\frac{b + h}{b}\sqrt{\frac{h}{b + h}} = F\sqrt{\frac{2\,g\,p}{\gamma}}\sqrt{\left(1 + \frac{h}{b}\right)\frac{h}{b}},$$

E.G., for atmospheric air

$$Q = 396\ F\sqrt{(1 + \delta\,\tau)\left(1 + \frac{h}{b}\right)\frac{h}{b}} \text{ cubic meters.}$$

$$Q = 1299\ F\sqrt{(1 + \delta\,\tau)\left(1 + \frac{h}{b}\right)\frac{h}{b}} \text{ cubic feet.}$$

EXAMPLE.—The air in a large reservoir is at a temperature of 120° *C* and at a pressure corresponding to a height of the manometer of 5 inches, while the barometer marks 29,2 inches; what will be the discharge through an orifice 1½ inches in diameter?

The theoretical velocity of efflux is

$$v = 1299\sqrt{(1 + 0{,}00367\,.\,120)\,\frac{5}{34{,}2}} = 1299\sqrt{\frac{1{,}4404\,.\,5}{34{,}2}} = 596 \text{ feet, and}$$

the cross-section of the orifice is

$$F = \frac{\pi\,d^2}{4} = \frac{\pi}{4}\,.\left(\frac{1}{8}\right)^2 = \frac{\pi}{256} = 0{,}01227 \text{ square feet;}$$

hence the theoretical discharge, measured at the pressure in the reservoir, is

$$Q_1 = F\,v = 596\,.\,0{,}01227 = 7{,}818 \text{ cubic feet,}$$

and, on the contrary, at the exterior pressure the volume is

$$Q = \frac{b + h}{b}\,Q = \frac{34{,}2}{29{,}2}\,.\,7{,}818 = 8{,}565 \text{ cubic feet.}$$

§ 460. Efflux according to Mariotte's Law.—If we suppose that the air does not change its temperature during the discharge, we can assume that it expands according to the law of Mariotte (see § 387), and therefore that the quantity of air Q in passing from the pressure p to the pressure p_1 performs the work $Q\,p\,l\left(\frac{p_1}{p}\right)$. If we put this work equal to the energy $\frac{v^2}{2g}\,Q\,\gamma$ stored by $Q\,\gamma$ during the efflux, we obtain the following formula

$$\frac{v^2}{2g}\,Q\,\gamma = l\left(\frac{p_1}{p}\right)Q\,p, \text{ or}$$

$$\frac{v^2}{2g} = \frac{p}{\gamma}\,l\left(\frac{p_1}{p}\right);$$

hence the velocity of efflux is

$$v = \sqrt{2g\frac{p}{\gamma}\,l\left(\frac{p_1}{p}\right)};$$

Now, as in the foregoing paragraph, for the metrical system of measures $\frac{p}{\gamma} = \frac{1 + \delta\,\tau}{1,2514}$; hence we have here also

$$v = 396\sqrt{(1 + \delta\,\tau)\,l\left(\frac{p_1}{p}\right)} = 396\sqrt{(1 + \delta\,\tau)\,l\left(\frac{b + h}{b}\right)}\text{ meters,}$$

and

$$v = 1299\sqrt{(1 + \delta\,\tau)\,l\left(\frac{p_1}{p}\right)} = 1299\sqrt{(1 + \delta\,\tau)\,l\left(\frac{b + h}{b}\right)}\text{ feet,}$$

in which b denotes the height of the barometer in the exterior air and h the height of the manometer for the confined air, τ the temperature of the latter in degrees centigrade and $\delta = 0,00367$ the well-known coefficient of expansion of air. Now the theoretical discharge per second is

$$Q = F\,v = F\sqrt{2g\frac{p}{\gamma}\,l\left(\frac{p_1}{p}\right)}$$

$$= 1299\,F\sqrt{(1 + \delta\,\tau)\,l\left(\frac{b + h}{b}\right)}\text{ cubic feet,}$$

or, when reduced to the pressure of the air in the reservoir,

$$Q_1 = \frac{p}{p_1}\,Q = \frac{p}{p_1}\,F\sqrt{2g\frac{p}{\gamma}\,l\,.\left(\frac{p_1}{p}\right)}$$

$$= \frac{b}{b + h}\,F\sqrt{2g\frac{p}{\gamma}\,l\left(\frac{b + h}{b}\right)}$$

$$= 1299\,F\,\frac{b}{b + h}\sqrt{(1 + \delta\,\tau)\,l\left(\frac{b + h}{b}\right)}.$$

If the excess of pressure of the air in the reservoir, or $\frac{h}{b}$, is very small, we can put

$$l\left(\frac{b+h}{b}\right) = l\left(1+\frac{h}{b}\right) = \frac{h}{b} - \frac{1}{2}\left(\frac{h}{b}\right)^2$$

(see the Ingenieur, page 81), and therefore, approximatively,

$$Q = F\sqrt{2g\frac{p}{\gamma}\left(1-\frac{h}{2\,b}\right)\frac{h}{b}},$$

while according to the first formula for the efflux (see § 459)

$$Q = F\sqrt{2g\frac{p}{\gamma}\left(1+\frac{h}{b}\right)\frac{h}{b}}$$

We see that if we assume that air in flowing out expands according to Mariotte's law, we obtain a smaller discharge than when we consider that the air acts exactly like water and does not expand at all. This difference diminishes with $\frac{h}{b}$, and in both cases for very small values of $\frac{h}{b}$, we have

$$Q = F\sqrt{2g\frac{p}{\gamma}\cdot\frac{h}{b}} = 1299\,F\sqrt{(1+\delta\,\tau)\,\frac{h}{b}} \text{ cubic feet.}$$

§ **461. Work Done by the Heat.**—The logarithmic expression, found in § 388, for the work done during the compression or expansion of air is correct only, when we assume that, while the change of volume or density is taking place, the temperature of the air does not alter; but this is correct only, when the change takes place so slowly that the heat in the confined air has time enough to communicate any excess to the walls of the vessel and to the exterior air. But if the change of density takes place so quickly that it is accompanied by a change of temperature, when the air is compressed, the temperature is elevated and when it is expanded, it is lowered. Under these circumstances the tension cannot change according to the law of Mariotte alone. If p and p_1 are the pressures, γ and γ_1 the heavinesses and τ and τ_1 the temperatures of the same air, we have, according to § 392, the formula

$$\frac{p_1}{p} = \frac{1+\delta\,\tau_1}{1+\delta\,\tau}\cdot\frac{\gamma_1}{\gamma}.$$

Now if during the sudden change of pressure the temperature varies in the ratio

$$\frac{1+\delta\,\tau_1}{1+\delta\,\tau} = \left(\frac{\gamma_1}{\gamma}\right)^{\frac{2}{5}},$$

we can put

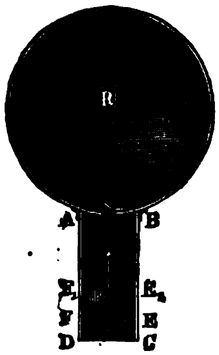
$$\frac{p_1}{p} = \left(\frac{1 + \delta\,\tau_1}{1 + \delta\,\tau}\right)^3 = \left(\frac{\gamma_1}{\gamma}\right)^{\frac{3}{2}},$$

or

$$\frac{\gamma_1}{\gamma} = \left(\frac{1 + \delta\,\tau_1}{1 + \delta\,\tau}\right)^2 = \left(\frac{p_1}{p}\right)^{\frac{2}{3}},$$

If in a cylinder $A\,C$, Fig. 794, a prism of air, whose initial height is $E\,B = s$, whose initial tension is p and whose heaviness is γ, is cut off by a piston $E\,F$, and if, by suddenly raising the piston a distance x, we cause the density of this mass of air to become y and its tension to become z, we have, according to the last formula,

FIG. 794.



$$\frac{z}{p} = \left(\frac{y}{\gamma}\right)^{\frac{3}{2}} = \left(\frac{s}{s - x}\right)^{\frac{3}{2}},$$

and therefore

$$z = \left(\frac{s}{s + x}\right)^{\frac{3}{2}} p.$$

In order to move the piston, whose area we will for simplicity put equal to the unit of surface, through an element σ of its path the work, which must be done, is

$$z\,\sigma \doteq \left(\frac{s}{s - x}\right)^{\frac{3}{2}} p\,\sigma = p\,\sigma\,s^{\frac{3}{2}}\,(s - x)^{-\frac{3}{2}}.$$

Substituting instead of x successively $1\,\sigma, 2\,\sigma, 3\,\sigma \ldots$ and putting $s = n\,\sigma$ and the height of the prism of air, when the piston has described the space $E\,E_1$, $E_1\,B = s_1 = m\,\sigma$, we have for the work done by the piston in moving the distance $E\,E_1$

$$A_1 = p\,\sigma\,s^{\frac{3}{2}}\left[s^{-\frac{3}{2}} + (s - \sigma)^{-\frac{3}{2}} + (s - 2\,\sigma)^{-\frac{3}{2}} + \ldots + (s - m\,\sigma)^{-\frac{3}{2}}\right]$$

$$= p\,\sigma\,s^{\frac{3}{2}}\left\{\begin{array}{l}(\sigma)^{-\frac{3}{2}} + (2\,\sigma)^{-\frac{3}{2}} + (3\,\sigma)^{-\frac{3}{2}} + \ldots + (n\,\sigma)^{-\frac{3}{2}} \\ -\left[(\sigma)^{-\frac{3}{2}} + (2\,\sigma)^{-\frac{3}{2}} + (3\,\sigma)^{-\frac{3}{2}} + \ldots + (m\,\sigma)^{-\frac{3}{2}}\right]\end{array}\right\}$$

$$= \frac{p\,s^{\frac{3}{2}}}{\sigma^{\frac{1}{2}}}\left\{\begin{array}{l}1^{-\frac{3}{2}} + 2^{-\frac{3}{2}} + 3^{-\frac{3}{2}} + \ldots + m^{-\frac{3}{2}} + \ldots + n^{-\frac{3}{2}} \\ -\left(1^{-\frac{3}{2}} + 2^{-\frac{3}{2}} + 3^{-\frac{3}{2}} + \ldots + m^{-\frac{3}{2}}\right)\end{array}\right\}.$$

Now, according to page 88 of the Ingenieur, when m and n are infinitely great numbers, we have

$$1^{-\frac{3}{2}} + 2^{-\frac{3}{2}} + 3^{-\frac{3}{2}} + \ldots + m^{-\frac{3}{2}} = \frac{m^{-\frac{1}{2}}}{-\frac{1}{2}} = -\frac{2}{m^{\frac{1}{2}}}$$

and

$$1^{-\frac{3}{2}} + 2^{-\frac{3}{2}} + 3^{-\frac{3}{2}} + \ldots + n^{-\frac{3}{2}} = -\frac{2}{n^{\frac{1}{2}}};$$

hence

$$A_1 = \frac{p\,s^{\frac{3}{2}}}{\sigma^{\frac{1}{2}}}\left(\frac{2}{m^{\frac{1}{2}}} - \frac{2}{n^{\frac{1}{2}}}\right) = 2\,p\,s^{\frac{3}{2}}\left(\frac{1}{s_1^{\frac{1}{2}}} - \frac{1}{s^{\frac{1}{2}}}\right)$$
$$= 2\,p\,s\left[\left(\frac{s}{s_1}\right)^{\frac{1}{2}} - 1\right].$$

If by raising the piston another distance s we wish to force the compressed mass of air $A\ E_1$ into a space R, where the pressure is

$$p_1 = p\left(\frac{s}{s_1}\right)^{\frac{3}{2}},$$

the work to be done will be

$$A_2 = p_1\,s_1 = \frac{p\,s^{\frac{3}{2}}}{s_1^{\frac{1}{2}}};$$

the exterior air presses upon the piston during the whole of its course with a force p and transmits to it the mechanical effect $A_3 = p\,s$. Hence the total mechanical effect necessary to compress the volume of air $(1\,.\,s)$ and force it into the space R is

$$A = A_1 + A_2 - A_3$$
$$= 2\,p\,s\left[\left(\frac{s}{s_1}\right)^{\frac{1}{2}} - 1\right] + \frac{p\,s^{\frac{3}{2}}}{s_1^{\frac{1}{2}}} - p\,s = 3\,p\,s\left[\left(\frac{s}{s_1}\right)^{\frac{1}{2}} - 1\right],$$

and consequently the work done in compressing a volume of air from the pressure p to p_1 is

$$A = 3\,V p\left[\left(\frac{s}{s_1}\right)^{\frac{1}{2}} - 1\right] = 3\,V p\left[\left(\frac{p_1}{p}\right)^{\frac{1}{3}} - 1\right] = 3\,V p\left(\sqrt[3]{\frac{p_1}{p}} - 1\right),$$

while, according to Mariotte's law, we should put

$$A = V p\,l\left(\frac{p_1}{p}\right),$$

and for perfectly incompressible fluids we have

$$A = V(p_1 - p) = V p\left(\frac{p_1}{p} - 1\right).$$

If, on the contrary, the quantity $V_1\,\gamma_1$ of air at the pressure p_1 is brought back by sudden expansion to the pressure p and the density

$$\gamma = \gamma_1\left(\frac{p}{p_1}\right)^{\frac{2}{3}},$$

or to the volume

$$V = V_1\left(\frac{p_1}{p}\right)^{\frac{2}{3}},$$

the work done by air is

$$A = 3\,V p\left[\left(\frac{p_1}{p}\right)^{\frac{1}{3}} - 1\right] = 3\,V_1\,p_1\left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right].$$

EXAMPLE.—If a blowing engine converts per second 10 cubic feet of air at a pressure $b = 28$ inches of the barometer into a blast at the pressure $b + h = 30$ inches, it requires, according to the formula,

$$A = 3\,Vp\left[\left(\frac{p_1}{p}\right)^{\frac{1}{3}} - 1\right],$$

since the pressure per square foot is

$$p = 144\,.\,0{,}4913\,b = 144\,.\,0{,}4913\,.\,28 = 1981 \text{ pounds},$$

the mechanical effect

$$A = 30\,.\,1981\left(\sqrt[3]{\frac{30}{28}} - 1\right) = 59430\left(\sqrt[3]{\frac{15}{14}} - 1\right) = 5943\,.\,0{,}2326$$

$$= 1382 \text{ foot-pounds.}$$

The logarithmic formula (see Example 1, § 388) gives $A = 1366{,}7$ foot-pounds, and that for water

$$A = Vp\left(\frac{p_1}{p} - 1\right) = 19810\left(\frac{15}{14} - 1\right) = \frac{19810}{14} = 1415 \text{ foot-pounds.}$$

§ 462. Efflux of Air, when the Cooling is taken into consideration.

—The energy $A = 3\,Q_1\,p_1\left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right]$, which is restored during the sudden expansion of Q_1 to Q, can be put equal to the work $Q_1\,\gamma_1\,.\,\frac{v^2}{2\,g}$ done in overcoming the inertia of the mass $\frac{Q_1\,\gamma_1}{g}$ of air when the latter assumes the velocity v.

From the equation

$$Q_1\,\gamma_1\,\frac{v^2}{2\,g} = 3\,Q_1\,p_1\left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right],$$

we deduce the following formula for efflux:

$$\frac{v^2}{2\,g} = \frac{3\,p_1}{\gamma_1}\left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right], \text{ or}$$

$$v = \sqrt{2\,g\,.\,\frac{3\,p_1}{\gamma_1}\left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right]};$$

hence we have in meters

$$v = 154{,}8\sqrt{2\,g\,(1 + \delta\,\tau)\left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right]}$$

$$= 685{,}8\sqrt{(1 + \delta\,\tau)\left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right]},$$

and in English feet

$$v = 280{,}4\sqrt{2\,g\,(1 + \delta\,\tau)\left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right]}$$

$$= 2250\sqrt{(1 + \delta\,\tau)\left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right]} \text{ feet.}$$

The tension of the issuing air is that of the exterior air p; its heaviness is

$$\gamma_2 = \gamma_1 \left(\frac{p}{p_1}\right)^{\frac{2}{3}},$$

and its temperature is

$$\tau_2 = \tau_1 \left(\frac{p}{p_1}\right)^{\frac{1}{3}} \frac{\left(\frac{p}{p_1}\right)^{\frac{1}{3}} - 1}{\delta},$$

and the theoretical discharge from an orifice, whose area is F, is

$$Q_2 = F v = F \sqrt{2 g \, . \, \frac{3 p_1}{\gamma_1} \left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right]}$$

$$= 280{,}4 \, F \sqrt{2 g \, (1 + \delta \tau) \left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right]} \text{ cubic feet,}$$

in which p_1, γ_1 and τ_1 denote the pressure, heaviness and temperature of the confined air.

Reduced to the pressure in the reservoir, this discharge is

$$Q_1 = \frac{\gamma_2}{\gamma_1} \, . \, Q_2 = \left(\frac{p}{p_1}\right)^{\frac{2}{3}} Q_2 = F \left(\frac{p}{p_1}\right)^{\frac{2}{3}} \sqrt{2 g \, . \, 3 \frac{p_1}{\gamma_1} \left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right]},$$

and, finally, reduced to the pressure of the exterior air and to the temperature of the air in the vessel or to the heaviness $\gamma = \gamma_1 \left(\frac{p}{p_1}\right)$, it is

$$Q = \frac{p_1}{p} Q_1 = F \left(\frac{p_1}{p}\right)^{\frac{1}{3}} \sqrt{2 g \frac{3 p_1}{\gamma_1} \left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right]}$$

$$= F \sqrt{2 g \frac{3 p_1}{\gamma_1} \left(\frac{p_1}{p}\right)^{\frac{1}{3}} \left[\left(\frac{p_1}{p}\right)^{\frac{1}{3}} - 1\right]}.$$

If we put $\frac{p_1}{p} = \frac{b + h}{b}$, in which b denotes the height of the barometer in the exterior air and $b + h$ that of the barometer in the confined air, we obtain

$$Q = F \sqrt{2 g \frac{3 p_1}{\gamma_1} \left(\frac{b + h}{b}\right)^{\frac{1}{3}} \left[\left(\frac{b + h}{b}\right)^{\frac{1}{3}} - 1\right]}$$

$$= 280{,}4 \, F \sqrt{2 g \, (1 + \delta \tau) \left(\frac{b + h}{b}\right)^{\frac{1}{3}} \left[\left(\frac{b + h}{b}\right)^{\frac{1}{3}} - 1\right]}$$

$$= 2250 \, F \sqrt{(1 + \delta \tau) \left(\frac{b + h}{b}\right)^{\frac{1}{3}} \left[\left(\frac{b + h}{b}\right)^{\frac{1}{3}} - 1\right]} \text{ cubic feet.}$$

In most cases $\frac{h}{b}$ is very small, and we can put

$$\left(\frac{b + h}{b}\right)^{\frac{1}{3}} = \left(1 + \frac{h}{b}\right)^{\frac{1}{3}} = 1 + \tfrac{1}{3} \frac{h}{b} - \tfrac{1}{9} \left(\frac{h}{b}\right)^2 + \dots,$$

$$\left(\frac{b + h}{b}\right)^{\frac{1}{3}} - 1 = \tfrac{1}{3} \frac{h}{b} - \tfrac{1}{9} \left(\frac{h}{b}\right)^2 + \tfrac{5}{81} \left(\frac{h}{b}\right)^3$$

$$= \frac{h}{3\,b}\left[1 - \tfrac{1}{3}\frac{h}{b} + \tfrac{5}{27}\left(\frac{h}{b}\right)^2\right],$$

and therefore

$$Q = F\sqrt{2\,g\,\frac{p_1}{\gamma_1}\cdot\frac{h}{b}\left[1 + \tfrac{1}{3}\frac{h}{b} - \tfrac{1}{9}\left(\frac{h}{b}\right)^2\right]\left[1 - \tfrac{1}{3}\frac{h}{b} + \tfrac{5}{27}\left(\frac{h}{b}\right)^2\right]}$$

$$= F\sqrt{2\,g\,\frac{p_1}{\gamma_1}\cdot\frac{h}{b}\left[1 - \tfrac{1}{27}\left(\frac{h}{b}\right)^2\right]}$$

$$= F\left[1 - \tfrac{1}{54}\left(\frac{h}{b}\right)^2\right]\sqrt{2\,g\,\frac{p_1}{\gamma_1}\,\frac{h}{b}}.$$

In the application of this formula to fans, blowing engines, etc., in which cases $\frac{h}{b} < \frac{1}{2}$, the theoretical discharge, measured at the exterior pressure and the interior temperature, is simply

$$Q = F\sqrt{2\,g\,\frac{p_1}{\gamma_1}\,\frac{h}{b}}$$

$$= 89{,}39\,F\sqrt{2\,g\,(1 + \delta\,\tau)\frac{h}{b}} = 396\,F\sqrt{(1 + \delta\,\tau)\,\frac{h}{b}}\text{ cubic meters}$$

$$= 161{,}9\,F\sqrt{2\,g\,(1 + \delta\,\tau)\frac{h}{b}} = 1299\,F\sqrt{(1 + \delta\,\tau)\,\frac{h}{b}}\text{ cubic feet.}$$

EXAMPLE.—In the case treated in the Example of § 459, where $b = 29{,}2$, $h = 5$ inches, $\tau = 120°$ and $F = \frac{\pi\,d^2}{4} = 0{,}01227$ square feet, we have the discharge according to the last formula, measured at the pressure of the external air,

$$Q = 1299\,F\sqrt{1{,}4404\,.\,\frac{5}{29{,}2}} = 1299\,F\sqrt{0{,}2466}$$
$$= 645{,}1\,F = 645{,}1\,.\,0{,}01227 = 7{,}915\text{ cubic feet,}$$

while previously (§ 459) we found, according to the formula for water, $Q = 8{,}565$ cubic feet, and according to the logarithmic formula in § 460, we have

$$Q = 1299\,F\sqrt{1{,}4404\,l\,\frac{34{,}2}{29{,}2}} = 1299\,F\sqrt{0{,}2277}$$
$$= 619{,}9\,.\,0{,}01227 = 7{,}606\text{ cubic feet.}$$

§ **463. Efflux of Moving Air.**—The formulas for efflux already found are based upon the supposition that the pressure p or the height h of the manometer is measured at a place, where the air is at rest or moving very slowly; but if we measure p_1 and h_1 at a point, where the air is in motion, if, E.G., the manometer M_1 is in communication with the air in a pipe $C\,F$, Fig. 795, we must take into consideration, in determining the velocity of efflux, the vis viva of the approaching air. If c be the velocity of the air passing the orifice of the manometer, we must put

$$Q_1 \gamma_1 \frac{v^2}{2g} = Q_1 \gamma_1 \frac{c^2}{2g} + 3\, Q_1 p_1 \left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right]$$

If F denotes the cross-section of the orifice and G that of the tube or of the stream, which passes the orifice of the manometer, the discharge of air is $Q_1 \gamma_1 = G\, c\, \gamma_1 = F\, v\, \gamma_2$; hence

$$\frac{c}{v} = \frac{F \gamma_2}{G \gamma_1} = \frac{F}{G} \left(\frac{p}{p_1}\right)^{\frac{2}{3}} \text{ and}$$

$$Q_1 \gamma_1 \left[1 - \left(\frac{F}{G}\right)^2 \left(\frac{p}{p_1}\right)^{\frac{4}{3}}\right] \frac{v^2}{2g} = 3\, Q_1 p_1 \left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right],$$

and the *required velocity of efflux* is

$$v = \sqrt{\frac{2g \frac{3 p_1}{\gamma_1} \left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right]}{1 - \left(\frac{F}{G}\right)^2 \left(\frac{p}{p_1}\right)^{\frac{4}{3}}}},$$

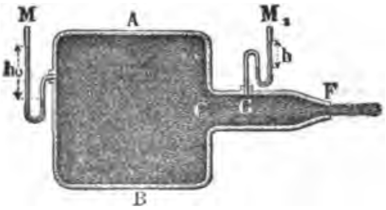
or approximatively, when p_1 is not much greater than p,

$$v = \sqrt{\frac{2g \frac{3 p_1}{\gamma_1} \left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right]}{1 - \left(\frac{F}{G}\right)^2}}$$

$$= 2250 \sqrt{\frac{(1 + \delta \tau) \left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right]}{1 - \left(\frac{F}{G}\right)^2}} \text{ feet.}$$

Here, as in the case of the efflux of water, the velocity of efflux increases with the ratio $\frac{F}{G}$ of the cross-section of the orifice to that G of the pipe or moving stream of air. We see from this that, under the same circumstances, the height p_1 of the manometer decreases as the diameter of the tube diminishes, or as the velocity of the air in the pipe increases.

FIG. 795.



If we denote by p_0 the tension in the reservoir, where the air is at rest, we have also

$$\frac{v^2}{2g} = \frac{3\,p_1}{\gamma_1}\left[1 - \left(\frac{p}{p_0}\right)^{\frac{1}{3}}\right],$$

and if we eliminate v from the two expressions, we obtain

$$\frac{1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}}{1 - \left(\frac{p}{p_0}\right)^{\frac{1}{3}}} = 1 - \left(\frac{F}{G}\right)^2 \left(\frac{p}{p_1}\right)^{\frac{4}{3}}, \text{approximatively} = 1 - \left(\frac{F}{G}\right)^2$$

If b denotes the height of the barometer in the free air, h that of the manometer connected with the reservoir and F the area of the orifice of efflux, we have, finally, the theoretical discharge, measured when its heaviness is

$$\gamma = \left(\frac{p}{p_1}\right)\gamma_1 = \frac{b\,\gamma_1}{b + h},$$

$$Q = F\sqrt{\frac{2g\,\frac{p_1}{\gamma}\,\frac{h}{b}}{1 - \left(\frac{F}{G}\right)^2}} = 161,9\,F\sqrt{\frac{2g\,(1+\delta\,\tau)\,\frac{h}{b}}{1 - \left(\frac{F}{G}\right)^2}}$$

$$= 1299\,F\sqrt{\frac{(1 + \delta\,\tau)\,\frac{h}{b}}{1 - \left(\frac{F}{G}\right)^2}}$$

EXAMPLE.—The height of a quicksilver manometer, which is placed upon a pipe 3½ inches in diameter through which air is passing, is 2½ inches, while the air is discharged through a circular orifice 2 inches in diameter at the end of the pipe: what is the velocity of discharge, assuming the barometer in the external air to stand at 27½ inches and the air in the pipe to be at a temperature of 10° C? Here

$$\sqrt{1 + \delta\,\tau} = \sqrt{1,0367} = 1,018, \sqrt{\frac{h}{b}} = \sqrt{\tfrac{5}{55}} = \sqrt{\tfrac{1}{11}} = 0,3015 \text{ and}$$

$$F = \pi\,r^2 = 3,141 : 144 = 0,02181 \text{ and}$$

$$\sqrt{1 - \left(\frac{F}{G}\right)^2} = \frac{\sqrt{49^2 - 16^2}}{49} = \frac{46,314}{49} = 0,9452;$$

hence the discharge is

$$Q = 1299\,F \,.\, \frac{1,018\,.\,0,3015}{0,9452} = 421,8\,F = 9,20 \text{ cubic feet.}$$

For the corresponding tension p_0 in the reservoir, we have

$$1 - \left(\frac{p}{p_0}\right)^{\frac{1}{3}} = \left[1 - \left(\frac{p}{p_1}\right)^{\frac{1}{3}}\right] : \left[1 - \left(\frac{F}{G}\right)^2\right] = (1 - \sqrt[3]{\tfrac{11}{12}}) : 0,9452^2$$

$$= \frac{0,0287}{0,8934} = 0,03212; \text{ hence}$$

$$\sqrt[3]{\frac{p}{p_0}} = 0,90788, p_0 = 1,108\,p \text{ and } b + h_0 = 1,108\,b$$

and consequently the height of the manometer in the reservoir is

$$h_0 = 0,108\,b = 0,108 \,.\, 27,5 = 2,83 \text{ inches.}$$

§ **464. Coefficients of Efflux.**—The *phenomena of contraction*, which we have studied for the efflux of water, are also met with in the efflux of air from vessels. If the orifice of efflux is *in a thin plate*, the stream of air has a smaller cross-section than the orifice, and the effective discharge Q_1 is consequently smaller than the theoretical Q, or the product $F v$ of the cross-section F of the orifice and the theoretical velocity v. This diminution of the discharge is owing principally, as we can observe in a stream of smoke, to the contraction of the stream of air, and we can, therefore, as in the case of water (see § 406), call the ratio $a = \frac{F_1}{F}$ of the cross-section F_1 of the stream of air to that F of the orifice

the *coefficient of contraction*,

the ratio $\phi = \frac{v_1}{v}$ of the effective velocity v_1 to the theoretical v (see § 408)

the *coefficient of velocity*.

and the ratio $\mu = \frac{Q_1}{Q} = \frac{F_1 v_1}{F v} = a\,\phi$ of the effective discharge Q_1 to the theoretical discharge Q

the *coefficient of efflux*.

As in the case of water the coefficient of velocity ϕ for the efflux of air through an *orifice* in a *thin plate* is nearly = 1, and therefore, so long as we have no measurements of the stream of air, we must put the *coefficient of efflux* $\mu = a\,\phi$ equal to the *coefficient of contraction* a. The older experiments upon the efflux of air through orifices in a thin plate vary very considerably from each other. The experiments of Koch, calculated according to the formula for water by Buff, gave for circular orifices from 3 to 6 lines in diameter, when the height of the water manometer was from 0,2 to 6,2 feet, $\mu = 0,60$ to $0,50$; on the contrary, the experiments of d'Aubuisson, calculated in the same way, give for circular orifices 1 to 3 centimeters in diameter, when the height of the water manometer is between 0,027 and 0,144 meters, $\mu = 0,65$ to $0,64$. Poncelet also found, upon calculating the experiments of Pecqueur by the same formula, for an orifice 1 centimeter in diameter, under an excess of pressure of 1 atmosphere, or of a column of water 10 meters high, $\mu = 0,563$, and for a similar one 1,5 centimeters wide, $\mu = 0,566$. The more extended experiments of the author, calculated according to the last formula

$$Q = F\left[1 - \tfrac{1}{54}\left(\frac{h}{b}\right)^2\right]\sqrt{2g\frac{p_1}{\gamma_1}\frac{h}{b}},$$

gave the following results:

1) When the diameter of the orifice $d = 1$ centimeter and the ratio of the pressures was

$\frac{p_1}{p} = \frac{b+h}{b} =$	1,05	1,09	1,43	1,65	1,89	2,15
$\mu =$	0,555	0,589	0,692	0,724	0,754	0,788

2) When the diameter of the orifice $d = 2,14$ centimeters, for

$\frac{b+h}{b} =$	1,05	1,09	1,36	1,67	2,01
$\mu =$	0,558	0,573	0,634	0,678	0,723

3) When the diameter of the orifice $d = 1,725$ centimeters, for

$\frac{b+h}{b} =$	1,08	1,37	1,63
$\mu =$	0,563	0,631	0,665

4) When the diameter of the orifice $d = 2$ centimeters, for

$\frac{b+h}{b} =$	1,08	1,39
$\mu =$	0,578	0,641

The coefficient of contraction for efflux through an orifice in a thin plate increases sensibly with the head. But if the formula for water is employed, there is much less variation; this formula gives μ nearly $\sqrt{\frac{p}{p_1}}$, E.G. for $\frac{p_1}{p} = 2$; $\sqrt{\tfrac{1}{2}} = 0,707$ times as great as the last formula. According to the first table, for $d = 1$ and $\frac{p_1}{p} = 2$, $\mu = \frac{0{,}754 + 0{,}788}{2} = 0{,}771$; hence, according to the water formula, $\mu = 0{,}707 \,.\, 0{,}771 = 0{,}555$, which is nearly the same value as Poncelet found.

For efflux through a circular orifice 1 centimeter in diameter, situated *in a conically convergent wall*, the angle of convergence being 100 degrees, the author found for

$\frac{b+h}{b} =$	1,31	1,66
$\mu =$	0,752	0,793

In like manner with the same orifice in a *conically divergent wall*, the angle of divergence being 100 degrees, the author obtained for

$\frac{b+h}{b} =$	1,30	1,66
$\mu =$	0,589	0,663

§ **465.** The variability of the coefficient of contraction $\alpha = \mu$ for the efflux of air through an orifice in a thin plate also affects, according to the well-known formula

$$\mu = \phi = \frac{1}{\sqrt{1+\zeta}} = \frac{1}{\sqrt{1 + \left(\frac{1}{\alpha} - 1\right)^2}}, \text{ (see § 422)},$$

the coefficient of efflux for short pipes. According to the experiments of Koch, cited above, we have for such tubes 3 to 4 lines in diameter and from 4 to 6 times their diameter in length, when the pressure is 0,3 to 6,2 feet of the water manometer, $\mu = 0{,}74$ to 0,72, while, on the contrary, d'Aubuisson gives for similar tubes, 1 to 3 centimeters in diameter, 3 to 4 times as long as wide, and under a pressure equal to 0,027 to 0,141 meters of the water manometer, $\mu = 0{,}92$ to 0,93; and Poncelet found for cylindrical pipes 1 centimeter in diameter and from $2\frac{1}{2}$ to 10 centimeters long, under twice the atmospheric pressure, $\mu = 0$ 632 to 0,650.

The experiments made by the author, on the contrary, have led to the following results:

1) A *short cylindrical tube* or ajutage, 1 centimeter in diameter and 3 centimeters long, gave for

$\frac{b+h}{b} =$	1,05	1,10	1,30
$\mu =$	0,730	0,771	0,830

2) A similar tube, 1,414 centimeters in diameter and *three times as long as wide*, gave for

$\frac{b+h}{b} =$	1,41	1,69
$\mu =$	0,813	0,822

3) A similar pipe, 2,44 *centimeters wide* and three times as long, gave for

$$\frac{b+h}{b} = 1{,}74, \mu = 0{,}833.$$

The increase of the coefficient of efflux as the pressure increases is explained by the simultaneous increase of the coefficient of contraction.

The short pipe (1), when its inlet orifice was *slightly rounded* off, gave as a mean value for its coefficient of efflux $\mu = 0{,}927$, which is much greater than that for a similar pipe which is not rounded off.

4) A short pipe, *with its inlet orifice well rounded* off, 1 centimeter wide and 1,6 centimeters long, gave for

$\frac{b+h}{b} =$	1,24	1,38	1,59	1,85	2,14
$\mu =$	0,979	0,986	0,965	0,971	0,978

The advantage of the formula for efflux

$$Q = \mu F \sqrt{2 g \frac{p_1}{\gamma} \frac{b}{p}} \text{ over the others}$$

is shown by the fact that this coefficient approaches very nearly (as it should do) unity.

The older formula gives of course for great pressures much smaller values for μ.

On the contrary, the logarithmic formula (see § 460) gives much greater values which may sometimes even exceed unity.

A *short conical pipe, rounded off at the inlet orifice,* gave nearly the same values for μ, and a short conical tube, *which was not rounded off,* and which was 1 centimeter in diameter and 4 centimeters long, and whose angle of convergence was 7° 9′, gave for

$\frac{b+h}{b} =$	1,08	1,27	1,65
$\mu =$	0,910	0,922	0,964

Koch and Buff found with a similar tube, whose exterior diameter was 2,72 lines and the angle of convergence of whose sides was 6°, under a head of 0,3 to 6,2 feet of the water manometer $\mu = 0,73$ to 0,85, and according to d'Aubuisson a similar pipe, whose orifice was 1,5 centimeters in diameter, gave under a pressure measured by a height of from 0,027 to 0,144 meters of the water manometer, $\mu = 0,94$. The old or water formula was employed in the calculations.

The complete nozzle *A C*, Fig. 736, § 434, consisting of a conical tube with an angle of convergence of 6°, which was 14,5 centimeters long, 1 centimeter wide at the outlet and 3,8 centimeters wide at the inlet, which was well rounded off, gave for

$\frac{b+h}{b} =$	1,08	1,45	2,16
$\mu =$	0,932	0,960	0,984

By experiments upon the influx of air into vessels, Saint-Venant and Wantzel found for a short *mouth-piece, rounded* off internally in the form of a quarter of a circle, when the calculations were made according to the new formula, $\mu = 0,98$, and *for an orifice in a thin plate,* $\mu = 0,61$.

If the pressures are small, as is the case in the ordinary fan, where $\frac{h}{b} < \frac{1}{8}$, we can substitute, according to what precedes, when we employ the new formula for efflux

$$Q = \mu F \sqrt{2 g \frac{p_1}{\gamma} \cdot \frac{h}{b}} = 1299 \mu F \sqrt{(1 + 0,004\, \tau) \frac{h}{b}} \text{ cubic feet,}$$

as a mean

1) for an orifice in a *thin plate,* $\mu = 0,56$,

2) for a *short cylindrical pipe*, $\mu = 0{,}75$,

3) for a well *rounded off conical mouth-piece*, $\mu = 0{,}98$,

4) for a *conical pipe*, whose angle of convergence is about 6°, $\mu = 0{,}92$.

EXAMPLE.—If the sum of the areas of two conical tuyeres of a blowing machine is 8 square inches, the temperature in the reservoir is 15°, the height of the manometer in the regulator is 8 inches and the height of the barometer in the exterior air is 29 inches, we have the effective discharge, measured at the pressure of the exterior air,

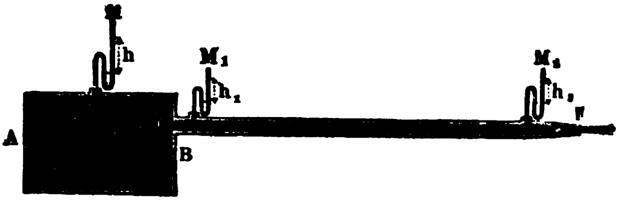
$$Q = 1299\,\mu\,F\,\sqrt{(1 + 0{,}004\,\tau)\,\frac{b}{h}}$$

$$= 1299\,.\,0{,}92\,.\,\frac{8}{144}\sqrt{(1 + 0{,}004\,.\,15)\,\frac{8}{29}} = 24{,}9\,\sqrt{\frac{1{,}06\,.\,8}{29}}$$

$$= 24{,}9\,.\,0{,}331 = 8{,}242 \text{ cubic feet.}$$

§ 466. Coefficient of Friction of Air.—If air moves through a *long pipe C F*, Fig. 796, it has, like water, a *resistance of friction* to overcome, and this resistance can be measured by the height of a column of air, which is determined by the expression

FIG. 796.



$$z = \zeta\,.\,\frac{l}{d}\,.\,\frac{v^2}{2g}$$

in which, as in the case of water pipes, l denotes the length, d the diameter of the pipe, v the velocity of the air, and ζ the coefficient of resistance of friction, to be determined by experiment.

Girard's experiments upon the movement of air in pipes gave a coefficient of resistance $\zeta = 0{,}0256$, those of d'Aubuisson, as a mean, $\zeta = 0{,}0238$, while according to the experiments of Buff the mean value of $\zeta = 0{,}0375$. Poncelet, on the contrary, found from the data furnished by the experiments of Pecqueur, when the ratio of pressure is $\frac{p_1}{p} = 2$, $\mu = 0{,}0237$.

The experiments of the author, calculated according to the new formula, gave the following results:

1) A *brass tube*, 1 centimeter wide and 2 meters long, gave for

velocities of from 25 to 150 meters ζ gradually decreasing from 0,027260 to 0,01482.

2) A *glass tube* of the same length, when the velocities were about the same, gave $\zeta = 0{,}02738$ to $0{,}01390$.

3) A *brass tube*, 1,41 centimeters wide and 3 meters long, gave $\zeta = 0{,}02578$ to $0{,}01214$.

4) and a similar *glass tube*, $\zeta = 0{,}02663$ to $0{,}009408$.

5) Finally, a *zinc tube*, 2,4 centimeters wide and 10 meters long, gave, for velocities of from 25 to 80 meters, $\zeta = 0{,}2303$ to $0{,}01296$.

From what precedes we may conclude that it is only when velocities are about 25 meters or 80 feet, that the coefficient of resistance ζ can be put $= 0{,}024$, and that it becomes smaller and smaller as the velocity of the air in the pipe increases.

Approximatively we can write, when the velocity is expressed in meters, $\zeta = \frac{0{,}120}{\sqrt{v}}$ or when it is expressed in feet $\zeta = \frac{0{,}217}{\sqrt{v}}$. The general relations of the flow of air in pipes are very similar to those of water.

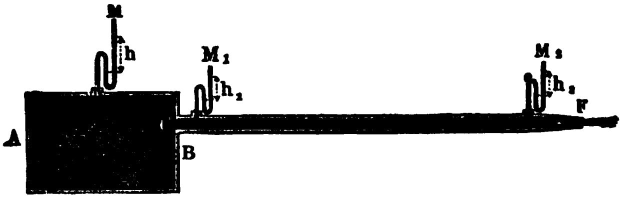
The resistance, caused by *elbows* and *bends*, is to be treated in the same way as in the case of water.

In the author's experiments a *rectangular elbow*, 1 centimeter in diameter, gave $\zeta = 1{,}61$, and a similar one, 1,41 centimeters in diameter, gave $\zeta = 1{,}24$, and a pipe like the former, when bent in the shape of a quarter of a circle, gave $\zeta = 0{,}485$, and one like the latter, bent in the same way, gave $\zeta = 0{,}471$.

§ 467. Motion of Air in Long Pipes.—By the aid of the coefficient ζ of the resistance of friction of a pipe $B\,F$, we can calculate the velocity of efflux and the discharge for a given length and width of the pipe.

If h_2 is the height of the manometer M_2 at the end of the pipe $C\,F$, Fig. 797, directly behind the mouth-piece F, whose coefficient of resistance is $\zeta = \frac{1}{\mu_1^2} - 1$, and if d denote the diameter of the

FIG. 797.



pipe and d_1 that of the orifice, whose area is therefore $F_1 = \frac{\pi d_1^2}{4}$, we have, according to what precedes, the discharge

$$Q = \mu_1 F_1 \sqrt{\frac{2g \frac{p_1}{\gamma_1} \cdot \frac{h_2}{b}}{1 - \left(\frac{d_1}{d}\right)^4}} = 1299\, \mu_1 \pi \frac{d_1^2}{4} \sqrt{\frac{(1 + d\tau)\frac{h_2}{b}}{1 - \left(\frac{d_1}{d}\right)^4}}, \text{cub. ft.}$$

or, inversely, for the height h_2 of the manometer

$$\frac{p_1}{\gamma_1} \cdot \frac{h_2}{b} = \left[1 - \left(\frac{d_1}{d}\right)^4\right] \frac{1}{2g} \left(\frac{Q}{\mu_1 F_1}\right)^2.$$

But the height of the manometer at the entrance of the pipe is

$$h_1 = h_2 + \zeta \frac{l}{d} \frac{v^2}{2g},$$

l denoting the length of the pipe between M_1 and M_2, and v the velocity of the air in this pipe; hence we have

$$\frac{p_1}{\gamma_1} \cdot \frac{h_1}{b} = \left[1 - \left(\frac{d_1}{d}\right)^4\right] \cdot \frac{1}{2g} \left(\frac{Q}{\mu_1 F_1}\right)^2 + \zeta \frac{l}{d} \frac{v^2}{2g}, \text{ or,}$$

substituting $v = \left(\frac{d_1}{d}\right)^2 v_1$ and $v_1 = \frac{Q}{F_1}$,

$$\frac{p_1}{\gamma_1} \cdot \frac{h_1}{b} = \left(\left[1 - \left(\frac{d_1}{d}\right)^4\right] \frac{1}{\mu_1^2} + \zeta \frac{l}{d} \left(\frac{d_1}{d}\right)^4\right) \cdot \frac{1}{2g} \left(\frac{Q}{F_1}\right)^2;$$

hence the discharge is

$$Q = F_1 \sqrt{\frac{2g \frac{p_1}{\gamma_1} \cdot \frac{h_1}{b}}{\left[1 - \left(\frac{d_1}{d}\right)^4\right] \frac{1}{\mu_1^2} + \zeta \frac{l}{d} \left(\frac{d_1}{d}\right)^4}},$$

$$= 1299 \frac{\pi d_1^2}{4} \sqrt{\frac{(1 + d\tau)\frac{h_1}{b}}{\left[1 - \left(\frac{d_1}{d}\right)^4\right] \frac{1}{\mu_1^2} + \zeta \frac{l}{d} \left(\frac{d_1}{d}\right)^4}} \text{ cubic feet.}$$

If, finally, the height h of the manometer M in the reservoir $A\,B$ is known, we have, when we denote the coefficient of resistance for the entrance C by ζ_0 and substitute $\frac{1}{\mu_1^2} = 1 + \zeta_1$, since at the entrance into the pipe the head $\zeta_0 \frac{v^2}{2g}$ is lost,

$$\frac{p_1}{\gamma_1} \cdot \frac{h}{b} = \left[\left(\zeta_0 + \zeta \frac{l}{d}\right) \left(\frac{d_1}{d}\right)^4 + 1 + \zeta_1\right] \frac{1}{2g} \left(\frac{Q}{F_1}\right)^2,$$

and consequently the discharge

$$Q = F_1 \sqrt{\frac{2g \frac{p_1}{\gamma_1} \cdot \frac{h}{b}}{\left(\zeta_0 + \zeta \frac{l}{d}\right)\left(\frac{d_1}{d}\right)^4 + 1 + \zeta_1}}$$

$$= 1299 \frac{\pi d_1^2}{4} \sqrt{\frac{(1 + 0{,}04\,\tau)\frac{h}{b}}{\left(\zeta_0 + \zeta \frac{l}{d}\right)\left(\frac{d_1}{d}\right)^4 + 1 + \zeta_1}} \text{ cubic feet.}$$

If the point *where the air enters the pipe* is a distance s *below* or *above* the point where it is discharged from it, we must subtract from or add to the quantity $\frac{p_1}{\gamma_1} \cdot \frac{h}{b}$ in the numerator under the radical sign a quantity s.

EXAMPLE.—The height of a quicksilver manometer, which is placed upon a regulator at the head of a system of air pipes 320 feet long and 4 inches in diameter, is 3,1 inches, the height of the barometer in the free air is 29 inches, the width of orifice in the conically convergent end of the pipe is $d_1 = 2$ inches, and the temperature of the compressed air in the regulator is $\tau = 20°$ C.; what quantity of air is delivered through these pipes?

Here $(1 + 0{,}004\,\tau)\frac{b}{h} = 1{,}08 \cdot \frac{3{,}1}{29} = 0{,}11545,$

$$\zeta_0 = \frac{1}{\mu_0^2} - 1 = \frac{1}{0{,}75^2} - 1 = \frac{16}{9} - 1 = \frac{7}{9} = 0{,}778,$$

$$\zeta \frac{l}{d} = 0{,}024 \cdot 320 \cdot 3 = 23{,}04, \left(\frac{d_1}{d}\right)^4 = \left(\frac{2}{4}\right)^4 = \frac{1}{16} = 0{,}0625,$$

$$\zeta_1 = \frac{1}{\mu_1^2} - 1 = \frac{1}{0{,}92^2} - 1 = 0{,}330, \text{ and}$$

$$F_1 = \frac{\pi d_1^2}{4} = \frac{\pi}{4} \cdot (\tfrac{1}{6})^2 = \frac{3{,}1416}{144} = 0{,}021817 \text{ square feet;}$$

hence the required discharge is

$$Q = 1299 \cdot 0{,}021817 \sqrt{\frac{0{,}11545}{(0{,}778 + 23{,}04)\,0{,}0625 + 1{,}330}}$$

$$= 28{,}34 \sqrt{\frac{0{,}11545}{1{,}489 + 1.330}} = 28{,}34 \sqrt{0{,}040954} = 5{,}735 \text{ cubic feet.}$$

§ **468. Efflux when the Pressure Diminishes.**—If there is no influx of air into a *reservoir*, from which an uninterrupted discharge of air is taking place through an orifice in it, the density and tension gradually diminish, and consequently the velocity of efflux becomes less and less. The relations of this diminution to the time and to the discharge can be determined in the following manner.

Let the volume of the reservoir be V, the initial height of the manometer be $= h_0$, and its height at the end of a certain time t be $= h_1$, and let that of the barometer in the free air be $= b$; then the quantity of air originally in the reservoir, reduced to the pressure of the exterior air, is

$$= \frac{V(b + h_0)}{b},$$

and at the end of the time t it is

$$= \frac{V(b + h_1)}{b};$$

hence the discharge in the time t, reduced to the external pressure, is

$$V_1 = \frac{V(b + h_0)}{b} - \frac{V(b + h_1)}{b} = \frac{V(h_0 - h_1)}{b}.$$

But we have also

$$V_1 = \mu F t \sqrt{2 g \frac{p_1}{\gamma_1}} \cdot \frac{x}{b},$$

x denoting the mean height of the barometer during the time t of efflux; hence

$$t = \frac{V(h_0 - h_1)}{\mu F \sqrt{2 g \frac{p_1}{\gamma_1} b x}} = \frac{V(h_0 - h_1)}{\mu F \sqrt{2 g \frac{p_1}{\gamma_1} b}} (x)^{-\frac{1}{2}}.$$

Now if we put $h_0 = m\sigma$ and $h_1 = n\sigma$, we have the mean value

$$(x)^{-\frac{1}{2}} = \frac{(\sigma)^{-\frac{1}{2}}}{m-n}(1^{-\frac{1}{2}} + 2^{-\frac{1}{2}} + \ldots + m^{-\frac{1}{2}}) - (1^{-\frac{1}{2}} + 2^{-\frac{1}{2}} + \ldots + n^{-\frac{1}{2}})$$

$$= \frac{(\sigma)^{-\frac{1}{2}}}{m-n}\left(\frac{m^{\frac{1}{2}}}{\frac{1}{2}} - \frac{n^{\frac{1}{2}}}{\frac{1}{2}}\right) = \frac{2(\sigma)^{-\frac{1}{2}}}{m-n}\left(\sqrt{\frac{h_0}{\sigma}} - \sqrt{\frac{h_1}{\sigma}}\right)$$

$$= \frac{2(\sqrt{h_0} - \sqrt{h_1})}{(m-n)\sigma} = \frac{2(\sqrt{h_0} - \sqrt{h_1})}{h_0 - h_1} \text{ (see Ingenieur, p. 88);}$$

hence the required time of efflux is

$$t = \frac{2V(\sqrt{h_0} - \sqrt{h_1})}{\mu F \sqrt{2 g \frac{p_1}{\gamma_1} b}} = \frac{2V}{\mu F \sqrt{2 g \frac{p_1}{\gamma_1}}}\left(\sqrt{\frac{h_0}{b}} - \sqrt{\frac{h_1}{b}}\right).$$

This determination is sufficiently correct only when the reservoir (V) is large, or when the orifice of efflux, as well as the pressure, is small, in which case the cooling of the air in the reservoir is very slight.

EXAMPLE.—A cylindrical regulator 50 feet long and 5 feet in diameter is filled with air at a pressure corresponding to the height $h = 10$ inches of the manometer and at a temperature of 6° C. Now if the air issues from an orifice 1 inch in diameter into a space where the barometer stands at 27 inches, the question arises, in what time will the manometer sink to 7 inches and what will be the discharge in that time? The volume of the regulator or boiler is

$$V = \frac{\pi}{4} \,.\, 5^2 \,.\, 50 = 1250 \,.\, \frac{\pi}{4} = 981{,}75 \text{ cubic feet, and}$$

$$\sqrt{\frac{h_0}{b}} - \sqrt{\frac{h_1}{b}} = \sqrt{\frac{10}{27}} - \sqrt{\frac{7}{27}} = 0{,}09942,$$

$$\sqrt{2\,g\,\frac{p_1}{\gamma_1}} = 1299\,\sqrt{1 + 0{,}00367 \,.\, \tau} = 1299\,\sqrt{1{,}02202} = 1313 \text{ and}$$

$$F = \frac{\pi}{4}\,(\tfrac{1}{12})^2 = \frac{\pi}{576} = 0{,}005454 \text{ square feet.}$$

Now if we put the coefficient of efflux $\mu = 0{,}95$, we have the required duration of the efflux

$$t = \frac{2 \,.\, 981{,}75 \,.\, 0{,}09942}{0{,}95 \,.\, 0{,}005454 \,.\, 1313} = 28{,}69 \text{ seconds.}$$

Remark.—A more general theory of the efflux of air and steam will be given in the second volume.

Final Remark.—Experiments upon the efflux of air have been made by Young, Schmidt, Lagerhjelm, Koch, d'Aubuisson, Buff, and more recently by Saint Venant, Wantzel, and Pecqueur. In reference to the experiments of Young and Schmidt, see Gilbert's Annalen, Vol. 32, 1801, and Vol. 6, 1820, and Poggendorf's Annalen, Vol. 2, 1824; for those of Koch and Buff, see the "Studien des Götting'schen Vereines bergmännischer Freunde," Vol. 1, 1824; Vol. 3, 1833; Vol. 4, 1837; and Vol. 5, 1838; also Poggendorf's Annalen, Vol. 27, 1836, and Vol. 40, 1837. See also Gerstner's "Mechanik," Vol. 3, and Hülsse's "Algemeine Maschinenencyklopädie," Article "Ausfluss." Lagerhjelm's experiments are discussed in the Swedish work "Hydrauliska Försök af Lagerhjelm, Forselles och Kallstenius," 1 Delen, Stockholm, 1818. The experiments of d'Aubuisson are to be found in the "Annales des Mines," Vol. 11, 1825; Vol. 13, 1826; Vols. 3 and 4, 1828; and also in d'Aubuisson's "Traité d'Hydraulique." The experiments of Saint-Venant and Wantzel are to be found in the "Comptes rendus hebd. des séances de l'Académie des Sciences, à Paris, 1839." The latest French experiments are discussed by Poncelet in a "note sur les experiences de M. Pecqueur relatives à l'écoulement de l'air dans les tubes, etc.," which is contained in the Comptes rendus, and an abstract of it is to be found in the Polytechnische Centralblatt, Vol. 6, 1845. From these experiments Poncelet concludes that air follows the same laws of efflux as water. The greater number of these experiments were made with very narrow orifices, for which reason they scarcely fulfill the requirements of practice. Unfortunately these experiments do not agree as well as could be wished, and the coefficients found by d'Aubuisson differ very sensibly from those calculated from Koch's experiments. Comparative experiments upon the efflux and influx of air and upon the efflux of water are given in the author's "Experimental-Hydraulik." The results of the latest experiments of the author, which were made upon a large scale, are given in the 5th volume of the Civilingenieur.

CHAPTER VII.

OF THE MOTION OF WATER IN CANALS AND RIVERS.

§ 469. **Running Water.**—The theory of the motion of water in *canals* and *rivers* forms the second part of hydraulics. Water flows either in a *natural* or in an *artificial bed* (Fr. lit; Ger. Bett). In the first case the channel is a river, creek, rivulet, etc., in the second case it is a canal, ditch, race, trough, etc. In the theory of the motion of running water this difference is of but little importance.

The *bed of the stream* consists of the *bottom of the channel* (Fr. font du lit; Ger. Grundbett or Sohle) and of the two *banks or shores* (Fr. bords; Ger. Ufer). If we pass a plane through the stream of water at right angles to the direction, in which it is flowing, we obtain a *transverse section* (Fr. section; Ger. Querschnitt). The line bounding this section is the *tranverse profile* which is composed of the *water profile* or *wetted perimeter* and of the *air profile.* A vertical plane in the direction of the stream gives us the longitudinal *section or profile* (Fr. profil; Ger. Profil) of the latter. The *slope of the stream* (Fr. pente; Ger. Abhang) is the angle formed by its surface with the horizon. The relative slope is the fall in the unit of distance. The slope is determined for any definite distance by the *fall* (Fr. chute; Ger. Gefälle), which is the vertical distance of one of the extremities of a certain portion of the stream above the other. In the portion $A\ D = l$, Fig. 798, $B\ C$ is the bottom of the channel, $D\ H = h$ the fall and the angle $D\ A\ H = \delta$ is the slope. The relative slope is

FIG. 798.

$$\sin.\ \delta = \frac{h}{l}, \text{ or approximatively } \delta = \frac{h}{l}.$$

REMARK.—The fall of creeks and rivers varies very much. The Elb falls in a German mile (4¾ English miles) from Hohenelbe to Podiebrad 57 feet, from there to Leitmeritz 9 feet, from there to Mühlberg 2,5 feet. Mountain streams fall from 8 to 80 feet per mile. For particulars see "Vergleichende hydrographische Tabellen, etc., von Stranz." The fall in canals and other artificial channels is much smaller. The relative slope is generally less than 0,001, it is often 0,0001 and even less. More details upon this subject will be found in the second part.

§ 470. Different Velocities in a Cross-section.—The velocity of the water is far from being uniform in all points of the same transverse section. The adhesion of the water to the bed of the channel and the cohesion of the molecules of water cause the particles of water nearest to the sides and bed of the channel to be most hindered in their motion. For this reason, the velocity decreases from the surface towards the bed of the channel and it is a minimum at the shores and bottom. The maximum velocity in a straight river is generally found in the middle or in that portion of the surface, where the water is the deepest. That portion of the river, where the water has its maximum velocity, is called the *line of current* or *axis of the stream* and the deepest portion of the bed is called the *mid-channel*.

When the stream bends, the axis of the stream is general near the concave shore.

The mean velocity of the water in a cross-section, according to § 396, is

$$c = \frac{Q}{F} = \frac{\text{Discharge per second}}{\text{Area of the transverse section}}.$$

We can also determine the mean velocity from velocities c_1, c_2, c_3 etc., in the different portions of the transverse section and the areas F_1, F_2, F_3 etc., of the latter. We have here

$$Q = F_1 c_1 + F_2 c_2 + F_3 c_3 + \ldots$$

and, therefore, also

$$c = \frac{F_1 c_1 + F_2 c_2 + \ldots}{F_1 + F_2 + \ldots}.$$

Besides the mean velocity we introduce the ***mean depth of water***, I.E., that depth a, which a transverse section would have, if its area was the same and the depth was uniform instead of being variable and equal to a_1, a_2, a_3, etc. Here we have

$$a = \frac{F}{b} = \frac{\text{Area of the transverse section}}{\text{Width of the transverse section}}.$$

If the widths of the elements corresponding to the depths a_1, a_2, a_3, etc., Fig. 799, are b_1, b_2, b_3, etc., we have

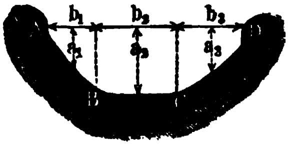
$$F = a_1 b_1 + a_2 b_2 + \ldots,$$

and consequently

$$a = \frac{a_1 b_1 + a_2 b_2 + \ldots}{b_1 + b_2 + \ldots}.$$

Finally, the mean velocity is

FIG. 799.



$$c = \frac{a_1 b_1 c_1 + a_2 b_2 c_2 + \dots}{a_1 b_1 + a_2 b_2 + \dots},$$

and, when the widths b_1, b_2, etc., of the portions are the same,

$$c = \frac{a_1 c_1 + a_2 c_2 + \dots}{a_1 + a_2 + \dots}.$$

A river or creek is in a state of *permanency* (Fr. permanence; Ger. Beharrungszustande) or it has a fixed regimen, when the same quantity of water passes through each of its cross-section in the same time, I.E., if Q or the product $F c$ of the area of the cross-section and the mean velocity is constant for the whole length of the portion of the river under consideration. Hence we have the simple law: *when the motion of the water is permanent the mean velocities of two transverse sections are to each other inversely as the areas of these sections.*

EXAMPLE—1) In the transverse section $A\,B\,C\,D$, Fig. 799, of a canal, we have found the widths of the divisions to be

$$b_1 = 3{,}1 \text{ feet}, \; b_2 = 5{,}4 \text{ feet}, \; b_3 = 4{,}3 \text{ feet},$$

the mean depths to be

$$a_1 = 2{,}5 \text{ feet}, \; a_2 = 4{,}5 \text{ feet}, \; a_3 = 3{,}0 \text{ feet}$$

and the corresponding mean velocities to be

$$c_1 = 2{,}9 \text{ feet}, \; c_2 = 3{,}7 \text{ feet}, \; c_3 = 3{,}2 \text{ feet}.$$

Here we can put the area of the section

$$F = 3{,}1 \,.\, 2{,}5 + 5{,}4 \,.\, 4{,}5 + 4{,}3 \,.\, 3{,}0 = 44{,}95 \text{ square feet}$$

and the discharge

$$Q = 3{,}1 \,.\, 2{,}5 \,.\, 2{,}9 + 5{,}4 \,.\, 4{,}5 \,.\, 3{,}7 + 4{,}3 \,.\, 3{,}0 \,.\, 3{,}2 = 153{,}665 \text{ cubic feet},$$

from which we obtain the mean velocity

$$c = \frac{Q}{F} = \frac{153{,}665}{44{,}95} = 3{,}419 \text{ feet}.$$

2) If a ditch should carry 4,5 cubic feet of water with a mean velocity of 2 feet per second, we must make the area of its transverse section $\frac{4{,}5}{2} = 2{,}25$ square feet.

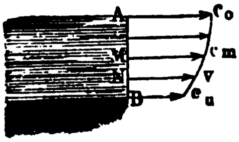
3) If the same river is at one place 560 feet wide and as an average 9 feet deep, and if it moves with a mean velocity of 2¼ feet, the mean velocity at another place, where it is 320 feet wide and as a mean 7,5 feet deep, is

$$c = \frac{560 \,.\, 9}{320 \,.\, 7{,}5} \cdot 2{,}25 = 4{,}725 \text{ feet}.$$

§ 471. Mean Velocity.—If we divide the depth of the water at any point into equal parts and lay off the corresponding velocities as ordinates, we obtain a scale $A\,B$, Fig 800, of the velocities of the stream. Although it is very probable that the law of this scale, or of the change of velocity, is expressed by a curve, as

E.G. according to Gerstner, by an ellipse, etc., yet without risking a very great error we can substitute a straight line, I.E., assume that the velocity diminishes regularly with the depth; for this diminution of the velocity is always slight. According to the experiments of Ximenes, Brünnings and Funk, the *mean velocity in a perpendicular line is*

FIG. 800.



$$c_m = 0,915\ c_0,$$

c_0 denoting the maximum velocity or that of the surface of the water. The diminution of the velocity from the surface to the middle *M* is therefore

$$c_0 - c_m = (1 - 0,915)\ c_0 = 0,085\ c_0,$$

and we can put the velocity at the bottom, or at the foot of the perpendicular,

$$c_n = c_0 - 2 .\ 0,085\ c_0 = (1 - 0,170)\ c_0 = 0,83\ c_0.$$

If the total depth is a, we have, if we assume the scale of velocity to be represented by a straight line, for a depth $A\ N = x$ below the water the velocity

$$v = c_0 - (c_0 - c_n)\frac{x}{a} = \left(1 - 0,17\ \frac{x}{a}\right) c_0.$$

Now if c_0, c_1, c_2 are the velocities at the surface of a profile, whose depth is not very variable, we have the corresponding velocities at the mean depth

$$0,915\ c_0,\ 0,915\ c_1,\ 0,915\ c_2,$$

and therefore the mean velocity in the whole transverse section

$$c = 0,915\ \frac{c_0 + c_1 + c_2 + \ldots + c_n}{n + 1}.$$

If, finally, we assume that the velocity diminishes from the line of current or axis of the stream towards the shores in the same manner as towards the bottom, we can put the mean superficial velocity $\frac{c_0 + c_1 + \ldots + c_n}{n + 1} = 0,915\ c_0$;

thus we obtain the *mean velocity of the whole transverse section*.

$$c = 0,915\ .\ 0,915\ .\ c_0 = 0,837\ c_0,$$

I.E., 83 to 84 per cent. of the maximum velocity.

Prony deduced from the experiments of du Buat, which, however, were made in small ditches, the following formula, which is perhaps more correct in such cases,

$$c_m = \left(\frac{2,372 + c_0}{3,153 + c_0}\right) c_0 \text{ meter} = \left(\frac{7,78 + c_0}{10,34 + c_0}\right) c_0 \text{ feet.}$$

Hence for mean velocities of 3 feet we have

$$c_m = 0{,}81\ c_0.$$

If the flow of the water is *impeded* by a contraction of the transverse section, the level of the water will be raised, and c_m becomes still greater.

EXAMPLE.—If the velocity of the water in the axis of a river is 4 feet, and if its depth 6 feet, we have the mean velocity in the corresponding perpendicular

$$c_m = 0{,}915\ .\ 4 = 3{,}66 \text{ feet},$$

the velocity at the bottom

$$= 0{,}83\ .\ 4 = 3{,}32 \text{ feet},$$

the velocity 2 feet from the surface

$$v = (1 - 0{,}17\ .\ \tfrac{2}{6})\ 4 = (1 - 0{,}057)\ .\ 4 = 3{,}772 \text{ feet}$$

and, finally, the mean velocity of the entire transverse section

$$c = 0{,}837\ .\ 4 = 3{,}348 \text{ feet};$$

on the contrary, according to Prony, we would have

$$c = \frac{11{,}78}{14{,}34}\ .\ 4 = \frac{23{,}56}{7{,}17} = 3{,}29 \text{ feet}.$$

REMARK.—This and the following subjects are treated at length in the Allgemeine Maschinenencyklopädie, Article "Bewegung des Wassers." New experiments and new views upon the same subject are to be found in the following work: "Lahmeyer, Erfahrungsresultate uber die Bewegung des Wassers in Flussbetten und Canälen, Braunschweig, 1845." According to Baumgarten's observations (see Annales des Ponts et Chaussées, Paris, 1848, and also polytechnisches Centralblatt, No. 14, 1849) the values given by this formula, when the velocities are great (above 1,5 meters), are too large and we must put in such cases

$$c_m = \left(\frac{2{,}372 + c_0}{3{,}153 + c_0}\right)\ .\ 0{,}8\ c_0 \text{ meters}.$$

Owing to the resistance of the air the maximum velocity of the water is to be found a little below the surface of the water. From this point of maximum velocity the velocity diminishes as the square of the depth; hence the scale of velocity corresponds to a parabola. In like manner, according to Boileau (see his Traité sur la mesure des eaux), the velocity decreases as the square of the distance from the axis of the stream. If c_0 denotes the velocity in the axis of the stream, the velocity at the horizontal distance x from it will be

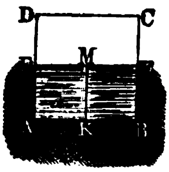
$$c_x = c_0 - \mu x^2,$$

in which μ denotes an empirical number, which is different for different streams.

§ 472. Most Advantageous Transverse Profile.—The *resistance*, offered by the bed of the stream in consequence of the adhesion, viscosity and friction of the water, is proportional to the surface of contact, and consequently to the wetted perimeter *p*, or to that portion of the profile which forms the bed. Now since the number of filaments of water passed by any transverse section increases with its area, the resistance to each filament is inversely proportional to the area, and consequently to the quotient $\frac{p}{F}$ of the wetted perimeter divided by the area *F* of the entire transverse section. In order to have the ***least resistance from friction,*** we must give the profile such a shape that $\frac{p}{F}$ shall be as small as possible, I.E., that the wetted perimeter *p* shall be a minimum for a given area, or that the area shall be a maximum for a given wetted perimeter *p*. When the apparatus which conducts the water is closed on all sides as in the case of pipes, *p* is the perimeter of the entire transverse section. Now among all figures of the same number of sides, the regular one, and among all the regular ones, the one with the greatest number of sides has the smallest perimeter for a given area; hence in conduits closed on all sides the resistance is smaller the more regular the shape of their transverse section is, and the greater the number of sides is. Since the *circle* is a regular figure of infinite number of sides, the resistance of friction is the smallest when the transverse section is of that form. When the aqueduct is open on top, the case is different; for the upper surface is free, or in contact with the air alone, which, so long as it is still, offers little or no resistance to the water. We must, therefore, in determining this resistance of friction, neglect the air profile.

In practice we employ in canals, ditches, troughs and flumes only *rectangular* and *trapezoidal* profiles. A horizontal line *E F*, Fig. 801, passing through the centre *M* of the square *A C*, divides the area and perimeter into two equal parts, and what has been said of the square is true for these halves; hence, among all rectangular profiles, the *half square A E*, or that which is *twice as wide as high*, is the one which causes the smallest resistance of friction.

FIG. 801.



In like manner, the *regular hexagon A C E*, Fig. 802, is divided by a horizontal line *C F* into two equal trapezoids, each of which, like the entire hexagon, has the greatest relative area, and consequently among all trapezoidal profiles, the half of the regular hexagon, or the trapezoid *A B C F*, with the angle of slope $B\ C\ M = 60°$, is the one which causes the least resistance of friction.

In like manner, *the half* of a *regular octagon* $A\ D\ E$, Fig. 803, the half of a *regular decagon*, etc., and finally the half circle $A\ D\ B$, Fig. 804, are, under the proper circumstances, the most advantageous profiles for canals, etc. The trapezoidal, or half hexagonal, cross-section causes less resistance than the half square or rectangle, the ratio of whose sides is 1 : 2; the relative perimeter of the hexagon is smaller than that of the square. The half decagon offers still less resistance, and with the semicircle the latter is a minimum. The circular and square profiles are employed only for troughs made of iron, stone, or wood. The trapezoid is employed in canals, which are dug out or walled up. Other forms are rarely used, owing to the difficulty of constructing them.

FIG. 802.

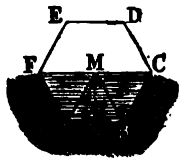


FIG. 803.

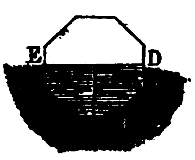
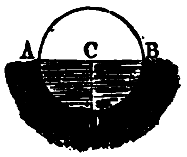


FIG. 804.



§ **473.** When canals are not walled up, but only dug in the earth or sand, an angle of slope of 60° is too great or the relative slope *cotg.* $60° = 0{,}57735$ too small; for the banks would not be sufficiently stable; we are therefore compelled to employ trapezoidal transverse profiles, in which the inclination of the side to the base is smaller than 60°, perhaps only 45° or even less. For a trapezoidal cross-section $A\ B\ C\ D$, Fig. 805, which has the same area and perimeter as the half square, the relative slope is $= \frac{4}{3}$, and the angle of slope is 36° 52′. If we divide the height $B\ E$ into three equal parts, the bottom $B\ C$ is equal to two of them, the parallel top $A\ D$ is equal to 10 and each side $A\ B = C\ D$ is $= 5$ parts. In many cases we make the relative slope $= 2$; in which case the angle is 26° 34′, and sometimes it exceeds even 2.

In any case the angle of slope $B\ A\ E = \theta$, Fig. 806, or the slope $\frac{A\ E}{B\ E} = cotang.\ \theta$ is to be considered as a given quantity, dependent upon the nature of the ground in which the canal is excavated, and therefore we have only to determine the dimensions of the profile which will offer the least resistance. Putting the width BC of

FIG. 805.

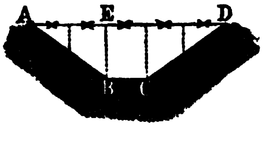
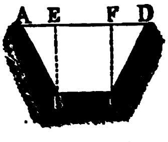


FIG. 806.



the bottom $= b$, the depth $BE = a$ and the slope $\frac{AE}{BE} = v$, we have the wetted perimeter of the profile $p =$

$$AB + BC + CD = b + 2\sqrt{a^2 + v^2 a^2} = b + 2a\sqrt{1 + v^2},$$

and the area of the same

$$F = ab + vaa = a(b + va),$$

or inversely

$$b = \frac{F}{a} - va,$$

whence the ratio

$$\frac{p}{F} = \frac{1}{a} + \frac{a}{F}(2\sqrt{v^2 + 1} - v).$$

Substituting instead of a, $a + x$, in which x is a small quantity, we have

$$\frac{p}{F} = \frac{1}{a + x} + \frac{(a + x)}{F}(2\sqrt{v^2 + 1} - v)$$

$$= \frac{1}{a}\left(1 - \frac{x}{a} + \frac{x^2}{a^2}\right) + \frac{a + x}{F}(2\sqrt{v^2 + 1} - v)$$

$$= \frac{1}{a} + \frac{a}{F}(2\sqrt{v^2 + 1} - v) + \left(\frac{2\sqrt{v^2 + 1} - v}{F} - \frac{1}{a^2}\right)x + \frac{x^2}{a^3}.$$

In order that this value, not only for a positive but also for a negative value of x, shall be greater than the first value

$$\frac{1}{a} + \frac{a}{F}(2\sqrt{v^2 + 1} - v),$$

or that $\frac{p}{F}$ shall be a minimum, it is necessary that the members with the factor x shall disappear or that

$$\frac{2\sqrt{v^2 + 1} - v}{F} - \frac{1}{a^2} = 0,$$

whence the required depth of the canal is

$$a^2 = \frac{F}{2\sqrt{v^2 + 1} - v},$$

or, since $v = cotang.\, \theta$ and $\sqrt{v^2 + 1} = \frac{1}{sin.\, \theta}$,

$$a^2 = \frac{F\, sin.\, \theta}{2 - cos.\, \theta}.$$

Hence for a given angle of slope θ and for a given area, the *most advantageous form for the transverse profile* is determined by the formulas

$$a = \sqrt{\frac{F\, sin.\, \theta}{2 - cos.\, \theta}} \text{ and } b = \frac{F}{a} - a\, cotang.\, \theta.$$

Consequently the width $A\ D$ of the top is

$$b_1 = b + 2\, \nu\, a = \frac{F}{a} + a\, cotang.\, \theta,$$

and the ratio

$$\frac{p}{F} = \frac{b}{F} + \frac{2\, a}{F\, sin.\, \theta} = \frac{1}{a} + \frac{(2 - cos.\, \theta)\, a}{F\, sin.\, \theta} = \frac{2}{a}.$$

EXAMPLE.—What dimensions should be given to the transverse profile of a canal, when the angle of slope of its banks is to be 40° and when it is to carry a quantity $Q = 75$ cubic feet of water with a mean velocity of 3 feet.

Here

$F = \frac{Q}{v} = \frac{75}{3} = 25$ square feet, and therefore the required depth is

$$a = \sqrt{\frac{25\, sin.\, 40^\circ}{2 - cos.\, 40^\circ}} = 5\sqrt{\frac{0,64279}{1,23396}} = 3,609 \text{ feet},$$

the width at the bottom is

$$b = \frac{25}{3,609} - 3,609\, cotang.\, 40^\circ = 6,927 - 4,301 = 2,626 \text{ feet},$$

the horizontal projection of the slope of the shore is

$$\nu\, a = a\, cotang.\, \theta = 3,609\, cotang.\, 40^\circ = 4,301,$$

the width on top is

$$b_1 = b + 2\, a\, cotang.\, \theta = 6,927 + 4,301 = 11,228 \text{ feet},$$

the wetted perimeter is

$$p = b + \frac{2\, a}{sin.\, \theta} = 2,626 + \frac{7,218}{sin.\, 40^\circ} = 13,855 \text{ feet},$$

and the ratio which determines the resistance of friction is

$$\frac{p}{F} = \frac{2}{a} = \frac{2}{3,609} = 0,5542.$$

We have for a transverse profile in the shape of the half of a regular hexagon, where $\theta = 60^\circ$, $a = 3,80$ feet, $b = 4,39$, $b_1 = 8,78$ and $p = 13,16$ feet, and therefore $\frac{p}{F} = \frac{13,16}{25} = 0,526$.

§ 474. Table of the Most Advantageous Transverse Profiles.

—The following table gives the dimensions of the most *advantageous transverse profiles* for different angles of slope and for given transverse sections:

Angle of slope θ.	Relative slope ν.	DIMENSIONS OF THE TRANSVERSE PROFILES.				Quotient $\frac{p}{F} = \frac{m}{\sqrt{F}}$
		Depth a.	Width of bottom b.	Horizontal projection of slope νa.	Width at the top $b + 2 \nu a$.	
90°	0	$0{,}707 \sqrt{F}$	$1{,}414 \sqrt{F}$	0	$1{,}414 \sqrt{F}$	$\frac{2{,}828}{\sqrt{F}}$
60°	0,577	$0{,}760 \sqrt{F}$	$0{,}877 \sqrt{F}$	$0{,}439 \sqrt{F}$	$1{,}755 \sqrt{F}$	$\frac{2{,}632}{\sqrt{F}}$
45°	1,000	$0{,}740 \sqrt{F}$	$0{,}613 \sqrt{F}$	$0{,}740 \sqrt{F}$	$2{,}092 \sqrt{F}$	$\frac{2{,}704}{\sqrt{F}}$
40°	1,192	$0{,}722 \sqrt{F}$	$0{,}525 \sqrt{F}$	$0{,}860 \sqrt{F}$	$2{,}246 \sqrt{F}$	$\frac{2{,}771}{\sqrt{F}}$
36° 52′	1,333	$0{,}707 \sqrt{F}$	$0{,}471 \sqrt{F}$	$0{,}943 \sqrt{F}$	$2{,}357 \sqrt{F}$	$\frac{2{,}828}{\sqrt{F}}$
35°	1,402	$0{,}697 \sqrt{F}$	$0{,}439 \sqrt{F}$	$0{,}995 \sqrt{F}$	$2{,}430 \sqrt{F}$	$\frac{2{,}870}{\sqrt{F}}$
30°	1,732	$0{,}664 \sqrt{F}$	$0{,}356 \sqrt{F}$	$1{,}150 \sqrt{F}$	$2{,}656 \sqrt{F}$	$\frac{3{,}012}{\sqrt{F}}$
26° 34′	2,000	$0{,}636 \sqrt{F}$	$0{,}300 \sqrt{F}$	$1{,}272 \sqrt{F}$	$2{,}844 \sqrt{F}$	$\frac{3{,}144}{\sqrt{F}}$
Semicircle	—	$0{,}798 \sqrt{F}$	—	—	$1{,}596 \sqrt{F}$	$\frac{2{,}507}{\sqrt{F}}$

We see from the above table that the quotient $\frac{p}{F}$ is a minimum and $= \frac{2{,}507}{\sqrt{F}}$ for the semicircle, that it is greater for the half hexagon and still greater for the half square, and for the trapezoid with its sides sloping at an angle of 36° 52′, etc.

Example.—What dimensions are to be given to a transverse profile, whose area is to be 40 feet, when the banks are to slope at an angle of 35°? According to the foregoing table

the depth is $a = 0{,}697 \sqrt{40} = 4{,}408$ feet,

the lower breadth is $b = 0{,}439 \sqrt{40} = 2{,}777$ feet,

the horizontal projection of the slope $\nu a = 0{,}995 \sqrt{40} = 6{,}293$ feet,

the upper breadth $b_1 = 15{,}368$,

and the quotient is

$$\frac{p}{F} = \frac{2,870}{\sqrt{40}} = 0,4538.$$

§ 475. Uniform Motion.—The motion of water in channels is for a certain distance either *uniform* or *variable ;* it is uniform, when the mean velocity in all the cross-sections is constant, and, on on the contrary, it is variable, when the mean velocity and also the area of the cross-sections change. We will now treat of *uniform motion.*

Fig. 807.

When the motion of water is uniform for a distance $A\,D = l$, Fig. 807, the entire fall h is employed in overcoming the friction upon the bed, and the water flows away with the same velocity, with which it arrived, I.E., a height due to a velocity is neither absorbed nor set free. If we measure this friction by the height of a column of water, we can put the latter equal to the fall. The height due to the resistance of friction increases with the quotient $\frac{p}{F}$, with l and with the square of the mean velocity c (§ 427); hence the formula

$$1)\ h = \zeta \frac{l\,p}{F} \cdot \frac{c^2}{2\,g}$$

holds good, in which ζ is an empirical number, which is called the *coefficient of the resistance of friction.*

By inversion we have

$$2)\ c = \sqrt{\frac{F}{\zeta \,.\, l\,p} \,.\, 2\,g\,h}.$$

To determine the fall from the length, the transverse profile and the velocity, or inversely, to determine the velocity from the fall, the length and the transverse profile, it is necessary to know the coefficient of friction ζ. According to Eytelwein's calculation of the 91 experiments of du Buat, Brünings, Funk and Woltmann, $\zeta = 0,007565$, and therefore

$$h = 0,007565 \,.\, \frac{l\,p}{F} \cdot \frac{c^2}{2\,g}.$$

If we put $g = 9,809$ meters or $32,2$ feet, we obtain for the metrical system

$$h = 0,0003856\, \frac{l\,p}{F} \,.\, c^2 \text{ and } c = 50,9 \sqrt{\frac{F\,h}{p\,l}},$$

and for the English system of measure

$$h = 0{,}00011747\ \frac{l\,p}{F}\,c^2 \text{ and } c = 92{,}26\sqrt{\frac{F\,h}{p\,l}}.$$

For conduit pipes $\frac{l\,p}{F} = \frac{\pi\,l\,d}{\frac{1}{4}\,\pi\,d^2} = \frac{4\,l}{d}$; hence the formula for pipes is

$$h = 0{,}03026\ \frac{l}{d}\cdot\frac{v^2}{2\,g},$$

while we found more correctly (§ 428) for medium velocities in the same

$$h = 0{,}025\ \frac{l}{d}\cdot\frac{v^2}{2\,g}.$$

The friction upon river beds is, therefore, as might be expected, greater than in metal conduit pipes.

Example—1) How much fall must a canal, whose length is $l = 2600$ feet, whose lower width is $b = 3$ feet, whose upper width is $b_1 = 7$ feet and whose depth is $a = 3$ feet, have in order to carry 40 cubic feet of water per second? Here

$$p = 3 + 2\sqrt{2^2 + 3^2} = 10{,}211,\ F = \frac{(7 + 3)\,3}{2} = 15 \text{ and } c = \frac{40}{15} = \frac{8}{3};$$

hence the required fall is

$$h = 0{,}00011747\ .\ \frac{2600\ .\ 10{,}211}{15}\left(\frac{8}{3}\right)^2 = \frac{0{,}305422\ .\ 10{,}211\ .\ 64}{15\ .\ 9} = 1{,}48 \text{ feet.}$$

2) What quantity of water will be delivered by a canal 5800 feet long, when the fall is 3 feet, its depth 5 feet, its lower breadth 4 feet and its upper breadth 12 feet? Here

$$\frac{p}{F} = \frac{4 + 2\sqrt{5^2 + 4^2}}{5\ .\ 8} = \frac{16{,}806}{40} = 0{,}42015;$$

hence the velocity is

$$c = 92{,}26\sqrt{\frac{3}{0{,}42015\ .\ 5800}} = \frac{92{,}26}{\sqrt{0{,}14005\ .\ 5800}} = \frac{92{,}26}{\sqrt{812{,}29}}$$

$$= \frac{92{,}26}{28{,}5} = 3{,}237 \text{ feet,}$$

and the quantity delivered is

$$Q = F\,c = 40\ .\ 3{,}237 = 129{,}48 \text{ cubic feet.}$$

§ 476. Coefficients of Friction.—The coefficient of friction, for which we assumed in the foregoing paragraph the mean value 0,007565, is not constant for rivers, creeks, etc., but, as in the case of pipes, increases slightly, when the velocity diminishes, and decreases, when the velocity increases. We must therefore put

$$\zeta = \zeta_1\left(1 + \frac{a}{c}\right) \text{ or } \zeta_1\left(1 + \frac{a}{\sqrt{c}}\right)$$

or to some similar formula.

The author in the article quoted in the remark of § 471 found from 255 experiments, most of which were made by himself, for English measures

$$\zeta = 0{,}007409\left(1 + \frac{0{,}1920}{c}\right),$$

and for the metrical system of measures

$$\zeta = 0{,}007409\left(1 + \frac{0{,}05853}{c}\right).$$

We see that this formula gives for a velocity $c = 8\frac{3}{4}$ feet the above-quoted mean value $\zeta = 0{,}007565$. In order to facilitate calculation, the following tables for the metrical system have been prepared:

Velocity $c =$	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	meters.
Coefficient of resistance $\zeta = 0{,}0$	1175	0958	0885	0849	0828	0813	0803	0795	0789	

Velocity $c =$	1	1,2	1,5	2	3	4	5 meters.
Coefficient of resistance $\zeta = 0{,}0$	0784	0777	0771	0763	0755	0752	0750

For English system of measures we can employ the following table.

Velocity $c =$	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1	1½	2	3	5	7	10	15 feet.
Coefficient of resistance $\zeta = 0{,}0$	1215	1097	1025	0978	0944	0918	0899	0883	0836	0812	0788	0769	0761	07551	07504

These tables are directly applicable to all cases, where the velocity c is given and the fall is required, and when formula No. 1 of the foregoing paragraph is employed. If the velocity c is unknown, or if that is the required quantity, the tables can only be employed directly when we have an approximate value of c. The simplest way to proceed is to determine c approximatively by one of the formulas

$$c = 50{,}9\sqrt{\frac{Fh}{pl}} \text{ meters or } c = 92{,}26\sqrt{\frac{Fh}{pl}} \text{ feet,}$$

then find ζ by means of the table, and substitute the value so found in the formula

$$\frac{c^2}{2g} = \frac{h}{\zeta} \cdot \frac{F}{lp}, \text{ or}$$

$$c = \sqrt{\frac{F}{\zeta l p} \cdot 2 g h}.$$

From the velocity c we determine the quantity of water $Q = F c$.

If the quantity of water and the fall are given, as is often the case in laying out canals, and it is required to determine the transverse profile, we must substitute $\frac{p}{F} = \frac{m}{\sqrt{F}}$ (see table, § 474) and $c = \frac{Q}{F}$ in the formula

$$h = 0{,}007565 \frac{l p}{F} \cdot \frac{c^2}{2 g}, \text{ and put}$$

$$h = 0{,}007565 \frac{m l Q^2}{2 g F^{\frac{5}{2}}}, \text{ from which we obtain}$$

$$F = \left(0{,}007565 \frac{m l Q^2}{2 g h}\right)^{\frac{2}{5}}, \text{ I.E. in meters}$$

$$F = 0{,}0431 \left(\frac{m l Q^2}{h}\right)^{\frac{2}{5}}, \text{ and in English feet}$$

$$F = 0{,}0268 \left(\frac{m l Q^2}{h}\right)^{\frac{2}{5}}.$$

From this we obtain the approximative value

$$c = \frac{Q}{F};$$

if we take the corresponding value of ζ from the table, we can calculate more accurately

$$F = \left(\zeta \cdot \frac{m l Q^2}{2 g h}\right)^{\frac{2}{5}},$$

from which we deduce more correct values for

$$c = \frac{Q}{F} \text{ and } p = m \sqrt{F},$$

as well as for a, b, etc.

Example—1) What must be the fall of a canal 1500 feet long, whose lower breadth is two feet, whose upper breadth is 8 feet, and whose depth is 4 feet, when it is required to convey 70 cubic feet of water per second? Here

$$p = 2 + 2\sqrt{4^2 + 3^2} = 12, F = 5 \cdot 4 = 20, c = \tfrac{70}{20} = 3{,}5;$$

hence

$$\zeta = 0{,}00784 \text{ and}$$

$$h = 0{,}00784 \frac{1500 \cdot 12}{20} \cdot \frac{3{,}5^2}{2 g} = \frac{86{,}436}{2 g} = 1{,}34 \text{ feet.}$$

2) What quantity of water is carried by a creek 40 feet wide, whose mean depth is $4\frac{1}{2}$ feet, and whose wetted perimeter is 46 feet, when it falls 10 inches in 750 feet? Here we have approximatively

$$c = 92{,}26 \,.\, \sqrt{\frac{40 \,.\, 4{,}5 \,.\, 10}{46 \,.\, 750 \,.\, 12}} = \frac{92{,}26}{\sqrt{230}} = 6{,}1 \text{ feet};$$

hence we can assume $\zeta = 0{,}00765.$

We have now more correctly

$$\frac{c^2}{2g} = \frac{F h}{\zeta l p} = \frac{4{,}5 \,.\, 40 \,.\, 10}{0{,}00765 \,.\, 46 \,.\, 750 \,.\, 12} = \frac{1}{1{,}7595} = 0{,}5683 \text{ and } c = 6{,}05 \text{ feet.}$$

The quantity of water carried is

$$Q = F c = 4{,}5 \,.\, 40 \,.\, 6{,}05 = 1089 \text{ cubic feet.}$$

3) It is required to excavate a ditch 3650 feet long, which, with a total fall of one foot, shall carry 12 cubic feet of water per second. What must be the dimensions of the transverse section when the form is a regular hexagon? Here $m = 2{,}632$ (see table, § 474); hence we have approximatively

$$F = 0{,}0268\,(2{,}632 \,.\, 3650 \,.\, 144)^{\frac{2}{5}} = 7{,}66 \text{ square feet, and}$$

$$c = \frac{12}{7{,}66} = 1{,}567.$$

Here we must take $\zeta = 0{,}0083$, and, therefore,

$$F = \left(0{,}0083 \,.\, 2{,}632 \,.\, \frac{3650 \,.\, 144}{64{,}4}\right)^{\frac{2}{5}} = 7{,}95 \text{ square feet.}$$

From this we obtain the depth

$a = 0{,}760 \sqrt{F} = 2{,}14$, the lower width

$b = 0{,}877 \sqrt{F} = 2{,}47$, and the upper width

$b_1 = 2 \,.\, 2{,}47 = 4{,}94.$

REMARK—1) According to Saint Venant, we can put accurately enough

$$h = 0{,}000401 \frac{p}{F} \,.\, v^{\frac{21}{11}} = 0{,}000401 \,.\, 2g \,.\, v^{-\frac{1}{11}} \,.\, \frac{p}{F} \,.\, \frac{v^2}{2g} \text{ meters};$$

hence the coefficient of resistance is

$$\zeta = 0{,}000401 \,.\, 2g \,.\, v^{-\frac{1}{11}} = 0{,}007887\, v^{-\frac{1}{11}},$$

E.G. for $v = 1$ meter

$$\zeta = 0{,}007887$$

and for $v = \frac{1}{4}$ meter

$$\zeta = 0{,}007887 \sqrt[11]{4} = 0{,}007887 \,.\, 1{,}134 = 0{,}008945.$$

(Compare § 428, Remark 3.)

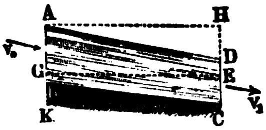
2) A table, which abridges these calculations, is given in the Ingenieur, pages 460 and 461.

§ 477. Variable Motion.—The theory of the *variable motion* of water in channels can be referred to the theory of uniform motion, when we consider the resistance of friction upon a small portion of the length of the river to be constant and put the corresponding head

$$= \zeta . \frac{l p}{F} . \frac{v^2}{2 g}.$$

We must also take into consideration the *vis viva* corresponding to the change of velocity.

FIG. 808.



Let $A\ B\ C\ D$, Fig. 808, be a short portion of the channel of a river, whose length $A\ D = l$, and whose fall $D\ H = h$; let v_0 be the velocity of approach and v_1 that with which the water flows away. If we apply the laws of efflux to an element D at the surface of the water, we have for its velocity v_1

$$\frac{v_1^2}{2 g} = h + \frac{v_0^2}{2 g};$$

but an element E, which is situated below the water, has on one side, it is true, a greater head $A\ G = E\ H$, but it is pressed back by the water below it with a head $D\ E$; hence the effective fall, which produces motion, is only $D\ H = E\ H - E\ D$, and consequently the following formula holds good for any element:

$$h = \frac{v_1^2 - v_0^2}{2 g};$$

if we add the resistance of friction, we obtain

$$h = \frac{v_1^2 - v_0^2}{2 g} + \zeta . \frac{l p}{F} . \frac{v^2}{2 g},$$

in which p, F and v denote the mean values of the wetted perimeter, the transverse section and the velocity. If F_0 denote the area of the upper and F_1 that of the lower transverse section, we can put

$$F = \frac{F_0 + F_1}{2} \text{ and } Q = F_0\, v_0 = F_1\, v_1, \text{ whence}$$

$$\frac{v_1^2 - v_0^2}{2 g} = \frac{1}{2 g}\left[\left(\frac{Q}{F_1}\right)^2 - \left(\frac{Q}{F_0}\right)^2\right] = \left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2 g}, \text{ and}$$

$$\frac{v^2}{F} = \frac{v_0^2 + v_1^2}{F_0 + F_1} = \left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{F_0 + F_1},$$

from which we obtain

$$1)\ h = \left[\frac{1}{F_1^2} - \frac{1}{F_0^2} + \zeta \frac{l p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\right]\frac{Q^2}{2 g}, \text{ and}$$

$$2)\ Q = \frac{\sqrt{2 g h}}{\sqrt{\frac{1}{F_1^2} - \frac{1}{F_0^2} + \zeta \frac{l p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)}}.$$

By the aid of formula 1) we can calculate from the quantity of water carried, the length and transverse sections of a section of river or canal the corresponding fall h, and by the aid of formula 2) from the fall, length and cross-section the quantity of water carried. In order to obtain greater accuracy we should calculate these for several small portions of the channel of the river and then take the arithmetical mean of the results. If the total fall only is known, we must substitute this value instead of h in the last formula and instead of

$$\frac{1}{F_1^2} - \frac{1}{F_0^2}, \ \frac{1}{F_n^2} - \frac{1}{F_0^2},$$

in which F_n denotes the area of the last cross-section, and instead of

$$\zeta \cdot \frac{l p}{F_0 + F_1} \left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right),$$

the sum of all the similar values for the different portions of the channel of the river.

Example.—A creek falls 9,6 inches in 300 feet, the mean value of its wetted perimeter is 40 feet, the area of its upper transverse section is 70 square feet, and that of its lower is 60 square feet. Required the discharge of this brook. Here

$$Q = \frac{8,025 \sqrt{0,8}}{\sqrt{\frac{1}{60^2} - \frac{1}{70^2} + 0,007565 \, . \, \frac{300 \, . \, 40}{130} \left(\frac{1}{60^2} + \frac{1}{70^2}\right)}},$$

$$= \frac{7,178}{\sqrt{0,0000731 + 0,0003365}} = \frac{7,178}{\sqrt{0,0004096}} = 354\tfrac{1}{2} \text{ cubic feet.}$$

The mean velocity is $\frac{2\ Q}{F_0 + F_1} = \frac{709}{130} = 5,45$ feet; hence it is more correct to put $\zeta = 0,00768$ instead of 0,007565,
and therefore we have more accurately

$$Q = \frac{7,178}{\sqrt{0,0000731 + 0,0003416}} = 352,5.$$

If the same stream has at another place the same amount of water in it and falls 11 inches in 450 feet, and if the area of its upper transverse section is 50 and that of its lower 60 feet, the mean length of its wetted perimeter being 36 feet, we have

$$Q = \frac{8,025 \sqrt{0,9167}}{\sqrt{\frac{1}{60^2} - \frac{1}{50^2} + 0,00768 \, . \, \frac{450 \, . \, 36}{110} \left(\frac{1}{60^2} + \frac{1}{50^2}\right)}}$$

$$= 8,025 \sqrt{\frac{0,9167}{-0,0001222 + 0,0007549}} = 305\tfrac{1}{2} \text{ cubic feet.}$$

The mean of the values is

$$Q = \frac{354\tfrac{1}{2} + 305\tfrac{1}{2}}{2} = 330 \text{ cubic feet.}$$

§ 478. In order to obtain the formula for the *depth of the water*, let us put the upper depth $= a_0$ and the lower $= a_1$, the slope of the bed $= a$, and consequently the fall of the bed $= l\ sin.\ a$. The fall of the stream is then

$$h = a_0 - a_1 + l\ sin.\ a;$$

hence we have the equation

$$a_0 - a_1 - \left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g} = \left[\zeta \frac{p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{2g} - sin.\ a\right] l,$$

whence we deduce

$$l = \frac{a_0 - a_1 - \left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g}}{\zeta \frac{p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{2g} - sin.\ a.}$$

By the aid of this formula we can determine the distance l, which corresponds to a given change $a_0 - a_1$ in depth. If the inverse problem is to be solved, we must resort to the method of approximation, I.E., we must calculate first the lengths l_1 and l_2 corresponding to the assumed changes $a_0 - a_1$ and $a_1 - a_2$ of depth, and then we must find by a proportion the change of depth corresponding to the given length l (see Ingenieur, Arithmetic, § 16, V, page 76).

The formula can be simplified, when the width b of the stream is constant. In this case we can put

$$\left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g} = \frac{F_0^2 - F_1^2}{F_1^2 F_0^2} \cdot \frac{Q^2}{2g} = \frac{(F_0 - F_1)(F_0 + F_1)}{F_1^2} \cdot \frac{v_0^2}{2g}$$

$$= \frac{(a_0 - a_1)(a_0 + a_1)}{a_1^2} \cdot \frac{v_0^2}{2g} \text{ approximatively } = 2\,\frac{(a_0 - a_1)}{a_0} \cdot \frac{v_0^2}{2g},$$

and in like manner

$$\frac{p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{2g} = \frac{p\,(F_0^2 + F_1^2)}{(F_0 + F_1)\,F_1^2} \cdot \frac{v_0^2}{2g}$$

approximatively $= \frac{p}{a_0 b} \cdot \frac{v_0^2}{2g}$, from which we obtain

$$l = \frac{(a_0 - a_1)\left(1 - \frac{2}{a_0} \cdot \frac{v_0^2}{2g}\right)}{\zeta \cdot \frac{p}{a_0 b} \cdot \frac{v_0^2}{2g} - sin.\ a},$$

and consequently

$$\frac{a_0 - a_1}{l} = \frac{\zeta \cdot \frac{p}{a_0 b} \cdot \frac{v_0^2}{2g} - sin.\ a}{1 - \frac{2}{a_0} \cdot \frac{v_0^2}{2g}}$$

By the aid of this formula we can obtain directly, for a given distance l, the corresponding change $(a_0 - a_1)$ of depth of the stream.

EXAMPLE.—A horizontal ditch 800 feet long and 5 feet wide is required to convey 20 cubic feet of water per second; the depth of water at the entrance is 2 feet, what will be its depth at the end of the ditch? Let us divide the entire length of the ditch into two equal sections and determine by the last formula the fall for each of them.

Here $\sin. \alpha = 0$, $l = \frac{800}{2} = 400$ and $b = 5$; for the first section, $v_0 = \frac{20}{2\,.\,5} = 2$; hence $\zeta = 0{,}00812$ and $a_0 = 2$; now since $p = 8\tfrac{1}{2}$, it follows that

$$a_0 - a_1 = \left(\frac{0{,}00812\,.\,\frac{8{,}5}{10}\,.\,\frac{4}{2g}}{1 - \frac{2}{2}\,.\,\frac{4}{2g}}\right).\,400 = \frac{2{,}7608}{15{,}1} = 0{,}183 \text{ feet.}$$

Now for the second section $a_1 = 2 - 0{,}183 = 1{,}817$ and p is about $= 8{,}2$, $v_1 = \frac{20}{9{,}085} = 2{,}201$; the fall in the second section will be

$$a_1 - a_2 = \left(\frac{0{,}00812\,.\,\frac{8{,}2}{9{,}085}\,.\,\frac{2{,}201^2}{2\,g}}{1 - \frac{2}{1{,}817}\,.\,\frac{2{,}201^2}{2\,g}}\right).\,400 = \frac{0{,}2205}{0{,}9172} = 0{,}240;$$

hence the entire fall is

$$= 0{,}183 + 0{,}240 = 0{,}423$$

and the depth of water at the lower end is

$$= 2 - 0{,}423 = 1{,}577 \text{ feet} = 18{,}92 \text{ inches.}$$

§ 479. **Floods and Freshets.**—When the *water level* in rivers or canals changes, it is accompanied by changes in the velocity and in the quantity of water carried. A rise of the water level not only increases the cross-section, but also causes a greater velocity and, therefore, for a double reason a greater discharge; in like manner a fall of the water level causes both a diminution of velocity and of cross-section, and consequently a two-fold diminution of the quantity of water. If the original depth $= a$ and the present one $= a_1$ and the upper width of the canal $= b$, we can put the increase of the cross-section $= b\,(a_1 - a)$; hence the cross-section, when the water level has risen a distance $a_1 - a$, is

$$F_1 = F + b\,(a_1 - a) \text{ and consequently}$$

$$\frac{F_1}{F} = 1 + \frac{b\,(a_1 - a)}{F},$$

and we can put approximatively

$$\sqrt{\frac{F_1}{F}} = 1 + \frac{b\,(a_1 - a)}{2\,F}.$$

If the original wetted perimeter $= p$, the present one $= p_1$ and the angle of slope of the banks $= \theta$, we have

$$p_1 = p + \frac{2(a_1 - a)}{sin.\ \theta}, \text{ whence}$$

$$\frac{p_1}{p} = 1 + \frac{2(a_1 - a)}{p\ sin.\ \theta} \text{ and}$$

$$\sqrt{\frac{p_1}{p}} = 1 + \frac{a_1 - a}{p\ sin.\ \theta}, \text{ or}$$

$$\sqrt{\frac{p}{p_1}} = 1 - \frac{a_1 - a}{p\ sin.\ \theta}.$$

Now the velocity for the original depth of water is $c = 92{,}26\sqrt{\frac{F h}{p\, l}}$, and for the present depth it is $c_1 = 92{,}26\sqrt{\frac{F_1}{p_1} \cdot \frac{h}{l}}$; hence

$$\frac{c_1}{c} = \sqrt{\frac{F_1}{F}} \cdot \sqrt{\frac{p}{p_1}} = \left(1 + \frac{b(a_1 - a)}{2F}\right)\left(1 - \frac{a_1 - a}{p\ sin.\ \theta}\right)$$

$$= 1 + (a_1 - a)\left(\frac{b}{2F} - \frac{1}{p\ sin.\ \theta}\right),$$

and the *relative change of velocity is*

$$1)\ \frac{c_1 - c}{c} = (a_1 - a)\left(\frac{b}{2F} - \frac{1}{p\ sin.\ \theta}\right).$$

On the contrary, the ratio of the quantity of water carried by the river is

$$\frac{Q_1}{Q} = \frac{F_1 c_1}{F c} = \left(1 + \frac{b(a_1 - a)}{F}\right)\left[1 + (a_1 - a)\left(\frac{b}{2F} - \frac{1}{p\ sin.\ \theta}\right)\right]$$

$$= 1 + (a_1 - a)\left(\frac{3b}{2F} - \frac{1}{p\ sin.\ \theta}\right),$$

and the relative *increase in the quantity of water* is

$$2)\ \frac{Q_1 - Q}{Q} = (a_1 - a)\left(\frac{3b}{2F} - \frac{1}{p\ sin.\ \theta}\right).$$

We can put less accurately, but in many cases, particularly for wide canals with little slope, sufficiently so, $F = a\,b$ and neglect $\frac{1}{p\ sin.\ \theta}$, in which case we have more simply

$$\frac{c_1 - c}{c} = \frac{1}{2}\,\frac{a_1 - a}{a} \text{ and } \frac{Q_1 - Q}{Q} = \frac{3}{2}\,\frac{a_1 - a}{a}.$$

According to these formulas *the relative change in the velocity is half as great and that in the quantity of water* $\frac{3}{2}$ *as great as the relative change in the depth of the water.*

The foregoing formulas are only applicable, when the motion of the water in its *channel* is permanent, in which case the depth of the water is constant, but they do not hold good when the depth of the water is variable. The mean velocity in the same transverse section is *greater*, when the water is *rising*, and *smaller*, when the water is *falling* than when the depth of the water is constant; hence in the first case more water and in the second case less water passes through than when the motion is permanent.

EXAMPLE—1) If the depth of the water increases $\frac{1}{10}$, the velocity is increased $\frac{1}{20}$ and the quantity of water $\frac{3}{20}$ of its original value.

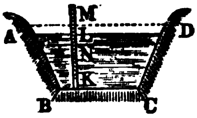
2) If the depth decreases 8 per cent., the velocity is diminished 4 per cent. and the quantity of water 12 per cent.

3) By the aid of the more accurate formula

$$\frac{Q_1 - Q}{Q} = (a_1 - a)\left(\frac{3b}{2F} - \frac{1}{p \sin. \theta}\right),$$

we can construct a water-level scale $K M$, Fig. 809, upon which we can read off the quantity of water passing in a canal for any depth $K L$, when we know the quantity of water for a certain mean depth. If $b = 9$ feet, $b_1 = 3$, $a = 3$, and $\theta = 45°$, we have

FIG. 809.



$$F = \frac{(9 + 3)\ 3}{2} = 18 \text{ square feet,}$$

$$p = 3 + 2\ .\ 3\ .\ \sqrt{2} = 11{,}485, \text{ and}$$

$$\sin. \theta = \sqrt{\tfrac{1}{2}} = 0{,}707; \text{ hence}$$

$$\frac{Q_1 - Q}{Q} = \left(\frac{3\ .\ 9}{2\ .\ 18} - \frac{1}{11{,}485\ .\ 0{,}707}\right)(a_1 - a) = (0{,}750 - 0{,}128)\ (a_1 - a)$$
$$= 0{,}627\ (a_1 - a).$$

If the quantity of water corresponding to the mean water level is $Q =$ 40 cubic feet, we have

$$Q_1 = 40 + 40\ .\ 0{,}627\ (a_1 - a) = 40 + 25\ (a_1 - a).$$

If $a_1 - a = 0{,}04$ feet $= 5{,}76$ lines, $Q_1 = 41$ cu. ft.; if $a_1 - a = 0{,}08$ feet $= 11{,}52$ lines, $Q_1 = 42$ cu. ft.; if $a_1 - a = -0{,}04$ feet, $Q_1 = 39$ cu. ft., etc., a scale, whose divisions are $L M = L N = 5{,}76$ lines apart, would give the quantity of water to a cubic foot. The accuracy of course diminishes as the difference of the depth of water from the mean depth increases.

REMARK.—The construction of mill-races, canals for bringing water, as well as the location of dams, weirs, etc., will be treated of at length in the second volume.

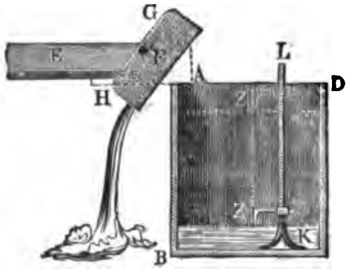
FINAL REMARK.—The author has discussed at length the subject of the motion of water in canals and rivers in the Allgemeine Encyklopädie, Vol. II, Article "Bewegung des Wassers in Canälen und Flüssen," and has given there a list of the treatises upon this subject up to 1844. Rittinger's tabulated synopsis of the experiments upon the motion of water in canals is contained in the "Zeitschrift des österreichischen Ingenieurvereins," year 1855.

CHAPTER VIII.

HYDROMETRY, OR THE THEORY OF MEASURING WATER.

§ **480. Gauging.**—The discharge of a running stream within a certain time is measured either by *gauged vessels*, by a discharging *apparatus*, or by hydrometers. The most simple method is that by means of gauged vessels, but this is only applicable to small quantities of water. The vessel is most frequently composed of boards, and is therefore parallelopipedical in form, and to increase its strength, it is generally bound with iron hoops. The manner of calculating the exact contents of this vessel is given in the Ingenieur, page 208. The water is brought to the vessel by a trough *E F*, Fig. 810, at the end of which is placed a double clack, by means of which the water can be made to flow into the vessel or alongside of it. In order to determine accurately the depth of the body of water, we employ a scale *K L*. If, before we begin the measurement, we lower the pointer *Z* until it touches the surface of the water, which, perhaps, may only cover the bottom, and read off on the scale the depth of the water, we obtain the depth $Z\ Z_1$ of the water to be measured by subtracting the former reading from that given by the scale, when the pointer *Z*, after the completion of the observation, is brought into contact with the top of the water. The clack must of course be so arranged before the experiment that water shall discharge alongside of the vessel. If we are satisfied that the influx into the trough has become constant, we observe the time upon a watch held in the hand and turn the clack around so that the water will discharge into the vessel; when the vessel is full, or partially so, we observe again upon the watch the time and return the clack to its original position. From the mean cross-section *F* and the depth $Z\ Z_1 = s$ of the body of water, we calculate the total discharge $V = F\ s$, which,

FIG. 810.

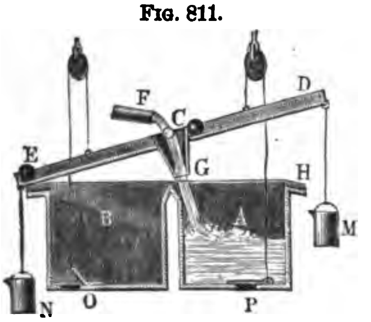


when divided by the duration of the influx, which is the difference t of the two times observed upon the watch, gives the discharge per second

$$Q = \frac{Fs}{t}.$$

REMARK.—If we wish to know at any time the discharge of a variable stream of water, we can employ the apparatus represented in Fig. 811, which is often met with in salt works. Here there are two measuring vessels A and B, which are alternately filled and emptied. The water, which is brought to them by the pipe F, passes through a short tube $C\ G$, which is rigidly connected with the lever $D\ E$ which turns about C. If one of the vessels, as, E.G., A, is filled, the water flows through a small trough H and fills the little bucket M, which then draws the lever down and the pipe $C\ G$ comes into such a position as to carry the water into B. The valves O and P are opened by cords passing around pulleys and attached to the lever. The opening of the valves is assisted by iron balls, which give the last impulse to the lever when it is descending. The buckets M and N contain small orifices, through which they empty themselves after the lever has turned. A counter attached to the apparatus gives the number of strokes, which can be read off at any time. Other apparatuses of the same sort, which were employed by Brown, are described in Dingler's Polyt. Journal, Vol. 115. In reference to a new apparatus for measuring water by Noeggerath, see Polyt. Centralblatt, 1856, No. 5. Compare also the works of Francis, Lesbros, etc., which have been cited. See also further on, § 506.

FIG. 811.



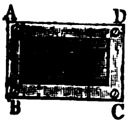
§ **481. Regulators of Efflux.**—Very often small and medium discharges are measured by causing them to pass through *a known orifice* under a known head. From the area F of the orifice and the head h we determine, by the aid of the coefficient of efflux, the discharge per second

$$Q = \mu\, F \sqrt{2 g h}.$$

Poncelet's orifices are the best for this purpose; for their coefficients of efflux are known (§ 410) with great accuracy for different heads, but they are only applicable, when the discharge is a medium one. The author employs in his measurements of water four such orifices, one 5, one 10, one 15 and one 20 centimeters high and all

20 centimeters wide. These orifices are cut out of brass plates, which are placed upon wooden frames such as *A C*, Fig. 812, and these frames can be fastened to any wall by means of four strong iron screws. But in many cases we must employ much larger orifices for which the coefficients of efflux have not been determined so certainly; and very often we can only employ overfalls or notches, which are even less accurate. But in any case we should endeavor to produce both perfect and complete contraction. Hence, if the orifice is in a thick wall, we should bevel it off upon the outside. The corrections to be applied for partial and incomplete contraction have been sufficiently explained in §§ 416, 417.

FIG. 812.



In order to measure the quantity of water in a trough, we first put the mouth-piece in its place and then wait until the head becomes permanent. In order to measure the head, we can employ either the fixed scale *K L* with a pointer, Fig. 813, or the movable one *E F*, Fig. 814. If we wish to observe the efflux directly at the sluice gate, it is advisable to attach to the guides two brass *scales B C* and *D E*, Fig. 815, with the pointers *F* and *G* by means of which we are able to read off with more certainty the height of the orifice. It is always better, when measuring water, to employ a new sluice gate and new guides which are properly beveled outwards.

FIG. 813.

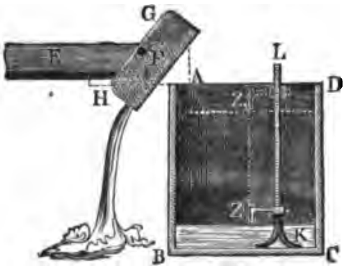


FIG. 814.

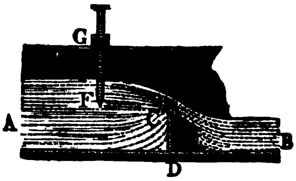
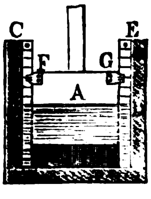


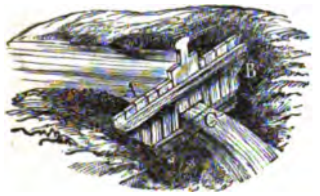
FIG. 815.



The most simple way of measuring the water in a trough is to place a *board C F*, Fig. 701, *beveled at the top*, across it and to measure the overfall which is produced. If the ditch or trough is long and nearly horizontal, considerable time will elapse before the flow becomes permanent, and it is, therefore, advisable before beginning the measurement to put in another board, which will prevent for some time the efflux of the water and thus hasten its rise to the height necessary for a permanent flow.

In order to measure the discharge of a creek, we can construct a dam *A B*, Fig. 816, of boards and allow the water to flow through an opening *C* in it, or we can employ a simple overfall or weir (this subject will be treated more at length in the second volume).

FIG. 816.



REMARK.—The most simple method of determining the head is to observe the position of the pointer, first, when its point touches the surface of the water, while the flow is permanent, and secondly, when it touches the surface of the still water which is on a level with the top of the sill. The difference of the two observed heights is the head of water or the height of the water above the sill. We must be careful in observing the last height of the pointer to pay attention to the action of the capillary attraction, in consequence of which the level of the water may be 1,87 lines above or below the sill, before efflux over the latter will begin (see § 380).

§ **482.** We can easily measure the quantity of water carried by a canal or trough *A B*, Figs. 817 and 818, in the following manner: a board, ***the lower edge of which has been beveled***, is inserted in the trough in such a manner as to leave an opening between it and the bottom of the latter, through which the water will pass. This method has an advantage over that in which overfalls are employed, viz.: the water, which is dammed back, comes to rest better, and we can, therefore, measure the head more accurately. When it is possible to have a *free efflux*, as in Fig. 817, we should prefer it, since greater accuracy can be obtained, but when the

FIG. 817.

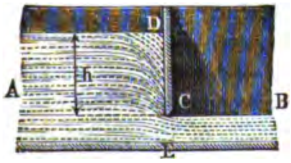
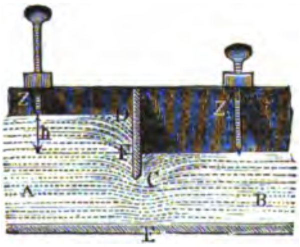


FIG. 818.



quantities of water are large, it is not possible to prevent the water from rising, and we are obliged to be satisfied with an efflux under water, such as is represented in Fig. 818. If the trough ends but a short distance from the orifice, I.E., if it forms a *shoot*, the water flows through it almost freely and we have one of the cases of Lesbros' experiments (§ 418). If a denote the height and b the width of the orifice, h the head measured to the middle of the orifice and μ the coefficient of efflux, taken from Table II, § 418, we have the discharge

$$Q = \mu a b \sqrt{2 g h}.$$

If, on the contrary, the trough is long, or if the water, which is flowing away, is so obstructed that its surface becomes horizontal, the water will pass all portions of the cross-section of the orifice with the same velocity, which is that corresponding to a head equal to the difference of level of the water A above and the water B below the orifice, and we have only to substitute in the latter formula for h the difference of level.

If the water flows into the air, or if the surface of the lower water, as in Fig. 817, does not rise above the upper edge of the orifice, we must substitute for an orifice with beveled or with rounded edges

$$\mu = 0{,}965,$$

and consequently, when the depth of the stream is a and its width b,

$$Q = 0{,}965\, a b \sqrt{2 g h},$$

or more accurately, when a_1 is the depth of the approaching water and a that of the water flowing away (see § 398),

$$Q = 0{,}965\, a b \sqrt{\frac{2 g h}{1 - \left(\frac{a}{a_1}\right)^2}}.$$

When the *efflux takes place under water*, in which case the lower surface of the water is above the upper edge of the orifice (see Fig. 818), an eddy is formed behind the wall of the orifice, by which the efflux is considerably interfered with. According to several experiments of the author, for an orifice with a *sharp edge* we must put, as a mean value, $\mu = 0{,}462$,

and, on the contrary, when the edge is *rounded off in the shape of a quadrant*, $\mu = 0{,}717$.

EXAMPLE.—In order to find the discharge of a trough $A\,B$, Fig. 818, a sharp-edged board $C\,D$ was placed in it and an efflux under water was thus produced; the following observations were then made. Width of orifice or trough $b = 3$ feet, height of orifice or distance $C\,E$ of the edge C

of the board above the bottom of the trough $a = 6$ inches, length of the pointer Z above the orifice $h_1 = 0{,}445$ feet, length of the pointer Z_1 below the orifice $h_2 = 1{,}078$. Hence the difference of level is

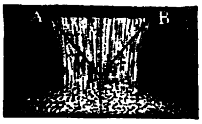
$$h = h_2 - h_1 = 1{,}078 - 0{,}445 = 0{,}628 \text{ feet,}$$

and the required discharge is

$$Q = 0{,}46\dot{2} \,.\, 8{,}025 \,.\, 3 \,.\, 0{,}5 \sqrt{h_2 - h_1} = 5{,}56 \sqrt{0{,}628} = 4{,}40 \text{ cubic feet.}$$

§ 483. If the coefficient of efflux were always the same for similar cross-sections, the triangular or two-sided notch $A\ B\ C$, Fig. 819, would have a great advantage over the notch with a horizontal sill; but this assumption, as we have seen in the case of circular apertures, is not correct for small orifices, and only approximatively so for large ones. Professor Thomson, of Belfast, recommends such notches as useful for measuring water. From the width $A\ B = b$ and the height $C\ D = h$, we obtain the discharge

Fig. 819.



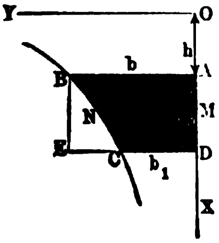
$$Q = \tfrac{8}{15} \frac{\mu\, b\, h}{2} \sqrt{2\, g\, h} \text{ (see § 402),}$$

and if we put, with Prof. Thomson, the coefficient of efflux $\mu = 0{,}619$,

$$Q = 0{,}33 \frac{b\, h}{2} \sqrt{2\, g\, h} = 0{,}132\, b\, h^{\frac{5}{2}} \text{ cubic feet.}$$

Orifices, so shaped that the discharge through them shall be *proportional to their height*, are useful in measuring water. If they are provided with a sluice-gate the height of the opening is the measure of the discharge. Let the head above the upper edge of such an orifice $A\ B\ C\ D$, Fig. 820, be $O\ A = h$, the length of this edge be $A\ B = b$, that of the lower edge, $C\ D = b_1$, and the height of the orifice, $A\ D = a$. Horizontal lines at the distance $\frac{a}{n}$ from each other will divide this orifice into strips of equal height, and the discharge $\frac{Q}{n}$ through each of them should be the same. For the upper strip, whose width is b and for which the head is h, we have

Fig. 820.



$$\frac{Q}{n} = \frac{b\, a}{n} \sqrt{2\, g\, h},$$

and, on the contrary, for another strip at a distance $O\ M = x$ below the surface of the water, whose width $M\ N = y$,

$$\frac{Q}{n} = \frac{y\,a}{n}\sqrt{2\,g\,x};$$

equating these two values of $\frac{Q}{n}$, we obtain

$$y\sqrt{x} = b\sqrt{h}, \text{ or}$$

$$\frac{y}{b} = \sqrt{\frac{h}{x}}.$$

The curve $B\ N\ C$, which bounds the orifice on the side, belongs to one of the system of curves discussed in Article 9 of the Introduction to the Calculus; its asymptotes are the horizontal line $O\ Y$ and the vertical one $O\ X$.

From Q, h and a we obtain

1) the upper width of the orifice $b = \frac{Q}{a\sqrt{2\,g\,h}}$,

2) the width of orifice at the depth x, $y = b\sqrt{\frac{h}{x}}$,

3) the lower width of the orifice $b_1 = b\sqrt{\frac{h}{h+a}}$.

The area of the orifice is

$$F = 2\,b\left(\sqrt{h\,(h+a)} - h\right),$$

and consequently the mean head is

$$z = \frac{1}{2\,g}\left(\frac{Q}{F}\right)^2 = \left(\frac{a}{\sqrt{h\,(h+a)} - h}\right)^2 \cdot \frac{h}{2}.$$

If the orifice is provided with a sliding gate, when it is raised a distance $D\ M = a_1$, it gives an orifice of efflux $M\ C$, the discharge through which is $Q_1 = \frac{a_1}{a}\,Q$.

§ **484. Prony's Method.**—As considerable time often elapses before the flow of the water, which has been dammed back, becomes permanent, the following method, *proposed by Prony*, can often be employed with advantage. We begin by closing the orifice completely by means of a sluice-gate, and we wait until the water has risen to a certain height, or as high as circumstances will permit; we then open the gate enough to allow more water to be discharged than is arriving, and we measure the height of the water at equal intervals of time, which should be as small as possible; finally, we close the orifice again perfectly and observe the time t_1 in which the water rises to its former height. Now during the lapse of the time $t + t_1$ the same quantity of water has of course arrived and been discharged; hence the quantity of water which arrives in the time $t + t_1$ is equal to the discharge in the time t. If the heads, while the level of the water was sinking, were h_0, h_1, h_2, h_3, and h_4, we have the mean velocity

$$v = \frac{\sqrt{2g}}{12}(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}) \text{ (see § 453)},$$

and if the area of the opening of the sluice is F, the discharge in the time t is

$$V = \frac{\mu F t \sqrt{2g}}{12}(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4});$$

hence the *quantity of water* arriving in a second is

$$Q = \frac{V}{t + t_1} = \frac{\mu F t \sqrt{2g}}{12(t + t_1)}(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}).$$

EXAMPLE.—In order to measure the quantity of water in a brook, which we wish to employ to turn a water-wheel, the stream was dammed up by a wall of boards, as is represented in Fig. 816, and after opening the rectangular orifice in it, we made the following observations: initial head, 2 feet; after 30″, 1,8 feet; after 60″, 1,55 feet; after 90″, 1,3 feet; after 120″, 1,15 feet; after 150,″, 1,05 feet; and after 180″, 0,9 feet; width of the orifice = 2 fect, height = ½ foot, time required for the water to rise to former level 110″. In the first place the mean velocity is

$$\begin{aligned} v &= \frac{8{,}025}{18}(\sqrt{2} + 4\sqrt{1{,}8} + 2\sqrt{1{,}55} + 4\sqrt{1{,}3} + 2\sqrt{1{,}15} + 4\sqrt{1{,}05} + \sqrt{0{,}9}) \\ &= 0{,}4458\,(1{,}414 + 5{,}364 + 2{,}490 + 4{,}561 + 2{,}145 + 4{,}099 + 0{,}949) \\ &= 0{,}4458 \,.\, 21{,}022 = 9{,}372 \text{ feet.} \end{aligned}$$

But $F = 2 \,.\, \frac{1}{2} = 1$ square foot, the theoretical discharge is, therefore, = 9,372 cubic feet. If we assume that the coefficient of efflux = 0,61, we obtain the required quantity of water

$$Q = \frac{0{,}61 \,.\, 180}{180 + 110} \,.\, 9{,}372 = 3{,}548 \text{ cubic feet.}$$

§ **485**. **Water-inch.**—When we are required to *measure small quantities of water*, we often allow it to discharge under a given head through circular orifices in a thin plate, which are one inch in diameter. We call the discharge through such an orifice, under the smallest pressure, I.E. when the surface of the water is one line above the uppermost part of the orifice, a *water-inch* (Fr. pouce d'eau; Ger. Wasserzoll or Brunnenzoll). The French assume that a water-inch (old Paris measure) corresponds to a discharge in 24 hours of 19,1953 cubic meters, or

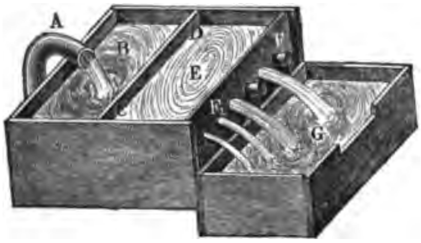
in 1 hour, 0,7998 cubic meters, and
in 1 minute, 0,01333 cubic meters;

but the older data, given by Mariotte, Couplet, and Bossut, differ considerably from the above. According to Hagen, the water-inch (for Prussian measure) discharges 520 cubic feet in 24 hours, or 0,3611 cubic feet in a minute. Prony's double *water modulus* (or "nouveau pouce d'eau"), which corresponds to an orifice 2 centimeters in diameter, under a pressure of 5 centimeters, and which discharges 20 cubic meters in 24 hours, has not been adopted generally.

The observations can be made with more certainty when we have a greater head; it is simpler to make this head equal to the diameter 1 inch of the orifice. According to Bornemann and Röting, such a *water-inch* passes daily 642,8 cubic feet (Prussian) of water (see the Ingenieur, page 463).

The apparatus, by which we measure the water with the aid of the water-inch, is represented in Fig. 821. The water to be measured is discharged from the pipe A into a box B, from which it passes through holes, made in the partition C D below the level of the water, into the box E; from it the water is discharged through circular orifices F one inch in diameter, which are cut out of sheet iron, into the reservoir G. To preserve the level of the water 1 line above the top of the orifice we must have a sufficient number of holes, a portion of which are closed by stoppers. We employ for more accurate determinations in addition the orifice F_1 which allows $\frac{1}{2}$, $\frac{1}{4}$ of a water-inch to pass through. When the quantity of water is very great, we divide it into several portions and measure in this way but one portion, as, E.G., a tenth. This division is easily accomplished by conducting the water into a reservoir with a certain number of orifices on the same level and catching the water delivered from one of the orifices only in the above apparatus.

FIG. 821.



REMARK.—We can also employ cocks and other regulating apparatuses for measuring water, when we know the coefficients of resistance corresponding to every position. If h is the head, F the cross-section of the pipe and μ the coefficient of efflux for the cock, when fully open, we have the discharge

$$Q = \mu F \sqrt{2 g h},$$

or inversely

$$\mu = \frac{Q}{F \sqrt{2 g h}} \text{ and } \frac{1}{\mu^2} = \left(\frac{F}{Q}\right)^2 . 2 g h.$$

Now if we put the coefficient of resistance for a certain position of the cock, which may be taken from one of the tables given previously, $= \zeta$, we have the corresponding discharge

$$Q_1 = F \sqrt{\frac{2\,g\,h}{\frac{1}{\mu^2} + \zeta}} = \frac{\mu\,F\sqrt{2\,g\,h}}{\sqrt{1 + \mu^2\,\zeta}} = \frac{Q}{\sqrt{1 + \mu^2\,\zeta}}$$

$$= \frac{Q}{\sqrt{1 + \zeta\left(\frac{Q}{F}\right)^2 \frac{1}{2\,g\,h}}}$$

We can construct from the above formula a convenient table, and we have only to glance at it when we wish to know the discharge corresponding to a certain position of the cock. If, E.G., $\mu = 0{,}7$ and $F_1 = 4$ square inches, we have

$$Q_1 = \frac{0{,}7\,.\,4\,.\,12\,.\,8{,}025\,\sqrt{h}}{\sqrt{1 + 0{,}49\,\zeta}} = 269{,}64 \sqrt{\frac{h}{1 + 0{,}49\,\zeta}} \text{ cubic inches,}$$

or, if h is constant and $= 1$ foot,

$$Q = \frac{269{,}64}{\sqrt{1 + 0{,}49\,\zeta}}.$$

Now if the cock is turned 5°, 10°, 15°, 20°, 25°, etc., the coefficients of resistance are 0,057, 0,293, 0,758, 1,559, 3,095, and the corresponding discharges are 266, 252,1, 230,2, 203, 170 cubic inches.

§ **486**. In order to regulate the efflux through an orifice F, Fig. 822, we employ either a cock or valve A, Fig. 822, which is

FIG. 822.

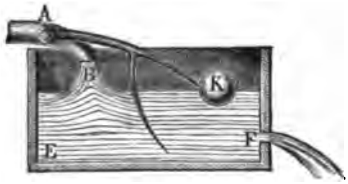
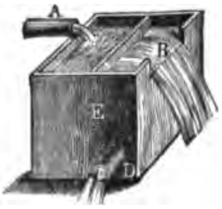


FIG. 823.



regulated by means of a lever and a float K, so that the same quantity of water is discharged through B as through F.

The discharge of water from a reservoir $B\,D\,E$, Fig. 823, through a lower orifice or tube D can be regulated by means of a wide overfall B, since a moderate change in the quantity of water, discharged through A, will produce but a slight change in the height of the water above the sill B; hence the augmentation of the head of the orifice of efflux will be inconsiderable.

Let F denote the area of the orifice D, h the height of the sill of the overfall above the middle of the orifice and h_1 the height of the surface of the water above the same sill. We have the discharge through D

$$Q = \mu F \sqrt{2 g (h + h_1)},$$

when the coefficient of efflux is μ. Substituting the head h_1 above the weir, which can be determined from the discharge Q_1, the width b_1 and the coefficient of efflux μ_1 by means of the equation

$$Q_1 = \tfrac{2}{3} \mu_1 b_1 \sqrt{2 g h_1^3}, \text{ or by the formula}$$

$$h_1 = \left[\frac{1}{2 g} \left(\frac{\frac{3}{2} Q_1}{\mu_1 b_1}\right)^2\right]^{\frac{1}{3}},$$

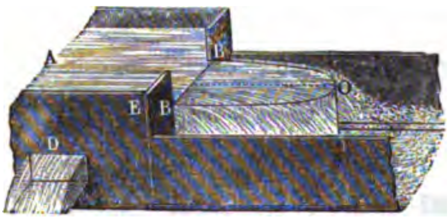
we obtain the expression

$$Q = \mu F \sqrt{2 g h + \left(\frac{3 g Q_1}{\mu_1 b_1}\right)^{\frac{2}{3}}},$$

from which it is easy to see that Q varies less with Q_1, the greater the value of h is and the greater the width b_1 of the overfall is.

The width b of the overfall can be easily increased by giving it a curved form like $B\ O\ B$, Fig. 824. The discharge through the orifice D is then almost constant, although the quantity of water flowing in may be very variable; for the height of the water above the long curved sill is always small compared with the height of this sill above the orifice of efflux.

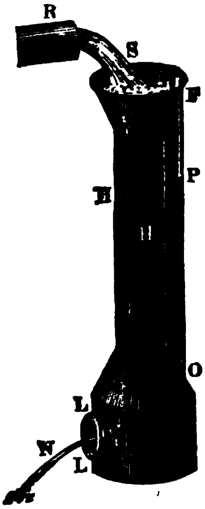
FIG. 824.



REMARK.—Such an apparatus for dividing the water was constructed of sheet iron for the *Wernergraben* at Freiberg by *Oberkunstmeister* Schwamkrug. It discharges through a rectangular orifice D, which is 5 feet long and 1 foot high, almost invariably 40 cubic feet of water per second, while the remaining water passes over the overfall, the sill of which lies 2 feet above the upper edge of the orifice, and flows on in the ditch to where it is wanted.

§ 487. **Hydrometric Goblet.**—We can employ to measure small quantities of running water the small vessel, represented in Fig. 825, which I have called the *hydrometric goblet*. This instrument consists of a tube B, 12 inches long and 3 inches in diameter with a funnel-shaped mouth-piece A, and of a vessel D, 6 inches high and 6 inches wide, which is united to B by an intermediate conical piece *C*. This vessel has an orifice *L L* in the side, in which we can insert mouth-pieces containing different sized circular orifices in a thin plate. The instrument is held by means of the handle *H* under the water *S*, which is being discharged, E.G., from the pipe *R*, and the water thus caught is allowed to discharge itself through the orifices *L L*. In order to tranquilize the water in the vessel a fine sieve or wire gauze is placed in the reservoir *D*, and in order to observe the head of the water a glass tube *O P*, which is placed close to a brass scale and ends a half an inch from the bottom of the vessel, is added to it. From the observed head, the known cross-section of the orifice and the corresponding coefficient of efflux, we can calculate the discharge by means of the formula

FIG. 825.



$$Q = \mu F \sqrt{2 g h}.$$

If we prepare a small table, we can spare ourselves this calculation and the only operation, which we are required to perform, is a simple interpolation between the values in the table. If *d* is the diameter of the orifice,

$$F = \frac{\pi d^2}{4}, \text{ and therefore}$$

$$Q = \frac{\mu \pi}{4} d^2 \sqrt{2 g h} = \frac{\mu \pi}{4} \sqrt{2 g} . d^2 \sqrt{h}.$$

The discharge Q is double, when the cross-section or d^2 is double, or when the head is four times as great. If we so arrange the instrument that the maximum head shall be four times the minimum; if, E.G., the former is 12 and the latter 3 inches, and if we employ a series of orifices whose diameters form the geometrical series

$d, \sqrt{2}\, d, 2\, d, 2 \sqrt{2}\, d, 4\, d,$ etc.
I.E. $d, 1{,}414\, d, 2\, d, 2{,}828\, d, 4\, d,$ etc.,

we obtain a means of measuring all discharges from the minimum given by the smallest orifice with the diameter *d* under the smallest head, to the maximum, given by the largest orifice with the diameter $\sqrt{n}\, .\, d$ under the greatest head $4\, h$.

If we assume for

	I.	II.	III.	IV.	V.	VI.	VII.
$d =$	$\frac{1}{8}$ $= 0,1250$	$\frac{1}{8}\sqrt{2}$ $= 0,1768$	$\frac{1}{4}$ $= 0,2500$	$\frac{1}{4}\sqrt{2}$ $= 0,3535$	$\frac{1}{2}$ $= 0,5000$	$\frac{1}{2}\sqrt{2}$ $= 0,7071$	1 inch $= 1,0000$
$\mu =$	0,690	0,675	0,660	0,647	0,635	0,627	0,620

we can calculate the following useful table.

Table of the hourly discharge in cubic feet for the following orifices.

Head h in inches.	I.	II.	III.	IV.	V.	VI.	VII.
3	0,85	1,66	3,25	6,37	12,51	24,70	48,85
4	0,98	1,92	3,75	7,36	14,44	28,52	56,40
5	1,10	2,14	4,19	8,23	16,15	31,89	63,06
6	1,20	2,35	4,60	9,01	17,69	34,93	69,08
7	1,30	2,54	4,96	9,73	19,10	37,73	74,61
8	1,39	2,72	5,31	10,41	20,42	40,33	79,77
9	1,47	2,88	5,63	11,04	21,66	42,78	84,60
10	1,55	3,03	5,93	11,65	22,84	45,09	89,18
11	1,63	3,18	6,22	12,20	23,95	47,30	93,53
12	1,70	3,32	6,50	12,74	25,01	49,40	97,69
13	1,77	3,46	6,77	13,26	26,04	51,42	101,68

The manner of using this table is shown by the following example.

EXAMPLE.—In order to determine the quantity of water furnished by a spring, the water from it was caught in a hydrometric goblet, and it was found that a state of permanency occurred when the efflux took place through the orifice V (one half inch in diameter) under a head of 10,4 inches. According to the table for $h = 10$ inches

$$Q = 22,84 \text{ cubic feet per hour,}$$

and for $h = 11$ inches

$$Q = 23,95 \text{ cubic feet,}$$

consequently the difference for one inch is 1,11 cubic feet, and for 0,4 inches

$0,4 \,.\, 1,11 = 0,444$. Hence the discharge under the head $h = 10,4$ inches is

$$Q = 22,84 + 0,444 = 28,284 \text{ cubic feet.}$$

§ 488. **Floating Bodies.**—The *discharge* of *large creeks, canals and rivers* can only be measured by means of *hydrometers*, which indicate the velocity. The simplest of these instruments are floating bodies (Fr. flotteurs; Ger. Schwimmer). We can use any floating body for this purpose, but it is safer to employ bodies of medium size and of but little less specific gravity than the water itself. Bodies whose volumes are about $\frac{1}{10}$ of a foot are quite large enough. Very large bodies do not easily assume the velocity of the water, and very small bodies, particularly when they project much above the level of the water, are easily disturbed in their motion by accidental circumstances, such as the wind, etc. A simple piece of wood is often employed, but it is better to cover the wood with a light-colored paint; hollow floats, such as glass bottles, sheet-iron balls, etc., are better; for we can fill them partially with water. Floating balls are, however, most generally employed. They are made of sheet brass and are from 4 to 12 inches in diameter; to prevent their being lost sight of, they are covered with a coat of light-colored oil paint. Such a *floating ball A*, Fig. 826, gives the velocity at the surface only, and often only that in the axis of the stream. By uniting two balls A and B, we can find also the velocity at different depths. In this case one ball, which is to be submerged, is filled with water, and the other contains enough to prevent more than a small portion of it from projecting above the level of the water. The two balls are united by a string, wire or thin wire chain. We first determine by a single ball the superficial velocity c_0, and we then determine the mean velocity c of the two connected balls; now if we denote the velocity at the depth of the second ball by c_1, we can put

FIG. 826.

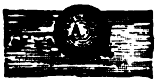
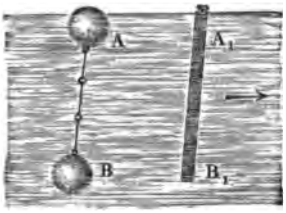


FIG. 827.



$$c = \frac{c_0 + c_1}{2}, \text{ and, therefore, inversely, } c_1 = 2\,c - c_0.$$

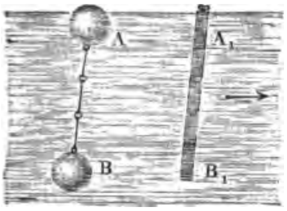
If we unite the balls successively by longer and longer pieces of wire, we obtain in this way the velocities at greater and greater depths. The mean velocity of a perpendicular is determined by allowing the second ball to swim near the bottom and putting

$$c = \frac{c_0 + c_1}{2};$$

it is more accurate, however, to take the mean of all the observed velocities in the perpendicular as the mean velocity.

To obtain the mean velocity in a perpendicular, a floating staff $A_1 B_1$, represented in Fig. 828, is often employed, and it is very convenient, when it is used for measurements in canals and ditches, to have it made of short pieces which can be screwed together. The one used by the author is composed of 15 hollow pieces, each one decimeter long. In order to make it float nearly perpendicularly, the lower part is filled with enough shot to prevent more than the head from projecting above the water. The number of pieces to be screwed together depends, of course, upon the depth of the canal.

FIG. 828.



We observe, when using the floating staff and the connected balls, that, when the movement of water in channels is not impeded, the velocity at the surface is greater than that at the bottom; for the top of the staff and the uppermost ball are always in advance. It is only when the channel is contracted, as, E.G., by piers of bridges, that the opposite phenomenon is observed.

REMARK.--Generally, and particularly with large floating bodies such as ships, etc, the velocity of the floating body is somewhat greater than that of the water; this is owing less to the fact that the body, in floating, slides down an inclined plane formed by the surface of the water, than to the fact that it does not participate, or at least only partially so, in the irregular internal motion of the water; this variation is, however, so slight, when the floating bodies are small, as to be negligible.

§ 489. **Determination of the Velocity and of the Cross-section.**—We find the velocity of a floating ball by observing by means of a good watch with a second-hand or by means of a half-second pendulum (§ 327) the time t, in which it describes the distance $A\ B = s$, Fig. 829, which has been previously measured and staked off on the shore. The required velocity of the sphere is then $c = \frac{s}{t}$. In order that the time t shall correspond exactly to the distance measured on the shore, it is necessary to put two rods *C* and *D*, by means of a suitable instrument, in such a position upon the other side of the river that the lines *C A* and *D B* shall be perpendicular to *A B*. Placing ourselves behind *A*, we note the instant the float *K*, which has been placed in the water some distance above, arrives at the line *A C*, and then passing behind *B*, we observe upon the watch the instant that the float arrives at the line *B D*; by subtracting the time of the first observation from that of the second, we obtain the time *t*, in which the space *s* is described. In order to determine the discharge $Q = F c$, we must know, besides the mean velocity *c*, the area *F* of the cross-section. To find this area, it is necessary to know the width and the mean depth of the water. The depth is measured by a graduated *sounding-rod* *A B*, Fig. 830, the cross-section of which is elongated and the foot of which is formed by board; when the depth is great, we can make use of a *sounding-chain*, to the end of which an iron plate is attached, which, when the measurement is being made, lies upon the bottom. The width and the abscissas or distances from the shore corresponding to the depths measured are easily found for canals and small creeks *E F G*, Fig. 831, by stretching a measuring chain *A B* or laying a rod, etc., across the stream. When the river is wide, we make use of a plane-table *M*, placed at a proper distance *A O* from the cross-section *E F*, Fig. 852, to be measured. If *a o* upon the plane-table is the reduced distance *A O* of the fixed points *A* and *O* from each other, and if we have placed *a o* in the direction *A O*, and thus made the direction *a f* of the width, which had been drawn, previously to putting the plane-table in position, parallel to the line *A F* to be measured off, each line of sight towards the points *E*, *F*, *G*, etc., in the transverse profile cuts off upon the table the corresponding points *e*, *f*, *g*, and

FIG. 829.

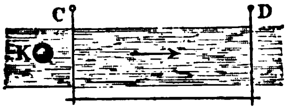


FIG. 830.

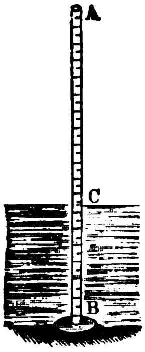
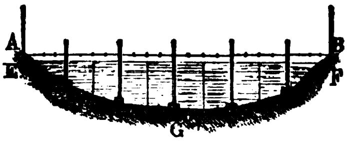
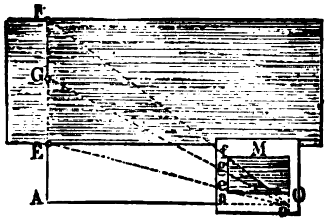


FIG. 831.



a e, *a f*, *a g*, etc., are the distances *A E*, *A F*, *A G*, etc., upon the reduced scale. When using the sounding-rod to measure the depth, it is, therefore, not necessary to measure the distance of the corresponding points from the shore; for the engineer, who is at the plane-table, can sight at the sounding-rod, when it is placed in the line *E F*.

FIG. 832.



Now if the width *E F*, Fig. 831, of a transverse profile is made up of the portions b_1, b_2, b_3, etc., and if the mean depths of these portions are a_1, a_2, a_3, and the mean velocities c_1, c_2, c_3, etc., we have the area of the cross-section

$$F = a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots,$$

the discharge

$$Q = a_1 b_1 c_1 + a_2 b_2 c_2 + a_3 b_3 c_3 + \ldots,$$

and finally the mean velocity

$$c = \frac{Q}{F} = \frac{a_1 b_1 c_1 + a_2 b_2 c_2 + \ldots}{a_1 b_1 + a_2 b_2 + \ldots}.$$

EXAMPLE.—Upon a pretty straight and constant portion of a river the following observations were made:

	Feet.	Feet.	Feet.	Feet.	Feet.
At the centre of the divisions of the width	5	12	20	15	7
the depths were	3	6	11	8	4
the mean velocities were	1,9	2,3	2,8	2,4	2,1

The area of the cross-section is

$$F = 5 . 3 + 12 . 6 + 20 . 11 + 15 . 8 + 7 . 4 = 455 \text{ square feet,}$$

the discharge is

$$Q = 15 . 1,9 + 72 . 2,3 + 220 . 2,8 + 120 . 2,4 + 28 . 2,1 = 1156,9 \text{ cubic feet,}$$

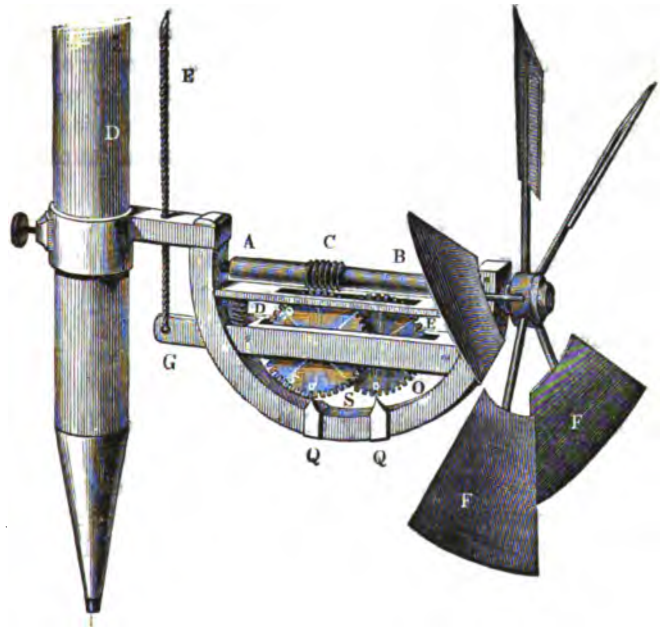
and the mean velocity is

$$c = \frac{1156,9}{455} = 2,54 \text{ feet.}$$

§ 490. Woltmann's Mill or Tachometer.—The best hydrometer is *Woltmann's tachometer* or *Woltmann's Mill* (Fr. Moulinet de Woltmann; Ger. hydrometrisches Flügelrad von Woltmann), Fig. 833. It consists of a horizontal shaft *A B* with from 2 to 5 surfaces or *vanes* *F*, inclined to the direction of the axis; when immersed in water and held opposite to the direction of motion, it indicates by the number of its revolutions the velocity of the running water. To enable us to count the number of revolutions the shaft has cut upon it a certain number of threads of an endless screw *C*, which work into the teeth of a cog-wheel *D*, which indicates, by means of a pointer and figures engraved upon the wheel, the number of revolutions of the wheel *F*. As we often wish to register a great number of revolutions the shaft of the cog-wheel carries a pinion, which takes into another cog-wheel *E*, upon which we can read off, as upon the hour-hand of a watch, multiples (E.G., five or tenfold) of the number of revolutions of the vanes. If, for example, both cog-wheels have 50 teeth and the pinion has 10, the second wheel will turn one tooth, while the first moves five, or the shaft of the vane wheel makes five turns. When the pointer of the first wheel is at $27 = 25 + 2$ and that of the second at 32, the corresponding number of revolutions of the vane-wheel is

$$= 32\,.\,5 + 2 = 162.$$

FIG. 833.



The entire instrument with a sheet iron vane is screwed to a pole, so that it may easily be immersed and held in the water. In order to prevent the gearing from turning except during the time of the observation, its shafts run in bearings placed upon a lever $G\ O$, which is pressed down by means of a spring, so that the teeth of the first cog-wheel do not take into the endless screw except when the string $G\ E$ is drawn upwards. The number of revolutions in a given time is not exactly proportional to the velocity of the water; hence we cannot put $v = \alpha \cdot u$, in which u is the number of revolutions, v the velocity and α an empirical number, but we must put

$$v = v_0 + \alpha u,$$

or more accurately

$$v = v_0 + \alpha u + \beta u^2 \ldots,$$

or still more accurately

$$v = \alpha u + \sqrt{v_0^2 + \beta u^2},$$

in which v_0 denotes the velocity of the water, when it ceases to move the vanes, and α and β are numbers to be determined by experiment. The constants v_0, α, β must be determined for each particular instrument. By their aid a single observation gives the velocity, but it is always safer to make at least two and then take their mean value as the true one.

EXAMPLE.—If for a tachometer $v_0 = 0{,}110$ feet, $\alpha = 0{,}480$ and $\beta = 0$, then $v = 0{,}11 + 0{,}48\ u$, and if we have found the number of revolutions of the fan to be 210 in 80 seconds, the corresponding velocity of the water is

$$v = 0{,}11 + 0{,}48 \cdot \frac{210}{80} = 0{,}11 + 1{,}26 = 1{,}37 \text{ feet.}$$

REMARK 1.—The constants v_0, α and β depend principally upon the angle of impact, I.E., upon the angle formed by the surface of the vanes with direction of the motion of the water and also with the direction of the axis of the wheel. If we wish to make, when the velocities are small, pretty accurate observations, it is advisable to make the angle of impact large, I.E., about 70°. It is also desirable to have vane-wheels of different sizes and of different angles of impact, so that when the depth or velocity of the water is greater or smaller we can employ one or the other.

REMARK 2.—If the tachometer had no resistance to overcome in turning, the vanes $A\ B$, Fig. 834, would describe the space $C\ C_1 = C\ D$ $tang.\ C\ D\ C_1$ while the water describes $C\ D$; hence, if we denote by v the velocity of the water and by δ the angle of impact $O\ C\ B = C\ D\ C_1$, we have under this supposition the mean velocity of rotation of the vane-wheel

$$v_1 = v \, tang. \, \delta,$$

from which it is easy to see that, when r denotes the mean radius of the vane-wheel, the number of revolutions is

$$u = \frac{v_1}{2 \pi r} = \frac{v \, tang. \, \delta}{2 \pi r},$$

and that, consequently, it is directly proportional to the velocity v of the water and to the tangent of the angle of impact and inversely to the mean radius of the vane-wheel.

FIG. 834.

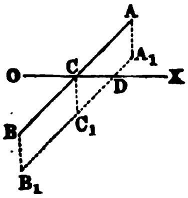
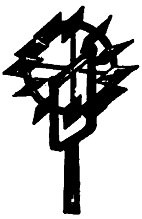


FIG. 835.



REMARK 3.—In order to determine the superficial velocity of water we also employ a small wheel made of metal, like the one represented in Fig. 835, and we allow only the lower part to be immersed in the water. The number of revolutions is given by a train of wheels, exactly as in the tachometer.

§ 491. In order to determine the *constants* or *the coefficients of a tachometer*, it is necessary to hold the instrument in running water, the velocity of which is known, and to observe the number of revolutions. Although only as many observations as there are constants are required, yet it is safer to make as many observations as possible, particularly with very different velocities, and to employ the *method of the least squares* (see Introduction to the Calculus, Art. 36) and thus do away with the accidental errors of observation. The velocity of the water may be determined by a floating sphere, or we may catch the water in a gauged vessel and divide the quantity of water caught by the cross-section. If the floating sphere is employed, the air must be still and the water must move uniformly and in a straight line. The vane-wheel must be immersed at several points along the path described by the floating sphere, and to insure perfect accuracy, the diameter of the sphere should be about equal to that of the vane-wheel.

The second method of determination by catching the water, in which the mill is immersed, in a gauged vessel possesses many advantages. For this purpose, and for adjusting hydrometers generally, it is very desirable to have at one's disposition a *hydraulic observatory*, which consists of a gauged vessel, a trough, and a discharging vessel or reservoir. We can then give the water any desired velocity; for we can regulate not only the entrance of the water into the trough, but also, by inserting boards, we can regulate at will the velocity in it. In making the observation, we have but to insert the tachometer at different parts of the cross-section of the trough, to measure the depth of this section by a scale, and then to gauge the quantity of water, which has passed through in a given time (§ 480). The area of the cross-section is obtained by multiplying the mean depth by the mean width, and the discharge Q is calculated from the mean cross-section of the receiving reservoir and the depth of the water, which has flowed into it, by means of the formula

$$Q = \frac{G s}{t};$$

finally, from Q and F we deduce the mean velocity of the water

$$v = \frac{Q}{F} = \frac{G s}{F t}$$

The corresponding number u of revolutions of the vane-wheel is the mean of all the revolutions observed when we inserted the instrument in different parts of the transverse profile.

If by experiment we have determined a series v_1, v_2, v_3, etc., of mean velocities and the corresponding numbers of revolutions, we obtain, by substituting them in the formula

$$v = v_0 + \alpha u,$$

or in the more accurate one

$$v = \alpha u + \sqrt{v_0^2 + \beta u^2},$$

as many equations of conditions for the constants v_0, α, β, as we made observations, and we can find from them the constants themselves either by employing the method given in Art. 36 of the Introduction to the Calculus, or by dividing these equations into as many groups as there are unknown constants, and combining them by addition into as many equations of condition as are necessary for the determination of v_0, α and β.

If we assume the passive resistances of the instrument to be small enough to be neglected, we can put $v = \alpha u$ and determine α by moving the instrument forward in still water and observing the number $n = u t$ of revolutions made in describing the space $s = v t$; then

$$\alpha = \frac{v}{u} = \frac{v t}{u t} = \frac{s}{n}.$$

REMARK—1) If we employ the simple formula with two constants, we can put, according to the method of least squares,

$$v_0 = \frac{\Sigma (y^2)\, \Sigma (x) - \Sigma (x y)\, \Sigma (y)}{\Sigma (x^2)\, \Sigma (y^2) - [\Sigma (x y)]^2} \text{ and } \alpha = \frac{\Sigma (x^2)\, \Sigma (y) - \Sigma (x y)\, \Sigma (x)}{\Sigma (x^2)\, \Sigma (y^2) - [\Sigma (x y)]^2}$$

in which $x = \frac{1}{v}$ and $y = \frac{u}{v}$, and the sign Σ denotes the sum of all the values of the same kind as that which follows it, E.G.

$$\Sigma\,(x) = \frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3} + \ldots,$$

$$\Sigma\,(x\,y) = \frac{1}{v_1} \cdot \frac{u_1}{v_1} + \frac{1}{v_2} \cdot \frac{u_2}{v_2} + \frac{1}{v_3} \cdot \frac{u_3}{v_3} + \ldots$$

EXAMPLE.—We have observed with a small tachometer that for the velocities

0,163, 0,205, 0,298, 0,366, 0,610 meters

the number of revolutions per second were

0,600, 0,835, 1,467, 1,805, 3,142,

and we wish to determine the constants corresponding to this instrument. By the aid of the formula given in the Remark, we obtain, since

$$\Sigma\,(x) = \frac{1}{0,163} + \frac{1}{0,205} + \ldots = 18,740,$$

$$\Sigma\,(y) = \frac{0,600}{0,163} + \frac{0,835}{0,205} + \ldots = 22,759,$$

$$\Sigma\,(x^2) = \left(\frac{1}{0,163}\right)^2 + \left(\frac{1}{0,205}\right)^2 + \ldots = 82,246,$$

$$\Sigma\,(y^2) = 105,233, \text{ and}$$

$$\Sigma\,(x\,y) = \frac{0,600}{(0,163)^2} + \frac{0,835}{(0,205)^2} + \ldots = 80,961,$$

$$v_0 = \frac{105,233 \,.\, 18,740 - 80,961 \,.\, 22,759}{82,846 \,.\, 105,233 - (80,961)^2} = \frac{129,5}{2162} = 0,060 \text{ and}$$

$$\alpha = \frac{368,3}{2162} = 0,1703;$$

hence the formula for this instrument is

$$v = 0,060 + 0,1703\,u.$$

Substituting $u = 0,6$, we obtain

$$v = 0,060 + 0,102 = 0,162;$$

$u = 0,835$ gives

$$v = 0,060 + 0,142 = 0,202;$$

$u = 1,467$,

$$v = 0,060 + 0,249 = 0,309;$$

$u = 1,805$,

$$v = 0,060 + 0,307 = 0,367;$$

and finally, $u = 3,142$,

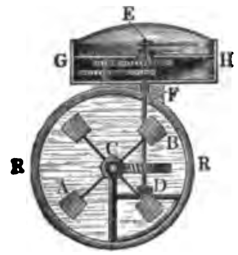
$$v = 0,060 + 0,535 = 0,595.$$

The calculated values therefore agree very well with the observed ones.

REMARK—2) We can also, according to Lapointe, insert the tachometer in a cylindrical pipe, and thus obtain the velocity of the water flowing through it. The counting apparatus can be placed outside of the pipe and connected with the vane-wheel by means of a shaft. Lapointe calls this instrument *une tube jaugeur* (see "Comptes rendues," T. XXV, 1848;

also Polytechn. Centralblatt, 1847).' Fig. 836 gives an ideal representation of the tachometer in a pipe. The vane-wheel in this case also puts a shaft *D E* in rotation by means of an endless screw; the former passes out of the pipe *R R*, in which the water to be measured flows, through a stuffing-box *F* into the case *G H* of the counting apparatus, the arrangement of which may be very varied.

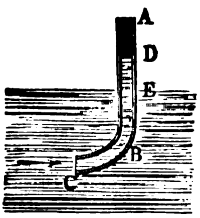
FIG. 836.



REMARK—8) The French have but lately begun to give sufficient attention to the tachometer. A complete treatise upon this instrument, by Baumgarten, is to be found in the "Annales des ponts et chaussées," T. XIV, 1847, and an abstract of it in the "Polytechnisches Centralblatt, 1849." Baumgarten recommends a screw-wheel and adds several remarks, which agree very well with our experiments, made many years ago. A new tachometer, without wheels and with a long screw, is described by Boileau in his "Traité de la mesure des eaux courantes."

§ **492**. **Pitot's Tube.**—The other hydrometers are more imperfect than the tachometer; for they are either less accurate or more difficult to use. The simplest instrument of this kind is Pitot's tube (Fr. la tube de Pitot; Ger. Pitot'sche Röhre). It consists of a bent glass tube *A B C*, Fig. 837, which is held in the water in such a manner that the lower part is horizontal and opposite to the motion of the water. By the impulse of the water a column of water will be forced into the tube and held above the level of the water, and this rise *D E* is proportional to the impulse or to the velocity of the water which produces it; this rise or difference of level can therefore serve to measure the velocity of the water. If the height *D E* above the exterior surface of the water $= h$ and the velocity of the water $= v$, we can put

FIG. 837.



$$h = \frac{v^2}{2 g \mu^2},$$

in which μ is an empirical number, or inversely

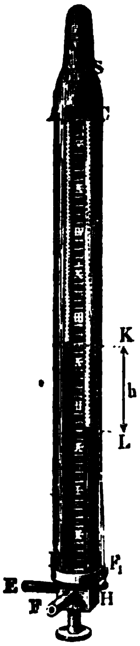
$$v = \mu \sqrt{2 g h}, \text{ or more simply}$$

$$v = \psi \sqrt{h}.$$

In order to find the constant ψ, we hold the instrument in the water where the velocity is known to be v_1; if the rise is $= h_1$, we have the constant $\psi = \frac{v_1}{\sqrt{h_1}}$, which can be employed in other cases, where the velocity is to be determined by this instrument.

In order to facilitate the reading off of the height h, the instrument is composed of two tubes $A\ B$ and $C\ D$, as is represented in Fig. 838; from one of the tubes a pipe proceeds in the direction of the stream, and from the other two pipes F and F_1 at right-angles to that direction, but by means of the same cock both tubes can be closed at once. If we draw the instrument out of the water, we can easily read off the difference of height $K\ L = h$ of the columns of water upon the scale placed between them. In order to prevent the water from oscillating in the tubes, it is necessary to make their mouths narrow; and in order that the cock may be shut quickly and certainly, it is provided with a crank and a rod $H\ S$, which is represented in the figure principally by a dotted line and terminates near the handle of the instrument.

FIG. 838.

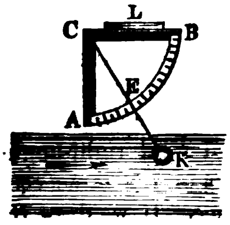


REMARK—1) Although Pitot's tube is not so accurate as the tachometer, yet, on account of its simplicity, it can be highly recommended. The author has discussed this instrument at length in the "Polytechnisches Centralblatt, 1847," and gives there a series of numbers, determined by experiment, and the values of the coefficient ψ deduced from them. With fine instruments, when the velocities were between 0,32 to 1,24 meters, we found $v = 3{,}545\ \sqrt{h}$ meters.

2) Duchemin recommends Pitot's tube with a float. Since the latter must be pretty wide, it dams the water back to a certain extent, so that it cannot be employed for narrow canals (see Duchemin: "Recherches expérim. sur les lois de la résistance des fluides"). Boileau describes in his work, cited in § 412, a new kind of Pitot's tube, which is provided with a small gauged vessel; the velocity is measured by the quantity of water pressed above the surface of the water.

§ 493. **Hydrometric Pendulum.**—The *hydrometric pendulum* (Fr. pendule hydrométrique; Ger. Stromquadrant or hydrometrisches Pendel) was principally employed by Ximenes, Michelotti, Gerstner, and Eytelwein to measure the velocity of running water. This instrument consists of a quadrant $A\ B$, Fig. 839, divided into degrees and parts of a degree, and of a string attached to its centre C, at the other end of which is fastened a metal or ivory ball K, 2 or 3 inches in diameter. The velocity of the water is given by the angle $A\ C\ E$ formed by the stretched string with the vertical, when the plane of the instrument is placed in the direction of the stream, and the ball is immersed in the water. Since the angle cannot easily exceed 40°, this instrument often has the form of a right-angled triangle, and the graduation is then marked upon the base. In order to place the zero line vertical, we can either place a level upon the instrument or we can employ the ball itself by allowing it to hang out of the water and then turning the instrument until the string corresponds with the zero line. For velocities less than 4 feet we can employ an ivory ball; for greater velocities, however, we must use heavy balls of metal. On account of the vibrations of the ball, not only in the direction of the motion of the water but also in that at right angles to it, it is always difficult to read off the angle, and the result is never free from uncertainty; this instrument cannot therefore be considered to be a perfect one.

FIG. 839.



The dependence of the angle of deviation, for a ball that is not deeply immersed, upon the velocity of the water can be determined in the following manner. The weight G of the ball and the impulse of the water $P = \mu\ F\ v^2$, which increases with the cross-section F of the ball and the square of the velocity v, give rise to a resultant R, which is counteracted by the string and is determined by the angle of deviation δ, for which we have

$$tang.\ \delta = \frac{P}{G} = \frac{\mu\ F\ v^2}{G},$$

or inversely

$$v^2 = \frac{G\ tang.\ \delta}{\mu\ F} \text{ and } v = \sqrt{\frac{G}{\mu\ F}} \cdot \sqrt{tang.\ \delta},$$

I.E.,

$$v = \psi\sqrt{tang.\ \delta},$$

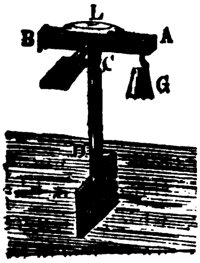
in which ψ is an empirical coefficient, which must be determined in the manner stated above (§ 491) before the instrument can be used.

§ 494. **Rheometer.**—The remaining hydrometers, such as Lorgna's water-lever, Ximenes' water-vane, Michelotti's hydraulic balance, Brunning's tachometer and Poletti's rheometer, etc., are difficult to use and partially uncertain. The principle of all of them is the same; they consist of a balance and of a surface, which is subjected to the impact of the water; the former serves to measure the impulse P of the water against the former, but since the impulse is $= \mu F v^2$, we have inversely

$$v = \sqrt{\frac{P}{\mu F}} = \psi \sqrt{P},$$

in which ψ is an empirical constant, dependent upon the magnitude of the surface subjected to the impulse of the water.

FIG. 840.



The Rheometer, which has been lately proposed by Poletti, does not differ essentially from Michelotti's balance and consists of a lever $A\,B$, Fig. 840, movable about a fixed axis C, and of a second arm $C\,D$, upon which a surface, or, according to Poletti, a simple rod, which is to be subjected to the impact, is screwed. In order to balance the force of impact of the water, shot or weights are put into the sheet iron box, which is suspended at A upon the lever, and to balance the empty apparatus in still water, weights are hung at B, the extreme end of the arm $C\,B$. From the weights added at G and the arms of the lever $C\,A = a$ and $C\,F = b$, we obtain by means of the formula $P\,b = G\,a$ the impulse

$$P = \frac{a}{b}\,G \text{ and } v = \sqrt{\frac{P}{\mu F}} = \sqrt{\frac{a\,G}{m\,b\,F}} = \psi \sqrt{G},$$

in which ψ denotes an empirical constant.

A hydrometer constructed upon the same principle, in which the impulse of the water is balanced by the force of a spring (hydromètre dynamométrique) is described by Boileau in his treatise upon the measurement of water.

REMARK 1.—The last-mentioned hydrometers are discussed at length in Eytelwein's "Handbuch der Mechanik," Vol. II, in Brunning's "Abhandlung über die Geschwindigkeit des fliessenden Wassers," in Venturoli's

"Elementi di Meccanica e d'Idraulica," Vol. II. Concerning Poletti's Rheometer, see Dingler's Polytechn. Journal, Vol. XX, 1826. Stevenson's hydrometer is Woltmann's tachometer, see Dingler's Journal, Vol. LXV, 1842. The water-meters and gas-meters constructed like reaction wheels will be treated in the following chapter.

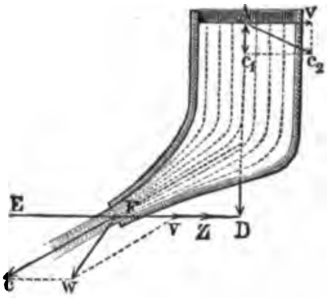
REMARK 2.—A work to be particularly recommended for practical purposes is the "Hydrometrie oder practische Anleitung zum Wassermessen von Bornemann, Freiberg, 1849." Boileau's work has already been mentioned several times (see § 412, etc.).

CHAPTER IX.

OF THE IMPULSE AND RESISTANCE OF FLUIDS.

§ 495. Reaction of Water.—The total pressure of the still water in a vessel is, according to § 362, reduced to a vertical force equal to the weight of the mass of water; but if the vessel *A F*, Fig. 841, has an opening *F*, through which the water issues, this force undergoes a change not only because a portion of the wall of the vessel is absent, but also because the water, which issues from the orifice, like every other body, which changes its conditions of motion, reacts by virtue of its inertia. The change in the motion of a body may consist either of a change of velocity, or of a change of direction, and, therefore, the *reaction* (Fr. réaction; Ger. Reaction) of the issuing water may be due not only to an acceleration but also to a constant change in the direction of the water, which is approaching the orifice.

FIG. 841.



We can make ourselves acquainted with the complete reaction of the water in a discharging vessel in the following manner.

Let c be the velocity of the water, which is issuing from the orifice F, c_1 the relative velocity of the water at the surface A,

G the area of this surface and h the head of water $A\ D$ at the orifice. Then we have

$$\frac{c^2}{2\,g} = h + \frac{c_1^2}{2\,g},$$

and the discharge

$$Q = F\,c = G\,c_1.$$

If we imagine the vase $A\ F$, Fig. 841, to move forward *in a horizontal direction* with a velocity v, we must put for the absolute velocity c_2 of the water entering the vessel

$$c_2^2 = c_1^2 + v^2,$$

and if the angle of inclination of the axis of the stream to the horizon is $E\,F\,c = a$, we have for the absolute velocity w of the effluent stream

$$w^2 = c^2 + v^2 - 2\,c\,v\,cos.\,a.$$

Now the actual energy of the water before efflux is

$$L_1 = \left(\frac{c_2^2}{2\,g} + h\right) Q\,\gamma = \left(\frac{c_1^2 + v^2}{2\,g} + h\right) Q\,\gamma$$

and that after efflux it is

$$L_2 = \frac{w^2}{2\,g}\,Q\,\gamma = \left(\frac{c^2 + v^2 - 2\,c\,v\,cos.\,a}{2\,g}\right) Q\,\gamma;$$

hence the energy withdrawn from the water and transmitted to the vessels is

$$L = L_1 - L_2 = \left(\frac{c_1^2 - c^2 + 2\,c\,v\,cos.\,a}{2\,g} + h\right) Q\,\gamma,$$

or, since $\frac{c^2}{2\,g} - \frac{c_1^2}{2\,g} = h$,

$$L = \frac{c\,v\,cos.\,a}{g}\,Q\,\gamma.$$

The *horizontal component* of the *reaction of the water* is

$$H = \frac{L}{v} = \frac{c\,cos.\,a}{g}\,Q\,\gamma.$$

Since $Q = F\,c$, we have also

$$H = \frac{c^2}{g}\,F\,\gamma\,cos.\,a = 2\,.\,\frac{c^2}{2\,g}\,F\,\gamma\,cos.\,a = 2\,h\,F\,\gamma\,cos.\,a,$$

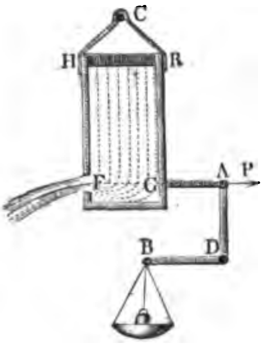
and therefore, when the direction of the stream is horizontal, as in Fig. 842,

$$H = 2\,h\,F\,\gamma.$$

Therefore, *the reaction of a horizontal stream is equal to the weight of a column of water, whose cross-section is that of the stream and whose height is double that* (2 h) *due to the velocity.*

REMARK.—Mr. Peter Ewart, an Englishman, has recently made experiments to prove the correctness of this law (see "Memoirs of the Manchester Philosophical Society," Vol. II, or the "Ingenieur, Zeitschrift für das gesammte Ingenieurwesen," Vol. I). He hung the vessel $H\ R\ F$ upon a horizontal axis C, Fig. 842, and measured the reaction by a bent lever $A\ D\ B$, upon which the vessel acted by means of a horizontal rod $A\ G$, which pressed against the vessel exactly opposite to the orifice F. For efflux through an orifice in a thin plate, he found

FIG. 842.



$$P = 1{,}14 \,.\, \frac{v^2}{2g} F \gamma.$$

If we put the cross-section

$$F_1 = 0{,}64\ F$$

and the effective velocity of discharge

$$v_1 = 0{,}96\ v$$

(see § 405), we obtain by the theoretical formula

$$P = 2 \,.\, \frac{v_1^{\ 2}}{2g} \,.\, F_1\, \gamma = 2 \,.\, 0{,}96^2 \,.\, 0{,}64 \,.\, \frac{v^2}{2g} F \gamma = 1{,}18 \frac{v^2}{2g} F \gamma,$$

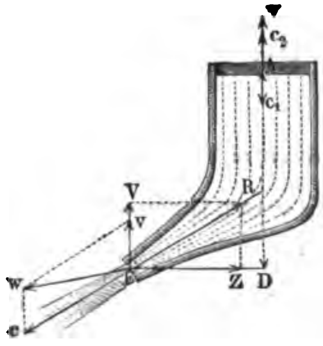
or about the same that was given by experiment. With an orifice shaped like the contracted stream, he found $P = 1{,}78 \dfrac{v^2}{2\,g} F \gamma$, and the coefficient of efflux or velocity $= 0{,}94$. Since in this case $F_1 = F$ and $v_1 = 0{,}94\ v$, we have theoretically

$$P = 2 \,.\, 0{,}94^2 \frac{v^2}{2\,g} F \gamma = 1{,}77 \,.\, \frac{v^2}{2\,g} F \gamma,$$

which agrees very well with the result of the experiment.

§ 496. If we imagine the discharging vessel $A\ F$, Fig. 843, to be moved vertically upwards with a velocity v, we have for the absolute velocity of the water which enters it

FIG. 843.



$$c_2 = v - c_1,$$

and, on the contrary, for that of the water issuing from it (the same notations being employed as in the foregoing paragraph)

$$w^2 = c^2 + v^2 + 2\,c\,v\,cos.\,(90^\circ + a)$$
$$= c^2 + v^2 - 2\,c\,v\,sin.\,a.$$

Hence the total energy of the volume of water Q per second is

$$L_1 = \left(\frac{(v-c)^2}{2\,g} + h\right) Q\,\gamma,$$

and, on the contrary, that of the water discharged is

$$L_2 = (c^2 + v^2 - 2\,c\,v\,sin.\,a)\ Q : 2\,g$$

consequently the mechanical effect imparted by the water to the vessel is

$$L = L_1 - L_2 = \left(\frac{2\, v\, c_1 + c_1^2 - c^2 + 2\, c\, v\, sin.\, a}{2\, g} + h\right) Q\, \gamma,$$

or, since $h = \frac{c^2}{2\, g} - \frac{c_1^2}{2\, g}$,

$$L = \frac{(c\, sin.\, a - c_1)\, v}{g}\, Q\, \gamma,$$

and the corresponding vertical force is

$$V = \frac{L}{v} = \left(\frac{c\, sin.\, a - c_1}{g}\right) Q\, \gamma = \left(sin.\, a - \frac{F}{G}\right) \frac{c}{g}\, Q\, \gamma$$

$$= \left(sin.\, a - \frac{F}{G}\right) \frac{c^2}{g}\, F\, \gamma = \left(sin.\, a - \frac{F}{G}\right) . \, 2\, h\, F\, \gamma.$$

If the orifice of efflux is small, compared to the surface G, we have $\frac{F}{G} = 0$, and, therefore, the *vertical component of the reaction*

$$V = 2\, h\, F\, \gamma\, sin.\, a.$$

According to the foregoing paragraph the horizontal component of this force was

$$H = 2\, h\, F\, \gamma\, cos.\, a;$$

hence the *total reaction of the water* is

$$R = \sqrt{V^2 + H^2} = 2\, h\, F\, \gamma,$$

and its direction is exactly opposite to that of the motion of the effluent water.

If $F = G$, I.E., if the water flows through a pipe of uniform width, we have $\frac{F}{G} = 1$, and therefore

$$V = (sin.\, a - 1) . \, 2\, h\, F\, \gamma = -\, (1 - sin.\, a) . \, 2\, h\, F\, \gamma;$$

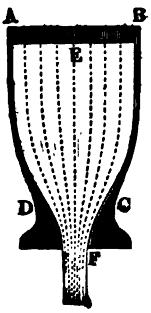
in this case V does not act upwards but downwards, and the total reaction is

$$R = \sqrt{V^2 + H^2} = \sqrt{cos.\, a^2 + (1 - sin.\, a)^2} . \, 2\, h\, F\, \gamma$$

$$= \sqrt{2\, (1 - sin.\, a)} . \, 2\, h\, F\, \gamma$$

$$= 4\, h\, F\, \gamma\, sin. \left(45° - \frac{a}{2}\right).$$

FIG. 844.



For $a = -\, 90°$, I.E., when the pipe forms a semicircle, $R = 4\, h\, F\, \gamma$

If $a = +\, 90°$, we have the case represented in Fig. 844, where $H = 0$ and

$$V = \frac{(c - c_1)}{g}\, Q\, \gamma = \left(1 - \frac{F}{G}\right) . \, 2\, h\, F\, \gamma,$$

consequently, for $\frac{F}{G} = 0$, we have

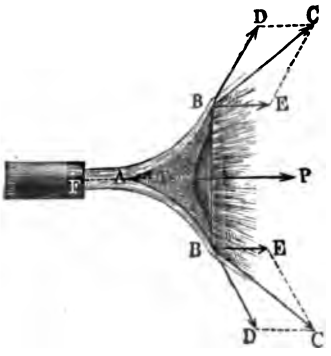
$$V = R = 2\,h\,F\,\gamma.$$

The total weight of the water in the vessel will be diminished that much, when the water is allowed to flow out.

§ 497. **Impulse and Resistance of Water.**—Water or any other fluid, when it impinges upon a solid body, imparts a force or impulse to it, and thus produces a change in its state of motion. The *resistance* (Fr. résistance; Ger. Widerstand), which water makes to the motion of a body, is not essentially different from impulse. The examination of these two forces constitutes the third chief division of hydraulics. We distinguish from each other first, *the impact of an isolated stream* (Fr. choc d'une veine de fluide; Ger. Stoss isolirter Wasserstrahlen); secondly, the *impact of a bounded stream* (Fr. choc d'un fluide défini; Ger. Stoss im begrenzten Wasser oder Gerinne); and thirdly, the *impact of an unlimited stream* (F. choc d'un fluide indéfini; Ger. Stoss im unbegrenzten Wasser). Impact of the first sort takes place when a stream discharged from a vessel encounters a body, as, E.G., the bucket of an overshot water-wheel; impact of the second sort occurs, when the water in a canal or trough strikes against a body which entirely fills the cross-section of the latter, as, E.G., the float of an under-shot water-wheel. Finally, impact of the third kind occurs, when running water strikes upon a body immersed in it and the cross-section of the latter is but a small part of that of the stream, as, E.G., the float of a wheel in an open current.

We distinguish also impact against *bodies at rest* and *bodies in motion*, against *curved* and *plane* surfaces; the latter may be either direct or oblique.

Fig. 845.



We will now consider a more general case, viz. the impact of an isolated stream against a *surface of revolution*, moving in the direction of the motion of the stream, which coincides with the direction of the axis of the surface.

§ 498. **Impact of an Isolated Stream.**—Let *B A B*, Fig. 845, be a *surface of revolution*, $A\ P$ its axis, and $F\ A$ a stream of water moving in the direction of the axis of the latter and impinging against it; let us put the velocity of the water $= c$, that of the surface $= v$, and the angle $B\ T\ P$, which the tangent $D\ T$ to the end B of the generatrix or each fibre $B\ D$ of the stream of water, which leaves the surface, makes with the direction $B\ E$ of the axis, $= a$, and let us assume that the water does not lose any vis viva in consequence of the friction while passing over the curved surface. The water impinges upon the surface with the velocity $c - v$ and then passes over the surface with that velocity and leaves it in a tangential direction $T\ B$, $T\ B$, etc., with the same velocity. From the tangential velocity $B\ D = c - v$ and from the velocity $B\ E = v$ in the direction of the axis, we obtain the absolute velocity $B\ C = c_1$ of the water, after it has impinged upon the surface, by the well-known formula

$$c_1 = \sqrt{(c - v)^2 + 2\,(c - v)\,v \cos. a + v^2}.$$

Now a discharge Q can produce by its vis viva a mechanical effect $\frac{c^2}{2\,g} \cdot Q\,\gamma$, when it loses its entire velocity c; hence the energy remaining in the water is $= \frac{c_1^2}{2\,g} \cdot Q\,\gamma$, that transmitted to the surface is

$$P\,v = \frac{c^2}{2\,g}\,Q\,\gamma - \frac{c_1^2}{2\,g}\,Q\,\gamma = \frac{c^2 - c_1^2}{2\,g} \cdot Q\,\gamma$$

$$= \frac{[c^2 - (c - v)^2 - 2\,(c - v)\,v \cos. a - v^2]}{2\,g}\,Q\,\gamma$$

$$= \frac{2\,c\,v - 2\,v^2 - 2\,(c - v)\,v \cos. a}{2\,g}\,Q\,\gamma, \text{ I.E.}$$

$$P\,v = (1 - \cos. a)\,\frac{(c - v)\,v}{g}\,Q\,\gamma,$$

and the *force or impulse* in the direction of the axis is

$$P = (1 - \cos. a)\,\frac{c - v}{g}\,Q\,\gamma.$$

If the surface moves with a velocity v, which is in the *opposite direction* to that of the water, we will have

$$P = (1 - \cos. a)\,\frac{(c + v)}{g}\,Q\,\gamma,$$

and if the surface *does not move* or if $v = 0$, the impulse or hydraulic pressure in the direction of the axis is

$$P = (1 - \cos. a)\,\frac{c}{g} \cdot Q\,\gamma.$$

From this it follows *that the impulse of one and the same mass of water, when the other circumstances are the same, is proportional to the relative velocity* $c \mp v$ *of the water.*

If the area of the cross-section of the stream is F, the volume of the impinging water is $F(c \mp v)$; hence

$$P = (1 - cos.\ a) \frac{(c \mp v)^2}{g} F \gamma;$$

or for $v = 0$,

$$P = (1 - cos.\ a) \frac{c^2}{g} F \gamma.$$

If the cross-section of the stream remains the same, the impulse against a surface at rest increases with the square of the velocity of the water.

§ **499. Impact against Plane Surfaces.**—The impulse of the same stream of water depends principally upon the angle a, at which the water moves off from the axis after the impact; it is null when this angle $= 0$, and, on the contrary, a maximum and

$$= 2 \frac{(c \mp v)}{g} Q \gamma$$

when this angle is 180° or when its cosine $= -1$, in which case, as is represented in Fig. 846, the water quits the surface in a direction opposite to that in which it struck it. In general the impact is greater against *concave* than against *convex surfaces;* for in the former case the angle is obtuse and its cosine negative and $1 -$ *cos.* a becomes $1 +$ *cos.* a.

Fig. 846.

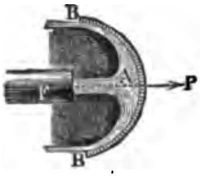
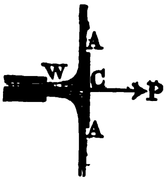


Fig. 847.



Usually the surface is, as is represented in Fig. 847, plane and therefore $a = 90°$ or $cos.\ a = 0$ and the impulse

$$P = \frac{(c \mp v)}{g} \cdot Q \gamma.$$

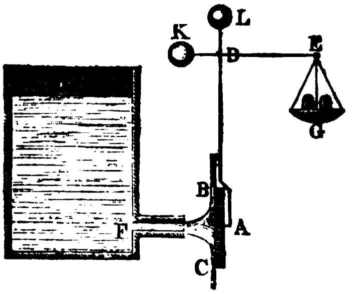
When the surface is at rest, we have

$$P = \frac{c}{g} Q \gamma = \frac{c^2}{g} F \gamma = 2 \cdot \frac{c^2}{2g} F \gamma = 2 F h \gamma.$$

The normal impulse of water against a plane surface is equal to the weight of a column of water, the cross-section of whose base is equal to the cross-section of the stream, and whose height is twice that due to the velocity $\left(2\ h = 2 \cdot \frac{c^2}{2g}\right)$.

The results of the experiments made upon this subject by Michelotti, Vince, Langsdorf, Bossut, Morosi and Bidone were about the same, when the cross-section of the impinged surface was at least 6 times that of the stream and when this surface was at a distance not less than twice the thickness of the stream from the orifice. The apparatus employed consisted of a lever like Poletti's Rheometer (§ 494), upon one end of which the stream impinged, the impulse was balanced at the other end by weights. The apparatus employed by Bidone is represented in Fig. 848. *B C* is the surface subjected to the action of the stream, *G* the scale-pan for receiving the weights, *D* the axis of rotation, and *K* and *L* are counter weights.

FIG. 848.



REMARK.—The most extensive experiments upon the impulse of water were made by Bidone (see "Memoire de la Reale Accademia delle Scienze di Torino," T. XL, 1838). They were made with a velocity of at least 27 feet and with brass plates of from 2 to 9 inches in diameter. Bidone generally found the normal impulse against a plane surface somewhat greater than $2\,F\,h\,\gamma$; but this increase is to be ascribed to the increase of the arm of the lever, in consequence of the falling back of the water. See Duchemin: Recherches expérimentales sur les lois de la résistance des fluides (translated into German by Schnuse). When the impinged surface was very near the orifice, Bidone found P to be only $1{,}5\,F\,h\,\gamma$. When the impinged surface was of the same size as the stream, in which case the angle of deviation α is acute, according to du Buat and Langsdorf, P is only $= F\,h\,\gamma$. Bidone and others have found that the impulse during the first instant was nearly twice the permanent impulse. Comparative experiments upon the impulse and reaction of water have been made by the author with a reaction wheel. See his "Experimentalhydraulik" and the "Civilingenieur," Vol. I, 1854.

By more recent experiments upon the impact of isolated streams of air and water (see Civilingenieur, Vol. VII, No. 5, and Vol. VIII, No. 1), the author found the effective impulse of an isolated stream of air or water against a normal plane to be 92 to 96 per cent. of the theoretical force $P = \frac{c\,Q\,\gamma}{g}$, that, on the contrary, the impulse of such a stream against a hollow surface of rotation by which the direction of the stream is made to deviate an angle $\delta = 134°$, is but 83 to 88 per cent. of the theoretical force $P = c\,(1 - cos.\,\delta)\,\frac{Q\,\gamma}{g}$.

§ 500. Maximum Work done by the Impulse.—The mechanical effect

$$P\,v = (1 - \cos.\,a)\,\frac{(c - v)\,v}{g}\,Q\,\gamma$$

depends principally upon the velocity v of the impinged surface; E.G. it is null not only for $v = c$, but also for $v = 0$; hence it follows that there must be a velocity, for which the work done by the impulse is a maximum. It is evident that this is the case when $(c - v)\,v$ is a maximum. If we consider c to be half the periphery of a rectangle and v to be its base, we have its height $= c - v$ and its area $= (c - v)\,v$; now the square is that rectangle, which has the greatest area for a given periphery; hence $(c - v)\,v$ is a maximum, when $(c - v) = v$, I.E., $v = \frac{c}{2}$, and we obtain the maximum mechanical effect of the impulse, when the surface moves in the direction of the stream with half the velocity of the latter; the work done is then

$$P\,v = (1 - \cos.\,a)\,.\,\tfrac{1}{2}\,.\,\frac{c^2}{2g}\,.\,Q\,\gamma = (1 - \cos.\,a)\,.\,\tfrac{1}{2}\,Q\,h\,\gamma.$$

Now if $a = 180°$, I.E., if the motion of the water is reversed by the impact, we have the work done

$$= 2\,.\,\tfrac{1}{2}\,Q\,h\,\gamma = Q\,h\,\gamma;$$

but if $a = 90°$, I.E., if the stream strikes against a plane surface, the work done is but $\frac{1}{2}\,Q\,h\,\gamma$, in this case the water transmits to the surface but one-half of its actual energy, or but one-half of the mechanical effect corresponding to its vis viva.

EXAMPLE—1) If a stream of water, the area of whose cross-section is 40 square inches, delivers 5 cubic feet per second and strikes normally against a plane surface, which moves away with a velocity of 12 feet, the impulse is

$$P = \frac{(c - v)}{g}\,Q\,\gamma = \left(\frac{5\,.\,144}{40} - 12\right).\,0{,}031\,.\,5\,.\,62{,}5 = 6\,.\,0{,}031\,.\,312{,}5$$

$$= 58{,}125 \text{ pounds,}$$

and the mechanical effect transmitted to the surface is

$$P\,v = 58{,}125\,.\,12 = 697{,}5 \text{ foot-pounds.}$$

The maximum effect is obtained, when

$$v = \frac{c}{2} = \tfrac{1}{2}\,.\,\frac{5\,.\,144}{40} = 9 \text{ feet,}$$

and it is

$$L = \tfrac{1}{2}\,.\,\frac{c^2}{2\,g}\,.\,Q\,\gamma = \tfrac{1}{2}\,.\,18^2\,.\,0{,}0155\,.\,5\,.\,62\tfrac{1}{2} = 81\,.\,0{,}155\,.\,62{,}5 = 784{,}6875$$

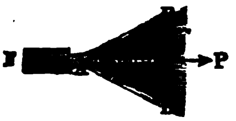
foot-pounds;

the corresponding impulse or hydraulic pressure is

$$P = \frac{784{,}6875}{9} = 87{,}19 \text{ pounds.}$$

2) If a stream $F A$, Fig. 849, the area of whose cross-section is 64 square inches, impinges with a velocity of 40 feet upon an immovable cone, whose angle of convergence $B A B = 100°$, the hydraulic pressure in the direction of the stream is

FIG. 849.



$$P = (1 - cos.\, \alpha)\, \frac{c}{g}\, Q\, \gamma$$

$$= (1 - cos.\, 50°)\, .\, 40\, .\, 0{,}031\, .\, \frac{64}{144}\, .\, 40\, .\, 62{,}5$$

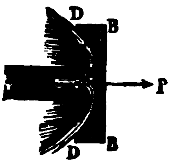
$$= (1 - 0{,}64279)\, .\, 1{,}24\, .\, \frac{10000}{9}$$

$$= 0{,}\, 35721\, .\, 1377{,}8 = 492{,}16 \text{ pounds.}$$

§ 501. Impact of a Bounded and of an Unlimited Stream.

—If we surround the periphery of a plane surface $B B$, Fig. 850, with borders $B D, B D$ (Fr. rebords; Ger. Leisten), which project beyond the surface struck by the water, the water will be deviated from its course at an obtuse angle as in the case of concave surfaces, and the impulse is greater than when the surface is plane. The action of this impact depends principally upon the height of the border and upon the ratio of the cross-section of the stream to that of the enclosed surface. In an experiment, where the stream was one inch thick and the cylindrical border 3 inches in diameter and 3½ lines high, the water flowed from the surface in nearly the opposite direction and the impulse was

FIG. 850.

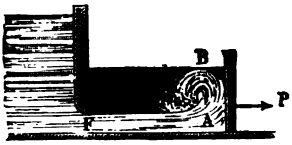


$$3{,}93\, \frac{c^2}{2\, g}\, F \gamma;$$

in all other cases this force was smaller. It is impossible ever to attain the theoretical maximum value $4\, \frac{c^2}{2\, g}\, F\, \gamma$ in consequence of the friction of the water upon the surface and upon the border.

In the case of the impact of the bounded stream $F A B$, Fig. 851, there is also a border; it is, however, only partial and includes but a portion of the periphery; it limits, moreover, both the stream and the impinged surface. The impinging stream is turned in the direction of the portion of the periphery, which has no border, and is therefore deviated 90° from its original direction; hence the formula, which we found for the isolated stream,

FIG. 851.



$$P = \frac{(c - v)}{g} Q \gamma = \left(\frac{c - v}{g}\right) c F \gamma,$$

holds good here. If the surface $B B$, Fig. 847, against which the stream strikes, moves away with a velocity v in a direction, which forms an angle δ with the original direction of the stream, the velocity of this surface in the direction of the impact is

$$v_1 = v \cos. \delta;$$

hence the impulse is

$$P = \frac{(c - v \cos. \delta)}{g} Q \gamma$$

and the work done by it per second is

$$L = P v_1 = \frac{(c - v \cos. \delta) v \cos. \delta}{g} Q \gamma.$$

The principal application of this formula is to the impact of an *unlimited stream*, in which case

$$Q = F (c - v \cos. \delta), \text{ and therefore}$$

$$P = \frac{(c - v \cos. \delta)^2}{g} F \gamma.$$

§ **502. Oblique Impact.**—There are several cases of oblique impact, viz.: where the water after impact flows away in one, in two or in more directions. If, as in the case of the impact of a bounded stream, the surface $A B$, Fig. 852, has a border upon three sides so that the water can flow away in one direction only, we have the hydraulic pressure of the water against the surface in the direction of the stream

$$P = (1 - \cos. \alpha) \frac{(c - v)}{g} Q \gamma.$$

FIG. 852.

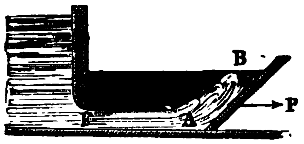
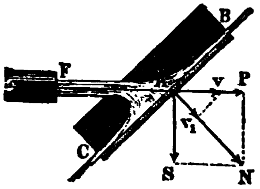


FIG. 853.



But if the impinged plane $B C$, Fig. 853, has a border upon two opposite sides only, the stream divides itself into two unequal parts, the angle of deviation α of the larger part Q_1 is less than that $180° - \alpha$ of the smaller part Q_2 and the total impulse in the direction of the stream is

$$P = (1 - \cos. \alpha) . \frac{c - v}{g} Q_1 \gamma + (1 + \cos. \alpha) . \frac{c - v}{g} Q_2 \gamma$$

$$= \left(\frac{c - v}{g}\right) [(1 - \cos. a)\, Q_1 + (1 + \cos. a)\, Q_2]\, \gamma.$$

But the conditions of equilibrium of the two portions of the stream require that the pressures

$$\frac{(c - v)}{g}\, (1 - \cos. a)\, Q_1\, \gamma \text{ and } \frac{(c - v)}{g}\, (1 + \cos. a)\, Q_2\, \gamma \quad .$$

shall be equal to each other; hence

$$(1 - \cos. a)\, Q_1 = (1 + \cos. a)\, Q_2,$$

or, since $Q = Q_1 + Q_2$, we can put

$$(1 - \cos. a)\, Q_1 = (1 + \cos. a)\, (Q - Q_1), \text{ i.e.}$$

$$Q_1 = \left(\frac{1 + \cos. a}{2}\right) Q \text{ and } Q_2 = \left(\frac{1 - \cos. a}{2}\right) Q,$$

so that the total impulse in the direction of the stream is

$$P = \frac{(c - v)}{g} \,.\, 2\, (1 - \cos. a)\, \frac{(1 + \cos. a)\, Q}{2}\, \gamma$$

$$= \frac{(c - v)}{g}\, (1 - \cos.^2 a)\, Q\, \gamma, \text{ i.e.}$$

$$P = \frac{c - v}{g}\, \sin.^2 a\, Q\, \gamma.$$

Dividing the work done by the impulse in a second

$$L = P\, v = \frac{(c - v)}{g}\, v\, \sin.^2 a \,.\, Q\, \gamma$$

by the velocity $\overline{A\, v_1} = v_1 = v\, \sin. a$, with which the surface recedes in a normal direction, we obtain the normal impulse

$$N = \frac{(c - v)\, v\, \sin.^2 a}{g\, v\, \sin. a} \,.\, Q\, \gamma = \frac{(c - v)}{g}\, \sin. a \,.\, Q\, \gamma,$$

which consists of the *parallel impulse*

$$P = N \sin. a = \frac{(c - v)}{g}\, \sin.^2 a \,.\, Q\, \gamma,$$

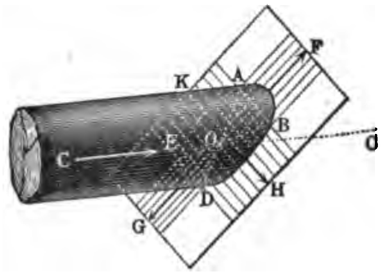
and of a *lateral impulse*

$$S = N \cos. a = \frac{(c - v)}{g}\, \sin. a \cos. a \,.\, Q\, \gamma = \frac{c - v}{2\, g}\, \sin. 2\, a\, Q\, \gamma.$$

The normal impulse is proportional to the sine, the parallel impulse to the square of the sine of angle of incidence, and the lateral impulse to the sine of double this angle.

If, finally, the oblique surface, which is struck, has no border, the water can flow away in all directions and the impulse is still greater; for a is the smallest angle which the fibres of water can make with the axis; hence every fibre which does not move in the normal plane exerts a greater pressure than those which do. If we assume that the angles of deviation of one portion Q_1, which corresponds to the sectors $A\ O\ B$ and $D\ O\ E$, Fig. 854, are $C\ O\ F = a$ and $C\ O\ G = 180° - a$, that those of the other portion Q_2, which corresponds to the sectors $A\ O\ E$ and $B\ O\ D$, are $C\ O\ K = C\ O\ H = 90°$, and that the two portions produce equal parallel impulses, we can put

FIG. 854.



$$P = \frac{c-v}{g}\, Q_1 \gamma\, sin.^2 a + \frac{c-v}{g}\, Q_2 \gamma,$$

and, since $Q_1\, sin.^2 a = Q_2$ and $Q = Q_1 + Q_2$, it follows that

$$Q_1 (1 + sin.^2 a) = Q,$$

and that the *total parallel impulse* is

$$P = \left(\frac{c-v}{g}\right) \frac{2\, Q\, \gamma\, sin.^2 a}{1 + sin.^2 a} = \frac{2\, sin.^2 a}{1 + sin.^2 a} \cdot \frac{c-v}{g} \cdot Q\, \gamma.$$

Although this assumption is only approximatively correct, yet the results of the latest experiments by Bidone agree very well with it.

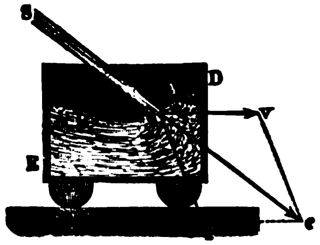
REMARK.—Prof. Brock, in his Mechanics, page 614, finds for oblique impact against a circular surface

$$P = \left(\frac{\pi}{2} - a\right) tang.\, a \left(\frac{c-v}{g}\right) Q\, \gamma, \text{ and}$$

$$N = tang.\, a\; l.\; cotg.\, \frac{a}{2} \left(\frac{c-v}{g}\right) Q\, \gamma.$$

§ 503. **Impact of Water in Water.**—If a certain quantity Q of water discharges with a velocity $\overline{A\,c} = c$ into a vessel $D\,E$, Fig. 855, which is moving with a velocity $A\,v = v$, a part only $L_1 = \frac{Q c_1^2}{2g} \gamma$ of its actual energy $L_0 = \frac{Q\, c^2}{2g} \gamma$ will be expended in producing and maintaining the eddy $A\ B$, which is due to the loss of velocity c_1. If we denote by a the angle $v\,A\,c$, made by the direction of the stream with that of the motion of the vessel, we have

FIG. 855.



$$c_1^2 = c^2 + v^2 - 2\, c\, v\, cos.\, a,$$

and, therefore, the mechanical effect lost in consequence of the eddy

$$L_1 = \frac{Q\,(c^2 + v^2 - 2\,c\,v\,cos.\,a)}{2\,g}.$$

As the volume Q of water participates in the motion of the vessel, its velocity v is the same as that of the latter, and the energy, which it still possesses, is $L_2 = \frac{Q\,v^2}{2\,g}\,\gamma$; hence the *energy* which is transmitted to the vessel and expended in moving it forward, is

$$L = L_0 - L_1 - L_2$$

$$= \left(\frac{c^2 - (c^2 + v^2 - 2\,c\,v\,cos.\,a) - v^2}{2\,g}\right) Q\,\gamma = \frac{2\,c\,v\,cos.\,a - 2\,v^2}{2\,g}\,Q\,\gamma$$

$$= \frac{(c\,cos.\,a - v)\,v}{g}\,Q\,\gamma,$$

and the *force* with which the vessel is urged forward in the direction of its motion by the water which flows into it is

$$P = \frac{L}{v} = \left(\frac{c\,cos.\,a - v}{g}\right) Q\,\gamma.$$

Now the discharge per second, which impinges against the vessel, is $Q = F\,c$, F denoting the cross-section of the stream at its entrance; hence we have

$$P = \frac{(c\,cos.\,a - v)\,c}{g}\,F\,\gamma,$$

and for the case when the vessel is at rest, or when $v = 0$,

$$P = \frac{c^2\,cos.\,a}{g}\,F\gamma = 2\,\frac{c^2}{2\,g}\,F\,\gamma\,cos.\,a = 2\,F\,h\,\gamma\,cos.\,a,$$

in which h denotes the height $\frac{c^2}{2\,g}$ due to the velocity.

The mechanical effect is a maximum for $v = \frac{1}{2}\,c\,cos.\,a$ and it is

$$L_m = \tfrac{1}{2}\,\frac{c^2\,cos.^2\,a}{2\,g}\,Q\,\gamma = \tfrac{1}{2}\,Q\,h\,\gamma\,cos.^2\,a.$$

If the direction of the stream is the same as that of the motion of the vessel, $a = 0$, and we have

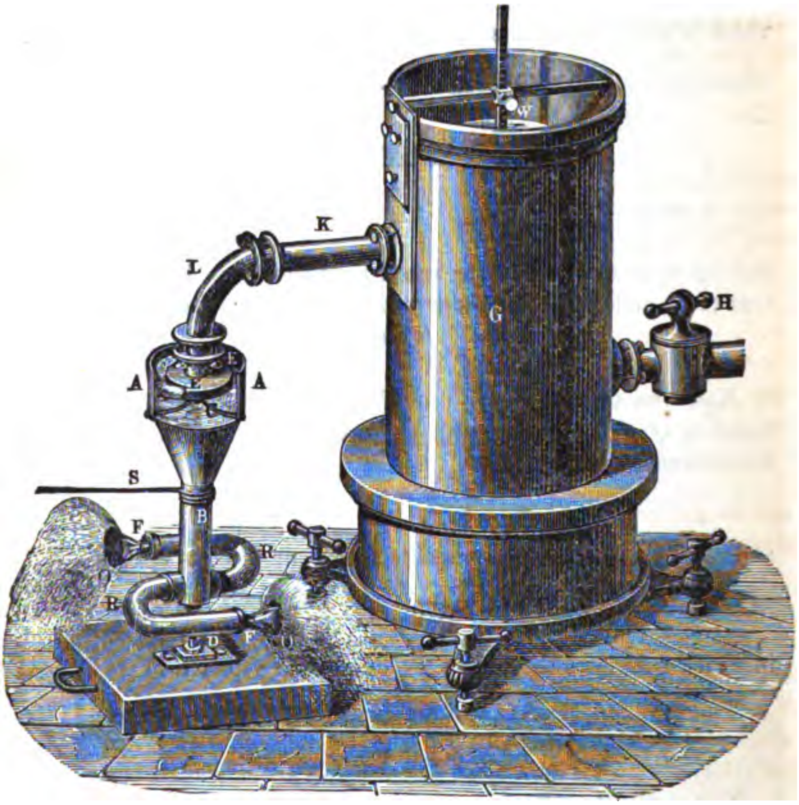
$$L = \frac{(c - v)\,v}{g}\,Q\,\gamma \text{ and}$$

$$L_m = \tfrac{1}{2}\,Q\,h\,\gamma.$$

In this case but half the total energy $Q\,h\,\gamma$ of the water is utilized (compare § 500).

§ 504. **Experiments with Reaction Wheels.**—The best method of proving the above theory of the impact and reaction of water is to make use of a reaction wheel *A A B*, Fig. 856, with a vertical axis of rotation *C D* (see the author's "Experimental-Hydraulik," § 48, etc.). The water which turns the machine enters into the receiver *A A* of the wheel nearly tangentially through two

FIG. 8



lateral canals *E*, *E*, and is discharged through two lateral orifices *F*, *F* in the ends of the revolving tubes *R*, *R*. In order to maintain the efflux of water constant and the rotating force invariable, the pipe which conveys the water to the reservoir *G* is provided with a cock *H*; from the reservoir the water is conveyed by the pipe *K L* to the chamber *E E*, into which the canals *E*, *E* open. While the machine is in operation, the cock *H* must be turned in such a manner that the surface of the water in the reservoir *G* shall always touch the end of the pointer *Z*.

When we wish to determine the reaction of the effluent water, a thin string S, to one end of which a weight is attached, is passed over a pulley and then wrapped round the central tube R. The quantity of water discharged is measured in the reservoir, from which the water flows into the pipe with the cock H, by observing the area A of the surface of the water and the distance a which it sinks during the experiment. If the duration of the observation is $= t$, we have the discharge per second

$$Q = \frac{A\,a}{t},$$

and if the fall, I.E. the vertical distance between the surface of the water in the reservoir G and the orifice of discharge of the wheel $= h$, the total energy of the water discharged per second is

$$L = Q\,h\,\gamma = \frac{A\,a\,h\,\gamma}{t}.$$

Now if the machine has raised the weight G a distance s in the time t, the work really done by the wheel in a second is

$$L_1 = \frac{G\,s}{t},$$

and we can now compare these two values, the second of which is always the smaller.

§ 505 Theory of the Reaction Wheel.—The total fall h in such a wheel consists of the fall h_1 from the surface of the water to the point E, where the water enters the wheel, and of the fall h_2 from the latter point to the orifice, by which the water leaves the wheel. From h_1 we calculate, by means of the formula $c_1 = \sqrt{2\,g\,h_1}$, the velocity with which the water enters the wheel, and from h_2, according to § 304, by means of the formula

$$c = \sqrt{2\,g\,h_2 + v^2 - v_1^2}$$

the velocity with which it quits it, when the velocities of rotation v_1 and v of the wheel at the points of entrance and exit are known. Since the direction of this reaction of the water, which acts as the rotating force, is opposite to that of the velocity of discharge, the absolute velocity of the water upon leaving the wheel is

$$w = c - v,$$

and its square

$$w^2 = c^2 - 2\,c\,v + v^2 = 2\,g\,h_2 - 2\,c\,v + 2\,v^2 - v_1^2;$$

hence the energy of the effluent water is

$$L_2 = Q\,\gamma \cdot \frac{w^2}{2\,g} = Q\,\gamma\left(h_2 - \frac{(c - v)\,v}{g} - \frac{v_1^2}{2\,g}\right).$$

The water, which enters the wheel with the relative velocity $w_1 = c_1 - v_1$, loses (according to § 436) by the impact the energy

$$L_1 = Q\,\gamma\,\frac{(c_1 - v_1)^2}{2\,g} = Q\,\gamma\left(h_1 - \frac{c_1\,v_1}{g} + \frac{v_1^2}{2\,g}\right),$$

and consequently of the total energy

$$Q\,h\,\gamma = Q\,(h_1 + h_2)\,\gamma,$$

only the portion

$$L = Q\gamma\,(h - h_1 - h_2) + Q\gamma\left(\frac{(c - v)\,v}{g} + \frac{c_1\,v_1}{g}\right) = Q\gamma\left(\frac{(c-v)\,v}{g} + \frac{c_1 v_1}{g}\right)$$

is transmitted to the wheel.

In order to obtain the greatest amount of work from the wheel we must have $w = 0$ or $v = c$ and $w_1 = 0$ or $v_1 = c_1$, and therefore

$$\frac{v_1^2}{2g} = h_2 \text{ or } v_1 = \sqrt{2\,g\,h_2}, \text{ as well as}$$

$$\frac{v_1^2}{2\,g} = h_1 \text{ or } v_1 = \sqrt{2\,g\,h_1}.$$

In this case, therefore, $h_1 = h_2 = \frac{1}{2}\,h$ and the corresponding maximum effect of the machine is

$$L_m = Q\,\gamma \cdot \frac{c_1\,v_1}{g} = Q\,\gamma \cdot \frac{v_1^2}{g} = 2\,Q\,h_1\,\gamma = Q\,h\,\gamma,$$

I.E., equal to the *total energy* of the water.

If r_1 denotes the distance of the point of entrance and r that of the orifice of exit of the wheel from the axis, we have

$$\frac{v_1}{v} = \frac{r_1}{r}, \text{ whence } v_1 = \frac{r_1}{r}\,v,$$

and, in general, the rate of work of the wheel

$$L = Q\,\gamma\left(c - v + \frac{r_1}{r}\,c_1\right)\frac{v}{g};$$

so that the rotating force, measured at the distance r, is

$$P = \frac{L}{v} = \frac{Q\,\gamma}{g}\left(c - v + \frac{r_1}{r}\,c_1\right).$$

If the arm of the suspended weight or load is a, which in the apparatus represented is very nearly the radius of the central tube B, we have $G\,a = P\,r$, and, therefore, the weight to be attached and to be raised during the rotation of the wheel is

$$G = \frac{r}{a}\,P = \frac{Q\,\gamma}{g\,a}\,[(c - v)\,r + c_1\,r_1],$$

or for $c = v$ and $c_1 = v_1$,

$$G = \frac{Q\,\gamma}{g\,a}\,c_1\,r_1 = \frac{Q\,\gamma}{g\,a}\,v_1\,r_1.$$

If F denote the area of the orifices of efflux and F_1 that of those of influx, we have

$$Q = F\,c = F_1\,c_1, \text{ and therefore}$$

$$F_1 = \frac{Q}{c_1} = \frac{Q}{\sqrt{2\,g\,h_1}}, \text{ and}$$

$$F = \frac{Q}{c} = \frac{Q}{\sqrt{2\,g\,h_2 + v^2 - v_1^2}} = F_1\sqrt{\frac{2\,g\,h_1}{2\,g\,h_2 + v^2 - v_1^2}}.$$

For $v = c$ and $v_1 = c_1$, in which case $h_1 = h_2 = \frac{1}{2}\,h$, we have $Q = F\,v$, and therefore

$$P = \frac{Q\,h\,\gamma}{v} = F\,h\,\gamma;$$

on the contrary, for $v = 0$, $Q = F\,\sqrt{2\,g\,h_2}$, and therefore

$$P = \frac{F\,c\,\gamma}{g}\left(c + \frac{r_1}{r}\,c_1\right).$$

If we allow the water to enter the wheel slowly, we can put $c_1 = 0$ and $h_1 = 0$ and the *force of the reaction* in the last case becomes

$$P = \frac{F\,c^2\,\gamma}{g} = \frac{2\,F\,c^2\,\gamma}{2\,g} = 2\,F\,h_2\,\gamma = 2\,F\,h\,\gamma,$$

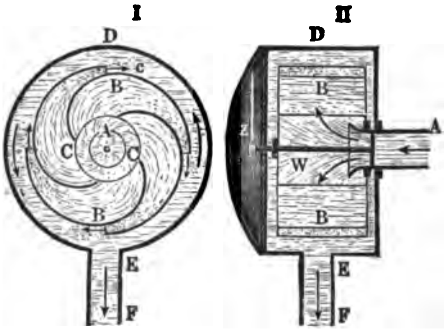
as we found above.

Since in these calculations we neglected the passive resistances, the experiments with the machine represented do not give the values for the force found above, but values which are a few per cent. less. However, the results of experiments carefully made with such a wheel agree very well with the theory just demonstrated.

When we wish to make use of this machine to test the *theory of the impact of water*, we begin by removing the chamber $E\,E$ so as to allow the water to enter near the centre without any velocity of rotation, and we then fasten opposite to the orifices in the revolving tubes the plates O, O, small vessels, etc., which are subjected to the impact of the water discharged. The rotating force is then equal to the difference between the reaction within the wheel and the impulse without it. We find, in accordance with the theory, that the wheel stands still, when the stream issuing from it impinges upon a plane plate at right angles to the direction of the water, or when it flows into a vessel filled with water. If the stream strikes obliquely against plane-plates or against convex surfaces, the wheel moves in the direction of the reaction, and if it is received by a concave surface, the wheel turns in the direction in which the water issues from the orifice.

§ **506. Water-meters.**—More recently *water-meters* (Fr. compteurs hydrauliques; Ger. Wassermesser) have been much used for measuring running water. They are put in motion by the reaction of the water discharged, and consist essentially of a reaction wheel or turbine. An ideal representation of the cross-section of such a wheel is given in Fig. 857. The water to be measured flows through a tube *A* into the centre of the wheel *B B*, and passes through 4 canals *C B*, *C B* . . . to the exterior circumference, where it is discharged into the case *D E*, from which it is conveyed away by a tube *E F*. The shaft *W* of this wheel carries a pointer *Z*, or rather a train of wheel-work, which indicates the number of revolutions of the wheel, and by it the volume of the water, which flows through it in any given time; for this volume is proportional to the number of revolutions. If *h* denotes the height of a column of water which measures the loss of pressure of the water in passing through the wheel, *Q* the discharge per second, *c* the velocity of efflux, and *v* the velocity of the wheel in the opposite direction, we have $c^2 - v^2 = 2\ g\ h$, and the rate of work of the wheel

FIG. 857.



$$L = \frac{(c - v)}{g} v\ Q \gamma \text{(see § 505)}.$$

If *R* is the resistance of the wheel, in consequence of the friction on the bearings, etc., we can put $L = R\ v$, and from it we obtain the formula

$$R = \left(\frac{c - v}{g}\right) Q\ \gamma,$$

or, if *F* denotes the sum of the areas of all the orifices of efflux, so that $Q = F\,c$ or $c = \frac{Q}{F}$, we can put

$$R = \left(\frac{Q}{F} - v\right)\frac{Q\ \gamma}{g}, \text{ from which we obtain}$$

$$v = \frac{Q}{F} - \frac{g\,R}{Q\,\gamma}.$$

If R were null, or at least very small, we could put $v = \frac{Q}{F}$, or assume the velocity v of rotation to be proportional to the discharge Q, which indeed it should be. If, on the contrary, $R = \psi\, v$, or if the resistance of the wheel increase with v, we will have

$$v + \frac{\psi g v}{Q \gamma} = \frac{Q}{F}, \text{ or}$$

$$v = \frac{Q}{F\left(1 + \frac{\psi g}{Q \gamma}\right)}, \text{ approximatively } = \frac{Q}{F}\left(1 - \frac{\psi g}{Q \gamma}\right).$$

If, then, the resistance R of the wheel is not very small, the velocity of rotation of the wheel is less than when R is null or negligible, and the instrument indicates too small a discharge.

If we put $v = 0$, we obtain for a discharge Q_0 the corresponding velocity of efflux

$$c_0 = \frac{g R}{Q_0 \gamma},$$

and we can then put, approximatively at least,

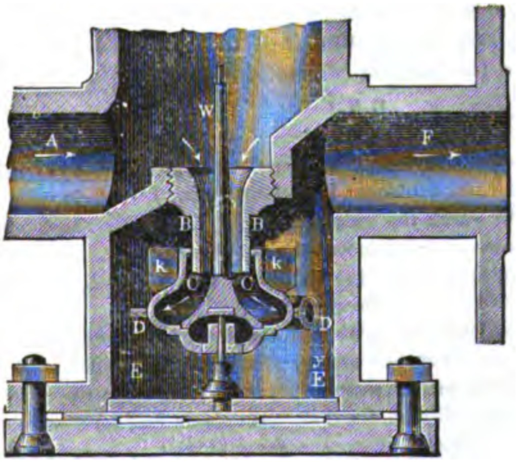
$$v = c - c_0 \text{ and}$$

$$Q = F(v + c_0) = \frac{\pi F r u}{30} + Q_0 = \mu u + Q_0,$$

r denoting the radius of the wheel, u the number of its rotations and μ a coefficient to be determined by experiment.

Within the last few years Siemens's water-meter has come into very general use; its principal parts are represented in cross-section in Fig. 858. The water which enters from A passes

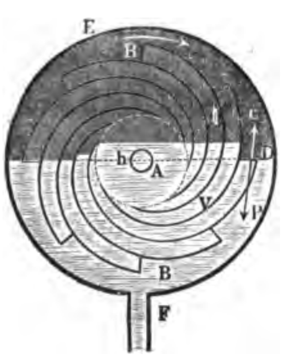
FIG. 858.



through the pipe $B\ B$ into the wheel $C\ C$ and is carried by the revolving tube $D\ D$ into the case $E\ E$, from which it is carried off by the pipe F. The shaft W of the wheel passes upwards through a stuffing-box and sets a train of wheel-work in motion by means of an endless screw fastened to its end. The wings k, k upon the wheel assist in regulating its motion of rotation by the resistance which they experience in moving in the water.

The reaction wheel can be constructed in such a manner that every time it makes a revolution it will allow a *certain quantity of water* to pass through. To accomplish this object, the wheel $B\ A\ B$, Fig. 859, is partially immersed in water, so that, when turning, the spiral tubes are alternately filled with air and water. Here also the water is conducted by a pipe into the centre of the wheel, and from thence by spiral pipes into the free space of the case $E\ F$, from which it flows away through the pipe F. The surface of the water in the interior of the wheel is at a distance h above that of the water in the case; hence, if the wheel turns in the direction indicated by the arrow, as soon as the orifice D arrives at the level of the water in the interior, the water begins to discharge, and in so doing reacts with a certain force P, by which the rotation of the wheel is maintained. If V is the volume of the water contained in one of the spiral pipes, and n the number of these canals, the discharge per second, when the number of rotations per minute of the volume of the water is u, is $Q = \frac{n\,u\,V}{60}$.

FIG. 859.



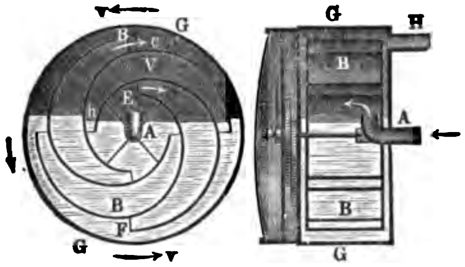
REMARK.—An account of Siemens' water-meter is given in the "Zeitschrift des Vereines deutscher Ingenieure," Vol. I, 1857, in which Jopling's water-meter (in which the water is gauged) is also described. See also the paper: "Siemens and Adamson's Patent Water Meter." A very peculiarly constructed water-meter of the nature of a reaction wheel is described in the "Génie industrielle," Tome XXI, No. 126, 1861, under the name: "Compteur hydraulique pour la mesure d'écoulement des liquides par Guyet." Two water-meters are described in the English work "Hydraulia," by W. Matthews. A *compteur hydraulique* used at the railroad station at Chartres is described in the "Bulletin de la Société d'encouragement," 51 year (1852) Uhler's apparatus for measuring fluids is treated of in

Dingler's Journal, Vol. 161. A description of an apparatus for measuring the quantity of spirit made in distilleries is contained in the "Mittheilungen des Gewerbevereines for Hannover," new series, 1861.

For a description of several kinds of water-meters, see "The Transactions of the Institution of Mechanical Engineers," 1856 (Tr.).

§ **507**. **Gas-meters.**—The so-called wet gas-meters (Fr. compteurs à gaz; Ger. Gasmesser or Gasuhren) are, like certain water-meters, small wheels with spiral canals, which are more than one-half immersed in water and are put in motion by the reaction of the gas passing through them; each spiral canal transfers a certain volume of gas from the inside to the outside. The essential parts of such a gas-meter are shown in the two sections of Fig. 860. The gas, which arrives, enters by a bent pipe *A* into the interior of the measuring wheel *B B*, in which it depresses the surface of the water a certain distance *h*, which depends upon the tension of the gas passing through the instrument. From this central chamber it enters successively the spiral canals, fills them almost entirely and, finally, passes out through the orifices at the circumference into the case *G G*, from which it is conducted by a pipe *H* to the point, where it is to be used. As we wish every spiral canal of the measuring wheel to carry over a certain definite quantity of gas at each revolution, we must so arrange the apparatus that at least one of the orifices of a canal shall always be under water; for in that case, when the gas is filling the canal, there is no efflux, and during the efflux no gas can enter it. The volume of gas V, passed by one spiral canal, is consequently a definite one, and we can, therefore, put the discharge per minute

FIG. 860.



$$Q = \frac{n\,u\,V}{60},$$

when the wheel makes n revolutions per minute. If we denote the height of the barometer in the gas leaving the machine by b, that in the gas entering it is $b + h$, and, therefore, according to Mariotte's law, the quantity of air in one spiral canal, measured at the pressure of the gas after it has left the measuring wheel, is

$$V_1 = \left(\frac{b+h}{b}\right) V,$$

consequently the quantity of gas, which passes from the wheel into the exterior case when the outlet of one of the spiral canals rises from the water, is

$$V_1 - V = \frac{h}{b} V.$$

When this quantity streams into the case the mechanical effect set free is

$$A = V p\, l \left(\frac{b+h}{b}\right)$$

(see § 388), and since $\frac{h}{b}$ is small, we can put

$$l\left(\frac{b+h}{b}\right) = l\left(1 + \frac{h}{b}\right) = \frac{h}{b};$$

hence, if the heaviness of the substance, with which the manometer is filled, is γ, we have $p = (b + h)\,\gamma = b\,\gamma$, and therefore $A = V h \gamma$.

One portion of this mechanical effect is expended in turning the wheel, and the rest in producing an eddy. The first portion is determined by the expression

$$A_1 = \frac{(c-v)\,v}{g} \cdot \frac{h}{b} V \gamma_1,$$

in which h denotes the mean height of the manometer, c the mean velocity of efflux, v the velocity of the wheel at its circumference and γ_1 the heaviness of the gas discharged. If R is the resistance of the wheel, reduced to its circumference, and r its radius, we have the required mechanical effect

$A_1 = R \cdot \frac{2\pi r}{n}$, and therefore we can put

$$\frac{(c-v)\,v}{g} \cdot \frac{h}{b} V \gamma_1 = \frac{2\pi r}{n} R, \text{ or since } 2\pi r = \frac{60\,v}{u},$$

$$\frac{c-v}{g} \cdot \frac{h}{b} V \gamma_1 = \frac{60\,R}{n\,u};$$

hence it follows that the velocity of rotation, corresponding to the distance h between the two surfaces of water, is

$$v = c - \frac{g\,b}{h V \gamma_1} \cdot \frac{60\,R}{n\,u}$$

and that the number of revolutions of the meter per minute is

$$u = \frac{30}{\pi r} \cdot \left(c - \frac{60\,g\,b\,R}{n\,u\,V h \gamma_1}\right).$$

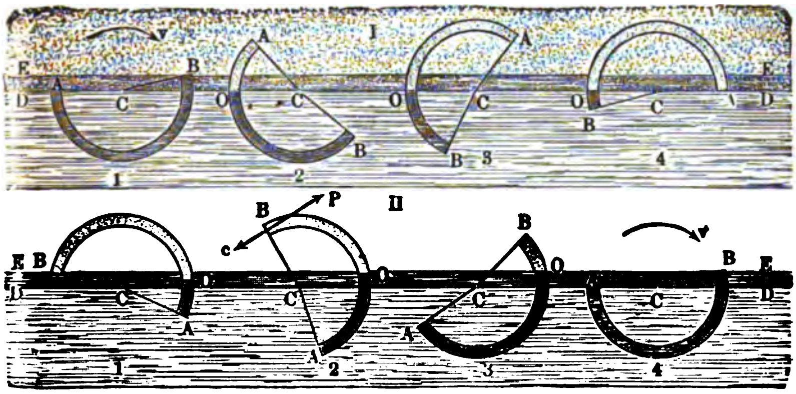
Approximatively we have $c = \sqrt{2\,g\,\frac{h\gamma}{\gamma_1}}$, when γ denotes the heaviness of the substance with which the manometer is filled. The volume of gas passing per minute is

$$Q = \frac{n\,u}{60}\,V,$$

and it is proportional to the number of revolutions u.

§ **508. Newer Gas-meters.**—Instead of placing the spiral canals of a gas-meter in a plane perpendicular to the axis, we can wind them round it like the thread of a screw. The action of such a gas-meter is shown by the two sections I and II, Fig. 861, in which $D\,D$ represents the surface of the water at the front and $E\,E$ that at the back of the measuring wheel, which is a horizontal drum. The orifice A of the spiral canal $A\ O\ B$ opens into the chamber, which is in front of the drum, and receives the gas, which is arriving; the orifice B, on the contrary, delivers the gas into the chamber at the back of the drum, from which it is carried off by a pipe. In Fig. 861, I, the different positions of a spiral canal, viewed from in front of the wheel, are represented. Fig. 861, II, on the contrary, represents the various positions of the canal as seen from the rear of the wheel. In consequence of the rotation of the wheel, in the direction indicated by the arrow, around the horizontal axis C, the inlet orifice A in (I, 1) is just emerging from the water in front, while the outlet B is just entering the water in the rear, in (I, 2) and (I, 3) the arcs $A\ O$, $A\ O$ of gas have entered through the orifice A, and in (I, 4) the orifice has re-entered the water, so that after a certain quantity V has been received into the canal, the entry of the gas is cut off. Shortly afterwards the orifice

Fig. 861.



B rises, as is represented in (II, 1), from the water in the rear of the drum and the discharge of the gas, which had previously been taken in, begins, and it is in full operation in the positions (II, 2) and (II, 3). When a new revolution begins, B re-enters the water in the rear of the drum, as is represented in (II, 4), and the gas again begins to fill the canal. During half a revolution of the spiral canal $A\ O\ B$, an arc of gas $A\ O$ (I, 4), which is at the greater tension $b + h$, enters the former and during the second half of the same it is transferred to the space beyond the wheel, where the pressure is less. In passing from the greater pressure to the less, the mechanical effect $A = V h \gamma$ is set free; a portion of this is expended in moving the wheel, as was shown in the foregoing paragraph. The general arrangement and action of such a gas-meter can be better understood from the ideal representation in Fig. 862. The gas is first introduced by means of a bent tube A into a chamber $B\ B$, which communicates in the middle around the axis of rotation C with the water in the case $E\ F\ G$, but upon the exterior circumference, where the spiral tubes enter it, it is air-tight. The drawing shows the spiral canal $H\ K$ to be receiving gas from $B\ B$ and the canal $L\ M$, which a short time before had received a certain volume of gas, to be discharging it at M into the upper space in the case $E\ F\ G$, from which it is carried away by the pipe F. By this arrangement of the meter the gas in the first chamber is cut-off entirely by the water from that in the rear chamber, and, therefore, the packing, which causes great loss of force, is rendered unnecessary. The other end D of the axis $C\ D$ of the wheel has a couple of turns of a screw cut upon it, by means of which the train of wheels of the counting apparatus is set in motion.

Fig. 862.

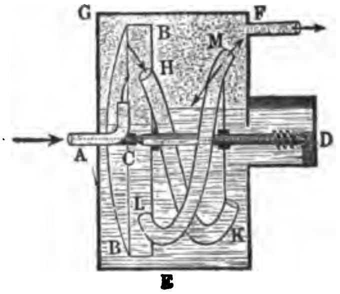
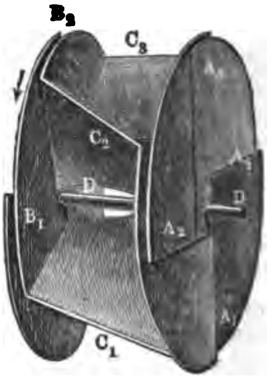


Fig. 863.



Crossley's gas-meters, which have come into very general use, are constructed according to the principles explained above; but their spiral canals are not tube-shaped, but real chambers or cells with spiral partitions and with triangular inlet and outlet orifices, which are made by bending out the end surfaces. Fig. 863 is a perspective view of such a wheel with the cover removed; it consists of 4 pieces of sheet iron like that represented in Fig. 864. A_1, A_2, A_3, A_4 are the inlet orifices, B_1, B_2 . . . the outlet orifices and C_1, C_2, C_3 . . . the partitions of the measuring wheel which turns around the axis *D D*. Fig. 865 is an elevation of the gas-meter with the exterior drum or case; we observe at *K* the bent tube, which conducts the gas into the chamber, and at *Z* the pipe, which carries off the gas from the upper space *A A* of the case of the meter. The gas does not flow directly into *K*, but the pipe *E* carries it first into a chamber *F*, from which it passes through the conical valve *i* into the chamber *G*, where it enters the upper part of the vertical pipe *H*, through which it is conducted into the bent tube *K*. The surface of the water in the chamber *G* reaches exactly to the top of the pipe *H*, through which the superfluous water overflows into a reservoir *L*. In order, on the other hand, to prevent the water from sinking too low, a float is placed in the chamber, which, when it sinks, carries the valve *i* with it and closes the opening, when the float has sunk a certain distance. The discharge of gas then ceases entirely, and we are thus noti-

FIG. 864.

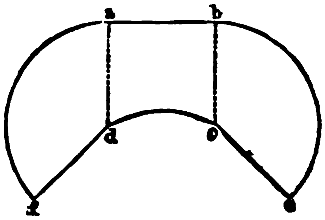
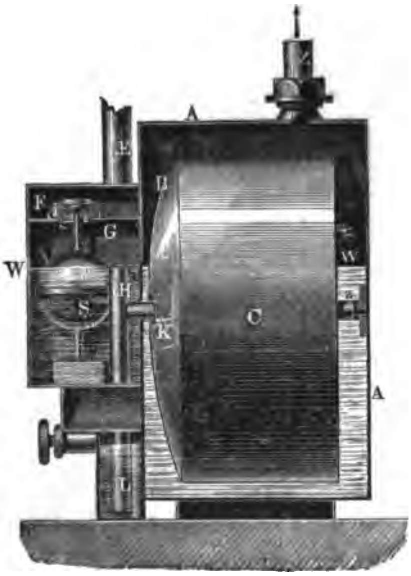


FIG. 865.

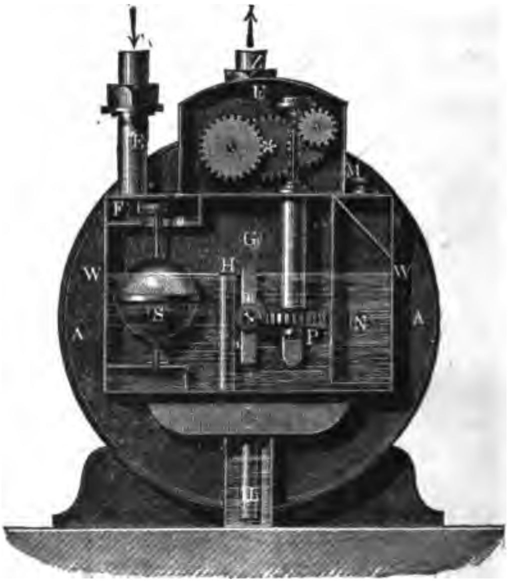


fied that it is necessary to fill the meter with water through an orifice *M*, that opens into a chamber *N*, which communicates, at the bottom only, with the water space.

Fig. 866 is transverse elevation of the front of such a meter, in which are to be seen not only the chamber *N* with the orifice *M*, but also the clockwork of the counting apparatus, which is set in motion by an endless screw upon the axle of the drum and a vertical shaft with a cog-wheel upon it.

An important resistance to the motion of Crosley's gas-meter is that occasioned by the entry and exit of the water through the narrow triangular orifices. We can calculate from the area F of an inlet or outlet orifice and from the discharge per second, which can be put equal to the volume Q of the gas, the velocity of exit and entrance $v_1 = \frac{Q}{F}$, and consequently the corresponding loss of mechanical effect per second

FIG. 866.



$$L_1 = 2\,\frac{v_1^2}{2g}\,Q\,\gamma = \left(\frac{Q}{F}\right)^2 \frac{Q\,\gamma}{g}.$$

REMARK.—Particulars upon the subject of gas-meters can be found in Schilling's "Handbuch der Steinkohlengasbeleuchtung," and Heeren's article "die Einrichtung der Gasuhren" in the "Mittheilungen des Gewerbevereins für das K. Hannover," year 1859. A new gas-meter by Hansen is described in the "Journal der Gasbeleuchtung," 1861.

§ 509. Action of Unlimited Fluids.—If a body has a motion of translation in an *unlimited fluid*, or if a body is placed in a *moving fluid*, it is subjected to a pressure, which is dependent upon the form and size of the body as well as upon the density of the fluid and the velocity of one or other of the masses; in the former case it is called the *resistance* and in the latter the *impulse* of the fluid. This hydraulic pressure is principally due to the inertia of the water, whose condition of motion is changed when it comes into contact with a rigid body, and also to the force of cohesion of the molecules of water, which are partially separated from and moved upon each other.

If a body *A C*, Fig. 867, is moved in *still water*, it pushes a certain quantity of water, the pressure of which is increased, before it. As the body progresses the quantity of water on one side is increased, while upon the other it is constantly flowing away, and the particles lying immediately contiguous to the surface *A B* assume a motion in the direction of this surface. If a *stream of water* encounters an obstacle *A C*, Fig. 868, which is at rest, the pressure of the water in front of it is increased, the molecules of water are diverted from their original direction and move along the front surface *A B*. When the particles of water have reached the edges of the front surface, they turn and follow the sides of the body, until they arrive at the back surface, where they do not immediately reunite, but assume first an eddying motion. We see that the general relations of the motion of the molecules, which surround the body, are the same for the impulse of water as for the resistance to a body moving in the water; but there is a difference in the eddies, when the body is short; for in the latter case the eddies occupy less space than in the former. The velocity of the molecules of water increases gradually from the centre of the front surface to the edges, where a contraction generally takes place and where the velocity is a maximum, it decreases as the water passes along the sides and becomes a minimum when the water arrives at the back surface and begins its eddying motion.

FIG. 867.

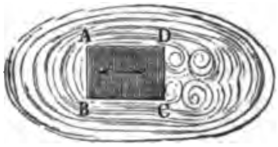
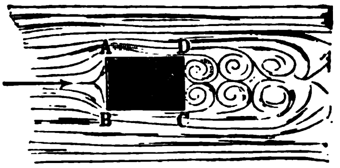


FIG. 868.



§ **510. Theory of Impulse and Resistance.**—The normal pressure of still or moving water upon a body moved or immersed in it is very different at different points of the body. This pressure is a maximum at the centre of the front surface and a minimum in the centre of the rear surface and at the beginning of the sides; for at the first point the water flows towards the body, and at the latter points it flows away from it. If the body is, as we will suppose in what follows, symmetrical in reference to the direction of motion, the pressures at right angles to this direction balance each other, and we must, therefore, consider only the pressures in the direction of the motion. But since the pressure upon the rear surface acts in an opposite direction to those upon the front surface, it follows *that the resulting impulse or resistance of the water is equal to the difference between the pressures upon the front and back surfaces.*

Although we cannot determine *a priori* the intensity of this pressure, yet, as the circumstances are very similar to those of the impact of an isolated stream, we can at least assume that the general law of the impact of an unlimited stream does not differ very much from that of an isolated stream. If F is the area of surface which an unbounded stream, whose heaviness is γ and whose velocity is v, encounters, we can put the corresponding impulse or hydraulic pressure

$$P = \zeta \frac{v^2}{2g} F \gamma,$$

in which ζ denotes an empirical number dependent upon the shape of the surface. This formula can be applied not only to the front, but also to the rear surface. But in the latter case, where the water tends to separate itself from the body, the expression becomes negative. Now if $F h \gamma$ is the hydrostatic pressure (§ 690) against the front and against the back surfaces of a body, the total pressure against the front surface is

$$P_1 = F h \gamma + \zeta_1 . \frac{v^2}{2g} F \gamma,$$

and that against the back surface is

$$P_2 = F h \gamma - \zeta_2 . \frac{v^2}{2g} F \gamma;$$

hence the resulting impulse or resistance of the water is

$$P = P_1 - P_2 = (\zeta_1 + \zeta_2) . \frac{v^2}{2g} F \gamma = \zeta . \frac{v^2}{2g} F \gamma,$$

when we put $\zeta_1 + \zeta_2 = \zeta$.

This general formula for the *impulse and resistance of an unlimited stream* is also applicable to the *impulse of wind* and to the *resistance of the air.* Here, however, besides the difference of the aërodynamic pressures upon the front and rear surfaces, a difference in the aërostatic pressure also exists, which is due to the fact that the air at the front surface has a greater heaviness (γ), in consequence of its greater tension, than that at the rear surface. For this reason, at least when the velocities are great, as in the case of musket and cannon balls, the coefficient of resistance of the air is greater than that of water.

REMARK.—A peculiar phenomenon attends the impulse and resistance of an unlimited medium (water or air), viz., a certain quantity of water or air attaches itself to the body, the influence of which is shown by the variable motion of the body, which, E.G., is very evident in the oscillations of a pendulum. The quantity of air or water which attaches itself to a sphere is 0,6 the volume of the sphere. For a prismatic body, moving in the direction of its axis, the ratio of these volumes is

$$= 0{,}13 + 0{,}705 \frac{\sqrt{F}}{l},$$

in which l denotes the length and F the cross-section of the body. This ratio, which was first determined by du Buat, has been fully confirmed by the later experiments of Bessel, Sabine and Bailly.

§ 511. Impulse and Resistance against Surfaces.—The coefficient ζ of resistance, or the number by which the height $\frac{v^2}{2g}$ due to the velocity must be multiplied, in order to obtain the height of the column of water which measures the hydraulic pressure, is very different for bodies of different form; it is determined approximatively only for plates, which are placed at right angles to the stream. According to du Buat's experiments and those of Thibault, we can put for the impulse of water and air against a plane surface at rest $\zeta = 1{,}86$, while, on the contrary, we can assume with less certainty for the resistance of the air and water to a plane surface in motion $\zeta = 1{,}25$. In both cases about two-thirds of the action is upon the front and about one-third upon the rear surface. The values, found for the resistance offered by the air to a body revolving in a circle by Borda, Hutton, and Thibault, vary much from each other. The latter found with a rotating plane surface, the area of which was 0,1 square meter, the resistance

$$P = 0{,}108\, F v^2, \text{ whence}$$

$$\zeta = 0{,}108 \,.\, \frac{2g}{\gamma} = 0{,}108 \,.\, \frac{19{,}62}{1{,}25} = 1{,}70.$$

This coefficient is, according to these experiments, almost constant, when the angle a formed by the surface with the direction of the motion is not less than 45°. When the angle is less than 45°, the coefficient diminishes with this angle of impact, and for $a = 10°$, ζ is only $= 0,53$. According to the researches of Didion, etc., we have for the resistance of rotating plane surfaces, whose areas are $0,2 \,.\, 0,2 = 0,04$ square meters,

$$\zeta = (0,1002 + 0,0434\, v^{-2}) \,.\, \frac{2\,g}{\gamma} = 1,573 + 0,681\, v^{-2},$$

in which v must be given in meters.

For a plane surface, whose area was one square meter, Didion found, when the motion was vertical, the coefficient of resistance

$$\zeta = (0,084 + 0,036\, v^{-2}) \,.\, \frac{2\,g}{\gamma} = 1,318 + 0,565\, v^{-2},$$

while Thibault, on the contrary, found for such surfaces, when their area was 0,1 to 0,2 square meters,

$$\zeta = (0,1188 + 0,036\, v^{-2}) \,.\, \frac{2\,g}{\gamma} = 1,865 + 0,565\, v^{-2}.$$

The foregoing formulas hold good only when the motion of the surface is uniform; if the motion is *variable*, they require an addition. If the velocity of a body which is moving in a resisting medium changes, the quantity of the fluid moved by the body or carried along with it varies; the resistance is, therefore, dependent upon the acceleration p. According to the experiments of Didion, etc., with a surface whose area was 1 square meter, and with one whose area was $\frac{1}{4}$ square meter, which were moved in a vertical line, the resistance was

$$P = (0,084\, v^2 + 0,036 + 0,164\, p)\, F; \text{ hence}$$

$$\zeta = [0,084 + (0,036 + 0,164\, p) v^{-2}] \,.\, \frac{2\,g}{\gamma}$$

$$= 1,318 + (0,565 + 2,574) v^{-2}.$$

We must also remember that for variable motion the mean square of the velocity is different from the square of the mean velocity.

The impulse and resistance of an unlimited medium is increased when the surfaces are hollowed out or provided with borders; but we have as yet no general data concerning the subject.

For a parachute, whose cross-section was 1,2 square meters and whose mean diameter was 1,27 meters and whose depth was 0,430 meter, Didion, etc., found for an accelerated motion, during which the hollow surface was in front,

$$P = (0,163\, v^2 + 0,070 + 0,142\, p)\, F, \text{ whence}$$

$$\zeta = 2,559 + (1,099 + 2,229\, p)\, v^{-2}.$$

§ 512. Impulse and Resistance against Bodies.—The impulse and resistance of water against *prismatical bodies,* whose axis coincides with the direction of motion, decrease when the lengths of the bodies increase. According to the experiments of du Buat and Duchemin, the impulse upon the front surface is constant, and the action upon the rear surface alone is variable. The coefficient $\zeta_1 = 1,186$ corresponds to the former; but when the relative lengths are

$$\frac{l}{\sqrt{F}} = 0, \quad 1, \quad 2, \quad 3,$$

the total action is

$$\zeta = 1,86;\ 1,47;\ 1,35;\ 1,33.$$

If the ratio between the length and the mean width $\sqrt{F}$ becomes greater, the coefficient ζ again increases in consequence of the friction of the water upon the sides of the body. The reverse is true of the resistance of the water. In this case, according to du Buat, the constant action against the front surface is $\zeta_1 = 1$, and the total action for

$$\frac{l}{\sqrt{F}} = 0, \quad 1, \quad 2, \quad 3, \text{ is}$$

$$\zeta = 1,25;\ 1,28;\ 1,31;\ 1,33;$$

so that for a prism three times as long as wide the *impulse* of the water is the same as the *resistance.*

The experiments of Newton, Borda, Hutton, Vince, Désaguilliers and others with round and angular bodies leave much uncertain and undetermined. It appears that for moderate velocities the coefficient of resistance of spheres can be put = 0,5 to 0,6. But when the velocities are greater and the motion takes place in the air, we can put, according to Robins and Hutton, for the velocities

$v = 1,\quad 5,\quad 25,\quad 100,\quad 200,\quad 300,\quad 400,\quad 500,\quad 600$ meters,
$\zeta = 0,59;\ 0,63;\ 0,67;\ 0,71;\ 0,77;\ 0,88;\ 0,99;\ 1,04;\ 1,01.$

Duchemin and Piobert have given particular formulas for the increase of this coefficient of resistance. According to Piobert the resistance to a musket ball in the air is

$P = 0,029\ (1 + 0,0023\ v)\ F v^2$ kilograms, whence
$\zeta = 0,451\ (1 + 0,0023\ v).$

For the impulse of water against a ball, Eytelwein found

$\zeta = 0,7886,$

while, on the contrary, according to the experiments of Piobert, etc., made with cannon balls 0,10 to 0,22 meters in diameter, the resistance to the balls in water is

$P = 23,8\ F v^2$ kilograms; hence we can put
$\zeta = 0,467.$

The coefficients of resistance for bodies *partially immersed* are different from those for bodies entirely surrounded by water. For a *floating prismatic body* five to six times as long as wide and moving in the direction of the axis, ζ should be put equal to 1,10. If the body is sharpened in front by two vertical planes like $A\ B\ C$, Fig. 869, ζ increases with the angle $A\ C\ A = \beta$, and we have

for $\beta =$	180°	156°	132°	108°	84°	60°	36°	12°
$\zeta =$	1,10	1,06	0,93	0,84	0,59	0,48	0,45	0,44

If, on the contrary, the rear portion $A\ C\ B$, Fig. 870, is sharpened, and if the angle $B\ C\ B = \beta$, we have

FIG. 869.

FIG. 870.

for $\beta =$	180°	138°	96°	48°	24°
$\zeta =$	1,10	1,03	0,98	0,95	0,92

When both front and rear portions of the floating body are sharpened, ζ becomes still smaller. For river steamboats, $\zeta = 0,12$ to 0,20 and for large ocean steamers, $\zeta = 0,05$ to 0,10.

REMARK.—This subject is treated at length by Poncelet in his "Introduction" cited above, and by Duchemin and Thibault in their "Recherches expèrimentales, etc." The subject of the resistance to floating bodies, particularly ships, and also that of the impulse of the wind against wheels, will be treated in the second and third volumes.

EXAMPLE.—If, according to Borda, we put the resistance and impulse at right angles to the axis of a cylinder $\frac{1}{2}$ that against a parallelopipedon, which has the same dimensions as it, we have the coefficient of resistance

$$\zeta = \tfrac{1}{2} \,.\, 1,28 = 0,64$$

and the impulse against the same

$$= \tfrac{1}{2} \,.\, 1,47 = 0,735.$$

If we apply these values to the human body, the area of the cross-section of which is 7 square feet, we find the resistance and impulse of the air against it

$$P = 0,64 \,.\, 0,0155 \,.\, 7 \,.\, 0,086\, v^2 = 0,00597\, v^2$$

and

$$P = 0,735 \,.\, 0,0155 \,.\, 7 \,.\, 0,086\, v^2 = 0,00686\, v^2.$$

For a velocity of 5 feet, the resistance of the air is, therefore, only 0,00597 . 25 = 0,1492 pounds, and the corresponding work done per second is = 5 . 0,1492 = 0,746 foot-pounds; for a velocity of 10 feet, the resistance is 4 times and the expenditure of mechanical effect 8 times as great, and for a velocity of 15 feet the resistance is 9 times and the work done 27 times greater. If a man moves with the velocity 5 feet against a wind, whose velocity is 50 feet, he has to overcome a resistance $0,00686 \,.\, 55^2 =$ 20,75 pounds, which corresponds to a velocity of 50 + 5 = 55 feet, and to perform an amount of work equal to 20,75 . 5 = 103,75 foot-pounds.

§ 513. Motion in Resisting Media.—The *laws of the motion of a body in a resisting medium* are not very simple; for the force in this case is variable, increasing with the square of the velocity. From the force P, which is drawing the body onwards, and from the resistance $P_1 = \zeta \cdot \frac{v^2}{2g} F\gamma$, offered by the medium, we obtain the motive force

$$P_0 = P - P_1 = P - \zeta \cdot \frac{v^2}{2g} F\gamma,$$

but since the mass of the body is $M = \frac{G}{g}$, its acceleration is

$$p = \frac{P_0}{M} = \left(P - \zeta \frac{v^2}{2g} F\gamma\right) : M = \left(\frac{P - \zeta \frac{v^2}{2g} F\gamma}{G}\right) \cdot g,$$

or if we denote $\frac{\zeta F\gamma}{2gP}$ by $\frac{1}{w^2}$, or put $\sqrt{\frac{2gP}{\zeta F\gamma}} = w$, we have

$$p = \left[1 - \left(\frac{v}{w}\right)^2\right] \frac{P}{G} g.$$

The maximum velocity which the body can assume is

$$v = w = \sqrt{\frac{2gP_1}{\zeta F\gamma}}$$

If the motive force P is constant, the motion approaches gradually a uniform one; for the acceleration becomes smaller and smaller as v increases.

Now the velocity v increases, when the acceleration is p, in an element of time τ a quantity $\kappa = p\,\tau$, hence we can put

$$\kappa = \left[1 - \left(\frac{v}{w}\right)^2\right] \frac{P}{G} g\,\tau, \text{ or inversely}$$

$$\tau = \frac{G}{P} \cdot \frac{\kappa}{g\left[1 - \left(\frac{v}{w}\right)^2\right]}.$$

In order to find the time, corresponding to a given variation of velocity, let us divide the difference $v_n - v_0$ between the initial and the final velocities in n parts and put such a part

$$\frac{v_n - v_0}{n} = \kappa$$

and then calculate from it the velocities

$$v_1 = v_0 + \kappa,\ v_2 = v_0 + 2\,\kappa,\ v_3 = v_0 + 3\,\kappa, \text{ etc.,}$$

substituting these values in Simpson's formula, we obtain the required time, when we assume four divisions,

$$1)\ t = \frac{G}{P} \cdot \frac{v_n - v_0}{12\,g}\left(\frac{1}{1 - \left(\frac{v_0}{w}\right)^2} + \frac{4}{1 - \left(\frac{v_1}{w}\right)^2} + \frac{2}{1 - \left(\frac{v_2}{w}\right)^2} + \frac{4}{1 - \left(\frac{v_3}{w}\right)^2} + \frac{1}{1 - \left(\frac{v_4}{w}\right)^2}\right).$$

The space described in an element τ of time (§ 19) is

$$\sigma = v\,\tau, \text{ or since we can put } \tau = \frac{\kappa}{p},$$

$$\sigma = \frac{v\,\kappa}{p}, \text{ or}$$

$$\sigma = \frac{v\,\kappa}{1 - \left(\frac{v}{w}\right)^2} \cdot \frac{G}{Pg}.$$

By employing Simpson's rule we find the space described, while the velocity changes from v_0 to v_n, to be

$$2)\ s = \frac{G}{P} \cdot \frac{v_n - v_0}{12\,g}\left(\frac{v_0}{1 - \left(\frac{v_0}{w}\right)^2} + \frac{4\,v_1}{1 - \left(\frac{v_1}{w}\right)^2} + \frac{2\,v_2}{1 - \left(\frac{v_2}{w}\right)^2} + \frac{4\,v_3}{1 - \left(\frac{v_3}{w}\right)^2} + \frac{v_4}{1 - \left(\frac{v_4}{w}\right)^2}\right).$$

The calculation is of course more accurate when we make 6, 8, or more divisions. This formula allows us to take into account the variability of the coefficient of resistance, which is necessary for high velocities. For the free fall of a body in air or water $P = G$, the apparent weight of the body, and for motion in a horizontal plane $P = 0$, or more correctly, equal to the friction $f\,G$. Since this is a resistance, it must be introduced as a negative quantity in the calculation; hence we must put

$$P_0 = -\,(P + P_1) \text{ and}$$
$$p = -\left[1 + \left(\frac{v}{w}\right)^2\right]\frac{P}{G}\,g.$$

Since in this case there can be no question of an increase, but only of a decrease of velocity, we must substitute $v_0 - v_n$ in the above formulas instead of $v_n - v_0$.

When a body is impelled by a force, such as its own weight, the motion approaches more and more to a uniform one, and after a certain time it may be considered as such, although it never will be really so. The acceleration p becomes $= 0$, when $\zeta\,\frac{v^2}{2g}\,F\gamma = P_0$, or when

$$v = \sqrt{\frac{2\,g\,P_0}{\zeta\,F\,\gamma}} = w.$$

A body falling freely in air approaches more and more to this result without ever attaining it.

EXAMPLE.—Piobert, Morin and Didion found for a parachute whose depth was 0,31 times the diameter of the opening, the coefficient of resistance $\zeta = 1{,}94 \,.\, 1{,}37 = 2{,}66$. From what height can a man weighing 150 pounds descend with such a parachute weighing 10 pounds and with a cross-section of 60 feet, without assuming a greater velocity than that he attains when he jumps down 10 feet? The latter velocity is $v = 8{,}025\sqrt{10} = 25{,}377$ feet, the force $P = G = 150 + 10 = 160$ lbs., the surface $F = 60$ feet, the heaviness $\gamma = 0{,}0807$ pounds and the coefficient of resistance $\zeta = 2{,}66$, hence

$$\frac{1}{w^2} = \frac{2{,}66 \,.\, 60 \,.\, 0{,}0807}{64{,}4 \,.\, 160} = \frac{1{,}33 \,.\, 3 \,.\, 0{,}0807}{64{,}4 \,.\, 4} = 0{,}00125$$

and $$\frac{v^2}{w^2} = 0{,}00125 \,.\, 25{,}377^2 = 0{,}805.$$

If we assume six divisions, we obtain

$$1 - \frac{v^2}{w^2} = 0{,}977639;\ 0{,}91055;\ 0{,}79875;\ 0{,}64222;\ 0{,}44097;\ 0{,}195,$$

and $$\frac{v}{1 - \frac{v^2}{w^2}} = 0;\ 4{,}326;\ 9{,}290;\ 15{,}886;\ 26{,}343;\ 47{,}958, \text{ and } 130{,}188;$$

hence, according to Simpson's rule, we have the mean value

$$= (1 \,.\, 0 + 4 \,.\, 4{,}326 + 2 \,.\, 9{,}290 + 4 \,.\, 15{,}886 + 2 \,.\, 26{,}343$$
$$+ 4 \,.\, 47{,}958 + 1 \,.\, 130{,}138) : (3 \,.\, 6) = \frac{474{,}084}{18} = 26{,}338;$$

and the required space, through which he can fall, is

$$s = \frac{v_n - v_0}{g} \text{ times the mean of } \frac{v}{1 - \frac{v^2}{w^2}} = \frac{25{,}377 - 0}{32{,}2} \,.\, 26{,}338 = 20{,}76 \text{ ft.}$$

The corresponding duration of the fall, since the mean value of $\frac{1}{1-\frac{v^2}{w^2}}$ is

$$= (1 . 0 + 4 . 1{,}023 + 2 . 1{,}098 + 4 : 1{,}252$$
$$+ 2 . 1{,}557 + 4 . 2{,}268 + 1 . 5{,}128) : 18 = 1{,}589, \text{ is}$$
$$t = \frac{25{,}377}{32{,}2} . 1{,}589 = 1{,}25 \text{ seconds.}$$

REMARK.—If the coefficient of resistance is constant, we obtain by the aid of the Calculus for the case of a body falling freely

$$v = \left(\frac{e^{\mu t} - 1}{e^{\mu t} + 1}\right) w = \left(\frac{e^{\mu t} - 1}{e^{\mu t} + 1}\right) \sqrt{2 g . \frac{G}{\zeta F \gamma}}$$

and

$$s = l\left(\frac{(e^{\mu t} + 1)^2}{4 e^{\mu t}}\right) \frac{w^2}{2 g} = l\left(\frac{(e^{\mu t} + 1)^2}{4 e^{\mu t}}\right) \frac{G}{\zeta F \gamma}$$
$$= l\left(\frac{w^2}{w^2 - v^2}\right) . \frac{w^2}{2 g},$$

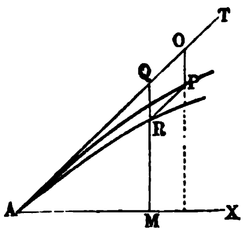
in which

$$\mu = \sqrt{2 g . \zeta \frac{F \gamma}{G}},$$

and e denotes the base of the Naperian system of logarithms and l the Naperian logarithm.

§ **514. Projectiles.**—We have already studied the *motion of projectiles* in vacuo and found in § 39 the path or trajectory to be a parabola. We can now investigate this motion in a resisting medium, E.G., the motion of a body projected in the air.

FIG. 871.



The path of a body projected through air is certainly not a parabola, as is the case when it is projected in vacuo, but an *unsymmetrical curve ;* the portion of the trajectory, where the body is rising, is not so steep as that where it is falling, as can be seen from what follows. During the instant τ the body, *which is rising* with a velocity v in the direction $A\ T$, Fig. 871, describes, in consequence of its inertia, the space

$$A\ O = s = v\ \tau,$$

and, in consequence of gravity, the vertical space

$$O\ P = h = \frac{g\ \tau^2}{2},$$

and the first space is diminished by the resistance $\zeta \frac{c^2}{2 g} F \gamma$ of the air an amount, which can be determined by the expression

$$O\,Q = \frac{\zeta \frac{v^2}{2g} F\gamma}{G} \cdot \frac{g\tau^2}{2} = \zeta \frac{F g}{2G} \cdot \frac{v^2 \tau^2}{2}.$$

If we put $\zeta \frac{F\gamma}{2G} = \mu$, we have more simply

$$O\,Q = \mu \frac{v^2 \tau^2}{2}.$$

The fourth corner R of the parallelogram $O\,P\,Q\,R$, constructed with $O\,P$ and $O\,Q$, gives the position which the body occupies at the end of the time τ, while P is the place which the body would have occupied at that moment, if the air offered no resistance. The path $A\,R$ of the projectile passes, therefore, below the parabola, which the body would have described in vacuo.

In like manner we have for a body *descending* with the initial velocity v in the direction $A\,T$, Fig. 872, the spaces described simultaneously in the time τ

$$A\,O = v\,\tau,$$

$$O\,P = g\,\frac{\tau^2}{2}, \text{ and}$$

$$O\,Q = \mu\,v^2\,\frac{\tau^2}{2},$$

and from the above we obtain again the position R occupied by the body at the end of this time, and the position P which it would have occupied, if its motion had taken place *in vacuo*. The path $A\,R$ described in this case passes also below the parabolic path $A\,P$, which the body would have followed, if the air opposed no resistance.

If the angle of inclination, at which a body rises with the initial

FIG. 872.

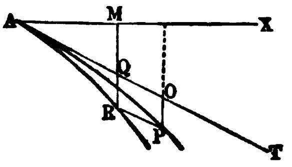
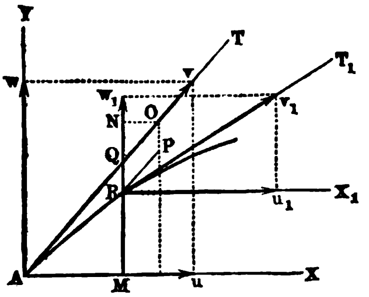


FIG. 873.



velocity v from A, is $T\,A\,X = \alpha$, Fig. 873, the initial co-ordinates or velocities in the direction of the axes are

$$u = v \cos. a \text{ and}$$
$$w = v \sin. a,$$

and we have for the position R of the moving body, after an instant τ, the abscissa

$$A\,M = x = A\,Q \cos. a = \left(v\,\tau - \frac{\mu\,v^2\,\tau^2}{2}\right) \cos. a$$
$$= \left(1 - \frac{\mu\,v\,\tau}{2}\right) v\,\tau \cos. a,$$

and the ordinate

$$M\,R = y = A\,Q \sin. a - Q\,R = \left(1 - \frac{\mu\,v\,\tau}{2}\right) v\,\tau \sin. a - \frac{g\,\tau^2}{2}.$$

The velocity in the direction of the abscissa is

$$\overline{R\,u_1} = u_1 = v \cos. a - \mu\,v^2\,\tau \cos. a = (1 - \mu\,v\,\tau)\,v \cos. a,$$

and that in the direction of the ordinate is

$$\overline{R\,w_1} = w_1 = v \sin. a - \mu\,v^2\,\tau \sin. a - g\,\tau = (1 - \mu\,v\,\tau)\,v \sin. a - g\,\tau.$$

From the two velocities we obtain the angle of inclination $T_1\,R\,X_1 = a_1$ of the path at R by means of the formula

$$tang.\, a_1 = \frac{w_1}{u_1} = tang.\, a - \frac{g\,\tau}{(1 - \mu\,v\,\tau)\,v \cos. a},$$

and the velocity in the direction of the curve is

$$\overline{R\,v_1} = v_1 = \sqrt{u_1^2 + w_1^2} = \sqrt{(1-\mu\,v\,\tau)^2 v^2 - 2\,(1-\mu\,v\,\tau)\,v\,g\,\tau \sin. a + g^2\,\tau^2}.$$

By repeated application of this formula, we can find the course of the whole trajectory of the projectile. If, E.G., we substitute in the above formulas for x and y, instead of a and v the values for a_1 and v_1 obtained from the last equation, we obtain the co-ordinates x_1 and y_1 of a new point referred to R, etc.

EXAMPLE.—A massive cast-iron cannon-ball, whose diameter is $2\,r = 4$ inches, is projected at an angle of elevation $a = 25°$ with a velocity $v = 1000$ feet; required the position of the same after $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}$, of a second, etc.

Since the weight of a cubic foot of air is 0,080728 pounds and that of a cubic foot of cast iron is 444 pounds, we have

$$\mu = \frac{F\,\gamma}{2\,G} \cdot \zeta = \frac{\pi\,r^2\,\gamma}{\frac{4}{3}\,\pi\,r^3\,\gamma_1}\,\zeta = \frac{3}{4}\,\frac{\gamma}{r\,\gamma_1}\,\zeta = \frac{3}{4}\,.\,6\,.\,\frac{0{,}080728}{444}\,\zeta = 0{,}000409094\,\zeta,$$

and, therefore, for $v = 1000$ feet, for which $\zeta = 0{,}9$ (see § 512), we have

$$\mu = 0{,}0003682.$$

If we take $\tau = 0{,}1$ seconds, we obtain

$$x = (1 - 0{,}0003682\,.\,1000\,.\,0{,}05)\,100 \cos. 25° = 0{,}98159\,.\,90{,}63 = 88{,}96 \text{ feet},$$

$$y = 0{,}98159\,.\,100 \sin. 25° - 32{,}2\,.\,\frac{0{,}01}{2} = 0{,}98159\,.\,42{,}26 - 0{,}16 = 41{,}32 \text{ feet},$$

and

$$tang.\, a_1 = tang.\, 25^\circ - \frac{32,2\,.\,0,1}{(1 - 0,03682)\,.\,906,3} = 0,46631 - \frac{3,22}{0,96318\,.\,906,3}$$
$$= 0,46631 - 0,00369 = 0,46262\,;$$

hence the angle of elevation is

$$a = 24^\circ\ 50',$$

and the velocity in the curve is

$$v_1 = \sqrt{(0,96318\,.\,1000)^2 - 2\,.\,0,96318\,.\,1000\,.\,32,2\,.\,0,04226 + (3,22)^2}$$
$$= \sqrt{927716 - 2621 + 10} = \sqrt{925105} = 961,82 \text{ feet.}$$

If we again take $\tau = 0,1$ second, we have, since for $v = 962$ feet, $\zeta = 0,88$, and consequently $\mu = 0,88\,.\,0,000409094 = 0,00036$,

$$x_1 = (1 - 0,00036\,.\,961,8\,.\,0,05)\,.\,96,18\, cos.\, 24^\circ\ 50'$$
$$= 0,9827\,.\,96,18\,.\,0,9075 = 85,77 \text{ feet,}$$
$$y_1 = 0,9827\,.\,96,18\, sin.\, 24^\circ\ 50' - 0,161 = 39,53 \text{ feet,}$$

and

$$tang.\, a_2 = tang.\, 24^\circ\ 50' - \frac{3,22}{0,96537\,.\,961,8\, cos.\, 24^\circ\ 50'}$$
$$= 0,46277 - 0,00382 = 0,45895,$$

whence

$$a = 24^\circ\ 39' \text{ and}$$

$$v = \sqrt{(0,96537\,.\,961,8)^2 - 2\,.\,0,96537\,.\,961,8\,.\,32,2\,.\,0,04200 + (3,22)^2}$$
$$= \sqrt{862099 - 2511 + 10} = \sqrt{859598} = 927,14 \text{ feet.}$$

Assuming once more $\tau = 0,1$ and $v = 927$ feet, we have $\zeta = 0,87$

$$\mu = 0,87\,.\,0,000409094 = 0,0003559,$$

and therefore

$$x_2 = (1 - 0,0003559\,.\,927,14\,.\,0,05)\,.\,92,71\, cos.\, 24^\circ\ 39' = 0,9835\,.\,92,71\,.\,0,9089$$
$$= 82,87 \text{ feet and}$$

$$y_2 = 0,9835\,.\,92,71\, sin.\, 24^\circ\ 39' - 0,156 = 37,87 \text{ feet.}$$

The position of the projectile in reference to the point of beginning is determined after 0,3 seconds by the co-ordinates

$x + x_1 + x_2 = 88,96 + 85,77 + 82,87 = 257,60$ feet and
$y + y_1 + y_2 = 41,32 + 39,53 + 37,87 = 118,72$ feet.

If the air offered no resistance and gravity did not act, we would have
$x + x_1 + x_2 = c\,t\, cos.\, a = 1000\,.\,0,3\,.\, cos.\, 25^\circ = 300\,.\,0,9063 = 271,89$ feet and
$y + y_1 + y_2 = c\,t\, sin.\, a = 300\,.\, sin.\, 25^\circ = 300\,.\,0,4226 = 126,78$ feet.
If we neglect the resistance of the air only, we have
$x + x_1 + x_2 = 271,89$ feet and

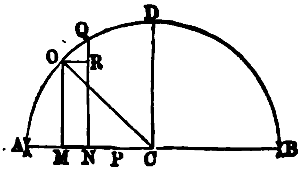
$$y + y_1 + y_2 = 126,78 - \frac{g\,t^2}{2} = 126,78 - 32,2\,.\,\frac{0,09}{2} = 126,78 - 1,449$$
$$= 125,33 \text{ feet.}$$

APPENDIX.

THE THEORY OF OSCILLATION.

(§ 1.) **Theory of Oscillation.**—A body has an *oscillatory or vibratory motion* (Fr. mouvement oscillatoire; Ger. schwingende Bewegung) or is in *oscillation* or *vibration* (Fr. oscillation; Ger. Schwingung), when it describes repeatedly the same path backwards and forwards in equal times. We meet with many examples of oscillatory motion in nature besides that of the pendulum. The most general cause of such a motion is a force which attracts or impels the oscillating body towards a certain point. Thus, E.G., gravity sets the pendulum in oscillation. If a body, previously at rest, can yield without impediment to the action of the force, which impels it towards a certain point, the oscillation takes place in a straight line; otherwise it will oscillate in a curve, as a pendulum does, where the action of gravity is continually interfered with, the body being united to a fixed point. In like manner, if the direction of the initial velocity of the body is different from that of the motive force, the oscillations will also take place in curved lines.

FIG. 874.



The simplest and most common case is that where *the force is proportional to the distance of the body from a certain point C.* Let C, Fig. 874, be the seat of the force, I.E. the position of the body when the force is $= 0$; let A be the point where the motion begins, and let M be the variable position of the body. If we denote the distance $C\,M$ by x, and by μ a constant, determined by experiment, we have the acceleration of the body at M

$$p = \mu\, x,$$

and since x decreases an amount $d\,x$, when the space $A\,M$ is increased by the same quantity, we have for the velocity v of the body (see § 20, III)

$$\tfrac{1}{2}\,v^2 = -\int p\,d\,x = -\,\mu\int x\,d\,x = -\,\frac{\mu\,x^2}{2} + Con.$$

But at A, $v = 0$ and x is a definite quantity $C\,A = a$; we have, therefore,

$$0 = -\,\frac{\mu\,a^2}{2} + Con., \text{ and}$$

$$v^2 = \mu\,(a^2 - x^2),$$

or the velocity itself

$$v = \sqrt{\mu\,(a^2 - x^2)}.$$

When the body arrives at C, $x = 0$ and v is a maximum, and its value is then

$$v = c = \sqrt{\mu\,a^2} = a\,\sqrt{\mu}.$$

Upon the other side of C, v gradually decreases, and at the distance $x = C\,B = -\,a$ from C it becomes again $= 0$; the body then returns with an increasing velocity to C. This return takes place in accordance with exactly the same law as the first motion; at C, $v = -\,c$, and at A, $v = 0$. Thus the motion repeats itself in the space $A\,B = 2\,a$, which for this reason is called the *amplitude of the oscillations* (Fr. amplitude des oscillations; Ger. die doppelte Schwingungsweite).

(§ **2.**) The time in which the oscillating body describes a certain space $A\,M = x_1$, Fig. 875, can be determined in the following manner. If in the element $d\,t$ of the time the element of the path $M\,N = d\,x_1 = -\,d\,x$ is described, we have (§ 20, I)

$$d\,x_1 = v\,d\,t, \text{ I.E. } d\,x = -\,\sqrt{\mu\,(a^2 - x^2)}\,d\,t,$$

and, therefore, inversely

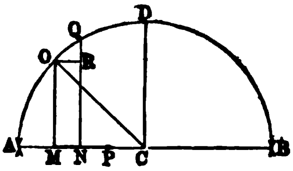
$$d\,t = -\,\frac{d\,x}{\sqrt{\mu\,(a^2 - x^2)}}$$

Now if we describe upon $A\,B$, with a radius $C\,A = C\,B = a$, a circle $A\,D\,B$, $\sqrt{a^2 - x^2}$ will be represented by the ordinate $M\,O = y$, and, therefore, we will have

$$d\,t = -\,\frac{d\,x}{\sqrt{\mu}\,.\,y}.$$

If we put the arc $D\,O$, corresponding to the abscissa $C\,M = x$, equal to s, and its differential $O\,Q = -\,d\,s$, we have, in consequence of the similarity of the triangles $O\,Q\,R$ and $O\,C\,M$, in which $O R = - d x$, $OQ = - ds$, $M O = y$, and $O C = a$, the proportion

FIG. 875.



$$\frac{d x}{d s} = \frac{y}{a}, \text{ and, therefore,}$$

$$\frac{d x}{y} = \frac{d s}{a}; \text{ hence it follows that}$$

$$d t = - \frac{d s}{\sqrt{\mu} \, . \, a}, \text{ and}$$

$$t = - \int \frac{d s}{\sqrt{\mu} \, . \, a} = - \frac{s}{\sqrt{\mu} \, . \, a} + Con.$$

But at the point A, where the motion begins, $t = 0$ and s is equal to the quadrant $D A = \frac{1}{2} \pi a$; consequently

$$0 = - \frac{\frac{1}{2} \pi a}{\sqrt{\mu} \, . \, a} + Con.,$$

and the time required by the body to come from A to M is

$$t = \frac{\frac{1}{2} \pi a}{\sqrt{\mu} \, . \, a} - \frac{s}{\sqrt{\mu} \, . \, a} = \frac{1}{\sqrt{\mu}} \left(\frac{\pi}{2} - \frac{s}{a} \right).$$

The period of *half an oscillation*, i.e. the time required by the body to pass from the point A to the position of rest C, for which $s = 0$, is

$$t = \frac{\pi}{2 \sqrt{\mu}},$$

and the period of a complete oscillation, or the time required to describe the whole distance $A B = 2 a$, is

$$t = \frac{\pi}{\sqrt{\mu}}.$$

After the time

$$t = \frac{2 \pi}{\sqrt{\mu}},$$

the body has made a double oscillation and returned to the point A.

The time required by the body to describe the space $2 A B = 4 a$ is the same, no matter from what point M we begin to count; for the time in which the body goes from M to B and back is

$$= 2 \, . \, \frac{\text{arc } O B}{\sqrt{\mu} \, . \, a},$$

and that in which it goes from M to A and back is

$$= 2 \, . \, \frac{\text{arc } O A}{\sqrt{\mu} \, . \, a};$$

consequently the time required to describe the space $2 M B + 2 M A$ is

$$= 2 \cdot \frac{\text{arc } (O\,B + O\,A)}{a\sqrt{\mu}} = \frac{2 \cdot \pi\, a}{a\sqrt{\mu}} = \frac{2\,\pi}{\sqrt{\mu}}.$$

We see that the period of an oscillation does not depend upon the amplitude. If we start from the point C, we can put the time, which corresponds to the distance $C\,M = x$,

$$t = \frac{s}{\sqrt{\mu} \cdot a},$$

or, since $s = a\, sin.^{-1} \frac{x}{a}$,

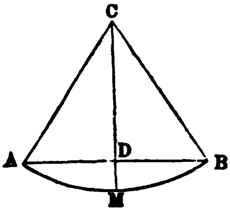
$$t = \frac{1}{\sqrt{\mu}}\, sin.^{-1} \frac{x}{a}, \text{ and inversely}$$

$$x = a\, sin.\, (t\sqrt{\mu}), \text{ and}$$

$$v = \sqrt{\mu}\sqrt{a^2 - a^2 [sin.\, (t\sqrt{\mu})]^2} = \sqrt{\mu} \cdot a\sqrt{1 - [sin.\, (t\sqrt{\mu})]^2}$$
$$= \sqrt{\mu} \cdot a\, cos.\, (t\sqrt{\mu}).$$

REMARK.—The foregoing theory of oscillation is applicable to the circular pendulum $C\,M$, Fig. 876, if the arcs in which it oscillates are small. At A the acceleration of the point, which is oscillating in the arc $A\,M\,B$, is

FIG. 876.



$$p = g\, sin.\, A\,C\,D = \frac{D\,A}{C\,A} \cdot g.$$

or, since for small displacements we can put $D\,A = M\,A$,

$$p = \frac{D\,A}{M\,A} \cdot g,$$

If we denote $C\,A$ by r and $M\,A$ by x, we obtain

$$p = \frac{g\,x}{r};$$

and by comparing it with the formula $p = \mu\, x$, we find

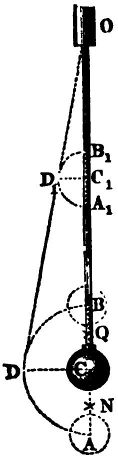
$$\mu = \frac{g}{r}.$$

Hence the period of an oscillation is

$$t = \frac{\pi}{\sqrt{\mu}} = \pi\sqrt{\frac{r}{g}} \text{ (compare § 821).}$$

(§ 3.) **Longitudinal Vibrations.**—The most common cause of oscillatory motion, which is then called *vibration*, is the *elasticity of bodies*. The most simple case is that presented by a rod, string or wire $O\ C$, Fig. 877, stretched by a weight G. If we move this weight from its *position of rest* C a certain distance $C\,A = a$ in the direction of the axis of the string and abandon it to itself, then, in consequence of the elasticity of the string, etc., it will be raised to C, where it arrives with a velocity c and above which it ascends, by virtue of its vis viva, to a point B, from which it falls again, etc. When at rest, the weight G is balanced by the elasticity $\frac{\lambda}{l} F E$ (see § 204) of the rod, and consequently the motive force is

FIG. 877.



$$P = \frac{\lambda}{l} F E - G = 0, \text{ or } \frac{\lambda}{l} F E = G.$$

But if the weight G is at a lower point N, whose distance from C is $C N = x$, the motive force becomes

$$P = \frac{\lambda + x}{l} F E - G = \frac{\lambda}{l} F E + \frac{x}{l} F E - G$$
$$= \frac{F E}{l} x,$$

and if it is at a higher point Q, this force is

$$P = G - \frac{\lambda - x}{l} F E = G - \frac{\lambda}{l} F E + \frac{x}{l} F E = \frac{F E}{l} x.$$

If we neglect the mass of the rod, the acceleration, with which the weight G returns towards C, is

$$p = \frac{P}{G} g = \frac{F E}{G l} g x, \text{ and consequently we have}$$

$$\mu = \frac{F E g}{G l},$$

when we put $p = \mu x$ and denote the length of the rod by l, its cross-section by F and its modulus of elasticity by E. As this formula corresponds to the case treated in the foregoing paragraph, the period of a simple vibration is

$$t = \frac{\pi}{\sqrt{\mu}} = \pi \sqrt{\frac{G l}{F E g}} = \frac{\pi}{\sqrt{g}} \sqrt{\frac{G l}{F E}}.$$

If instead of F we substitute the weight of the rod $G_1 = F l \gamma$ and instead of E the modulus of elasticity $L = \frac{E}{\gamma}$, expressed in units of length, we obtain

$$t = \frac{\pi l}{\sqrt{g}} \sqrt{\frac{G}{G_1 L}}.$$

If, on the contrary, we observe the period t of the simple vibrations, we can calculate the modulus of elasticity by putting

$$E = \frac{\pi^2}{g\,t^2} \cdot \frac{G\,l}{F} \text{ or } L = \frac{\pi^2\,l^2}{g\,t^2} \cdot \frac{G}{G_1}.$$

These formulas also hold good, when the vibrations of the rod are produced by simply attaching the weight (at B); in this case the semi-amplitude on each side of C is

$$a = \lambda = \frac{G}{F\,E}\,l,$$

while in the other case we assumed $a < \lambda$.

A *complete* vibration is a *double* oscillation.—[Tr.]

Example.—If an iron wire 20 feet long and 0,1 inch thick is put in longitudinal vibration by a weight $G = 100$ pounds and if the period of a complete vibration is $\frac{1}{9}$ of second, we have $t = \frac{1}{18}$, and consequently the modulus of elasticity

$$E = 0{,}031 \,.\, \pi^2 \,.\, 18^2 \,.\, \frac{100 \,.\, 20 \,.\, 4}{(0{,}1)^2 \,.\, \pi} = 0{,}031 \,.\, 800000 \,.\, 18^2 \,.\, \pi$$
$$= 24800 \,.\, 324 \,.\, \pi = 25000000 \text{ pounds.}$$

(§ 4) The foregoing formula is also applicable to the case, where the weight acts by *compression* upon a stiff prismatical rod. It also holds good, when the weight applied at the end of the rod has an *initial velocity* v. According to the principle of mechanical effect, when the height of fall of G is h, we have

$$G\,h + G\,\frac{v^2}{2\,g} = \frac{h}{l}\,F\,E \,.\, \frac{h}{2} = \frac{F\,E}{2\,l} \,.\, h^2, \text{ and, therefore,}$$

$$h = \frac{G\,l}{F\,E} + \sqrt{\left(\frac{G\,l}{F\,E}\right)^2 + \frac{2\,G\,l}{F\,E} \cdot \frac{v^2}{2\,g}}.$$

After the weight G has described this space, it has lost all its velocity, and in consequence of the elasticity it rises again to A, where it arrives with the velocity v. In consequence of its vis viva $G\,\frac{v^2}{2\,g}$, it compresses the rod and rises to a height h_1 before returning and beginning a new vibration. For this second distance we have

$$G\,\frac{v^2}{2\,g} = G\,h_1 + \frac{F\,E}{2\,l}\,h_1^2, \text{ and, therefore,}$$

$$h_1 = -\frac{G\,l}{F\,E} + \sqrt{\left(\frac{G\,l}{F\,E}\right)^2 + \frac{2\,G\,l}{F\,E} \cdot \frac{v^2}{2\,g}}.$$

By adding h and h_1 we obtain the total amplitude of the vibration

$$2\,a = h + h_1 = 2\sqrt{\left(\frac{G\,l}{F\,E}\right)^2 + \frac{2\,G\,l}{F\,E} \cdot \frac{v^2}{2\,g}};$$

hence the simple displacement is

$$a = \sqrt{\left(\frac{G\, l}{F\, E}\right)^2 + \frac{2\, G\, l}{F\, E} \cdot \frac{v^2}{2\, g}}.$$

Since in this case also $p = \frac{F\, E}{G\, l}\, g\, x = \mu\, x$, we have as above for the period of an oscillation or simple vibration

$$t = \frac{\pi}{\sqrt{g}} \sqrt{\frac{G\, l}{F\, E}}.$$

If the initial velocity v of the weight G_1 is caused by a *falling weight* G (Fig. 878), we have the case treated in § 348. If the weight G strikes with the velocity c, and if we suppose the impact to be inelastic, we have the initial velocity of $G + G_1$

$$v = \frac{G\, c}{G + G_1},$$

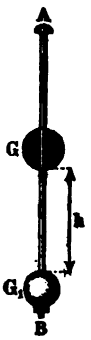
hence the maximum displacement is

$$a = \sqrt{\left(\frac{(G + G_1)\, l}{F\, E}\right)^2 + \frac{2\, G^2\, l}{(G + G_1)\, F\, E} \cdot \frac{c^2}{2\, g}},$$

and the period of a simple vibration is

$$t = \frac{\pi}{\sqrt{g}} \sqrt{\frac{(G + G_1)\, l}{F\, E}}.$$

FIG. 878.



The elements of the rod also participate in the vibrations of G or $G + G_1$, but their amplitude decreases as the position of the element approaches the point of suspension. For an element C_1, Fig. 877, situated at a distance $O\, C_1 = x$ from the point of suspension, the amplitude is

$$y = \frac{x}{l}\, a;$$

while the period of its vibration is the same as that of G; for it does not depend upon y or a. Hence the vibrations of all the elements of the rod are *isochronous*, but their amplitudes decrease gradually from C towards O.

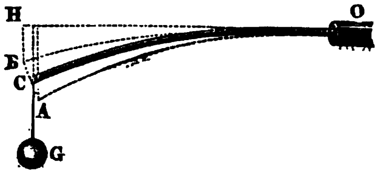
§ 5. **Transverse Vibrations.**—The *elasticity of flexure and of torsion* cause vibrations of the same nature as those just treated. If a rod or spring $O\, C$ (Fig. 879) is fixed at one end O and deflected at the other C by a weight G, we have, according to § 217, the deflection

$$H\, C = a = \frac{P\, l^3}{3\, W\, E};$$

inversely the force, with which the rod is bent, is

$$P = \frac{3\,W E\,a}{l^3}.$$

FIG. 879.



Now if this force is replaced by a weight G, attached at C, and if a is increased or diminished a distance $C\,A = C\,B = x$, we have the force, with which the rod will be driven back to its position of rest by its elasticity

$$P = \frac{3\,W E\,(a + x)}{l^3} - G = \frac{3\,W E\,(a + x)}{l^3} - \frac{3\,W E}{l^3}\,a = \frac{3\,W E}{l^3}\,x;$$

hence the acceleration is, when we consider the mass of G alone,

$$p = \frac{P}{G}\,g = \frac{3\,W E}{G\,l^3}\,g\,x, \text{ and, since } p = \mu\,x,$$

$$\mu = \frac{3\,W E}{G\,l^3}\,g.$$

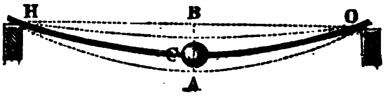
The relation between p and x allows us to employ the formulas of (§ 2), consequently the period of an oscillation or simple vibration is

$$t = \frac{\pi}{\sqrt{\mu}} = \frac{\pi}{\sqrt{g}}\sqrt{\frac{G\,l^3}{3\,W E}}.$$

If the rod $H\,O$, Fig. 880, is supported at both ends and loaded in the middle C with a weight G, we have, according to § 217,

$$a = \frac{P\,l^3}{48\,W E},$$

FIG. 880.



and, therefore, the duration of a simple vibration

$$t = \frac{\pi}{\sqrt{g}}\sqrt{\frac{G\,l^3}{48\,W E}}.$$

If we take the weight G_1 of the rod into consideration, we must substitute in the first case, Fig. 879, instead of G, $G + \frac{1}{4}\,G_1$, and in the second case, Fig. 880, instead of G, $G + \frac{1}{2}\,G_1$.

From the observed duration of an oscillation or simple vibration we can calculate the modulus of elasticity, in the first case by the formula

$$E = \left(\frac{\pi}{t}\right)^2 \left(\frac{G + \frac{1}{4}\,G_1}{3\,g\,W}\right) l^3$$

or, if $n = \frac{1}{t}$ denotes the number of simple vibrations per second,

$$E = (\pi\,n)^2 \left(\frac{G + \frac{1}{4}\,G_1}{3\,g\,W}\right) l^3.$$

EXAMPLE.—A pine rod 1 centimeter square is supported at two points 100 centimeters apart, and its centre is deflected a distance $a = 3{,}2$ centimeters by a weight $G = 1{,}87$ kilograms. According to this experiment the modulus of elasticity of pine is

$$E = \frac{P\,l^3}{48\,W a} = \frac{1{,}87\,.\,1000000}{48\,.\,\frac{1}{12}\,.\,3{,}2} = 107031 \text{ kilograms,}$$

while in the table on page 870 we find $E = 110000$.

The rod was then firmly fixed at one end, was loaded at the other with a weight $G = 0{,}31$, and put in vibration. It was found that the number of simple vibrations in 35 seconds was 100. The weight of the rod was $G_1 = 0{,}044$ kilograms; hence $G + \frac{1}{4}\,G_1 = 0{,}321$ kilograms and

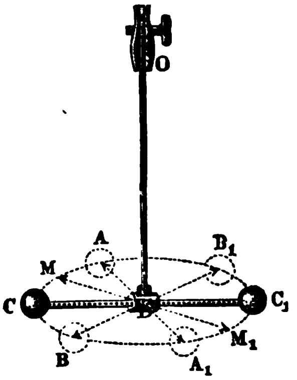
$$E = \left(\frac{\pi}{t}\right)^2 . \left(\frac{G + \frac{1}{4}\,G_1}{3\,g\,W}\right) l^3 = \left(\frac{3{,}141}{0{,}35}\right)^2 . \frac{321000}{981\,.\,\frac{3}{12}}$$

$$= 80{,}57\,.\,\frac{1281000}{981} = 105260 \text{ kilograms,}$$

or about the same value of E as was found by the experiment upon flexure.

§ 6. Vibrations Due to Torsion.—The formula $t = \frac{\pi}{\sqrt{\mu}}$ can also be applied to the *torsion balance or torsion rod* (Fr. balance de torsion; Ger. Torsionspendel), I.E. to a thread or rod $D\,O$, Fig. 881, oscillating about its axis, in consequence of its torsion. Generally the rod is provided with a loaded arm $C\,C_1$, by means of which the original torsion of the thread is produced, by bringing this arm from its position of rest $C\,C_1$ into the position $A\,A_1$. The torsion drives the arm back to $C\,C_1$, and the latter, by virtue of its inertia, moves further on until it comes into the position $B\,B_1$, from which it returns to $C\,C_1$ and $A\,A_1$, etc. We found previously (§ 262) the moment of torsion of a prismatic body to be

FIG. 881.



$$P\,a = \frac{\alpha\,W\,c}{l};$$

we know, therefore, from this formula, that it is inversely proportional to the length $O\,D = l$ of the rod and directly proportional to the angle of torsion $M\,D\,C = \alpha$; now if $\frac{G}{g}\,k^2$ is the moment of inertia of the arm $C\,D\,C_1$, $\frac{k^2}{a^2}\,\frac{G}{g}$ is the mass M reduced to the ends C and C_1 of the arm, and the acceleration of this point is

$$p = \frac{P}{M} = \frac{a W c}{l a} : \frac{k^2 G}{a^2 g} = \frac{\alpha a W C g}{G k^2 l}.$$

If we denote the arc $C M = a \alpha$, corresponding to the length of the arm $D A = D C = a$ and to the variable angle of displacement $C D M = \alpha$, by x, we obtain the expression

$$p = \frac{W C g}{G k^2 l} x, \text{ and we can again put } p = \mu x, \text{ or}$$

$$\mu = \frac{W C g}{G k^2 l}.$$

The period of an oscillation or simple vibration is, therefore,

$$t = \frac{\pi}{\sqrt{\mu}} = \frac{\pi}{\sqrt{g}} \sqrt{\frac{G k^2 l}{W C}},$$

no matter whether the amplitude $A C B = A_1 C_1 B_1$ is large or small.

Inversely, we have

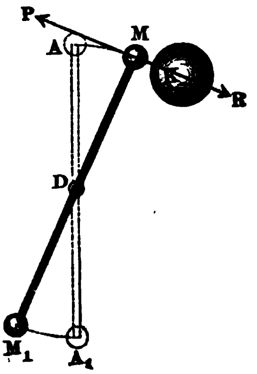
$$W C = \frac{\pi^2}{g t^2} G k^2 l,$$

and, therefore, the moment of torsion

$$P a = \frac{\pi^2}{g t^2} . a G k^2.$$

REMARK.—The above formulas for the vibrations produced by the elasticity of rigid bodies are not correct unless the displacement during the vibration is within the limit of elasticity. Great care should be taken to avoid as much as possible vibrations in the various parts of machines; for the energy expended upon them is lost to the machine. For this reason the parts should be united to each other with precision, and what is known as *lost motion* is to be avoided, as it gives rise to concussions and vibrations.

FIG. 882.



§ 7. Density of the Earth.—The theory of the torsion-rod can be directly applied to the determination of the *mean heaviness* or specific gravity ε of the earth. If we cause a heavy sphere K to approach the weight G, which is fastened upon the end of the arm $A D A_1$, Fig. 882, the latter will be attracted towards the former a certain distance $A M = x$; the attraction R of K balances the force of torsion P, when G occupies the position M; one of the above forces can, therefore, be determined from the other. Now if we remove the heavy sphere K and allow the torsion-rod to vibrate, we can observe the period of the vibrations, and from it we can calculate the force of torsion. According to the foregoing paragraph, the period of a simple vibration is

$$t = \frac{\pi}{\sqrt{\mu}},\ \mu = \frac{p}{x} \text{ and } p = \frac{\text{force of torsion}}{\text{mass of torsion-rod}} = \frac{P\,a^2}{G\,k^2}\,g,$$

when $G\,k^2$ denotes the moment of inertia and a the length of the arm of the torsion-rod; inversely, the twisting or attractive force is

$$P = \frac{G\,k^2\,p}{g\,a^2} = \frac{\mu\,G\,k^2\,x}{g\,a^2} = \frac{\pi^2}{g\,t^2} \cdot \frac{G\,k^2\,x}{a^2} = \frac{\pi^2}{g\,t^2} \cdot \frac{G\,k^2\,\alpha}{a},$$

and the moment of torsion corresponding to the angle of torsion α is

$$P\,a = \frac{\pi^2}{g\,t^2} \,.\, \alpha\,G\,k^2.$$

Now if the *forces*, with which the bodies *attract* each other, vary directly as their *masses* and inversely as the *squares of their distances* (see § 302, Example 3), we can compare the attraction P, exerted upon the body by K, with the weight Q of the small body which is placed upon the torsion rod; for the weight is the measure of attractive force of the earth; thus we obtain

$$\frac{P}{Q} = \frac{K : s^2}{E : r^2},$$

in which s denotes the distance $M\,K$ of the centres of the two masses G and K from each other, r the radius of the earth and E its weight. If we solve the above equation, we obtain the latter weight

$$E = \frac{K\,Q\,r^2}{P\,s^2},$$

and if we substitute $E = \frac{4}{3}\,\pi\,r^3 \,.\, \varepsilon\,\gamma$, we have the mean heaviness of the earth

$$\gamma_1 = \varepsilon\,\gamma = \frac{3\,E}{4\,\pi\,r^3} = \frac{3\,K\,Q\,r^2}{4\,\pi\,P\,r^3\,s^2} = \frac{3\,K\,Q}{4\,\pi\,P\,r\,s^2} = \frac{3\,K\,Q}{4\,\pi\,r\,s^2} \cdot \frac{g\,t^2\,a^2}{\pi^2\,G\,k^2\,x},$$

or if we introduce the length of the second pendulum $l = \frac{g}{\pi^2}$ (see § 323),

$$\gamma_1 = \varepsilon\,\gamma = \frac{3\,K\,l\,t^2}{4\,\pi\,r\,x\,s^2} \cdot \frac{Q\,a^2}{G\,k^2};$$

hence the mean specific gravity of the earth is

$$\varepsilon = \frac{3\,K\,l\,t^2}{4\,\pi\,r\,x\,s^2} \cdot \frac{Q\,a^2}{G\,k^2\,\gamma}.$$

If we put approximatively $G\,k^2 = Q\,a^2$, we obtain more simply

$$\varepsilon = \frac{3}{4}\,\frac{K\,l\,t^2}{\pi\,r\,x\,s^2\,\gamma}.$$

Cavendish found in the first place with the torsion rod, or Coulomb's torsion balance, as it is called, $\varepsilon = 5{,}48$; or, according to Hutton's revision, $\varepsilon = 5{,}42$.

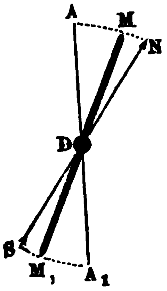
Reich found afterwards, with the aid of the mirror apparatus of Gauss and Poggendorff, $\varepsilon = 5{,}43$. Baily, on the contrary, found by experiments upon a larger scale, $\varepsilon = 5{,}675$.

When Reich repeated his experiments he found $\varepsilon = 5{,}583$. (See "Neue Versuche mit Drehwage, Leipzig, 1852.") The mean density of the earth is, therefore, according to these experiments, about equal to that of specular iron.

REMARK.—The following works may be consulted in reference to the manner in which the density of the earth was determined: "Gehler's physikal. Wörterbuch," Bd. III; the treatise of Reich "Versuche über die mittlere Dichtigkeit der Erde, Freiberg, 1838;" and that by Baily, "Experiments with the Torsion Rod for Determining of the Mean Density of the Earth, London, 1848."

§ 8. Magnetic Needle.—The torsion-balance may also be employed to find the directing force or the moment of rotation of a *magnet* or of a *magnetic needle* (Fr. aiguile aimantée; Ger. Magnetnadel). If we replace the transverse arm of the balance by a magnetic needle or by a bar magnet $M\ D\ M_1$, Fig. 883, it will assume a position in which the directing force is balanced by the twisting force. If the non-magnetic arm, when at rest in $A\ A_1$, forms an angle $A\ D\ N = a$ with the magnetic meridian $N\ S$, and if the bar magnet $M\ M_1$ assumes such a position that its axis forms an angle $M\ D\ N = \delta$ with the meridian $N\ S$, we have $R_1 = R\ sin.\ \delta$, in which formula R_1 denotes the component of the directing force R, which is parallel to $N\ S$. This component tends to turn the needle, and is balanced by the force of torsion. The latter force P, on the contrary, is proportional to the angle of torsion $M\ D\ A = a - \delta$, and we can, therefore, put

FIG. 888.



$$P = P_1\ (a - \delta);$$

hence we have $R\ sin.\ \delta = P\ (a - \delta)$, and consequently

$$R = \left(\frac{a - \delta}{sin.\ \delta}\right) P_1 = \left(\frac{a - \delta}{\delta}\right) P_1,$$

when the *variation* or angle of deviation δ is small.

Now according to the foregoing paragraph the force of torsion is expressed by the formula

$$P = \frac{\pi^2}{g\,t^2} \cdot \frac{G\ k^2\ x}{a^2} = \frac{\pi^2}{g\,t^2} \cdot \frac{G\ k^2\ a\ (a - \delta)}{a^2} = \frac{\pi^2}{g\,t^2} \cdot \frac{G\ k^2\ (a - \delta)}{a},$$

and we can calculate from the period t of an oscillation, etc., of the non-magnetic torsion-rod the ***directive force*** of the magnetic needle by the formula

$$R = \left(\frac{a-\delta}{\delta}\right) P_1 = \frac{P}{\delta} = \frac{a-\delta}{\delta} \cdot \frac{\pi^2}{g t^2} \cdot \frac{G k^2}{a}.$$

The moment of this force, when we assume that it is applied at a distance $D\,M = a$ from the axis of rotation and when the variation is $M\,D\,N = \delta$, is $R_1\,a = R\,a\,sin.\,\delta$, approximatively, for small variations,

$$= R\,a\,\delta = (a - \delta) \cdot \frac{\pi^2}{g t^2} \cdot G\,k^2.$$

This moment ($R\,a\,sin.\,\delta$) is a maximum and $= R\,a$ for $sin.\,\delta = 1$, I.E., when the magnetic needle is at right angles to the magnetic meridian, and, on the contrary, a minimum and $= 0$, when $\delta = 0$, I.E., when the axis of the magnet needle coincides with the magnetic meridian.

§ 9. **Magnetism.**—Since the directive force of the magnetic needle causes no pressure upon the axis, I.E., the needle has no tendency to move forward, but only a tendency to turn, when its axis does not coincide with the magnetic meridian, it follows that the entire action of the earth upon the magnet must consist of a couple $\frac{R}{2}, - \frac{R}{2}$, the maximum moment of which is $R\,a$. Now since every couple $\frac{R}{2}, - \frac{R}{2}$ can be replaced by an infinite number of other couples $\left(\frac{R_1}{2}, - \frac{R_1}{2}\right), \left(\frac{R_2}{2}, - \frac{R_2}{2}\right)$, etc., whose moments $R\,a$, $R_1\,a_1$, $R_2\,a_2$, etc., are equal to each other, it follows that neither R nor a, I.E., neither the directive force nor the point of application, but only the moment $R\,a$ is determined. This ***twisting moment*** depends, in addition, upon two factors, μ_1 and S, μ_1 corresponding to the ***magnetism of the earth*** and S to that of the bar or needle; hence we can put

$$R = \mu_1\,S \text{ and } R\,a = \mu_1\,S\,a.$$

The measure μ_1 of the magnetism of the earth for a needle vibrating horizontally (the case under consideration) is only the horizontal component of the intensity μ of the entire magnetism of the earth; for the vertical component μ_2 is counteracted by the support of the needle. If ι is the angle of ***dip*** or ***inclination*** or the angle formed by the magnetic axis of the earth with the horizon, we have the horizontal component

$$\mu_1 = \mu\,cos.\,\iota;$$

on the contrary, the vertical one

$$\mu_2 = \mu\,sin.\,\iota,$$

and, finally, the twisting moment of a magnetic needle is

$$R\, a\, sin.\, \delta = \mu\, cos.\, \iota\, .\, S\, a\, sin.\, \delta,$$

the maximum value of which is

$$R\, a = \mu\, S\, a\, cos.\, \iota.$$

§ 10. Oscillations of a Magnetic Needle.—We can calculate the moment of rotation of a magnetic needle from the period of its oscillations. If we move the suspended needle $M\,D\,M_1$, Fig. 884, from its position of rest, where the force of torsion and the directive force of the magnet are in equilibrium, so that its new position shall make a small angle $M\,D\,C = \phi$ with its former one, either the magnetic directing force R is increased by $R\,\phi$ and the force of torsion P_1 is diminished by $P_1\,\phi$, or the reverse takes place; in either case their resultant

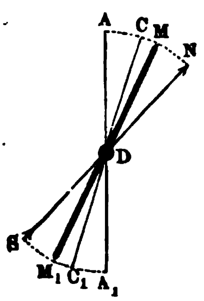
$$(R + P_1)\,\phi$$

or its moment

$$(R + P_1)\,\phi\, a = (R + P_1)\, x$$

drives the needle back to its position of rest.

Fig. 884.



If $G\,k^2$ is the moment of inertia of the needle, the acceleration, corresponding to this force, is

$$p = \frac{(R + P_1)\, a\, x}{G\, k^2}\, g;$$

if we put it $= \mu\, x$, we obtain

$$\mu = \left(\frac{R + P_1}{G\, k^2}\right) a\, g,$$

and the period of an oscillation is

$$t = \frac{\pi}{\sqrt{\mu}} = \pi \sqrt{\frac{G\, k^2}{(R + P_1)\, a\, g}}$$

$$= \frac{\pi}{\sqrt{g}} \sqrt{\frac{G\, k^2}{(R + P_1)\, a}},$$

or, if ν denotes the ratio $\frac{P_1}{R} = \frac{\delta}{a - \delta}$ of the force of torsion to the magnetic force,

$$t = \frac{\pi}{\sqrt{g}} \sqrt{\frac{G\, k^2}{(1 + \nu)\, R\, a}}.$$

If we have found t by observation, we can find by inversion the moment of rotation of the needle, which is

$$R\,a = \frac{\pi^2}{g\,t^2} \cdot \frac{G\,k^2}{1 + \nu}.$$

If the force of torsion is small, I.E., if the position of repose nearly coincides with that of the magnetic meridian, we can neglect ν and put

$$t = \frac{\pi}{\sqrt{g}} \sqrt{\frac{G\,k^2}{R\,a}} \text{ and}$$

$$R\,a = \frac{\pi^2}{g\,t^2} \,.\, G\,k^2.$$

We can also substitute for $R\,a$ its value, which has been given above, and express the moment of rotation by the formula

$$\mu\,S\,a\,cos.\,\iota = \frac{\pi^2}{g\,t^2} \,.\, G\,k^2.$$

For a *dipping needle*, which oscillates in the plane of the magnetic meridian, we have, on the contrary,

$$\mu\,S\,a = \frac{\pi^2}{g\,t^2} \,.\, G\,k^2,$$

and for a needle, whose axis lies in the magnetic meridian and which, therefore, tends to place itself in a vertical position we have

$$\mu\,S\,a\,sin.\,\iota = \frac{\pi^2}{g\,t^2} \,.\, G\,k^2.$$

In the formula $\mu\,S\,a\,cos.\,\iota = \frac{\pi^2}{g\,t^2} \,.\, G\,k^2$, $\mu\,S\,a\,cos.\,\iota$ is a product of four factors; however, since the inclination ι can be determined by observing a magnetic needle, and since $S\,a$ cannot be decomposed into two definite factors, we have to only resolve the product $\mu\,S\,a$ into the factors μ and $S\,a$. How this can be done by observing the declination of the needle will be shown in the sequel.

§ 11. **Law of Magnetic Attraction.**—The forces, with which the opposite poles of two magnets attract and the similar poles repel each other, *are inversely proportional to the squares of their distances from each other.* We can convince ourselves very easily of this fact by observing a small magnetic needle, which has been set in oscillation near a large bar magnet. The bar magnet is placed in a horizontal position and in the plane of the magnetic meridian, its north pole being directed to the north and the south pole towards the south; we then place a small variation compass in the prolongation of the axis of the bar magnet. If the distance s of the pivot of the needle from one pole of the bar magnet is much less than its distance from the other pole, we can disregard the action of the latter upon the needle and we can assume that, in consequence of the action of the nearer pole, the coefficient μ of the magnetic force of the earth is increased a certain amount κ_1 or κ_2. If the period of the oscillations of the needle is $= t$, when the bar magnet is removed, and, on the contrary, if it is $= t_1$, when the nearer pole of the bar magnet is at a distance s_1 from the pivot of the needle, and $= t_2$, when the latter distance is $= s_2$, we have

$$\mu_1 S a = \frac{\pi^2}{g t^2} . G k^2, (\mu_1 + \kappa_1) S a = \frac{\pi^2}{g t^2} G k^2 \text{ and } (\mu_1 + \kappa_2) S a = \frac{\pi^2}{g t_2^2} . G k^2,$$

whence we obtain by division

$$\frac{\mu_1 + \kappa_1}{\mu_1} = \frac{t^2}{t_1^2} \text{ and } \frac{\mu_1 + \kappa_2}{\mu_1} = \frac{t^2}{t_2^2};$$

resolving the last two equations, we obtain

$$\kappa_1 = \left(\frac{t^2 - t_1^2}{t_1^2}\right) \mu_1 \text{ and } \kappa_2 = \left(\frac{t^2 - t_2^2}{t_2^2}\right) \mu_1, \text{ and, finally,}$$

$$\kappa_1 : \kappa_2 = \frac{t^2 - t_1^2}{t_1^2} : \frac{t^2 - t_2^2}{t_2^2},$$

or, if we substitute instead of t, t_1 and t_2 the number of oscillations

$$n = \frac{60''}{t}, n_1 = \frac{60''}{t_1} \text{ and } n_2 = \frac{60''}{t_2},$$

$$\kappa_1 : \kappa_2 = n_1^2 - n^2 : n_2^2 - n^2.$$

If the action of the bar magnet upon the magnetic needle is inversely proportional to the square of the distance, we must have also

$$\kappa_1 : \kappa_2 = s_2^2 : s_1^2, \text{ and therefore}$$

$$\frac{n_1^2 - n^2}{n_2^2 - n^2} = \frac{s_2^2}{s_1^2},$$

which is confirmed by the observations.

§ **12**. The actions of a bar magnet $N S$ upon a magnetic needle $n s$ are simplest, when the bar magnet is placed at right-angles to the magnetic meridian in such a manner that the pivot of the compass $n s$, Fig. 885, lies either in the prolongation of $N S$ or in the line which is perpendicular to $N S$, Fig. 886, and passes through its middle C. If for the present we put the force, with which a pole of $N S$ acts upon a pole of $n s$, when their distance apart is unity, $= K$, we have in the first case, Fig. 885, when a denotes the length $N S$ and e the distance $C d$ between the centres C and d of the two bodies $N S$ and $n s$, the force, with which the north pole n is attracted by S,

$$P = \frac{K}{S n^2}, \text{ approximatively} = \frac{K}{(e - \frac{1}{2} a)^2},$$

and the force, with which *n* is repelled by *N*, is

FIG. 885.

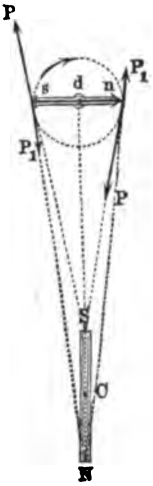
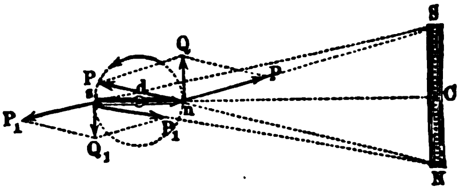


FIG. 886.



$$P = \frac{K}{N n^2} = \frac{K}{(e + \frac{1}{2} a)^2};$$

hence the resultant of P and P_1 is

$$Q = P - P_1 = K \left(\frac{1}{(e - \frac{1}{2} a)^2} - \frac{1}{(e + \frac{1}{2} a)^2} \right)$$

$$= \frac{(e + \frac{1}{2} a)^2 - (e - \frac{1}{2} a)^2}{(e + \frac{1}{2} a)^2 (e - \frac{1}{2} a)^2} K$$

$$= \frac{2 a e K}{(e + \frac{1}{2} a)^2 (e - \frac{1}{2} a)^2},$$

or, if $\frac{1}{2} a$ is small compared to e,

$$Q = \frac{2 a e K}{e^4} = \frac{2 a K}{e^3}.$$

In like manner we find the resultant of the attraction and repulsion of the south pole *s*

$$Q = - \frac{2 a K}{e^3};$$

hence the moment of the couple, formed by these forces, is

$$Q l = \frac{2 a l K}{e^3},$$

when l denotes the distance between the two poles of the needle.

For the second case (Fig. 886), on the contrary, the attraction and repulsion at *s* are

$$P_1 = \frac{K}{N s^2} = \frac{K}{S s^2}, \text{ and those at } n \text{ are}$$

$$P = \frac{K}{S n^2} = \frac{K}{N n^2};$$

hence the resultants are

$$Q_1 = 2 \,.\, \frac{C\,N}{N\,s} \,.\, P_1 = \frac{a\,P}{N\,s} = \frac{a\,K}{\overline{N\,s}^2} \text{ and } Q = \frac{a\,K}{\overline{N\,n}^2}.$$

Now if $\frac{1}{2}\,a$ and $\frac{1}{2}\,l$ are considerably smaller than e, we can substitute for $\overline{N\,s} = \overline{S\,s}$ and $\overline{N\,n} = \overline{S\,n}$ the mean value $\overline{N\,d} = \overline{S\,d}$ and for the latter the approximate value $\overline{C\,d} = e$; thus we obtain

$$Q = Q_1 = \frac{a\,K}{e^2},$$

and, therefore, the moment of the couple, formed by Q and Q_1,

$$Q\,l = \frac{a\,l\,K}{e^2},$$

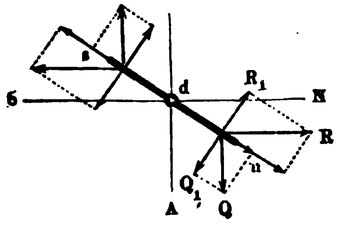
I.E., it is one-half as great as in the foregoing case, a result which is perfectly corroborated by observation.

But the force K is itself a product of the intensity κ of the magnetism of $n\,s$ and the intensity S of $\overline{N\,S}$, I.E., $K = \kappa\,S$; hence we have in the first case

$$Q = \frac{2\,\kappa\,S\,a}{e^2}, \text{ and in the second case } Q = \frac{\kappa\,S\,a}{e^2}.$$

§ **13. Determination of the Magnetism of the Earth.**—If in both the above-mentioned cases the magnetic needle $n\,s$ is abandoned to the action of the larger magnet, the former will assume a new position $n\,s$, Fig. 887, in which the force Q, with which the bar magnet acts upon the needle, is balanced by the force R, due to the magnetism of the earth. If δ is the variation $N\,d\,n = S\,d\,s$ of the needle from the magnetic meridian, we have for the components of Q and R, which balance each other,

FIG. 887.



$$Q_1 = Q\,cos.\,\delta \text{ and } R_1 = R\,sin.\,\delta;$$

hence

$$Q\,cos.\,\delta = R\,sin.\,\delta \text{ and}$$

$$tang.\,\delta = \frac{Q}{R},$$

or, if we put, according to the last paragraph, either

$$Q = \frac{2\,\kappa\,S\,a}{e^2} \text{ or } Q = \frac{\kappa\,S\,a}{e^2},$$

and, according to § 9 of the Appendix, $R = \mu_1\,\kappa$, we obtain either

$$tang.\ \delta = \frac{2\,\kappa\, S\, a}{\mu_1\, \kappa\, e^3} = \frac{2\, S\, a}{\mu_1\, e^3}, \text{ or } tang.\ \delta = \frac{S\, a}{\mu_1 e^3}.$$

By inversion we obtain the ratio of the magnetic moment of the bar to the *intensity of the magnetism of the earth;* for in the first case we have

$$\frac{S\, a}{\mu_1} = \tfrac{1}{2}\, e^3\ tang.\ \delta, \text{ and in the second case, } \frac{S\, a}{\mu_1} = e^3\ tang.\ \delta.$$

By observing the period of the oscillations of the bar magnet, we obtain (according to § 10) the product

$$\mu_1\, S\, a = \frac{\pi^2}{g\, t^2}\, G\, k^2;$$

by combining the two equations, we deduce the *magnetic moment of the bar*, which is

either $$S\, a = \frac{\pi}{t\, \sqrt{g}} \sqrt{\tfrac{1}{2}\, G\, k^2\, e^3\ tang.\ \delta}$$

or $$S\, a = \frac{\pi}{t\, \sqrt{g}} \sqrt{G\, k^2\, e^3\ tang.\ \delta},$$

and the measure of the horizontal component of the magnetism of the earth, which is either

$$\mu_1 = \frac{\pi}{t\, \sqrt{g}} \sqrt{\frac{2\, G\, k^2\ cotang.\ \delta}{e^3}} \text{ or } = \frac{\pi}{t\, \sqrt{g}} \sqrt{\frac{G\, k^2\ cotang.\ \delta}{e^3}},$$

the first formula being applicable to the case represented in Fig. 885, and the second to the case represented in Fig. 886. If we divide by the cosine of the angle of dip or inclination (ι), we obtain the total intensity of the magnetism of the earth

$$\mu = \frac{\mu_1}{cos.\ \iota}.$$

In order to obtain a clear idea of the coefficient or measure μ of the magnetism of the earth, we must put in the formulas

$$R\, a = \mu\, S\, a \text{ and } Q\, l = \frac{\kappa\, S\, l\, a}{e^3},\ a = l = e = 1,$$

and also $\kappa = S = 1$; thus we obtain $R\, a = \mu$ and $Q\, l = 1$; hence

1) the measure μ of the *intensity* of the *magnetism of the earth* is that moment, with which a magnetic needle, whose magnetic moment is = unity, will be turned by the magnetism of the earth; and

2) the *magnetic moment* of a magnetic needle is = unity when that needle communicates to another similar and equally powerful magnetic needle, placed in the position represented in Fig. 886 at the unit of distance from it, a moment = unity (1 millimeter-milli-

gram). According to Weber, if the acceleration of gravity were 1 millimeter, we would have

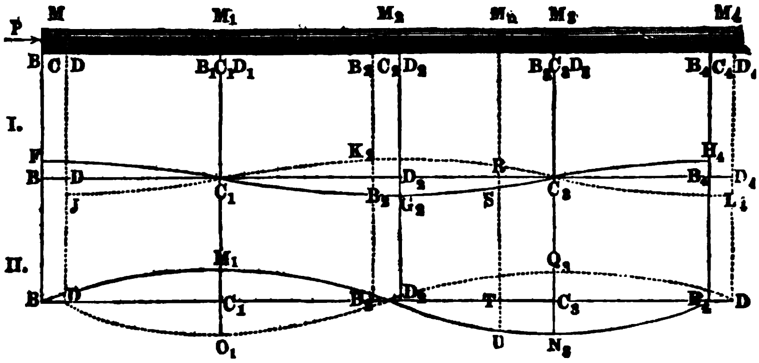
in Gottingen $\mu = 1{,}774$ millimeter-milligrams,
in Munich $\mu = 1{,}905$ " "
in Milan $\mu = 2{,}018$ " "

but, since the acceleration of gravity in Central Europe is 9810 millimeters, the true values are $\sqrt{9810} = 99$ times less.

Remark.—We would recommend to those who wish to make a more extended study of magnetism, besides Müller-Pouillet's "Lehrbuch der Physik;" Lamont's "Handbuch des Erdmagnetismus" (Berlin, 1849), and Gauss and Weber's "Resultate aus den Beobachtungen des magnetischen Vereins," Gottingen and Leipzig, 1837 to 1848; also the "Experimentalphysik" of Quintus Julius, and Mousson's "Physik auf Grundlage der Erfahrung," etc.

§ **14. Waves.**—In discussing the *longitudinal* and *transverse vibrations* of prismatical bodies, we have heretofore (§ 3, 4 and 5) neglected the mass of these bodies and considered only that of the weight, which produced the strain in the bodies. Hereafter, on the contrary, we will not consider any such weight, but suppose that the body is put in vibration by a sudden blow or by a force, which acts for an instant only; we must, therefore, take into account the inertia of the vibrating body alone. As the most simple case is that offered by *longitudinal vibrations*, we will, therefore, treat that first.

FIG. 888.

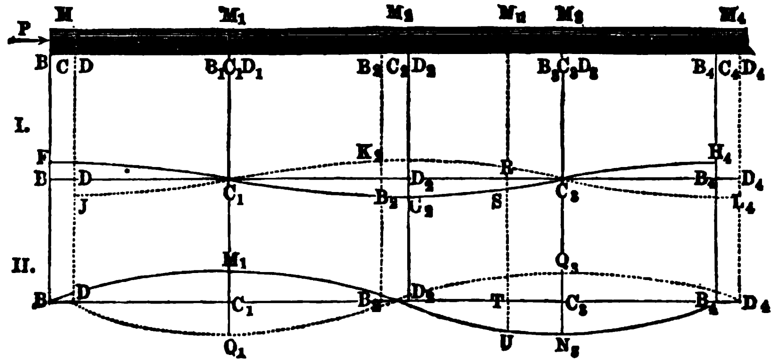


From what precedes, we know that all the parts of a prismatical rod $B\,M_4$, Fig. 888, are put in vibration, when this body is extended or compressed by a force P, acting in the direction of its axis. Not only the element M at the end, but also every other element M_1, M_2, M_3 .... of the rod vibrates back and forth in a certain space $B\ D$, $B_1\ D_1$, $B_2\ D_2$ . . . which is called the *amplitude of the vibration;* we can also assume, when the rod is very long, that this space is the same for all the elements. Although the time in which an element makes a vibration is the same for all parts of the rod, we cannot, therefore, assume that all these elements M, M_1, M_2 etc., are *simultaneously* in the same *phase of motion*, E.G., that they are all at the same time in the middle of a vibration, but we should rather suppose that time will be required to communicate the motion proceeding from M to the succeeding elements, and that the farther an element is situated from the origin P of the motion, the later it will enter upon the same phase of motion. It is, therefore, possible that at the instant, when the element M has made a complete vibration $B\ D$ forward and back, the element M_2 has made but one-half of its forward movement and has arrived at C_3, and that the element M_4 is just beginning a vibration. The latter will therefore vibrate isochronously with M. The velocity with which the same phase of motion advances in the body is called the *velocity of propagation* (Fr. vitesse de propagation; Ger. Fortpflanzungsgeschwindigkeit) of the vibrations of the body. The aggregate of all those elements between M and M_4, which are in the different phases of a complete vibration or which are included between two elements M and M_4, which are in the same phase, are called a *wave* (Fr. ondulation; Ger. Welle) of the vibrating body, and the distance $M\ M_4$ is called the *length of the wave.* A wave consists of a back part $B\ D_2$ which contains the returning elements, such as M_1, M_2 .... and of the *wave front* $D_2\ B_4$, which comprehends the advancing elements M_3, M_4 . . .; $B\ D_2$ is also called the *rarefied* and $D_2\ B_4$ the *condensed* portion, since all the elements in $B\ D_2$ are extended and those in $D_2\ B_4$ are compressed.

§ **15**. The *phases* of the *motion* and of the *velocity* in a wave can be very well represented by serpentine lines, such as $F\ C_1\ G_2\ C_3\ H_4$ and $B\ M_1\ D_2\ N_3\ B_4$, Fig. 889, I and II. At the moment when M begins a new vibration at B, its displacement is a maximum and its velocity is $= 0$; at the same time M_1 is in the position of rest, and consequently its displacement is $= 0$ and its velocity is a maximum; both of these facts are shown by the above curves; for the first curve (that of the displacement) (I) passes at B at a distance equal to the amplitude $B\ F = B\ C$ above the axis $B\ D_4$ and cuts this axis at C_1, while, on the contrary, the second curve (that of the velocity) (II) cuts the axis at B and at C_1 passes at a distance $C_1 M_1$, equal to the maximum velocity, above the axis. At the same moment the element M_2 is upon the other side of the position of rest C_2 and at the maximum distance from it, and its velocity, like that of M, is $= 0$; this is also shown by the two curves; for one passes at D_2 at a distance equal to the amplitude $D_2 G_2$ below the axis, and the other cuts it at that point, so that the ordinate which corresponds to the velocity is $= 0$. In like manner the phases of the motions and of the velocities of the elements M_3, M_4, etc., are represented by these curves. Since, E.G., the first curve cuts the axis at C_3 and the second passes below that point at a distance equal to the maximum value $C_3 N_3$, we know that the element M_3 at this moment passes through the position of rest with the maximum velocity in the positive direction. If we wish to know the phase of the motion of any other element M_n, situated between M, M_1, M_4, etc., at the moment when the element M_n begins a new vibration, we have only to let fall from it a perpendicular upon the corresponding curve. The portion $R\ S$ of this perpendicular lying between the curve and the axis corresponds to the displacement of this element, and the portion $T\ U$, between the second curve and its axis, gives its velocity. Since both ordinates are directed downwards, we know that both the displacement and the velocity are positive, I.E. their direction is that of the velocity of propagation.

If the element M were at D, I.E. about to begin its return motion, the displacements of the other elements of the wave would be represented by the dotted line $J\ C_1\ K_2\ C_3\ L_4$, and their velocities

FIG. 889.



by the ordinates of the dotted curve $D\ O_1\ B_2\ Q_3\ D_4$. The period of a double oscillation or that of a complete vibration, I.E. the time t, in which the space $B\ D\ +\ D\ B$ is described, is equal to the time in which a vibration is propagated through the length $M\ M_4 = l$ of a wave; if, therefore, c is the velocity of propagation, we have the total length of the wave

$$B\ B_4 = l = c\,.\,2\,t = 2\,c\,t.$$

The length of the back part of the wave is

$$B\ D_2 = l_1 = B\ B_2 + B_2\ D_2 = c\ t + \lambda,$$

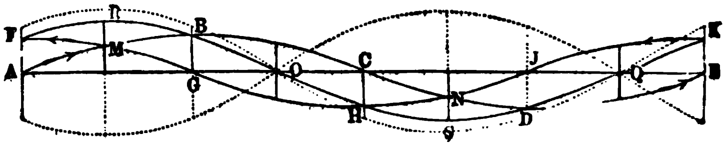
and that of the wave front is

$$D_2\ B_4 = l_2 = D_2\ D_4 - B_4\ D_4 = c\ t - \lambda,$$

in which λ denotes the amplitude of a vibration.

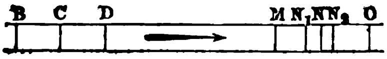
REMARK.—The phenomena accompanying the *interference* of waves can be shown by the aid of the curves of vibration. Let us consider two systems of equal waves, which are advancing in opposite directions, and let *A B C D E* and *F G H I K*, Fig. 890, be the curves, whose ordinates represent the displacements. The displacements of an element, which belongs to two waves, produce a mean displacement, which is determined in exactly the same manner as the resultant of two motions (see § 28), that is, by adding algebraically the two component displacements. Hence at the two points *M* and *N*, where the two curves meet each other, the ordinates are doubled, and, on the contrary, at the points *O* and *Q*, where the curves pass at equal distances from, but on opposite sides of the axis *A E*, the ordinates cancel each other, and the resultant of the two wave curves is a third curve *F R B O H S D Q K*, whose ordinates give the displacements of all the elements in the axis *A E*. While the two systems of waves *A B C* and *F G H* are moving towards each other, the position of the wave-curve *F R B O*, etc., of course changes; but it is easy to understand that the points of no motion *O* and *Q* do not change; for the ordinates of these points of the two component curves are always equal and opposite. These points are called the *nodes*.

FIG. 890.



(§ 16.) **Velocity of Propagation.**—The *velocity of propagation* of waves can be determined in the following manner. Let us imagine the vibrating body *B O*, Fig. 891, to be composed of an infinite number of elements, the cross-section of each being *A* and

FIG. 891.



its length $B\,C = C\,D = d\,x$, and let us assume that the phase of the motion of an element $B\,C = A\,d\,x$ is propagated completely to the following $C\,D = A\,d\,x$ in the elementary time $d\,t$, or that the phases of the motion are propagated in the direction of the axis of the body with the velocity $c = \frac{d\,x}{d\,t}$. Let us assume that the elements $B\,C$ and $C\,D$ oscillate from C to N in the time t, and thus come into the position $M\,N = d\,x_1$ and $N\,O = d\,x_2$, and let us denote the corresponding displacement $C\,N$ by y. If the surface of separation of the two elements, which before $d\,t$ seconds was at N_1, comes after $d\,t$ seconds to N_2, the corresponding spaces described by these elements are

$$N\,N_1 = d\,y_1 \text{ and } N\,N_2 = d\,y_2,$$

and their velocities are

$$v_1 = \frac{d\,y_1}{d\,t} \text{ and } v_2 = \frac{d\,y_2}{d\,t};$$

hence the retardation is

$$p = \frac{v_1 - v}{d\,t} = \frac{d\,y_1 - d\,y_2}{d\,t^2}.$$

Since $d\,t$ seconds before the moment, when the elements $B\,C$ and $C\,D$ occupied the positions $M\,N$ and $N\,O$, N_1 was in the same phase as O now is, we have $C\,N_1 = D\,O$; and since $d\,t$ seconds later N_2 is in the same phase as M, it follows also that $C\,N_2 = B\,M$. From these two equations we obtain

$N_1\,O = D\,O - D\,N_1 = D\,O - (C\,N_1 - C\,D) = C\,D$ and

$M\,N_2 = C\,N_2 - C\,M = C\,N_2 - (B\,M - B\,C) = B\,C$; hence

$N\,N_1 = d\,y_1 = N_1\,O - N\,O = C\,D - N\,O = d\,x - d\,x_2$ and

$N\,N_2 = d\,y_2 = M\,N_2 - M\,N = B\,C - M\,N = d\,x - d\,x_1$.

The element $d\,y$ of the space is equal to the compression $d\,x - d\,x_2$ of the element $N\,O$, and the element $d\,y_2$ of the space is equal to the compression $d\,x - d\,x_1$ of the element $M\,N$. If we denote by E the modulus of elasticity of the vibrating rod, the strains of the elements $M\,N$ and $N\,O$ produced by this compression are

$$S_1 = \left(\frac{d\,x - d\,x_1}{d\,x}\right) A\,E = \frac{d\,y_2}{d\,x} \,.\, A\,E \text{ and}$$

$$S_2 = \left(\frac{d\,x - d\,x_2}{d\,x}\right) A\,E = \frac{d\,y_1}{d\,x} \,.\, A\,E.$$

If we subtract the former from the latter, we obtain the retarding force

$$P = S_2 - S_1 = \left(\frac{d\,y_1 - d\,y_2}{d\,x}\right) A\ E.$$

If γ is the heaviness of the elements $B\ C$, $C\ D$, etc., of the rod, or $A\ d\ x \,.\, \gamma$ the weight, and $\frac{A\ d\ x\ .\ \gamma}{g}$ the mass of such an element, its acceleration at N_1 is

$$p = \frac{P}{M} = \left(\frac{d\,y_1 - d\,y_2}{d\,x}\right) A\ E \,.\, \frac{g}{A\ d\ x\ .\ \gamma} = \frac{g\ E}{\gamma} \,.\, \frac{d\,y_1 - d\,y_2}{d\,x^2};$$

equating the two values of p, we obtain

$$\frac{d\,y_1 - d\,y_2}{d\,t^2} = \frac{g\ E}{\gamma} \,.\, \frac{d\,y_1 - d\,y_2}{d\,x^2}, \text{ whence}$$

$$\frac{d\,x^2}{d\,t^2} = \frac{g\ E}{\gamma}, \text{ or } c^2 = \frac{g\ E}{\gamma};$$

hence the *velocity of propagation of the waves* (velocity of sound) is

$$c = \sqrt{\frac{g\ E}{\gamma}} = \sqrt{g\ L},$$

in which formula L denotes the modulus of elasticity expressed in units of length.

Example.—If we assume the modulus of elasticity of spruce wood to be $E = 1870000$ pounds and the weight of a cubic foot of it to be $= 30$ pounds, we obtain the velocity of propagation in it

$$c = \sqrt{\frac{144\ .\ 1870000}{30}\ .\ g} = \sqrt{48\ .\ 187000\ .\ g} = 17000 \text{ feet},$$

i.e. about 15 times as great as in air.

Remark.—This formula for the velocity of propagation is applicable not only to a stretched string, but also to water and to the air. If p denote the pressure of the air upon the unit of surface, we have, according to Mariotte's law, the tensions corresponding to the ratios of compression $\frac{d\,y_1}{d\,x}$ and $\frac{d\,y_2}{d\,x}$,

$$S_2 = \frac{p\,d\,x}{d\,x_2} = \frac{p\,d\,x}{d\,x - d\,y_1} \text{ and } S_1 = \frac{p\,d\,x}{d\,x_1} = \frac{p\,d\,x}{d\,x - d\,y_2},$$

and, therefore, the motive force upon an element, whose cross-section is A, is

$$P = A\,(S_2 - S_1) = \frac{(d\,y_1 - d\,y_2)\,A\,p\,d\,x}{(d\,x - d\,y_1)\,(d\,x - d\,y_2)};$$

now since $\frac{d\,y}{d\,x}$ is a small fraction, we can put $(d\,x - d\,y_1)\,(d\,x - d\,y_2) = d\,x^2$ and

$$P = \frac{(d\,y_1 - d\,y_2)\,A\,p}{d\,x}$$

This expression agrees exactly with the former one when we substitute p instead of E; hence the *velocity of sound in air* is

$$c = \sqrt{g \cdot \frac{p}{\gamma}}.$$

When the *theory of heat* is discussed in the second volume, it will be shown that a coefficient must be added to this formula in consequence of the change of temperature, which necessarily accompanies the change of density of the air. Since the heaviness of the air is proportional to the pressure p, they both disappear from the formula and the temperature alone remains. We generally assume for air

$$c = 333 \sqrt{1 + 0{,}00367 \cdot \tau} = 1092{,}5 \sqrt{1 + 0{,}00367 \cdot \tau} \text{ feet.}$$

Example.—If (according to the Remark of § 351), when a column of water is compressed by a force of 14,7 pounds, its volume is diminished 0,000050 of its original volume, its modulus of elasticity is

$$E = \frac{14{,}7}{0{,}000050} = 294000 \text{ pounds},$$

and the velocity of sound in water is

$$c = \sqrt{32{,}2 \cdot \frac{294000 \cdot 144}{62{,}425}} = \sqrt{32{,}2 \cdot \frac{1693440}{2{,}497}} = 4673 \text{ feet},$$

or about 4,3 times that in air.

(§ 17.) **Period of a Vibration.**—We can now find the *period of a vibration* by obtaining the equation, which expresses the dependence of the amplitude of the vibration upon the time and upon the abscissa x, which determines the position of the vibrating element when it is at rest. Now y is certainly a function of t as well as of x; we can, therefore, put $y = \phi(t)$ and $y = \psi(x)$.

By differentiating the first equation, we obtain the variable velocity of vibration

$$v = \frac{dy}{dt} = \phi_1(t),$$

and in like manner, by a second differentiation, the corresponding acceleration

$$p = \frac{dv}{dt} = \phi_2(t),$$

in which $\phi_1(t)$ and $\phi_2(t)$ denote other functions of t (compare § 19).

The second function gives the ratio

$$\frac{dy}{dx} = \psi_1(x),$$

which determines the strain; from it we obtain the latter

$$S = A\,E \cdot \frac{dy}{dx} = A\,E \cdot \psi_1(x);$$

hence the motive force of the element of the mass $d\,M = A\,d\,x \frac{\gamma}{g}$ is

$$dS = AE.\frac{d[\psi_1(x)]}{dx} = \frac{AE\psi_2(x)}{dx},$$

and the corresponding acceleration is

$$p = \frac{dS}{dM} = \frac{gE}{\gamma}\psi_2(x),$$

in which $\psi_1(x)$ and $\psi_2(x)$ denote other functions of x.

If we equate the two values of p, we obtain

$$\phi_2(t) = \frac{gE}{\gamma}.\psi_2(x), \text{ or, since } \frac{gE}{\gamma} = c^2,$$

$$\phi_2(t) = c^2.\psi_2(x).$$

The integral of this differential equation is

$$y = \phi(t) = \psi(x) = F(ct + x) + f(ct - x),$$

in which F and f are undetermined functions of the quantities contained in the parentheses; for

$$\phi_1(t) = \frac{d[\phi(t)]}{dt} = cF_1(ct + x) + cf_1(ct - x),$$

$$\phi_2(t) = \frac{d[\phi_1(t)]}{dt} = c^2F_2(ct + x) + c^2f_2(ct - x)$$

$$= c^2[F_2(ct + x) + f_2(ct - x)], \text{ and}$$

$$\psi_1(x) = \frac{d[\psi(x)]}{dt} = F_1(ct + x) - f_1(ct - x) \text{ and}$$

$$\psi_2(x) = \frac{d[\psi_1(x)]}{dt} = F_2(ct + x) + f_2(ct - x),$$

and, therefore, we have really

$$\phi_2(t) = c^2.\psi_2(x).$$

Although the function

$$y = F(ct + x) + f(ct - x)$$

is an indeterminate one, yet, when we have more definite data in regard to the vibrating body, it can be employed to determine the period of the vibrations. A few examples of how this may be done will now be given.

REMARK.—If we eliminate dt from the formulas $dy = v\,dt$ and $dx = c\,dt$, we obtain the expression $\frac{dy}{dx} = \frac{v}{c}$, or since $\frac{dy}{dx}$ expresses the condensation σ of the vibrating element of the body, we have $\sigma = \frac{v}{c}$; the simultaneous condensation at every point of the vibrating rod is proportional to the velocity of vibration of that point.

(§ 18.) **Determination of the Modulus of Elasticity.**—Let us assume that the vibrating body, whose length is l, is *fixed at both ends.* In this case we have not only for $x = 0$, but also for $x = l, y = 0$; hence

$$F(c\,t) + f(c\,t) = 0 \text{ and } F(c\,t + l) + f(c\,t - l) = 0.$$

From the first equation we obtain $f = -F$, which, when substituted in the second equation, gives

$$f(c\,t + l) - f(c\,t - l) = 0, \text{ i.e. } f(c\,t + l) = f(c\,t - l),$$

or, if we put $c\,t - l = c\,t_1$,

$$f(c\,t_1 + 2\,l) = f(c\,t_1).$$

The function, therefore, assumes the same value when $c\,t_1$ is increased by $2\,l$ or when the time is increased by $t_1 = \frac{2\,l}{c}$; hence the *period of a complete vibration* or *double oscillation* is

$$t_1 = \frac{2\,l}{c} = 2\,l\sqrt{\frac{\gamma}{g\,E}}.$$

If, in the second place, we assume the body to be *free at both ends,* we have for $x = 0$ and $x = l$, $S = 0$ and $\psi_1(x) = 0$; hence

$$F_1(c\,t) - f_1(c\,t) = 0 \text{ and } F_1(c\,t + l) - f_1(c\,t - l) = 0.$$

We have, therefore,

$$f_1 = F_1 \text{ and } f_1(c\,t + l) = f_1(c\,t - l), \text{ or } f_1(c\,t_1 + 2\,l) = f_1(c\,t_1),$$

and consequently the period of a complete vibration is

$$t_1 = \frac{2\,l}{c}.$$

If the body is *free at one end and fixed at the other,* we have for $x = 0$, $y = 0$, and for $x = l$, $S = 0$; hence

$$F(c\,t) + f(c\,t) = 0 \text{ and } F_1(c\,t + l) - f_1(c\,t - l) = 0,$$

from which it follows that $f = -F$ and $f_1 = -F_1$, and therefore

$$f_1(c\,t + l) + f_1(c\,t - l) = 0, \text{ or } f_1(c\,t_1 + 2\,l) = -f_1(c\,t_1).$$

We see from the latter formula that the body, after the time $t_1 = \frac{2\,l}{c}$, will assume the opposite state of motion, and that it will consequently make a complete vibration in double that time, $2\,t_1 = \frac{4\,l}{c}$. The period of the complete vibration is, therefore,

$$t_2 = \frac{4\,l}{c} = 4\,l\sqrt{\frac{\gamma}{g\,E}},$$

or double that in the first two cases.

By means of these formulas we can calculate from the period t of a complete vibration or from the number n of vibrations, which a prismatical body makes in a given time, the *modulus of elasticity* $E = \left(\frac{2\,l}{t}\right)^2 . \frac{\gamma}{g}$, and the velocity of propagation or the velocity of sound in it, $c = \frac{2\,l}{t}$.

Example.—An iron wire, which was 60 feet long and was fixed at both ends, was put in longitudinal vibration by means of friction in the direction of its axis. The number of complete vibrations was 1687 in a second; what was the modulus of elasticity of the wire and what was the velocity of propagation in it? According to one of the above formulas, we have for the modulus of elasticity, expressed in units of length,

$$L = \frac{1}{g}\left(\frac{2\,l}{t}\right)^2 = \frac{1}{g}(2\,n\,l)^2 = \frac{(1637\,.\,120)^2}{32{,}2\,.\,12} = 99870000 \text{ inches,}$$

and if a cubic inch of this iron weighs 0,28 pound, the modulus of elasticity, expressed in pounds, is

$$E = 99870000\,.\,0{,}28 = 27960000 \text{ pounds (compare the table, § 212).}$$

The velocity of propagation, or the velocity of sound in it, is

$$c = \sqrt{g\,L} = \sqrt{32{,}2\,.\,99870000\,.\,\tfrac{1}{12}} = \sqrt{16{,}1\,.\,16645000} = 16370 \text{ feet,}$$

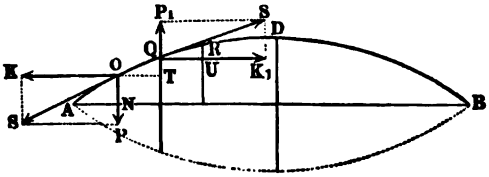
or, assuming the velocity of sound in the air to be $c = 1092$ feet, we have

$$c = \frac{16370}{1092} = 15.$$

If the vibrating wire is very long, the period of a vibration depends upon the length of the wave or upon the distance l between two nodes, and it is always $t_1 = \frac{2\,l}{c}$. This time determines the *pitch of the note* produced by the vibrating wire; the greater or smaller t_1 is, the lower or higher the note is. The intensity of the sound, on the contrary, increases with the amplitude of the vibration. For spherical waves, in which sound propagates itself in air and water, c and t remain unchanged, and it is only the amplitude of the vibration, or the intensity of the sound, which diminishes.

(§ 19.) **Transverse Vibrations of a String.**—The *transverse vibrations* of *a string* or elastic rod can be treated in the same manner as the longitudinal ones. As the simplest case is that of a *stretched string* (Fr. corde; Ger. Saite), we will discuss that first. Let $A\,D\,B$, Fig. 892, be any position of the vibrating string, A and B the two fixed points, $l = A\,B$ the length of the string, G its weight and S the tension, which is to be regarded as constant. Now if $A\,N = x$ and $N\,O = y$ be the co-ordinates of any point O of the string, and if we resolve the tension S at it into two components K and P, one parallel to $A\ B$, and the other perpendicular to it,

FIG. 892.



we can regard the latter as the motive force a one end O of the element $O\ Q$. If the arc $A\ O = s$ is increased by the element $O\ Q = d\ s$, and if the corresponding increase of the ordinate y is $Q\ T = d\ y$, P, S, $d\ y$ and $d\ s$ are the homologous sides of two similar triangles $O\ P\ S$ and $Q\ T\ O$, and we can put

$$\frac{P}{S} = \frac{Q\ T}{O\ Q} = \frac{d\ y}{d\ s}, \text{ or } P = \frac{d\ y}{d\ s} \cdot S.$$

But another force $P_1 = \frac{R\ U}{Q\ R} \cdot S = \frac{d\ y_1}{d\ s}\ S$, which is one of the components of the opposite tension, acts in the opposite direction upon the same element $O\ Q$; hence the motive force, which moves the element $O\ Q$ back to the axis $A\ B$, is

$$P - P_1 = \left(\frac{d\ y - d\ y_1}{d\ s}\right) S.$$

The mass M of this element is proportional to its length $O\ Q = d\ s$; now if we suppose the amplitude y of the oscillation to be small, we can assume that the mass is proportional to the element $O\ T = Q\ U = d\ x$ of the abscissa, or that $M = \frac{d\ x}{l} \cdot \frac{G}{g}$. If we make this assumption, we have the acceleration with which the element approaches its position of rest $A\ B$

$$p = \frac{P - P_1}{M} = \frac{d\ y - d\ y_1}{d\ s\,.\,d\ x} \cdot \frac{g\ S\ l}{G},$$

or, if we put $d\ s = d\ x$,

$$p = \frac{d\ y - d\ y_1}{d\ x^2} \cdot \frac{g\ S\ l}{G}.$$

Now y is some function of x, E.G. $\psi\ (x)$; hence $\frac{d\ y}{d\ x}$ is another function $\psi_1\ (x)$ and $\frac{d\ y - d\ y_1}{d\ x^2} = \frac{d\ d\ y}{d\ x^2} = \frac{d\,[\psi_1\ (x)]}{d\ x}$ is a third function $\psi_2\ (x)$ of this quantity, and

$$p = \psi_2(x) \cdot \frac{g S l}{G}.$$

Since y is also a function of the time t, I.E. $y = \phi(t)$, the velocity with which the element $O\,Q$ returns to its position of rest is $v = \frac{d y}{d t} = \phi_1(t)$, and the corresponding acceleration is

$$p = \frac{d \phi_1(t)}{d t} = \phi_2(t).$$

If we equate these two values of p, we obtain, as in § 17 of the Appendix, the differential equation

$$\phi_2(t) = \psi_2(x) \cdot \frac{g S l}{G} = c^2 \psi_2(x),$$

and we can put here, as we did there,

$$y = \phi(t) = \psi(x) = F(c t + x) + f(c t - x) \text{ and}$$
$$v = c\,[F_1(c t + x) + f_1(c t - x)].$$

Since here also for $x = 0$ and $x = l$, y and $v = 0$, we have again $f = -F$ and $f(c t + l) = f(c t - l)$, or $f(c t_1 + 2 l) = f(c t_1)$; hence the period of a complete vibration or double oscillation is

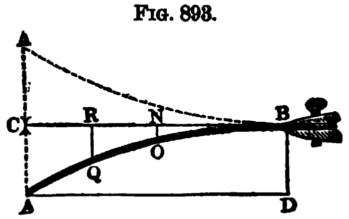
$$t_1 = \frac{2 l}{c} = 2 l \sqrt{\frac{G}{g S l}}, \text{ or, if we put } G = A\, l\, \gamma,$$
$$t_1 = 2 l \sqrt{\frac{A \gamma}{g S}}.$$

The period of vibration of a string is therefore directly proportional to the *length* l and to the *square root of the weight of the unit of length*, and it is *inversely proportional to the square root of the tension* S *of the string.*

EXAMPLE.—Since half the period of the vibration corresponds to that of the next octave, a string will give, according to this formula, the octave of the fundamental tone, when it is shortened one-half or supported in the middle, or when it is stretched four times as much, or when it is replaced by another whose unit of length weighs one-fourth as much as that of the first one.

(§ 20.) **Transverse Vibrations of a Rod.**—The *period of vibration* of an *elastic rod* or *spring* $A\,B$ (Fr. lame; Ger. Stab), Fig. 893, which is fixed at one end, can be determined in the following somewhat circuitous manner. According to § 226, if r denotes the radius of curvature of the rod at a certain point O,

FIG. 893.



determined by the co-ordinates $C N = x_1$ and $N O = y_1$, the moment of flexure of the arc $A\ O = s_1$ is

$$M = \frac{W E}{r}.$$

If we put the force, with which an element Q, which corresponds to the co-ordinates $C R = x$ and $R Q = y$, approaches the axis or position of rest $C B$, $= P\, d x$, or its moment

$= \overline{N R} \,.\, P\, d x = (x_1 - x)\, P\, d x$, we obtain

$$\frac{W E}{r} = \int_0^{x_1} (x_1 - x)\, P\, d x.$$

But $\int_0^{x_1} (x_1 - x)\, P\, d x = \int_0^{x_1} P\, x_1\, d x - \int_0^{x_1} P\, x\, d x$

$$= x_1 \int_0^{x_1} P\, d x - \int_0^{x_1} P\, x\, d x,$$

or, if we put $\int_0^{x_1} P\, d x = P_1$, and therefore

$$\int_0^{x_1} P\, x\, d x = \int_0^{x_1} P\, d x \,.\, x = P_1\, x_1 - \int_0^{x_1} P_1\, d x,$$

$\int_0^{x_1} (x_1 - x)\, P\, d x = \int_0^{x_1} P_1\, d x$; hence we have also

$$\frac{W E}{r} = \int_0^{x_1} P_1\, d x.$$

Now we know that $r = - \frac{d s^3}{d x^2\, d\, (tang.\, a)}$ (see Art. 33 of the Introduction to the Calculus), or, since we can put, when the deflection is small, $d s = d x$,

$$r = - \frac{d x}{d\ (tang.\ a)};\ \text{hence}$$

$$- W E \frac{d\, (tang.\ a)}{d x} = \int_0^{x_1} P_1\, d x,$$

by differentiating which, we obtain

$$- W E \,.\, d \left(\frac{d\ (tang.\ a)}{d x}\right) = P_1\, d x.$$

If we put $y = \psi\ (x)$, $tang.\, a = \frac{d y}{d x} = \psi_1\ (x)$, $\frac{d\, (tang.\, a)}{d x}$

$= \psi_2\ (x)$ and $d \left(\frac{d\ (tang.\ a)}{d x^2}\right) = \psi_3\ (x)$, we obtain the equation

$$P_1 = - W E \,.\, \psi_3\ (x),$$

by differentiating which again, we find

$$d P_1 = - W E\, d\, \psi_3\ (x),\ \text{i.e.}\ P\, d x = - W E\, d\, \psi_3\ (x),\ \text{or}$$

$$P = - W E \frac{d\, \psi_3\ (x)}{d x} = - W E\, \psi_4\ (x).$$

In order that the spring shall vibrate symmetrically, we can assume that P is proportional to y, or that $P = -Ky$; hence we have

$$W E \psi_4 (x) = K y, \text{ or } \psi_4 (x) = \frac{K}{W E} \cdot y = k^4 y,$$

when we denote $\frac{K}{W E}$ by k^4.

This differential equation $\psi_4 (x) = k^4 y$ corresponds to the equation $y = \psi (x) = A \cos. (k x) + B \sin. (k x) + C e^{kx} + D e^{-kx}$; for by successive differentiations we obtain

$$\psi_1 (x) = k \left[- A \sin. (k x) + B \cos. (k x) + C e^{kx} - D e^{-kx}\right],$$
$$\psi_2 (x) = k^2 \left[- A \cos. (k x) - B \sin. (k x) + C e^{kx} + D e^{-kx}\right],$$
$$\psi_3 (x) = k^3 \left[A \sin. (k x) - B \cos. (k x) + C e^{kx} - D e^{-kx}\right], \text{ and}$$
$$\psi_4 (x) = k^4 \left[A \cos. (k x) + B \sin. (k x) + C e^{kx} + D e^{-kx}\right],$$

so that we have really

$$\psi_4 (x) = k^4 y.$$

(§ **21.**) The *period of vibration t* of the *elastic rod* is found, as above, by substituting $p = \phi_2 (t) = \frac{\text{force}}{\text{mass}}$. But the force acting upon an element is

$$= P\, d x = - K y\, d x = - W E k^4 y\, d x,$$

and, when the cross-section is F and the heaviness is γ, the mass is

$$= F\, d x \frac{\gamma}{g}; \text{ hence}$$

$$\phi_2 (t) = - \frac{g W E k^4}{F \gamma} \cdot y = - \mu^2 y,$$

when we denote the expression $\frac{g W E k^4}{F \gamma}$ by μ^2.

This differential equation corresponds to the simple formula

$$y = \phi (t) = \sin. (\mu t + \tau),$$

in which τ expresses any arbitrary time of beginning; for by differentiation we obtain

$$v = \frac{d y}{d t} = \phi_1 (t) = \mu \,.\, \cos. (\mu t + \tau) \text{ and}$$

$$p = \frac{d v}{d t} = \phi_2 (t) = - \mu^2 \,.\, \sin. (\mu t + \tau), \text{ I.E.,}$$

$$\phi_2 (t) = - \mu^2 y.$$

If in the equation $y = \sin. (\mu t + \tau)$ we take $\tau = 0$, we obtain $y = \sin. (\mu t)$; hence for $\mu t = 0, \pi, 2 \pi$, etc., $y = 0$, and consequently

$t_1 = \frac{\pi}{\mu}$ is the period of a simple vibration and

$t = \frac{2\pi}{\mu} = \frac{2\pi}{k^2}\sqrt{\frac{F\gamma}{g W E}}$ is the period of a complete vibration.

In order to calculate the period of a vibration, we must know not only the quantity k, but also the ratio $\frac{F}{W}$

If the rod is cylindrical and its radius $= r$, we have

$$\frac{F}{W} = \frac{\pi r^2}{\frac{1}{4}\pi r^4} = \frac{4}{r^2} \text{ (see § 231)},$$

and if it is a parallelopipedon, whose width is b and whose height is h,

$$\frac{F}{W} = \frac{b h}{\frac{1}{12} b h^3} = \frac{12}{h^2} \text{ (see § 226)}.$$

We have, therefore, for the first rod

$$t = \frac{4\pi}{r k^2}\sqrt{\frac{\gamma}{g E}},$$

and for the second

$$t = \frac{4\pi}{h k^2}\sqrt{\frac{3\gamma}{g E}}.$$

The quantity k is found in the following manner from the equation

$$y = A \cos. (k x) + B \sin. (k x) + C e^{kx} + D e^{-kx}.$$

If we substitute in this formula the corresponding values $x = l$ and $y = 0$, we obtain

1) $0 = A \cos. (k l) + B \sin. (k l) + C e^{kl} + D e^{-kl}.$

If we perform the same operation in the equation

$$tang.\ a = \frac{d y}{d x} = \psi_1 (x), \text{ we obtain}$$

2) $0 = - A \sin. (k l) + B \cos. (k l) + C e^{kl} + D e^{-kl}.$

Since the moment of flexure at the end A of the rod $= 0$ and consequently the radius of curvature $r = \infty$, or $\psi_2 (x) = 0$ and $\phi_2 (x) = 0$, it follows that

$0 = - A \cos. 0 - B \sin. 0 + C e^0 + D e^{-0}$, I.E., $- A + C + D = 0$

and

$0 = A \sin. 0 - B \cos. 0 + C e^0 - D e^{-0}$, I.E., $- B + C - D = 0$,

whence 3) $A = C + D$ and

4) $B = C - D.$

If we eliminate A and B from these four equations, we have

$(C + D) \cos. (k l) + (C - D) \sin. (k l) + C e^{kl} + D e^{-kl} = 0$, and

$- (C + D) \sin. (k l) + (C - D) \cos. (k l) + C e^{kl} - D e^{-kl} = 0;$

from which we obtain by addition

$$C \cos.(k\,l) - D \sin.(k\,l) + C\,e^{k\,l} = 0,$$

and by subtraction

$$D \cos.(k\,l) + C \sin.(k\,l) + D\,e^{-k\,l} = 0, \text{ or}$$

$$C\,[\cos.(k\,l) + e^{k\,l}] = D \sin.(k\,l) \text{ and}$$

$$D\,[\cos.(k\,l) + e^{-k\,l}] = - C \sin.(k\,l);$$

hence we have by division

$$- \frac{\cos.(k\,l) + e^{k\,l}}{\sin.(k\,l)} = \frac{\sin.(k\,l)}{\cos.(k\,l) + e^{-k\,l}}, \text{ whence}$$

$$2 + \cos.(k\,l)\,(e^{k\,l} + e^{-k\,l}) = 0, \text{ or}$$

$$\cos.(k\,l) = - \frac{2}{e^{k\,l} + e^{-k\,l}}.$$

The smallest of the different values, which correspond to the different tones that the rod can give out and which depend upon the number of nodes, is $k\,l = 1,8751$; the greater are, on the contrary, nearly $k\,l = \frac{3\,\pi}{2}, \frac{5\,\pi}{2}, \frac{7\,\pi}{2}$, etc.

If we are required to find from the observed period t of the complete vibration the modulus of elasticity E, we have generally to consider but the smallest value; we must, therefore, put

$$k = \frac{1,8751}{l} \text{ and } k^2 = \frac{3,516}{l^2};$$

hence for a cylindrical rod

$$E = \frac{\gamma}{g}\left(\frac{4\,\pi}{r\,k^2\,t}\right)^2 = \frac{\gamma}{g}\left(\frac{4\,\pi\,l^2}{3,516\,r\,t}\right)^2 = 12,774\,\frac{\gamma}{g}\,\frac{l^4}{r^2\,t^2},$$

and for a parallelopipedical one

$$E = \frac{\gamma}{3\,g}\left(\frac{4\,\pi}{h\,k^2\,t}\right)^2 = \frac{\gamma}{3\,g}\left(\frac{4\,\pi\,l^2}{3,516\,h\,t}\right)^2 = 4,2579\,.\,\frac{\gamma}{g}\,\frac{l^4}{h^2\,t^2}.$$

Remark 1.—If we compare with each other the formulas

$$t = \frac{4\,\pi}{r\,k^2}\sqrt{\frac{\gamma}{g\,E}} \text{ and } t_1 = 2\,l_1\sqrt{\frac{\gamma}{g\,E}}$$

for the transverse and longitudinal vibrations of one and the same rod, we obtain the proportion

$$t : t_1 = \frac{l^2}{r} : \frac{3,516}{2\,\pi}\,l_1, \text{ i.e., } t : t_1 = \frac{l^2}{r} : 0,5596\,l_1.$$

Wertheim found by experiment that this proportion was correct for cast steel and brass.

Remark 2.—The transverse vibrations of an elastic rod are discussed by Seebeck in a "*Abhandlung der Leipziger Gesellschaft der Wissenschaften*," Leipzig, 1849, and also in the "Programme der technischen Bildungsanstalt in Dresden," for the year 1846. Wertheim's experiments upon the elasticity of the metals and of wood by means of transverse and longitudinal vibrations are discussed at length in "Poggendorff's Annalen," Ergänzungsband II, 1845.

Remark 3.—The period of vibration or rather the number of vibrations of a rod in a given time cannot generally be determined directly on account of their rapidity; we must, therefore, employ various artifices to do it. We can determine it either, as Chladni, Savart, etc., did, by the pitch of the note produced by the vibration, or we can employ the method first proposed by Duhamel, which consists in causing the rod to describe by means of a small point a wave-line upon a revolving glass plate, which is covered with lamp-black. A *chronometric apparatus*, to which a *flying pinion*, such as used in the striking works of town clocks, is attached, is employed to produce a regular motion of rotation. An account of this apparatus is to be found in Morin's "Description des appareils dynamometriques, etc., Paris, 1838," as well as in his "Notions fondamentales de mécanique." Wertheim determined the number of vibrations in a given time by allowing another body, such as a tuning-fork, whose number of vibrations was known, to vibrate at the same time with the rod to be examined. If we cause both bodies to trace wave-lines upon the lamp-black and then count the number of waves corresponding to the same central angle, the ratio of these numbers will give the ratio of the numbers of vibrations. The longitudinal vibrations are generally accompanied by small transverse ones; the rod describes, therefore, a corrugated wave-line. By counting the small waves contained in one large wave of the main wave-line, we can easily compare the number of longitudinal vibrations with the number of transverse ones.

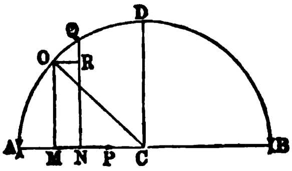
§ 22. **Resistance to Vibration.**—The forces, which cause the vibrations of a body, are very often accompanied by *passive resistances*, whose influence must be examined more particularly. If such a resistance is *constant*, as, E.G., the *friction* of a pendulum *upon its axis* or that of a magnetic needle upon its pivot, it has no influence upon the period of the oscillations, but their amplitude is diminished at every stroke. For the case in § 1 (Appendix), in which the motive force is proportional to the distance x from the position of rest or centre C of the motion $A\ B$, Fig. 894, we can put

$$p = \mu x = \mu (a - x_1),$$

in which x_1 denotes the space $A\ M$ described. If we take into consideration the diminution k of this space, in consequence of the friction, we have, when the body is describing the first half $A\ C$ of its path,

$$p = \mu (a - k - x_1),$$

Fig. 894.



and when it is describing the second half $C\,B$

$$p = -\mu\,[x_1 - (a + k)];$$

the influence of the friction k consists, therefore, in this alone, that for one-half of the path a must be replaced by $a - k$ and for the other by $a + k$, and that the whole space described in one oscillation must be changed from $2\,a$ to $2\,a - 2\,k$, I.E. the amplitude of the oscillation will be diminished a certain quantity $2\,k$ at each oscillation. Finally, since the amplitude does not enter into the formula

$$t = \frac{\gamma}{\sqrt{\mu}},$$

k can have no influence upon the period of the oscillations.

The case is different with the *resistance of the air*. The latter, when the velocities, as in the case of the pendulum, are small, is more nearly proportional to the simple velocity than to its square, as was shown by Bessel's researches upon the length of the simple pendulum (Abhandl. der Akademie der Wissensch. zu Berlin, 1826). This is explained by the fact that this resistance is increased principally by the condensation and rarefaction of the air in front and behind the vibrating body, which increase with the velocity v of the body (see § 510 and Appendix, § 17, Remark). In accordance with this assumption, we can put the acceleration of the vibrating body

$$p = -(\mu\,x + \nu\,v) \text{ or } p + \nu\,v + \mu\,x = 0,$$

when we assume the body to be moving from the point of repose and measure the space from that point.

If we put

$$x = f(t), v = \frac{d\,x}{d\,t} = f_1(t) \text{ and } p = \frac{d\,v}{d\,t} = f_2(t),$$

we can write also $f_2(t) + \nu\,f_1(t) + \mu\,f(t) = 0$, which corresponds to the integral equation

$$x = [b\,cos.\,(\psi\,t\,\sqrt{\mu}) + b_1\,sin.\,(\psi\,t\,\sqrt{\mu})]\;e^{-\frac{\nu\,t}{2}},$$

in which b and b_1 denote constants to be determined and $\psi = \sqrt{1 - \frac{\nu^2}{4\mu}}$. Now for $t = 0$, $x = 0$, whence $b = 0$; hence we have more simply

$$x = b_1\,sin.\,(\psi\,t\,\sqrt{\mu})\;e^{-\frac{\nu\,t}{2}}.$$

Since this value becomes $= 0$, when $\psi\,t\,\sqrt{\mu} = \pi$, the period of an oscillation or simple vibration is

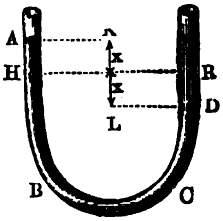
$$t = \frac{\pi}{\psi \sqrt{\mu}} = \frac{\pi}{\sqrt{\mu - \frac{\nu^2}{4}}}, \text{ I.E. } \frac{1}{\psi} = \frac{1}{\sqrt{1 - \frac{\nu^2}{4\,\mu}}}$$

times as great as if the resistance of the air were not present.

REMARK.—It is easy to explain why bodies which are set in vibration make smaller and smaller oscillations and finally come to rest. This effect is due to two causes, the resistance of the air and the imperfect elasticity of the vibrating body; in consequence of the latter fact, the contraction and expansion of the body, particularly within a short space of time, is not proportional to the forces acting upon it.

§ **23. Oscillation of Water.**—The simplest case of the *wave motion of water* is that presented by *its oscillations in two communicating tubes $A\ B\ C\ D$*, Fig. 895. Let us assume that both have the same cross-section, and let us imagine the surface of the water in one leg to be raised a certain distance $H A = x$ above the position it occupies when at rest, and that in the other leg to be depressed an equal distance $R D = x$. We have here the motive force

FIG. 895.



$$P = A \,.\, 2\, x\, \gamma,$$

and if l denotes the entire length $A\ B\ C\ D = H\ B\ C\ R$ of the water, the mass moved is $M = \frac{A\, l\, \gamma}{g}$; hence the acceleration with which the surface of the water rises or falls is

$$p = \frac{P}{M} = \frac{2\, A\, x\, \gamma}{A\, l\, \gamma}\, g = \frac{2\, g\, x}{l}.$$

Since this formula corresponds exactly to the law of oscillation $p = \mu\, x$, discussed in § 1 and § 2 of the Appendix, we have for the period of an oscillation

$$t = \frac{\pi}{\sqrt{\mu}} = \pi \sqrt{\frac{l}{2\, g}}.$$

Since the period of the oscillations of the simple pendulum, whose length is $\frac{l}{2}$, is

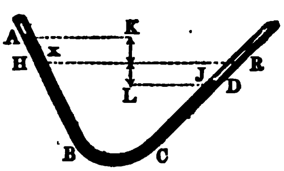
$$t = \pi \sqrt{\frac{l}{2\, g}},$$

the oscillations of the water in the communicating tubes are isochronous with those of this pendulum.

If both legs of the tube $A\ B\ C\ D$, Fig. 896, are inclined, I.E. if the axis of one of the tubes forms an angle α and that of the other an angle β with the horizon, the space $A\,H = D\,R = x$, which the surface of the water describes upwards in one and downwards in the other leg, corresponds to the difference of level

$$z = x \sin. \alpha + x \sin. \beta = x (\sin. \alpha + \sin. \beta)\cdot$$

FIG. 896.



hence the force is

$$P = A\,\gamma\,x\,(\sin. \alpha + \sin. \beta),$$

the acceleration is

$$p = \frac{g\,(\sin. \alpha + \sin. \beta)\,.\,x}{l},$$

and the period of the oscillations is

$$t = \pi \sqrt{\frac{l}{g\,(\sin. \alpha + \sin. \beta)}}.$$

If, finally, the tubes are of *different widths*, the determination of the period of the oscillations becomes much more complicated. Let A be the cross-section and l the length of the middle tube, α_1, A_1 and l_1 the angle of inclination, the cross-section and the length of one lateral tube, and α_2, A_2 and l_2 the angle of inclination, the cross-section and the length of the other; finally, let us suppose that the surface of the water in the axis of one tube has risen a distance x and that the surface of the water in the axis of the other has sunk a distance x_2. We have then

$$A_1\,x_1 = A_2\,x_2, \text{ whence } x_2 = \frac{A_1}{A_2}\,x_1,$$

and the motive force, reduced to A_1,

$$P = A_1\,(x_1 \sin. \alpha_1 + x_2 \sin. \alpha_2)\,\gamma = \frac{A_1\,\gamma}{A_2}\,(A_2 \sin. \alpha_1 + A_1 \sin. \alpha_2)\,x_1.$$

The mass of the water in the middle tube is constant and equal to $\frac{A\,l\,\gamma}{g}$, and, since the ratio of its velocity to that of the force is $\frac{A_1}{A}$, the mass reduced to the point of application is

$$= \left(\frac{A_1}{A}\right)^2 .\,A\,l\,\gamma.$$

The mass of the water in the first leg is

$$= \frac{A_1\,(l_1 + x_1)\,\gamma}{g}, \text{ and that in the second}$$

$$= \frac{A_2\,(l_2 - x_2)\,\gamma}{g}$$

or reduced to the point of application of the force

$$= \left(\frac{A_1}{A_2}\right)^2 \frac{A_2 (l_2 - x_2) \gamma}{g}.$$

Finally the mass moved by P is

$$M = A_1^2 \frac{\gamma}{g} \left(\frac{l}{A} + \frac{l_1 + x_1}{A_1} + \frac{l_2 - x_2}{A_2}\right)$$

$$= A_1^2 \frac{\gamma}{g} \left(\frac{l}{A} + \frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{x_1}{A_1} - \frac{A_1 x_1}{A_2^2}\right)$$

$$= \frac{A_1^2 \gamma}{g} \left[\frac{l}{A} + \frac{l_1}{A_1} + \frac{l_2}{A_2} + \left(\frac{1}{A_1^2} - \frac{1}{A_2^2}\right) A_1 x_1\right],$$

and the acceleration is

$$p = \frac{P}{M} = \frac{\left(\frac{sin.\, a_1}{A_1} + \frac{sin.\, a_2}{A_2}\right) g x_1}{\frac{l}{A} + \frac{l_1}{A_1} + \frac{l_2}{A_2} + \left(\frac{1}{A_1^2} - \frac{1}{A_2^2}\right) A_1 x_1}.$$

If the cross-sections of the two tubes were the same, we would have $A_1 = A_2$, and therefore

$$p = \frac{(sin.\, a_1 + sin.\, a_2)\, g x_1}{\left(\frac{l}{A} + \frac{l_1 + l_2}{A_1}\right) A_1} = \frac{(sin.\, a_1 + sin.\, a_2)\, g x_1}{\frac{A_1 l}{A} + l_1 + l_2}.$$

and the period of the oscillations

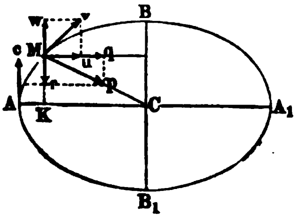
$$t = \pi \sqrt{\frac{A_1 l + A (l_1 + l_2)}{g A (sin.\, a_1 + sin.\, a_2)}}.$$

REMARK.—In consequence of the friction and of the resistance due to the bend in the tube, these formulas must, of course, be modified (compare Appendix, § 25).

§ 24. Elliptical Oscillations.

§ 24. Elliptical Oscillations.—If a body, which is driven with an acceleration $p = \mu z = \mu . \overline{C M}$ towards a fixed point C, Fig. 897, possesses an initial velocity c, whose direction differs from that of the force, the oscillations no longer take place in a straight line, but in an ellipse, as we will now proceed to prove. Let the direction of the motion at the point of beginning A be at right angles to the distance $C A = a$ and let the corresponding velocity be $= c$. If we pass the co-ordinate axes through C, one upon and the other at right angles to

FIG. 897.



$C\,A$, and denote the co-ordinates $C\,K$ and $K\,M$ by x and y, we have for the components q and r of $p = \mu\,z$, which are parallel to the axes, since $\frac{q}{p} = \frac{x}{z}$ and $\frac{r}{p} = \frac{y}{z}$,

$$q = \mu\,x \text{ and } r = \mu\,y.$$

If u and v are the components of the velocity w of the body M, which are parallel to the axis, we have, according to § 1 of the Appendix,

$$u = \sqrt{\mu\,(a^2 - x^2)};$$

and at the same time

$$c^2 - v^2 = z\int r\,d\,y = 2\,\mu\int y\,d\,y = \mu\,y^2, \text{ whence } v = \sqrt{c^2 - \mu\,y^2}.$$

Since for $y = b$, $v = 0$, it follows that

$$0 = c^2 - \mu\,b^2;\ \text{hence } c = b\,\sqrt{\mu} \text{ and } v = \sqrt{\mu\,(b^2 - y^2)}.$$

But now $u = \frac{d\,x}{d\,t}$ and $v = \frac{d\,y}{d\,t}$, and therefore

$$\frac{u}{v} = \frac{d\,x}{d\,y} = \sqrt{\frac{a^2 - x^2}{b^2 - y^2}} \text{ or } \frac{d\,x}{\sqrt{a^2 - x^2}} = \frac{d\,y}{\sqrt{b^2 - y^2}}, \text{ I.E.,}$$

$$\frac{d\left(\frac{x}{a}\right)}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} = \frac{d\left(\frac{y}{b}\right)}{\sqrt{1 - \left(\frac{y}{b}\right)^2}};$$

hence (according to Art. 26, V, of the Introduction to the Calculus)

$$sin.^{-1}\frac{x}{a} = sin.^{-1}\frac{y}{a} + Con.$$

or, since for $x = a$, $y = 0$,

$$sin.^{-1}\frac{a}{a} = sin.^{-1}\frac{0}{b} + Con., \text{ or}$$

$$sin.^{-1}\,1 = sin.^{-1}\,0 + Con., \text{ I.E., } \frac{\pi}{2} = Con. \text{ and}$$

$$sin.^{-1}\frac{x}{a} = sin.^{-1}\frac{y}{b} + \frac{\pi}{2}, \text{ or}$$

$$sin.^{-1}\frac{x}{a} - sin.^{-1}\frac{y}{b} = \frac{\pi}{2}.$$

When the difference of two arcs is $\frac{\pi}{2}$, the sine of one is equal to the cosine of the other, I.E.,

$$\frac{x}{a} = \sqrt{1 - \left(\frac{y}{b}\right)^2}, \text{ or } \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

Since this is the *equation of an ellipse*, it follows that a point, which is impelled or attracted towards C with an acceleration μz, will describe an ellipse, whose semi-axes are $C A = a$ and $C B = b$.

We have also

$$d t = \frac{d y}{v} = \frac{d y}{\sqrt{\mu (b^2 - y^2)}}; \text{ hence the time is}$$

$$t = \sqrt{\frac{1}{\mu}} sin.^{-1} \frac{y}{b}, \text{ or inversely,}$$

$$y = b \, sin. (t \sqrt{\mu}) \text{ and } x = a \, cos. (t \sqrt{\mu}).$$

The time, in which the body will describe a quadrant of the ellipse, is found by putting $y = b$, and it is

$$t_1 = \sqrt{\frac{1}{\mu}} sin.^{-1} \frac{b}{b} = \sqrt{\frac{1}{\mu}} sin.^{-1} 1 = \frac{\pi}{2 \sqrt{\mu}}.$$

The time, in which the body describes half the ellipse, is

$$2 t_1 = \frac{\pi}{\sqrt{\mu}},$$

and the period of a complete revolution or of a complete vibration is

$$4 t_1 = \frac{2 \pi}{\sqrt{\mu}},$$

or exactly the same as it would be, if the motion were a rectilinear reciprocating one. It follows also that

$$u = \sqrt{\mu (a^2 - x^2)} = \sqrt{\mu (a^2 - a^2 [cos. (t \sqrt{\mu})]^2)} = \mu \, a \, sin. (t \sqrt{\mu})$$

and

$$v = \sqrt{\mu (b^2 - y^2)} = \mu \, b \, cos. (t \sqrt{\mu});$$

hence the velocity of revolution is

$$w = \sqrt{u^2 + v^2} = \mu \sqrt{(a \, sin. \, t \sqrt{\mu})^2 + (b \, cos. \, t \sqrt{\mu})^2}.$$

Finally, we can put

$$x = \frac{a + b}{2} cos. (t \sqrt{\mu}) + \frac{a - b}{2} cos. (t \sqrt{\mu}) \text{ and}$$

$$y = \frac{a + b}{2} sin. (t \sqrt{\mu}) - \frac{a - b}{2} sin. (t \sqrt{\mu});$$

now since the first members

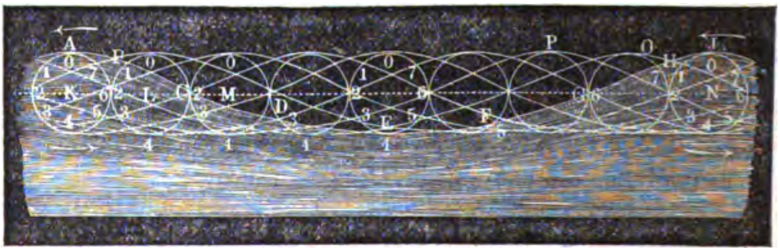
$$\frac{a + b}{2} cos. (t \sqrt{\mu}) \text{ and } \frac{a + b}{2} sin. (t \sqrt{\mu})$$

correspond to a uniform motion in a circle, whose radius is $\frac{a + b}{2}$, and since the two other members correspond to an opposite uniform motion in a circle, whose radius is $\frac{a-b}{2}$, we can also assume that the elliptical motion of the point is composed of two circular ones, I.E., that the point describes uniformly a circle, whose radius is $\frac{a-b}{2}$, while the centre of the latter moves uniformly in a circle, whose radius is $\frac{a+b}{2}$.

If $b = 0$, the oscillation takes place in a straight line, but we can imagine it to be composed of two equal opposite circular motions.

§ 25. Waves of Water.—According to the accurate observations of the Weber brothers, an example of *elliptical oscillation* is presented by the motion of waves of water (Fr. ondes; Ger. Wasserwellen). Not only every particle on the surface, but also every particle below it describes in the wave motion an ellipse. On account of the resistance on the bottom the ellipses below the surface of the water are smaller than those at it, and in general they decrease with the distance from that surface. The different elements in the surface of the water, as well as those in any other plane parallel to it, are at the same moment in different phases of motion; while an element A, Fig. 898, is beginning its path at (0), an element B is already at (1), a second C is at (2), a third D at (3), a fourth E at (4); at this moment the vertical section of the surface of the water is a *cycloidal* or *trochoidal* curve $A\ B\ C\ D\ E\ F\ G\ H\ J$. Before the wave motion began, the elements were at the centres $K, L \ldots N$ of their trajectories and formed the horizontal surface $K\ N$ of the water; during the wave motion, on the contrary, part of the elements are above and part are below this line, and all have, of course, a tendency to return to

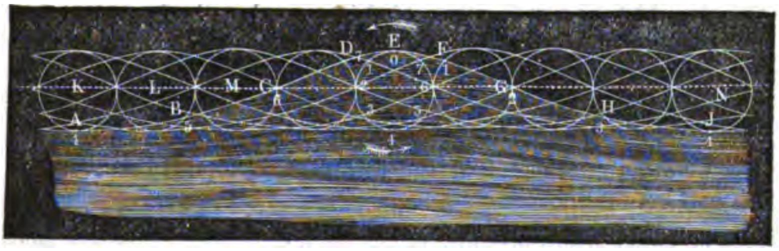
FIG. 898.



their positions of rest K, $L \ldots N$. The oscillations are, however, elliptical so long only as the waves remain unchanged; if they decrease gradually in magnitude, the path of each element becomes narrower and narrower and no longer forms an ellipse, but a spiral line. On the other hand, when the waves are forming or increasing in size, the elliptical trajectory is formed gradually from a spiral line.

After one instant A has moved in its trajectory to (1), B to (2), C to (3), etc., and the wave-form has been moved forward in consequence through the horizontal distance $K L$ between two elements; after a second instant A is at (2), B is at (3), C is at (4), and the wave-form has again advanced the distance $K L = L M$; thus, as the elements of the water revolve, the wave-form advances more and more, and when an element has made a complete revolution, the wave has advanced its own length $K N$. When an element has made half a revolution, as is shown in Fig. 899, the place of the *wave-crest* is occupied by a *trough* or *sinus*, and that of the latter by a crest. This advance of the wave-form does not, of course, consist in any particular motion of the water, but in the forward motion of the same phase, E.G., in the forward motion of the crest J (Fig. 898) of the wave to O, P, etc. If the period of a revolution t of an element of the water and the length $A J = s$ of a wave are known, we can calculate the *velocity of propagation* by means of the formula $c = \frac{s}{t}$.

FIG. 899.



The height of a wave, or the sum of the height of the crest and the depth of the trough is equal to the vertical axis $2\ b$ of the ellipse, in which the elements of the water revolve; the length $C\ G$ of the trough exceeds the half length of the wave by the length $2\ a$ of the horizontal axis of the ellipse, and the length of the crest is, of course, that much shorter than half the wave length. Hence the cross-section of the trough of a wave is larger than that of the wave-crest; now since this is impossible in consequence of the invariability of the volume of the water, the centre of the elliptical trajectory must be somewhat above the surface of the water when it is at rest.

§ 26. Webers' Experiments.—According to Webers' experiments, the path described by a particle of the water at the surface of a wave is a slightly compressed ellipse; according to Emy, on the contrary, the particles of water in sea-waves describe upright ellipses. Both axes of the elliptical path decrease as the depth below the surface increases, and according to Weber the horizontal axis decreases more rapidly than the vertical one. The wave appears not to be propagated in a vertical direction; elements vertically below each other are, according to the observations of the Weber brothers, in the same phase at the same time; on the contrary, those situated in a horizontal line form a complete series of the different phases of the motion. From the experiments cited above, it appears that the period of revolution of an element, or the time in which a wave is propagated its own length, depends principally upon the ratio of the two axes of the path. The greater the ratio of the horizontal axes $2\ a$ to the vertical one $2\ b$, the greater is the period of revolution. The particles, which lie deeper, describe their paths more quickly than those at the surface; from this we must conclude that the wave length diminishes towards the bottom.

The velocity of propagation $c = \frac{s}{t}$ of a wave depends, since the time of revolution t increases with the ratio $\frac{a}{b}$, not only upon the length s, but also upon the height b. If a wave is propagated between two parallel walls, E.G. in a canal, its width remains constant, its height b diminishes and its length increases in such a manner that the only change in the velocity of propagation is that resulting from the friction of the water upon the walls. If, on the contrary, a wave can propagate itself freely in all directions, and if it forms a wall which recedes into itself, its length and width are both increased at the expense of its height, and the wave becomes gradually flatter and flatter until in a short time the eye is no longer able to distinguish it. If such a wave is not originally circular it will gradually approach the circular form as it advances. According to Webers' experiments, the height diminishes in arithmetical progression when the wave advances in geometrical progression. The velocity of propagation of such a wave diminishes gradually, the farther the wave is propagated. If, on the contrary, a wave is propagated from without inwards and is contracted more and more in consequence, its height, length and velocity gradually increase. There is, therefore, a great difference between the waves of water and those of sound. In the latter the velocity of propagation depends upon the elasticity and density of the medium alone; in the former, on the contrary, it is a function of the length and height. If the undulations of the water are produced by a force which acts almost instantaneously, E.G., by the immersion and quick withdrawal of a solid body, the particles of the water describe elliptical paths which gradually decrease, or rather spiral lines, which draw themselves together more and more, and the periods of revolution become smaller and smaller. The origin of a whole series of waves, which become smaller and smaller, is to be attributed to these relations of motion. As the waves are propagated farther and farther, those which follow are increased in size by those which have preceded them, and those most in advance in a short time become so flat as to be invisible. This running together of the waves gives rise to systems of small waves, which present themselves like teeth upon the front surface of the main wave. These small waves or *teeth* advance, according to Poisson and Cauchy, with uniformly accelerated motion.

§ 27. **Hagen's Experiments.**—According to the latest investigations of *Geh. Oberbaurath Hagen* (see the "Seeufer-und Hafenbau von G. Hagen, Berlin, 1863," 1 Vol., which forms the third part of that author's "Wasserbaukunst;" also his treatise upon waves in water of uniform depth; Berlin, 1862), the particles of water of waves in deep water describe with *constant angular velocity* circles, whose diameters decrease as the depth increases, and at the bottom they are infinitely small. A filament of water, which when at rest is vertical, will oscillate, in consequence of the wave motion, backwards and forwards about this vertical line, its base remaining fixed very much as a stalk of wheat is moved by the wind. The line of the wave or the curve which unites the points, which are in the same phase of revolution and which, when the water is at rest, is a straight line, is therefore a *prolate cycloid*, that becomes more and more prolate as the depth increases; at the bottom it is nearly a straight line and at the surface it approaches the common cycloid. From the radius r of the common cycloid, whose value for high sea-waves rises to 50 feet, we obtain the length of the wave $l = 2 \pi r$, its velocity of propagation

$$c = \sqrt{2 g r} = \sqrt{\frac{g l}{r}},$$

the period of a wave

$$t = \frac{l}{c} = \pi \sqrt{\frac{2 r}{g}} = \sqrt{\frac{\pi l}{g}},$$

and the angular velocity with which the molecules of water describe their elliptical paths, $\omega = \frac{c}{r}$.

The centre of the circle, in which a particle which is situated lower down revolves, is determined from the radius z of this circle and from its distance y from the centre of the first circle, whose radius is r, by means of the formula

$$y = r\, l\left(\frac{r}{z}\right).$$

By inversion we obtain $z = r\, e^{-\frac{y}{r}}$, in which $e = 2{,}71828$ denotes the base of the Naperian system of logarithms. We can easily understand from this that the circles of oscillation decrease very rapidly with the depth; for $r = 10$ feet, at the depth $y = 50$ feet, $z = 10 \,.\, e^{-0{,}2} = 3{,}50$ feet, and at the depth $y = 200$ feet, $z = 10 \,.\, e^{-0{,}05} = 0{,}15$ feet.

When the waves are of small constant depth, as Mr. Scott Russel had already remarked, the horizontal motions of the particles of water, which lie above one another, are equally great; the filament of water, which was originally vertical, remains so during the wave motion, but its length and thickness vary. The different particles describe closed curves of equal horizontal diameters and of variable vertical ones, which decreases gradually with the depth; they are, however, ellipses only when we suppose that the height of the wave is infinitely small compared to the depth of the water.

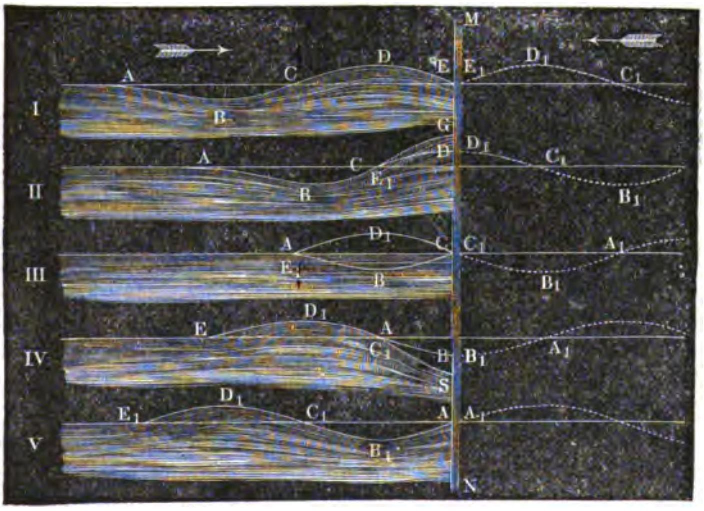
When the depth of the water is finite and the height of the waves is great, the laws of the motion of the waves are very complicated.

§ 28. **Interference of Waves of Water.**—If two *water-waves cross* each other, the same general phenomena occur as in the case of waves of air and other fluids; after they cross each other, each wave continues its motion as if they had not met; but according to Weber's observations, it is accompanied by a small loss of time, so that a wave requires a little more time to pass from one point to another when it passes through another wave than when it is propagated freely. If two *crests come together*, a crest twice as high as the first is produced, and in like manner when *two troughs* meet, a third, twice as deep, is formed. According to Weber's experiments, the ratio of the height of the simple wave to that of the compound one is 1 : 1,79. When two waves *interfere*, or when a wave-crest coincides with a trough of a wave, the two counterbalance each other, and the point where this occurs remains at the same level as the surface of the still water. The paths of the single particles, when two waves meet, become straight lines, which are vertical at the crest, but at a distance from it their positions are such that they are inclined towards the crest.

If a wave of water *impinges* against a *solid wall*, it will be reflected by it as if it came from a point as far behind the wall as that from which the wave started is in front of it, and the reflected wave will pass through the one which is arriving exactly in the same manner as any two waves, which cross each other, do.

In Fig. 900, I, II to V, the phenomena, which are presented

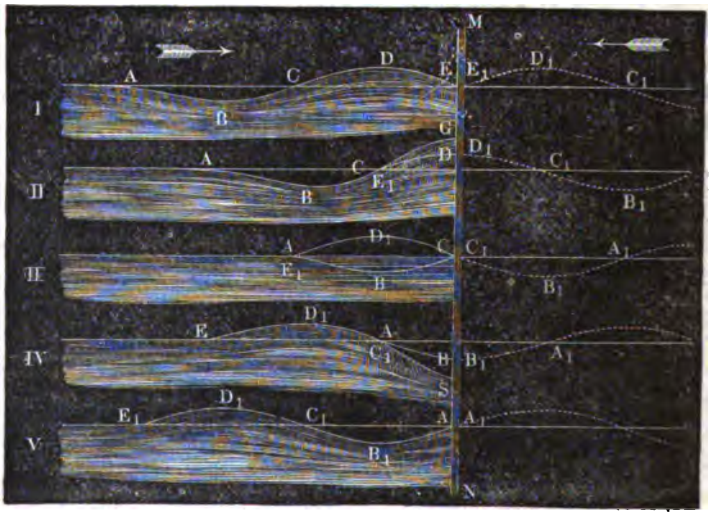
FIG. 900.



when a wave *A B C D E* is reflected by a rigid wall *M N*, are represented. In I the crest *C D E* of a wave is arriving at the wall

$M\ N$ and the reflection begins in the form of a wave $C_1\ D_1\ E_1$; in II the top of the crest D of the wave has arrived at the wall and has combined with the half $D_1\ E_1$ of the reflected crest of the wave; half a crest $C\ G$ of almost double the height is thus produced. In III the trough $A\ B\ C$ of the wave has just reached the wall, while the reflected crest $C_1\ D_1\ E_1$ is passing over it; an interference is thus produced which causes the wave to disappear entirely. In IV the bottom B of the trough of the approaching wave coincides with the bottom B_1 of the trough of the reflected wave; a trough $A\ S$ of double the depth is thus formed. Finally, in V the approaching wave $A\ B\ C\ D\ E$ is reflected completely by the wall $M\ N$ and thus changed into the wave $A_1\ B_1\ C_1\ D_1\ E_1$, which moves in the opposite direction.

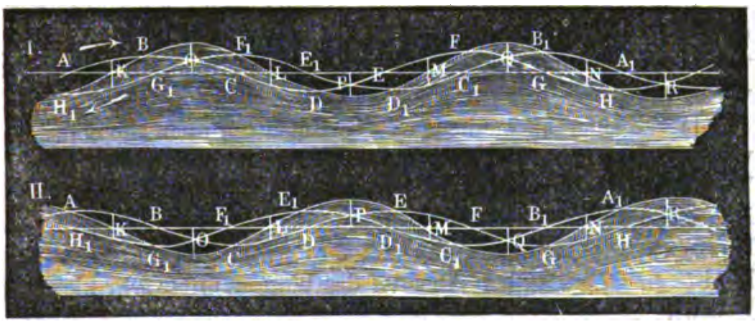
FIG. 901.



When the waves are reflected by a wall, the paths of the molecules undergo the same changes as when two waves cross each other; here also, in the neighborhood of the wall, the horizontal component of this motion is more and more balanced, and, on the contrary, the vertical one is increased more and more, so that near the wall the path becomes a vertical line, and farther from it an inclined one. If the wave strikes obliquely against the wall, it will be reflected, like every elastic body, at the same angle at which it struck. If a wave strikes but partially against an obstacle, the phenomena of *inflexion* are produced, new waves being formed at the extreme ends of the obstacle.

Finally, *stationary waves* of water, like those of a string or any other solid body, are formed when two waves of the same length, which originate at two points situated at a distance apart equal to 1, 3, 5, 7 ... times the fourth part of the length of a wave, cross each other. Let *A B C D E F G H*, Fig. 902, I and II, be one, and A_1 B_1 C_1 D_1 E_1 F_1 G_1 H_1 the other wave. At the points *K, L, M, N,* where the two systems of waves are at the same distance from, but on opposite sides of the middle line, the motions counteract each other and fixed points of interference are produced; on the contrary, above and below the points *O, P, Q, R,* where the two wave-lines cut each other and the paths are therefore doubled, the tops of the crests and the bottoms of the troughs are alternately formed.

FIG. 902.



REMARK.—The most complete treatise upon the motion of waves is the following: "Wellenlehre auf Experimente gegründet, etc.," by the brothers G. H. Weber and W. Weber, Leipzig, 1825. A good abstract of it is contained in the "Lehrbuch der Mechanischen Naturlehre," by August. Müller's "Lehrbuch der Physik und Meteorologie," Vol. I, can also be consulted. The treatises of Laplace, Lagrange, Flaugergues, Gerstner and Poisson are reviewed and criticised in Weber's work. Cauchy's "Wellen-Theorie" and Bidone's "Versuche" are discussed at length in "Gehler's Physikalisches Wörterbuch," Art. "Wellen." Emy's wave theory has been translated by Wiesenfeld and published under the title "Ueber die Bewegung der Wellen und über den Bau am Meere und im Meere," Vienna, 1839. Hagen's work has already been cited, § 27. The theory of water-waves has been treated by Airy in an article upon "Tides and Waves," in the Encyclopædia Metropolitana.

TRANSLATOR'S APPENDIX.

SINCE the last German edition of the present volume was issued the author has published in the "*Civilingenieur*" several articles upon subjects, which have been treated in the foregoing pages. As they contain much valuable information and give the results of a very great number of very careful experiments, a brief abstract of the matter contained in some of them will be given here. Those which will first be noticed are three articles upon the efflux of water, viz.:

(1) the different methods of experimenting upon the efflux of water under a constant head (Die verschiedenen Methoden der Versuche über den Ausfluss des Wassers unter constantem Drucke. X Band, 1 Heft);

(2) experiments upon the efflux of water under a very small head (Versuche über den Ausfluss des Wassers unter sehr kleinem Drucke. X Band, 3 und 4 Heft);

(3) the relations of compound efflux, considered theoretically and illustrated by experiment (Die zusammengesetzten Ausflussverhältnisse theoretisch entwickelt und durch Versuche erläutert. XI Band, 2 und 3 Heft).

Article No. 1 begins with a description of the various methods adopted by different experimenters to maintain a constant head in the main or discharging reservoir. Smeaton returned the water, which was discharged, to the reservoir by a hand-pump and thus maintained the water level constant in the former. Christian employed a large weighted cask, which was suspended by a rope; as the water was discharged from the reservoir, the cask was allowed to sink so as to displace exactly the same quantity of water as had flowed out of the reservoir. In Prony's experiments the escaping water was caught in a vessel, which was connected with two parallelopipedical cases (made of sheet-metal). The latter floated upon the water in the main reservoir, and the apparatus was so arranged that the increase in weight of the vessel caused the floats to displace exactly the same quantity of water as had been discharged. The impulse of the escaping water will interfere with the working of this apparatus, unless proper precautions are taken. Hachette (see his "Traité élémentaire des Machines") passed a hollow tube through the bottom of the reservoir; by sliding the tube up or down the level of the water in the reservoir could be changed. If the volume of the water, which entered the reservoir, exceeded the discharge, the excess escaped over the top of the tube. A slight variation of level, of course, took place. The author tried several different methods of obtaining the same result. The first, which to a certain extent resembles Smeaton's, was to feed the discharging reservoir from the main reservoir by means of a pipe, in which an ordinary *cock* was placed. An assistant is stationed at the cock, by turning which he maintains the surface of the water in the discharging reservoir at a constant level, which is marked by a fixed pointer in the reservoir. The second method he employed was Christian's. He used a hollow float made of sheet-metal; its weight could be regulated by filling it partially with sand. By allowing the float to sink as the water was discharged, the surface of the water was maintained at a constant level, which was indicated by a pointer. The volume of the float gives the discharge. This method is not so accurate as that last described (by means of a cock), and it is not so simple as it appears at first sight; for the size of the float must vary with that of the orifice. The *floating syphon* gives more accurate results than Prony's apparatus, described above. It consists essentially of a T-shaped syphon with two lateral pipes, by which the water enters, and of a larger central pipe, by which it leaves the apparatus. Each of the lateral pipes passes through a water-tight cylinder of sheet-metal, which is open on top and floats upon the water. These two floating cylinders support the syphon; by filling them partially with water we can immerse the inlet orifices of the syphon as deep as we please, and the outlet orifice can be brought to any desired distance below the level of the surface of the water in the reservoir. As the surface of the water in the reservoir sinks, the whole apparatus descends with it, and the head or distance of the outlet orifice below the level of the water remains constant.

The author has also applied the principle of Mariotte's flask to maintaining a constant head, or constant velocity of efflux. The discharging reservoir is a cylindrical vessel, which is provided with two orifices or openings, but which is in all other respects air-tight. One of these openings is in the top and the other is upon the side near the bottom. A tube or pipe, which is open at both ends, fits in the orifice in the top by means of an air-tight ground joint, in which it can slide up and down. The orifice in the side was so arranged that mouth-pieces of various kinds and sizes could be inserted in it. The vessel is first filled with water through the upper orifice and the pipe is then inserted and pushed down a certain distance, depending upon the head we wish to have; the orifice of efflux is then opened and the water in the tube sinks until air begins to pass under the bottom of the tube and rise to the top of the vessel. The head is now constant and is measured by the difference of level between the orifice of efflux and the bottom of the tube. In order to prevent the air, which enters through the tube, from causing too much disturbance, the bottom of the tube is surrounded by a cylinder of wire-gauze. A glass tube, which is open on top, enters the vessel at the bottom and is turned vertical upwards, serves to measure the pressure. The same principle can be applied in another form. An air-tight vessel, which is filled with water, has a pipe inserted in the side near the bottom; this pipe passes below the level of the water in the discharging vessel. Another pipe, which is smaller and is made principally of India-rubber, enters the air-tight vessel near the top, and the other end of it is placed so as just to touch the surface of the water in the discharging reservoir. If the level of the water in the latter sinks, air enters the tube and water is discharged from the air-tight vessel, in consequence of which the surface of the water in the discharging reservoir rises and seals the mouth of India-rubber tube and the flow of water into the main reservoir ceases. The objection to this method is the unsteadiness of the surface of the water, which renders it difficult to measure the head with accuracy. In order to render it more steady *Geh. Oberbaurath* Hagen had two small holes made in the side of the large tube just above the outlet and in addition employed an intermediate vessel.

A series of experiments, made with the aid of the different apparatus just described, gave the following results. The water was discharged through an orifice in a thin plate 1 centimeter in diameter.

TABLE.

Number of the experiment.	Nature of the head.	Description of the apparatus.	Head in meters.	Value of μ.
1	Gradually decreasing.	Author's ordinary apparatus for experiments upon efflux. .	$\begin{cases} h_1 = 0{,}1700 \\ h_2 = 0{,}0500 \end{cases}$	0,6647
2	Constant.	Level maintained by a cock	$h = 0{,}100$	0,6776
3	"	Level maintained by a floating body. . .	"	0,6576
4	"	Level maintained by Mariotte's flask. . .	"	0,6518
5	"	Level maintained by apparatus last described.	"	0,6654
	Average of the above five experiments.			0,6634

By the aid of one of the above-described apparatus, experiments upon efflux with constant influx can be made. The formula to be employed (see page 923) is

$$t = \frac{2\,G}{\mu\,F\sqrt{2g}}\left(\sqrt{h} - \sqrt{h_1} + \sqrt{k}\,l\left(\frac{\sqrt{h}-\sqrt{k}}{\sqrt{h_1}-\sqrt{k}}\right)\right).$$

The discharging reservoir which was used in these experiments was the apparatus represented upon page 927; by means of Mariotte's flask, the discharge per second into the former was maintained constant during each experiment. In these experiments the surface of the water in the discharging reservoir either rose or fell. By preliminary experiments, the coefficients of efflux for the orifices in both vessels were determined.

In the first experiment the surface of the water in the discharging reservoir rose. The observed duration of efflux was $t = 170{,}25$ seconds; that calculated by the above formula from the data given by the experiment was $t = 170{,}5$ seconds.

In the second experiment the surface of the water sank; the observed time was $t = 213{,}2$ seconds, the calculated was 213,9 seconds.

Another case, which often occurs in practice, is that represented in Fig. 776, page 908, when the reservoir $A\ C$ is very large compared to $G\ L$. The water passes from the large reservoir $A\ C$

through a pipe, into the reservoir *G L*, from which it is discharged through the orifice *F* into the air. By prolonging the discharge pipe of Mariotte's flask so that it will reach below the surface of the water in the discharging reservoir (Fig. 792), the level of which surface is variable during the experiment, we obtain an example of this case. The formula for the duration of efflux, which must be employed, is

$$t = \frac{G}{[(\mu\, F)^2 + (\mu_1\, F_1)^2]\, \sqrt{2\, g}} \Big(\mu\, F\Big[2\,(\sqrt{h} - \sqrt{x}) +$$
$$\sqrt{k}\, l\left(\frac{\sqrt{h} - \sqrt{k}}{\sqrt{h} + \sqrt{k}} \cdot \frac{\sqrt{x} + \sqrt{k}}{\sqrt{x} - \sqrt{k}}\right)\Big] + \mu_1\, F_1\Big[2\,(\sqrt{h_1} - \sqrt{y}) +$$
$$\sqrt{k_1}\, l\left(\frac{\sqrt{h_1} - \sqrt{k_1}}{\sqrt{h_1} + \sqrt{k_1}} \cdot \frac{\sqrt{y} + \sqrt{k_1}}{\sqrt{y} - \sqrt{k_1}}\right)\Big]\Big)$$

in which *G* denotes the cross-section of the main discharging reservoir, *F* the area of the orifice in the main reservoir, μ its coefficient of efflux, F_1 the cross-section of the outlet orifice of Mariotte's flask, μ_1 its coefficient of efflux, h_1 the height of the surface of the water in the main reservoir above the orifice in it, *h* the height of the constant water level in Mariotte's flask above the variable one in the main reservoir, *x* what *h* becomes in the time *t*, *y* what h_1 becomes in the time t_1, and $h_0 = h + h_1 = x + y$; *k* is the value of *x*, when the flow becomes permanent, i.e.

$$k = \frac{(\mu_1\, F_1)^2\, h_0}{(\mu\, F)^2 + (\mu_1\, F_1)^2},$$

and

$$k_1 = h_0 - k.$$

In the first experiment the surface of the water in the main reservoir sank; the observed value of *t* was 116,33 seconds and the calculated value was 116,67 seconds. In the second experiment the level of the water rose; the observed time was $t = 157{,}5$ seconds, and the calculated value of *t* was 158,18 seconds.

No. (2.) Experiments upon the Efflux of Water under a very small head.—From previous experiments by the author and others, we know that for an *orifice* in a thin plate one centimeter in diameter,

1, when the head is 103,578 meters, $\mu = 0{,}600$
2, " " 13,574 " $\mu = 0{,}632$
3, " " 0,909 " $\mu = 0{,}641$
4, " " 0,101 " $\mu = 0{,}665$,

and that for a brass tube 1 centimeter in diameter and 2 meters long, the coefficient of resistance

1, when the velocity is $v = 20{,}99$, is $\zeta = 0{,}01690$
2, " " $v = 12{,}32$, is $\zeta = 0{,}01784$
3, " " $v = 8{,}64$, is $\zeta = 0{,}01869$
4, " " $v = 2{,}02$, is $\zeta = 0{,}02725$
5, " " $v = 0{,}57$, is $\zeta = 0{,}03646$;

but we have no experiments which show how the coefficient of efflux increases, when the head is very small (E.G. 1 to 2 centimeters). It is also important to know how ζ increases, when the velocity of the water is very small (E.G. 0,1 meter). In the above-mentioned article the author gives a detailed description of a very extended series of experiments, undertaken for the purpose of discovering the above relations. The discharging reservoir was a wooden trough 2,25 meters long, 0,45 meters wide, and 0,190 meters deep. It was necessary to make the reservoir as long and wide as possible; for the surface of the water could, of course, sink but a very short distance during the experiment. The author then gives a description of the various methods and apparatus employed to determine with accuracy the cross-section of the orifices and the head of water. This portion of the article, although of the greatest interest, would be out of place here.

The table on page 1098 gives the results of the experiments with orifices in a thin plate and with other mouth-pieces.

The temperature of the water was between 15° and 18° Centigrade.

From the 8 experiments with orifices in a thin plate (No 1 to No. 5), whose diameters varied from 0,405 to 2,529 centimeters, we see that the contraction diminishes, when the head is small, as it does when the head is large, not only with the head, but also with the diameter of the orifice.

From the data given in the table on page 1098 and at the beginning of the article, the following table has been arranged.

Head h	0,020	0,101	0,909	13,574	103,578
Coefficient of efflux μ	0,711	0,665	0,641	0,632	0,600

The experiments under Nos. 6 and 7 show that in this case also the coefficient of contraction for an orifice in a thin conically convergent wall is greater than that for an orifice of the same size in a

No. of experiment.	Orifices or mouth-pieces.	Diameter of orifice d.	Cross-section of the orifice F.	Mean head h.	Velocity of efflux v.	Coeffic't of efflux μ.	Coeffic't of resistance ζ.
1	Circular orifices in a thin plate	0,405	0,12882	2,2394	0,6628	0,7197	—
				2,0823	0,6392	0,7287	—
				1,9236	0 6143	0,7410	—
2		1,010	0,80120	2,1907	0,6556	0,7088	—
				1,8504	0,6025	0,7177	—
3		1,408	1,55700	2,14066	0,6451	0,6861	—
4		1,985	3,09465	1,92988	0,6158	0,6657	—
5		2,529	5,02310	2,14915	0,6494	0,6572	—
6	Circular orifice in a conically convergent wall; angle of convergence = 100°	1,020	0,81710	2,17415	0,6531	0,8172	—
				1,86849	0,6047	0,8129	—
7	Circular orifice in a conically divergent wall; angle of divergence = 100°	1,020	0,81710	2,18507	0,6548	0,6418	—
				1,86786	0,6054	0,6519	—
8	Short conical mouth-piece	1,002	0,78850	2,17739	0.6289	0,9638	0,0765
				1,86568	0,5789	0,9589	0,0675
9	Cylindrical ajutages 3 times as long as wide	0.348	0,09940	2,2528	0,4582	0,6894	1,1041
				2,0637	0,4408	0,6895	1,1087
				1,9177	0,4215	0,6874	1,1164
10		0,485	0,18475	2,1787	0,4896	0,7499	0,7781
				1,8580	0,4506	0,7478	0,7885
				1,5855	0,4067	0,7416	0,8185
11		0,728	0,41625	2,1571	0,5277	0,8118	0,5178
				1,8290	0,4753	0,7945	0,5844
				1,5102	0,4168	0,7671	0,6995
12		1,014	0,80755	2,16678	0,5196	0,7988	0,5698
				1,81791	0,4761	0,7591	0.5659
13		1,402	1,54380	2,0907	0,5148	0,8016	0,5371
				1,6155	0,4867	0,7802	0.6427
14		1,927	2,91640	2,8126	0,5859	0,7958	0,5812
				1,9240	0,4701	0,7711	0,6818
15		2,487	4,85780	3,2251	0,6288	0,7678	0,6112
				2,0046	0,4645	0,7496	0,7797
16	Cylindrical ajutages, rounded off at the inlet orifice	1,014	0,80755	2,17680	0,5424	0,8814	0,4466
				1,86104	0,4965	0,8285	0,4747
17	Oblique ajutage, angle of deviation $\delta = 30°$	1,005	0,79325	2,14241	0,4460	0,6850	1.1068
				1,71523	0,3940	0,6808	1,1576
18	Interior ajutage	1,010	0,80120	2,17470	0,4657	0,7203	0,9274
19	Short conically convergent ajutage, angle of convergence = 7° 9′	1,004	0,79170	2,20780	0,6621	0,9162	0,1918
				1,90412	0,5525	0,9108	0,2056
20	The same, rounded off at the inlet, cylindrical at the outlet orifice	1,012	0,80440	2,19454	0,6048	0,9225	0,1751
				1,89447	0.5600	0,9041	0,2233
21	Long double conical mouth-piece	0,966	0,73290	2,20438	0,5941	0,9047	0,2218
				1,92460	0,5498	0,8942	0,2506
22	The same, shorter	1,404	1,54820	2,08075	0,5687	0,8918	0,2589
23	Short conically divergent tube, angle of divergence = 3° 24′	1,034	0,83970	2,19809	0,3605	0,5490	2,3072
				1,87685	0,3358	0,5547	2,2502
24	The same, wider, angle of divergence = 8° 4′	1,541	1,86510	2,11440	0,2607	0,4065	5,0521
25	The same, rounded off at the inlet orifice	1,518	1,80960	2,09467	0,3836	0,5993	1,7844

thin plate, and that it is less for an orifice in a conically divergent wall than for the latter. In experiment No. 18 a free contracted stream could not be obtained. The efflux took place with a filled tube and the stream pulsated quite violently.

It was also observed that the discharge was not increased as much by rounding off the inlet orifices of the ajutages, when the head was small as when it was great.

The table on page 1100 contains the results of experiments with long tubes made of glass, brass and zinc. Preliminary experiments were made to determine the coefficients of resistance of the inlet and outlet mouth-pieces combined. By subtracting the coefficients thus found from those obtained for the long tube and inlet and outlet mouth-pieces together, the author deduced the coefficient of resistance for the tube alone.

These experiments showed the coefficient of resistance ζ of the water to be very great, when the velocity is small. This coefficient ζ is nearly the same for glass and brass tubes.

To the table

for $v = 20{,}99$, $\zeta = 0{,}01690$
" $= 12{,}32$, $\zeta = 0{,}01784$
" $= 8{,}64$, $\zeta = 0{,}01869$
" $= 2{,}02$, $\zeta = 0{,}02725$
" $= 0{,}485$, $\zeta = 0{,}03453$,

we can now add

for $v = 0{,}2028$, $\zeta = 0{,}0587$
" $= 0{,}0890$, $\zeta = 0{,}1420$.

The third portion of the article is devoted to an account of a series of experiments upon the flow of water through bends and elbows under a very small head. The coefficient of resistance for the inlet and outlet portion was first determined as in the experiments, the results of which are given in the last table. The table on page 1101 contains the coefficients of resistance for the flow of water through elbows and bends under small heads.

We see from the last table that the coefficients of resistance of elbows are much greater than those for bends of the same diameter, when both cause the direction of the motion of the water to change 90° and when the radius of curvature of the axis of the bend is equal to the diameter of the tube.

The third article (No. 3), which is cited above, is very long, covering 68 columns of the Civilingenieur. As it would be impossible to condense the matter contained in it in the limited space

	Description of the tubes.		d Diameter of orifice. Centimeters.	h Mean head. Centimeters.	v_1 Mean velocity of the water in the tube. Meters.	ζ Coefficient of friction.
I	Glass tube No. 1	Length l_1 = 158,7 centimeters, Mean width d_1 = 0,29714 centimeters, Fall = 0,25 centimeters, Temperature of the water τ = 14° c.	0,387	5,53096 5,41190 5,29138 5,18291	0,0755 0,0730 0,0729 0,0708	0,3540 0,3708 0,3680 0,3769
	Glass tube No. 2	l_1 = 157,5 ct. m., d_1 = 0,47074 ct. m., Rise = 0,455 ct. m., τ = 18°.	0,483	3,6985 3,3650	0,1258 0,1128	0,1817 0,1496
	Glass tube No. 3	l_1 = 182,0 ct. m., d_1 = 0,71710 ct. m., Fall = 0,055 ct. m., τ = 17°.	0,723	3,2099 2,8847 2,5614	0,2094 0,1873 0,1661	0,05604 0,06886 0,07808
	Glass tube No. 4	l_1 = 203,5 ct. m., d_1 = 1,06598 ct. m., Fall = 0,78 ct. m., τ = 16°,5.	1,012	2,8744 2,3815	0,2351 0,2075	0,04165 0,04500
	Glass tube No. 5	l_1 = 1,70,6 ct. m., d_1 = 1,4302 ct. m., 1. Fall = 0,73 ct. m., τ = 18°,5, 2. Fall = 0,16 ct. m., τ = 15°,2.	1,402	2,8065 2,2867 1,8692 1,5088	0,2851 0,2212 0,1984 0,1718	0,04090 0,06078 0,06579 0,06747
II	Brass tube No. 1	l_1 = 200 ct. m., d_1 = 1,0878 ct. m. 1. Fall = 0,72 ct. m., τ = 13°,3, 2. Rise = 0,92 ct. m., τ = 16°,5.	1,012	2,8708 2,4754 2,6672 2,2346 1,3064 1,0881	0,1984 0,1757 0,2122 0,1896 0,0960 0,0819	0,06768 0,07099 0,04977 0,05269 0,13243 0,15171
	Brass tube No. 2	l_1 = 298,1 ct. m., d_1 = 1,4336 ct. m., Fall = 0,54 ct. m., τ = 18°,5.	1,402	2,5440 1,9178	0,2286 0,1838	0,03872 0,04434
III	Zinc tube No. 1	l_1 = 342 ct. m., d = 3,32633, Fall = 0,02 ct. m., τ = 15°,5.	3,315	3,3549	0,3600	0,03434
	Zinc tube No. 2	l_1 = 311 ct. m., d_1 2,4799, Rise = 0,34 ct. m., τ = 18°,5.	2,445	2,2259	0,2418	0,04251
	Zinc tube No. 3	l_1 = 673,8 ct. m., d_1 = 2,49908, Fall 0,86 ct. m., τ = 18°,5.	2,445	3,0894	0,3076	0,04316
	Zinc tube No. 4	l_1 = 1016 ct. m., d_1 = 2,49498 ct. m., Rise 0,82 ct. m., τ = 19°.	2,445	2,5940	0,1451	0,05187

Num'r of the experiment.	Nature of the tubes.	d Diameter of the orifice in centimeters.	h Mean head in centimeters.	v Mean velocity of the water in meters.	ζ Coefficient of friction for the elbow or bend.
1	Short cylindrical ajutage, rounded off inside with an *elbow* of 90° and another cylindrical mouth-piece beyond the elbow.	1,012	2,2105 1,9058	0,3160 0,2898	2,5137 2,6315
2	The same with a bend of 90°, instead of an elbow.	1,012	2,1762 1,7774	0,4227 0,3681	0,5657 0,7444
3	Two bends, one 90° and the other 180°, with a cylindrical ajutage, intermediate pipe and outlet pipe.	1,012	2,1828 1,7456	0,3285 0,2884	2,0928 2,3786
4	A wider elbow } with ajutage and mouth-piece like No. 1.	1,402	2,1994 1,9017	0,3290 0,3025	2,2349 2,3173
5	A wider bend } with ajutage and mouth-piece like No. 1.	1,402	2,1384 1,7288	0,4204 0,3658	0,6324 0,7332
6	A simple zinc tube, with bend at the end (90°). Fall = 0,23 ct. m.	2,445	2,3259	0,2338	0,2608
7	" " " double bend at the end (180°). Fall=0,725 ct. m.	2,445	2,7958	0,2508	0,6734
8	" " " rectangular elbow (90°). Fall = 0,35 ct. m.	2,445	2,4910	0,2268	1,4566

which is at our disposal, we will content ourselves with an enumeration of the subjects treated. They are—

(1.) The simultaneous discharge of water through two orifices, when the head diminishes.

(2.) The variable discharge of water from one vessel into a second, in which the orifice is submerged, while a constant quantity of water is continually discharged into the first vessel.

(3.) The variable efflux of water through a notch, either with or without influx.

(4.) Efflux of water from a prismatical vessel, with free influx into the latter from another prismatical vessel.

(5.) Efflux of water from a prismatical vessel, with influx under water from another prismatical reservoir.

These cases are treated at length; the formulas are first deduced and then tested by very careful experiments. Any one interested in the subject of hydraulics will find this article worthy of his most attentive perusal.

We would also call attention to the following articles by the author upon subjects connected with hydraulics.

"Hydrometric experiments upon the application of the formulas of Daniel Bernouilli (page 804) and Charles Borda (page 884), as well as upon the use of a new water-meter; also upon the friction of water in conical pipes and upon the play of jets d'eau" ("Hydrometrische Versuche über die Anwendung der Formeln von Daniel Bernouilli und Charles Borda, so wie über den Gebrauch eines neuen Wassermessers (einer Wasseruhr); ferner über die Reibung des Wassers in conischen Röhren und über das Spiel von springenden Wasserstrahlen," Civilingenieur, Band XIII, 1 Heft). "Comparative hydrometric measurements by means of a tachometer, a large rectangular orifice of efflux and a large overfall extending across the whole wall," ("Vergleichende hydrometrische Messungen mittels eines hydrometrischen Flügelrades, einer grösseren rectangulären Ausflussmündung und eines grösseren über die ganze Wand weggehenden Uberfalls," Civilingenieur, Band XIII, 5 and 6 Heft).

The latter article contains an account of Schwamkrug's *water-divider*, mentioned upon page 986.

I. "The quicksilver differential piezometer and its application to the determination of the difference of the pressure of the water in a set of conduit pipes."

II. "The water piezometer with a micrometer, as well as its application to the determination of the pressure of gas in pipes, etc."

III. "A supplement to the article cited above upon the different methods of experimenting upon efflux under a constant head." ("I. Das Quecksilber-Differentialpiezometer, etc. II. Das Wasserpiezometer mit Mikrometer, etc. III. Eine Ergänzung der Abhandlung über die verschiedenen Methoden der Ausflussversuche unter constantem Drucke." Civilingenieur, Band XV, 2 Heft.)

The translator would also call attention to two articles by the author upon "experimental mechanics," which form a part of a yet unpublished work upon that subject. The titles of the articles are:

(1.) "Experiments to accompany lectures upon the elasticity and strength of solid bodies" ("Versuche bei Vorträgen über Elasticität und Festigkeit fester Körper," Civilingenieur, Band IX, 5 Heft), and

(2) "Experiments to accompany lectures upon Mechanics" ("Versuche bei Vorträgen über Mechanik," Civilingenieur, Band XIV, 6 Heft).

The first article contains a description of the apparatus used by the author in experimenting before the students at Freiberg upon *flexure* and *torsion*. By means of this apparatus, which is very simple and easily constructed, the professor can show to the class almost all the phenomena of flexure and torsion. He can also determine the moduli of rupture and of elasticity not only by observing the deflection and angle of torsion, but also by allowing the body to be experimented upon to vibrate and counting the number of vibrations. The modulus of resilience and that of fragility can also be determined. No. (2) contains an account of some modifications of the above apparatus, by means of which experiments upon the theory of couples (including their composition and decomposition) can be made. This is followed by the description of a simple reversable pendulum, by means of which the value of g can be determined in the lecture-room with little difficulty. The author then takes up the subject of the elasticity of rigid bodies. He discusses four cases of double flexure: first, that of a prismatical rod of a rectangular cross-section, bent by a force, whose direction forms an angle δ (which is not 90°) with one of the sides of the cross-section; secondly, that when the cross-section of the rod is a right-angled triangle and the direction of the force is perpendicular to the base of the triangle; thirdly, that when the rod is acted upon by two forces, whose lines of action do not lie in the same plane; and fourthly, that when the beam is bent in the shape of an elbow and loaded at the extreme end with a weight (the crank is an example of this case). The article closes with an account of some experiments with compound girders.

Those engaged in teaching will find the last two articles full of valuable information; but a translation of them would occupy too much space here.

In conclusion, we would mention an article upon "the flexure of a homogeneous prismatical measuring rod, supported in two points, as well as the shortening of its length, produced by it, discussed in as elementary a manner as possible" ("die Biegung eines in zwei Punkten unterstützten homogenen prismatischen Messstabes, sowie die durch dieselbe hervorgebrachte Verkürzung seines Längenmaasses, auf möglichst einfache Weise ermittelt von Julius Weisbach," Civilingenieur, Band XII, 4 Heft).

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